Zero Represents Impossibility From the Viewpoint of Division by Zero

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Abstract: In this note, by using an elementary property of reproducing kernels, we will show that zero represents impossibility from the viewpoint of the division by zero.

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1 Introduction

In this note, by using an elementary property of reproducing kernels, we will show clearly that zero represents impossibility from the viewpoint of the division by zero. In particular, it seems that impossibility is not referred to clearly in connection with zero.

2 Essences of division by zero

We will show very elementary results on zero and so, in order to state the results in a self contained way, we state first the essences of division by zero.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will define

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [3, 4].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

3 Statement of results

We recall the fundamental of reproducing kernels. In general, a complex-valued function $k : E \times E \to \mathbb{C}$ is called a *positive definite quadratic form function* on an abstract set E, or shortly, *positive definite function*, when it satisfies the property that, for an arbitrary function $X : E \to \mathbf{C}$ and for any finite subset F of E,

$$\sum_{p,q \in F} X(p)\overline{X(q)}k(p,q) \geq 0.$$

Then we obtain the fundamental result:

For any positive definite quadratic form function $k : E \times E \to \mathbb{C}$, there exists a uniquely determined reproducing kernel Hilbert space $H_k = H_k(E)$ admitting the reproducing kernel k whose characterization is given by the two properties: (i) $k(\cdot,p) \in H_k$ for any $p \in E$ and, (ii) for any $f \in H_k$ and for any $p \in E$, $(f(\cdot), k(\cdot.p))_{H_k} = f(p)$.

The properties (i) and (ii) are called the reproducing property of the function $k(p,q)$ in the Hilbert space $H_k = H_k(E)$.

For the realization of the space $H_k = H_k(E)$ in terms of k we have many methods ([1, 2]).

As we see immediately from the reproducing property, if $k(p,p) \neq 0$, then the function

$$F_p(q) = \frac{k(q,p)}{k(p,p)} \tag{3.1}$$

is the uniquely determined extremal function in the space H_k that minimizes the norms among the functions normalized as taking the value one at p.

Meanwhile, if $k(p,p) = 0$, then $k(\cdot,p) \equiv 0$ and so, for any function $f(q)$ of H_k, $f(p) = 0$.

Now, we shall consider (3.1) in this case. If $k(p,p) = 0$, then $k(q,p) \equiv 0$ and so by the division by zero, we see that

$$F_p(q) = 0. \tag{3.2}$$

Apparently this result does not satisfy the condition $F_p(p) = 1$ and furthermore, in the space H_k there is no function satisfying $f(p) = 1$. Therefore, by (3.2) we can understand that **zero represents impossibility from the viewpoint of the division by zero.**

Even the fundamental result

$$\frac{1}{0} = 0,$$

we will be able to understand that it will mean that impossibility. With this sense, we can consider that **the division by zero is the discovery of the new meaning of zero.**

Indeed, for the mapping

$$W = \frac{1}{z},$$

we will consider the point at infinity as the image of the origin $z = 0$, by continuity, however, the point at infinity does not exist in the sense of intuitive one, and so it is represented by 0. To find the corresponding point for the origin $z = 0$ by the mapping is **impossible**. We can consider so.

Meanwhile, from the formula $a - a = 0$, zero represents void and empty. Zero will have deep and interesting properties and meanings, since zero represents even infinity in a sense. See [3, 4]. See also [5] for some general ideas for zero.

References

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