An Inconsistency

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Abstract. This paper proves an inconsistency within ZFC by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be deduced.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_n \) denote the natural numbers starting from \( n > 1 \) and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): Every even integer greater than 6 can be expressed as the sum of two different primes.

Theorem. Both SSGB and the negation \( \neg \text{SSGB} \) hold.

Proof. We define the set \( S_g := \{ (p_k, m_k, q_k) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \).

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( m_k \) of \( S_g \). So, by the definitions we have

\[
\text{SSGB} \iff \forall x \in \mathbb{N}_4 \exists (p_k, m_k, q_k) \in S_g \quad x = m.
\]

\[
\neg \text{SSGB} \iff \exists x \in \mathbb{N}_4 \forall (p_k, m_k, q_k) \in S_g \quad x \neq m.
\]

There are the following two properties of the set \( S_g \).

First, the whole range of \( \mathbb{N}_3 \) can be expressed by the triple components of \( S_g \) ("covering"), because every integer \( x \geq 3 \) can be written as some \( p_k \) with \( k = 1 \) when \( x \) is prime, as some \( p_k \) with \( k \neq 1 \) when \( x \) is composite and not a power of 2, or as \( (3 + 5)k / 2 \) when \( x \) is a power of 2; \( p \in \mathbb{P}_3, k \in \mathbb{N} \). So we have

\[
\text{(C)} \quad \forall x \in \mathbb{N}_3 \exists (p_k, m_k, q_k) \in S_g \quad x = p_k \lor x = m_k = 4k.
\]

Second, due to the definition of the set \( S_g \), all pairs \( (p, q) \) of distinct odd primes are used ("maximality"). So we have

\[
\text{(M)} \quad \forall p, q \in \mathbb{P}_3, p < q \forall k \in \mathbb{N} \quad (p_k, m_k, q_k) \in S_g, \text{ where } m = (p + q) / 2.
\]

In case of \( \neg \text{SSGB} \) there is at least one \( n \in \mathbb{N}_4 \) that is different from all the numbers \( m \) defined in \( S_g \). In case of SSGB there is no such \( n \).
If (C) or (M) did not hold, an \(n\) different from all \(m\) could exist for the reason that \(n\) is different from all \(S_g\) triple components \(p_k, m_k, q_k\) or for the reason that \(n\) is the arithmetic mean of a pair of primes not used in \(S_g\). This would not lead to a contradiction. However, since both (C) and (M) hold, these two possibilities are ruled out and we can proceed.

The following steps work regardless of the choice of \(n\) if there is more than one.

We split \(S_g\) into two complementary subsets: For any \(y \in \mathbb{N}_3\), \(S_g = S_g+(y) \cup S_g-(y)\), where
\[
S_g+(y) := \{ (p_k, m_k, q_k) \in S_g | \exists k' \in \mathbb{N} \quad p_k = yk' \lor m_k = yk' \lor q_k = yk' \}\]
\[
S_g-(y) := \{ (p_k, m_k, q_k) \in S_g | \forall k' \in \mathbb{N} \quad p_k \neq yk' \land m_k \neq yk' \land q_k \neq yk' \}.
\]

Let \(n \in \mathbb{N}_4\) be given by \(-\text{SSGB}\) as above. Then, we have
\[-\text{SSGB} \implies S_g = S_g+(n) \cup S_g-(n).
\]

More precisely, under the assumption \(-\text{SSGB}\) with the associated \(n\) the set \(S_g\) can be written as the union of the following triples.

(i) \(S_g\) triples of the form \((p_k = nk', m_k, q_k)\) with \(k = k'\) in case \(n\) is prime, due to (C)

(ii) \(S_g\) triples of the form \((p_k = nk', m_k, q_k)\) with \(k \neq k'\) in case \(n\) is composite and not a power of 2, due to (C)

(iii) \(S_g\) triples of the form \((3k, 4k = nk', 5k)\) in case \(n\) is a power of 2, due to (C)

(iv) all remaining \(S_g\) triples of the form \((p_k = nk', m_k, q_k), (p_k, m_k = nk', q_k)\) or \((p_k, m_k, q_k = nk')\)

and

(v) \(S_g\) triples of the form \((p_k \neq nk', m_k \neq nk', q_k \neq nk'), i.e. those \(S_g\) triples where none of the \(nk'\) equals a component.

So, \(S_g+(n)\) is the union of the triples of the above types (i) to (iv) and \(S_g-(n)\) is the union of the triples of type (v).

Since \(S_g = S_g+(n) \cup S_g-(n)\) is deduced from \(-\text{SSGB}\) and since \(S_g+(n) \cup S_g-(n)\) is independent of \(n\), we obtain
\[
(1) \quad -\text{SSGB} \implies S_g = S_g+(y_1) \cup S_g-(y_1), \text{ for any } y_1 \in \mathbb{N}_3,
\]
where the consequent is deduced from the antecedent.
Under the assumption SSGB there is no $n$ as above. Therefore, under this assumption, we can choose an arbitrary $y_2 \in \mathbb{N}_3$ such that $S_g = S_g + (y_2) \cup S_g - (y_2)$. That is, we obtain

(2) $SSGB \implies S_g = S_g + (y_2) \cup S_g - (y_2)$, for any $y_2 \in \mathbb{N}_3$,

where the consequent is deduced from the antecedent.

On the other hand, after defining

$S_1 := \{(pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds}\}$

$S_2 := \{(pk, mk, qk) \in S_g \mid SSGB \text{ holds}\}$,

since $S_g$ is non-empty we have

(3) $\neg SSGB \iff S_g = S_1$

(4) $SSGB \iff S_g = S_2$.

Using (3) and (4) in (1) and (2), we get

(1') $S_g = S_1 \implies S_g = S_g + (y_1) \cup S_g - (y_1)$, for any $y_1 \in \mathbb{N}_3$

(2') $S_g = S_2 \implies S_g = S_g + (y_2) \cup S_g - (y_2)$, for any $y_2 \in \mathbb{N}_3$,

where, in accordance with (1) and (2), in (1') the consequent is deduced from $S_g = S_1$ and in (2') the consequent is deduced from $S_g = S_2$.

Therefore, from (1') we conclude

(1'') $S_1 = S_g + (y_1) \cup S_g - (y_1)$, for any $y_1 \in \mathbb{N}_3$

and from (2') we conclude

(2'') $S_2 = S_g + (y_2) \cup S_g - (y_2)$, for any $y_2 \in \mathbb{N}_3$.

Since $S_g + (y) \cup S_g - (y) = S_g$ for every $y \in \mathbb{N}_3$, from (1'') and (2'') we obtain $S_1 = S_2$.

Thereby, (3) and (4) imply $\neg SSGB \iff SSGB$, which yields the contradiction $(SSGB \text{ and } \neg SSGB)$. □
**Note.** The proof is based on the following general principle.

Suppose there are non-empty sets A, B1, B2 and a proposition P such that

(1) \( \neg P \implies A = B1 \)

(2) \( P \implies A = B2 \),

where the consequents are deduced from the antecedents.

Suppose further \( B1 = B2 \). Then, we have a contradiction for the following reason.

After defining

\[
A1 := \{ a \in A \mid \neg P \text{ holds} \}
\]

\[
A2 := \{ a \in A \mid P \text{ holds} \},
\]

since A is non-empty we have

(3) \( \neg P \iff A = A1 \)

(4) \( P \iff A = A2 \).

Using (3) and (4) in (1) and (2), we get

(1') \( A = A1 \implies A = B1 \)

(2') \( A = A2 \implies A = B2 \),

where the consequents are deduced from the antecedents.

From (1') we conclude \( A1 = B1 \) and from (2') we conclude \( A2 = B2 \).

Since \( B1 = B2 \), we obtain \( A1 = A2 \).

Then, (3) and (4) imply \( \neg P \iff P \), which yields \( P \text{ and } \neg P \).