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Study on the Kinetics of a Special Particle Swarm

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For randomly-moving-particle swarm, the past researches only focused on its whole behavior and few people have studied the special particle swarm formed in it, which leading to the phenomenon and reasons for the spontaneous aggregation of particles in the special particle swarm being still unknown. For such a special particle swarm, we have previously studied the causes of its special relativity phenomenon. Here we show the causes of spontaneous aggregation of "randomly moving" particles. The diffusion kinetics of particles in a special circumstance (that is, in a moving reference frame $\mathcal{R}_u$ relative to the stationary reference frame $\mathcal{R}_0$) are studied theoretically. For the first time, the effects of the location aggregation and velocity direction aggregation of randomly moving particles on the diffusion coefficient are considered, and the corresponding generalized diffusion equation is deduced employing concise mathematical logic and Mathematica package.

Keywords: Randomly Moving Particles; Effects of Location Aggregation; Non-diffusion Particle Swarm; Generalized Diffusion Equation

1. Introduction

The kinetics of randomly moving particles have been extensively studied in the past. However, previous studies have been based on the case in which the means (velocity and density) of the particles in the target (sub) domain are equal to those in the total (parent) domain (Fig. 1) or the particle swarm in the sub- and parent domains are not distinguished[1, 2]. In fact, there are some special subparticle swarms with low probabilities in the particle swarm that are formed by random moving particles. For example, during a certain period, the sub-particle swarm ($\mathcal{R}_u$) with a constant velocity relative to the parent particle swarm belongs to this category (Fig. 1). These special subparticle swarms are accidental phenomena for the particles in the parent domain, but for the observers near these subparticle swarms, they are determined "gifts" from nature (survivor bias). These cases are also the more common existences we see and are meaningful to human beings (if the whole universe is regarded as composed of very small particles, the solar system in the Milky Way, the earth in the solar system and the atoms on the earth are similar to this kind of phenomenon). Therefore, it is necessary to study particle swarms in common but special cases.

These special particle swarms, as a portion of the total particle swarm in a completely random state, may be in a variety of different states. In a certain period and a fixed target domain (the volume is fixed and the location can move with the target particle swarm, the same as below), when a subparticle swarm is in a completely random (free) state, the location distribution of the particles in that state follows the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The velocity direction distribution is also consistent with the population (the norm of the average velocity follows the same Maxwell distribution). When a subparticle swarm remains in a special accidental state for a certain period, it is equivalent to the subparticle swarm being subject to some constraints and being in a non-completely-random state. According to the constraint situation of the subparticle swarm, we divide it into the following three types of constrained states. For the first

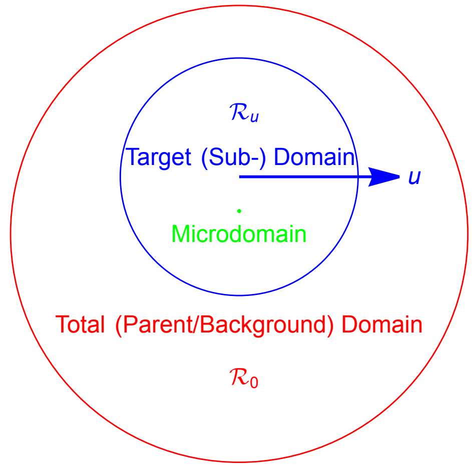

Figure 1. Relationship between the Total (Parent/Background) Domain (Red), Target (Sub-) Domain (Blue) and Microdomain (Green).

kind of constrained state, in a certain period and a fixed target domain, the location distribution of the particles follows a Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location, but the norms of the average velocities do not follow the Maxwell distribution. The special case of this state is that the average velocity norms of all counted particles are constant at u under the unchanged location distribution condition, which is called $\mathrm{I}u$ (Figure 2a). For the second kind of constrained state, in a certain period and a fixed target domain, the norms of the average particle velocities follow the Maxwell distribution, but the location distribution of the particles in the domain does not follow the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The special case of this state is that the number of particles in the fixed target domain is fixed under the condition that the velocity direction distribution remains unchanged. For the third kind of constrained state, in a certain period and a fixed target domain, the norms of the average particle velocities do not follow the Maxwell distribution, and the location distribution of the particles in the domain does not follow the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The special case of this state is that the number of particles is fixed and the average velocity norm of all particles is fixed as u in the fixed target domain, which is called $\mathrm{III}u$ (Fig. 2b). The abovementioned subparticle swarm ($\mathcal{R}_u$) with a constant average velocity during a certain period belongs to $\mathrm{III}u$.

When a subparticle swarm in the constrained state of $\mathrm{III}u$ ($\mathcal{R}_u$ or the target domain) is observed in the total domain ($\mathcal{R}_0$), it has the characteristics of location aggregation and velocity direction aggregation, which affect the diffusion rate constant of the particles. Therefore, the kinetic phenomena of this kind of particle swarm show some special properties. This article focused on the particle swarm in the constrained state of $\mathrm{III}u$, deduced the diffusion equation of the particles in this case and identified the formation conditions of a non-diffusion particle swarm. The basic structure of the whole article is as follows. Based on the defined physical model, the mathematical model was deduced step by step. Before the derivation, two verifications were carried out. First, it was confirmed that the physical model contained special relativistic effects; second, the Schrödinger equation was derived from the physical model under certain conditions. The process of the two checks also clarified how to derive the mathematical model, that is, the generalized diffusion equation. The process of deriving the generalized diffusion equation includes: (**i**) Vector decomposition. The decomposition of nonmoving

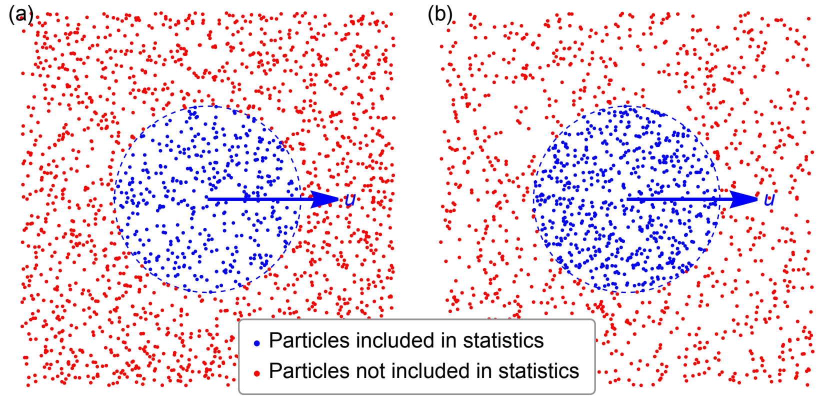

Figure 2. Relationships between the target (sub-) particles/domain and the total particles/domain. **a**, The constrained state of $\mathrm{I}u$: the number of blue particles follows the Poisson distribution based on time with the same strength as the Poisson distribution of the red particles based on location. **b**, The constrained state of $\mathrm{III}u$: the number of blue particles is fixed.

particles in space is extended to the decomposition of a 2-dimensional vector representing the sum of the 3-dimensional vector of moving particles at a certain point in space, which is the core of the whole derivation. (**ii**) The classic diffusion coefficient is reinterpreted and the essential key information is obtained. (**iii**) On the basis of (**i**) and (**ii**), the equations are assembled according to the classical diffusion principle to obtain the generalized diffusion equation. In addition, some important parts related to the equation are discussed and verified. The following is a detailed description.

2. Methods

In this article, the mathematical model was obtained by logical derivation based on the physical model. Mathematica 12.3.1.0 for Mac (*Wolfram Research Inc.*) was used for all of the mathematical calculations, and the hardware was a Mac mini (Z12P) with a macOS Monterey 12.0.1 operating system. The solutions to each specific problem can be found in the Supplementary Information.

3. Results and Discussions

3.1. Physical Model

It is assumed that there are countless identical point particles with certain masses in infinite 3-dimensional space. Their speed is c, the motion directions of each particle are evenly distributed in 3-dimensional space, and there is no interaction between these particles. Our research object is a subset of such particles. The particles in this subset are in the special case of the third kind of constrained state (i.e., $\mathrm{III}u$, the blue domain in Fig. 2b).

3.2. Special Relativistic Effects in the constrained state of $\mathrm{I}u$

In this article, the "point particles" described above are called "particles" or "1-particles", while larger finite-mass-level particles composed of k particles are called "k-particles". The k-particles or aggregates mentioned in this section are k-generalized-particles or aggregates. The k-particle term means that only k particles are counted, but it does not matter whether they truly gather together. The 1-particles can be represented by random vectors with equal norms that are equal to the same movement speeds in Euclidean space. Thus, the "random vectors" and "randomly moving particles (or velocities)" mentioned in this article have the same meaning.

My previous study[3] has proven that the vector group in the constrained state of $\mathrm{III}u$ formed by random vectors with equivalent norms has a special relativistic effect. That is, because of the statistical effect, when the centroid of the subparticle swarm moves at a speed of u in one direction, the particles or the generalized k-particles formed by the subparticles either lose a certain degree of freedom in other directions or the movement trends in other directions decrease, resulting in the effect of special relativity. Here, the slowing ratio $\dfrac{\sqrt{c^2-u^2}}{c}$ of the particles in $\mathcal{R}_u$ or generalized aggregates they form is recorded as $\Gamma[\cdot]$ or Γ (we call it the Γ, or $\Gamma[\cdot]$, effect). Although the particles in $\mathcal{R}_u$ are in the constrained state of $\mathrm{I}u$ when observed from $\mathcal{R}_0$, they are in a completely random state when observed from $\mathcal{R}_u$. Moreover, my previous study[3] has confirmed that all the physical laws are the same as when studying a k-generalized-particle in $\mathcal{R}_0$ observed from $\mathcal{R}_0$ and in $\mathcal{R}_u$ observed from $\mathcal{R}_u$. In the constrained state of $\mathrm{I}u$, the particles themselves or the generalized particles formed by the particles show the effect of special relativity; in the constrained state of $\mathrm{III}u$, the aggregation effect also includes the situation of location aggregation (But they are not related to each other). Here, these two (aggregation) effects combined with the simultaneous effects of the velocity direction and location aggregation are collectively called the statistical effect of random moving particles; such particles are in the constrained state of $\mathrm{III}u$. When these statistical effects work together, the generation conditions of a non-diffusion particle swarm can be obtained. This will be explained in detail below.

3.3. Establishment of the Classical Diffusion Equation in the constrained state of $\mathrm{I}u$

Regardless of how these particles move in 3-dimensional space, their trajectories are continuous, which will lead to diffusion (or agglomeration) behavior, which is the generalized diffusion of randomly moving particles in the constrained state of $\mathrm{III}u$. Considering particles of the same mass and speed, the generalized diffusivity of the corresponding random vectors is equivalent to the generalized diffusivity of random momenta (which are also vectors). It is considered that the scale of the "generalized diffusivity of vectors" is simply the scale that is most suitable for describing the invariant laws for randomly moving particles. More information will be lost if the scale is even slightly more macroscopic (e.g., the scale can be approximately described by real diffusion), and there will be no invariant statistical law to follow if the scale is even slightly more microscopic (for example, the scale described at the beginning of this paragraph). At this scale, the external behavior of the vectors in a tiny space cannot be considered isotropic. Before studying the particles in the constrained state of $\mathrm{III}u$, we first study the particles in the constrained state of $\mathrm{I}u$. For the time being, the Γ effect is not considered here; this is consistent with the scenario of a completely free state. Compared with the $\mathrm{III}u$ case, there is only diffusion without agglomeration, and other cases are consistent. According to the Maxwell distribution, the total vector in a certain domain always points in an uncertain direction, and the norm is directly proportional to $\sqrt{k}$, where k is the number of vectors (see Part 1 of the Supplementary Information for details). Although the direction of the total vector in a tiny space cannot be determined from the Maxwell distribution, we hope to use appropriate constraints to obtain the distribution rules governing the norm and direction of the total vector at any location in space.

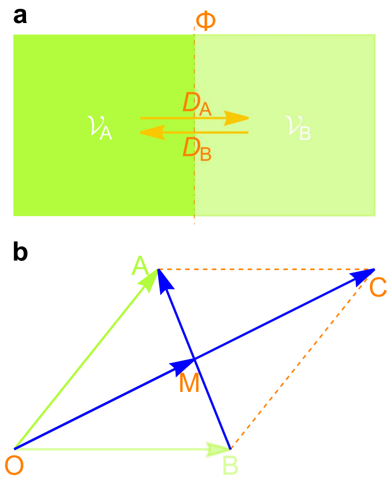

Figure 3. Illustration of the principle of the generation of a mutual diffusion potential in microdomains $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$.

First, we determine the constraints acting on spatial vectors (norms and directions). Let the density of the vector sum at some point $\mathcal{P}$ in space be denoted by $\boldsymbol{\mathcal{X}}$, which is a function of location and time, namely, $\boldsymbol{\mathcal{X}}(x, y, z, t)$. It is defined as follows: At a certain time t, let $\boldsymbol{\mathcal{Y}}(\mathcal{V})$ be a function of the sum of all vectors in the closed domain $\mathcal{V}$ containing $\mathcal{P}(x, y, z)$; then, $\boldsymbol{\mathcal{X}}(x, y, z, t) = \lim\limits_{\mathcal{V}\to\mathcal{P}} \dfrac{\boldsymbol{\mathcal{Y}}(\mathcal{V})}{\mathcal{V}}$ [in the following, $\boldsymbol{\mathcal{X}}$ is also a function of the spatial coordinates (x, y, z) and the time coordinate t].

$\boldsymbol{\mathcal{X}}$ is a statistical average vector. The relationship between $\boldsymbol{\mathcal{X}}$ and the number of vectors follows the Maxwell distribution. As illustrated in Fig. 3a, it is assumed that there are two microdomains $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$ of the same size along the normal direction on both sides of the segmentation surface Φ. If the sum of all vectors in $\mathcal{V}_\mathrm{A}$ is $\overrightarrow{\mathrm{OA}}$ and the sum of all vectors in $\mathcal{V}_\mathrm{B}$ is $\overrightarrow{\mathrm{OB}}$, then their sum is $\overrightarrow{\mathrm{OC}}$, and their difference is $\overrightarrow{\mathrm{BA}}$. Let the sum and difference vectors intersect at point M (Fig. 3b). Due to the characteristic that the velocity direction distribution is homogeneous and there is no need to consider the statistical effects owing to location aggregation here, considering the previous assumption that the domains $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$ on both sides of Φ are equal, after the particles randomly move and mix, both vectors must tend to approach their average value $\overrightarrow{\mathrm{OM}}$; that is, both $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ tend toward $\overrightarrow{\mathrm{OM}}$. The change rate of $\overrightarrow{\mathrm{OA}}$ or $\overrightarrow{\mathrm{OB}}$ to $\overrightarrow{\mathrm{OM}}$ depends on the difference between $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ and the diffusion (motion) rate of particles. Accordingly, the rate of change in $\boldsymbol{\mathcal{X}}$ along the normal direction at a particular point should be related to the time-dependent rate of change in $\boldsymbol{\mathcal{X}}$. This time-dependent rate of change is also affected by another inherent factor (i.e., the velocity of the particles forming $\boldsymbol{\mathcal{X}}$), the concrete value of which is temporally uncertain. Therefore, the above two rates of change should be directly proportional when the differences between particles caused by density (location aggregation of particles) are neglected.

In view of the similar calculus properties of vector and scalar, the derivation method for real diffusion is imitated here. If a domain $\mathcal{W}$ is enclosed by a closed surface Σ, then during the infinitesimal period $\mathrm{d}t$, the directional derivative $\dfrac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{N}}$ of $\boldsymbol{\mathcal{X}}$ along the normal direction of an infinitesimal area element $\mathrm{d}S$ on the surface Σ is directly proportional to the vector $\mathrm{d}\boldsymbol{\mathcal{X}}$ flowing through $\mathrm{d}S$ along the normal direction in the closed domain $\mathcal{W}$ enclosed by Σ (Fig. 4), under the assumption that the coefficient is a positive real number D.

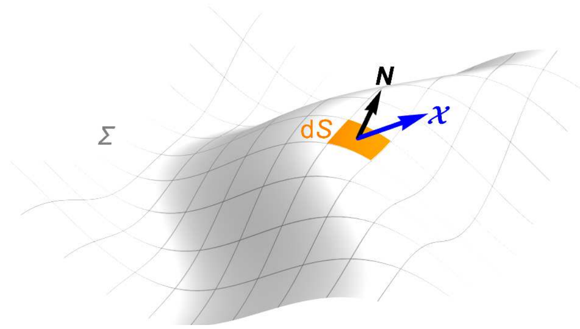

Figure 4. Illustration of the diffusion of the vector sum density $\boldsymbol{\mathcal{X}}$.

From time t_1 to time t_2, when the influence of the vector density on D is not considered (i.e., the diffusion coefficient is the same at every location), the variation of the vector sum $\boldsymbol{\mathcal{A}}$ inside the closed surface Σ is

$$\delta\boldsymbol{\mathcal{A}} = \int_{t_1}^{t_2}\left(\oiint_{\Sigma} D\frac{\partial\boldsymbol{\mathcal{X}}}{\partial\boldsymbol{N}}\mathrm{d}S\right)\mathrm{d}t. \tag{1}$$

According to the Gauss formula, Eq. 1 can also be written in the form

$$\delta\boldsymbol{\mathcal{A}} = \int_{t_1}^{t_2}\left(\iiint_{\mathcal{W}} D\Delta\boldsymbol{\mathcal{X}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)\mathrm{d}t, \tag{2}$$

where Δ is the Laplace operator, which describes the second derivative with respect to the location (x, y, z). The left-hand side of Eq. 1 (namely, $\delta\boldsymbol{\mathcal{A}}$) can also be written as

$$\delta\boldsymbol{\mathcal{A}} = \iiint_{\mathcal{W}}\left(\int_{t_1}^{t_2}\frac{\partial\boldsymbol{\mathcal{X}}}{\partial t}\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3}$$

By setting the right of Eq. 3 equal to the right of Eq. 2 and transforming the order of integration, we can obtain

$$\int_{t_1}^{t_2}\iiint_{\mathcal{W}}\frac{\partial\boldsymbol{\mathcal{X}}}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t = \int_{t_1}^{t_2}\iiint_{\mathcal{W}} D\Delta\boldsymbol{\mathcal{X}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t. \tag{4}$$

Based on the observation that t_1, t_2 and the domain $\mathcal{W}$ are all arbitrary, the following equation can be written:

$$\frac{\partial\boldsymbol{\mathcal{X}}}{\partial t} = D\Delta\boldsymbol{\mathcal{X}}. \tag{5}$$

To facilitate the task of vector decomposition in the following (the constrained state of IIIu), a 3-dimensional vector needs to be converted into a plane vector. Next, we determine the constraints acting on plane vectors. Although the operation in Eq. 5 is performed using 3-dimensional vectors, when differential operations are performed on a spatial vector, the (sum or) difference operations are always performed at two points on the vectors that are separated by an infinitesimal distance; thus, all 3-dimensional vectors can exhibit only relative 2-dimensional characteristics. Consequently, by solving this differential equation, only 2-dimensional constraints can be obtained. Therefore, only the derivatives of plane vectors are needed to act as the derivatives of the 3-dimensional vectors (in this case, plane vectors can retain the important information, such as the norms of the vectors and the included angle between them). Moreover, according to Sturm-Liouville theory, the function of plane vectors obtained by solving the partial differential equation expressed in terms of plane vectors is unique and corresponds to the 3-dimensional vectors obtained from a differential equation of the same form. It is assumed that the function of plane vectors describing the density of the vectors or momenta is $\boldsymbol{\mathcal{M}}(x, y, z, t)$, which corresponds to $\boldsymbol{\mathcal{X}}$ at the point (x, y, z, t) [unless otherwise stated, in the following, $\boldsymbol{\mathcal{M}}$ is a function of the spatial coordinates (x, y, z) and the time coordinate t]. Thus, the abovementioned $\boldsymbol{\mathcal{X}}$ can be replaced with $\boldsymbol{\mathcal{M}}$. After this replacement, it is obvious that the norm of the plane vector will not change, but its direction will be reoriented. Finally, Eq. 5 can be written as

$$\left|\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t}\right| = D\left|\Delta \boldsymbol{\mathcal{M}}\right|. \tag{6}$$

Now, let us determine the constraints on the direction of the plane vector $\boldsymbol{\mathcal{M}}$. In view of the continuity of the trajectories of infinitesimal particles, since $\boldsymbol{\mathcal{M}}$ is also characterized in terms of the statistical properties of an enormous number of particles, it should also be smooth. According to the theory of plane curves, the first and second derivatives of a plane vector in any direction in space are vertical. If an equation relating these derivatives is established following the above derivative relationship (Eq. 6), the direction needs to be adjusted to be consistent (otherwise the equations cannot be equal); then, the unique and definite relationship can be written in the form

$$\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = \mathbf{i}\, D\Delta \boldsymbol{\mathcal{M}}, \tag{7}$$

where $\mathbf{i}$ is the imaginary unit. By multiplying both sides of Eq. 7 by $\mathbf{i}$, the form of the Schrödinger equation (without an external field) can be obtained:

$$\mathbf{i}\,\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = -D\Delta \boldsymbol{\mathcal{M}}. \tag{8}$$

Eq. 8 describes the distribution of a moving particle swarm (including the direction of movement) in the constrained state of $\mathrm{I}u$ (not considering the Γ effect) or in a completely free state following the same diffusion coefficient; in other words, it is the classical (vector) diffusion equation. When u is small, the constrained state of $\mathrm{I}u$ can also be approximated to a completely free state (the Γ effect can be ignored). However, when u is large or there is both a location-constrained state (namely, the constrained state of $\mathrm{III}u$), the effect on diffusion is not clear. To more comprehensively describe this kind of diffusion process (which is called generalized diffusion), further analysis is needed.

3.4. Construction of the Generalized Diffusion Equation in the constrained state of $\mathrm{III}u$

To construct the generalized diffusion equation in the constrained state of $\mathrm{III}u$, we need to take into account many aspects, including whether the generalized diffusion coefficient *Đ* should vary and how to describe it to include the characteristics of the two kinds of constrained states.

When particles are in the constrained state of $\mathrm{I}u$ (not considering the Γ effect) or in a completely free state, they follow a diffusion equation with the same diffusion coefficient (the Schrödinger equation).

However, when such particles are in the constrained state of $\mathrm{III}u$, the effect of location aggregation on $Đ$ should be taken into account, and $Đ$ should vary with the value of the target vector. Suppose that, as illustrated in Fig. 3a, the vector sum density in the microdomain $\mathcal{V}_{\mathrm{A}}$ is greater than that in $\mathcal{V}_{\mathrm{B}}$. If both cases are in the constrained state of $\mathrm{III}u$, there is a greater consumption of degrees of freedom for the higher density in $\mathcal{V}_{\mathrm{A}}$. In terms of probability, less uncertainty is introduced into the unit volume, which inevitably affects the (average) particle movement speed. Therefore, the overall particle movement speed in $\mathcal{V}_{\mathrm{A}}$ decreases. As mentioned above (or in Eq. 21 below), the particle speed is what determines D; therefore, the law governing the diffusion rate towards the right (D_{A}) is not the same as the law governing the diffusion rate in $\mathcal{V}_{\mathrm{B}}$ towards the left (D_{B}) (under the assumption that $Đ$ is a combination of D_{A} and D_{B}). Therefore, it is necessary for the generalized diffusion coefficient to vary in time with the vector sum density to reflect this inequality.

In view of the above considerations, choosing the appropriate quantitative function to describe this phenomenon (with different laws) is the main problem to be solved in this article. First, the sum of momentum vectors in the microdomain is decomposed as follows.

3.4.1. Vector Decomposition

First, let us determine the distribution function for a certain number of nonmoving particles with equal probability (randomly) distributed in a certain domain, as follows: Suppose that the whole domain contains n particles in total. For convenience of description, the whole domain is also partitioned into n boxes of equal size. The gaps between boxes and the wall thickness are both 0. Now, let us determine the probability of k ($k \in \mathbb{N}_+$; the same is done below) particles in a local area containing $\mathcal{M}$ boxes (suppose that the particles are small enough to fall into the box, not the wall). In view of the statement described above, the probability of particles existing in each domain is the same. Accordingly, the total number of possible cases describing how n particles can be randomly distributed among n boxes is n^n, there are $\dbinom{n}{k}$ total ways that k particles can be randomly chosen from among n particles, there are $(n-\mathcal{M})^{n-k}$ total ways in which the k chosen particles can be randomly distributed among $\mathcal{M}$ boxes, and there are $\mathcal{M}^k$ total ways in which the remaining $n-k$ particles can be randomly distributed among the remaining $n-\mathcal{M}$ boxes. Therefore, the probability $P(\mathcal{M},k)$ of k particles existing in $\mathcal{M}$ boxes can be expressed as

$$P(\mathcal{M},k) = \frac{\dbinom{n}{k}\mathcal{M}^k\,(n-\mathcal{M})^{n-k}}{n^n}. \tag{9}$$

Suppose that the number n of particles in the whole domain is infinite; then, by taking the limit of Eq. 9 as $x \to +\infty$, we find that

$$P(\mathcal{M},k) = \frac{\mathrm{e}^{-\mathcal{M}}\mathcal{M}^k}{k!}, \tag{10}$$

again, where $\mathcal{M}$ denotes the number of boxes comprising the local domain of interest (the size of the volume in 3-dimensional space), k denotes the number of particles in that domain of $\mathcal{M}$ boxes, and P denotes the probability that k particles exist in that domain. Eq. 10 is the (location-based) Poisson distribution.

It is considered that this is the most appropriate method of partitioning a whole domain (the domain can be the whole universe or simply a broad range including the objects of investigation) into uniform boxes with the same number as that of particles. In addition to reducing the parameters involved and facilitating discussion, the reasons are as follows: if the boxes are slightly larger, they will not ensure the accuracy of the following vector decomposition; if they are slightly smaller, they will not adequately reflect the grouping effect of the particles. Therefore, in this article, the whole domain is divided into a number of uniform boxes equal to the number of particles it contains, and this partitioning serves as the basis for all of the following discussions. In this article, the whole domain (environment) is called the T-domain (it is the sub domain of sub domain in Fig. 1), and the local domain (target) is called the S-domain; the set of all particles contained in the T-domain is called the T-particle swarm (it is the sub particle swarm of sub particles in Fig. 2), and the subset of particles contained in the S-domain is called the S-particle swarm.

Next, we will investigate the equiprobability distribution of the nonmoving particle swarm in the abovementioned S-domain $\mathcal{V}$. In Eq. 10, $\mathcal{M}$ denotes the number of boxes (volume) spanned by some S-domain (which belonged to the domain in which the target particles are distributed). Put another way, when the T-domain is partitioned into uniform boxes following the above method, $\mathcal{M}$ can also denote the average relative density of the particles in the S-domain $\mathcal{V}$, where the reference density is the average density of the T-particle swarm in the T-domain. $\mathcal{M}$ represents the corresponding multiple of the average density, k denotes the number of particles in one box, and P is the probability of k particles existing in that box. Thus, the distribution of the S-particle swarm in $\mathcal{V}$ is a Poisson distribution with density intensity $\mathcal{M}$. Next, we will analyze the Poisson distribution formula given in Eq. 10. In fact, it is the proportion of each term determined by k (when $\mathrm{e}^{\mathcal{M}}$ is expanded as a power series) to the value of $\mathrm{e}^{\mathcal{M}}$. The meaning here is that it is also the proportion of the number of boxes containing k particles each to the total number of boxes in $\mathcal{V}$ when the S-particle swarm of relative density $\mathcal{M}$ is distributed among the reference boxes determined by the above criteria and spanned by the S-domain $\mathcal{V}$ (supposing that the number of boxes spanned by $\mathcal{V}$ is sufficiently large). According to mathematical analysis, we can see that the power series expansion for this case is unique, and obviously, this ratio distribution is also unique. If the right-hand side of Eq. 10 is multiplied by k, the result, denoted by $R(\mathcal{M}, k)$, takes the following form:

$$R(\mathcal{M}, k) = \frac{\mathrm{e}^{-\mathcal{M}}\mathcal{M}^k}{(k-1)!}. \tag{11}$$

In this way, termwise addition (by k) based on this expression offers a possible form for the decomposition of $\mathcal{M}$ into infinite items. Because the power series expansion above is unique, this decomposition form of the containing power series is also unique. According to the previous statement of physical meaning, the meaning of Eq. 11 is the relative density contributed by the particles in the boxes that contain k particles each to the total relative density $\mathcal{M}$ (the average relative density in $\mathcal{V}$) after the particles of relative density $\mathcal{M}$ are dispersed among the (infinitely many) reference boxes spanned by $\mathcal{V}$ with equal probability. Multiplying Eq. 11 by the number of boxes contained in $\mathcal{V}$ yields the total number of particles in the boxes containing k particles each. Since the distribution of particles in this form is definite (following the Poisson distribution), from this point of view, the decomposition of the relative density $\mathcal{M}$ in this (containing power series) form is also unique.

If $\boldsymbol{\mathcal{M}}$ is a complex number (or plane vector), Eq. 11 can be written in vector form as follows:

$$R(\boldsymbol{\mathcal{M}}, k) = \frac{\mathrm{e}^{-\boldsymbol{\mathcal{M}}}\boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{12}$$

The form obtained by dividing Eq. 12 by k is still the ratio of each term (complex) determined by k (when $\mathrm{e}^{\boldsymbol{\mathcal{M}}}$ is expanded as a power series) to the complex of $\mathrm{e}^{\boldsymbol{\mathcal{M}}}$. There is one more dimension here, and the power series expansion is still unique. Similarly, the termwise addition of Eq. 12 also provides a decomposition form for the vector $\boldsymbol{\mathcal{M}}$. This decomposition form of the containing power series is also unique.

Now, we study the distribution of the velocity of the moving S-particle swarm in the abovementioned

S-domain $\mathcal{V}$. If the particles of the T-particle swarm are moving randomly in the T-domain, the distribution of the S-particle swarm in a time slice in a sufficiently small S-domain (when the particle speed is fast enough) can also be approximately regarded as the equiprobable distribution. At the human scale (it will be proved with self-consistency that, in fact, in any scale range), the number of S-particles in almost every "microdomain" of the universe can be regarded as approaching infinity; therefore, the number distribution of particles in the moving S-particle swarm in a certain microdomain $\mathcal{V}$ can be described by Eq. 10. The moving particles in each type of box partitioned by k in one S-domain $\mathcal{V}$ can form a component vector, and these components can be added together to form the total vector in $\mathcal{V}$. Once the total 3-dimensional vector $\boldsymbol{\mathcal{Y}}$ of the moving S-particle swarm in $\mathcal{V}$, which includes the specific number of (equivalent) particles, is determined (that is, the average speed u of the system is determined), the norm (mathematical expectations) of each component vector should be (approximately) directly proportional to the number of particles forming it when the number of particles is large (see Part 2 of the Supplementary Information for details). It should be noted that even for $k = 1$, the number of samples in $\mathcal{V}$ should be very large. Therefore, the ratios between the norms (mathematical expectations) of the component vectors in various boxes partitioned by k are uniquely determined by the form of (containing) the power series determined by Eq. 10. As the limiting value $\boldsymbol{\mathcal{X}}$ of the quotient of $\boldsymbol{\mathcal{Y}}$ and $\mathcal{V}$, it can still be considered as a sum of 3-dimensional vectors in the S-domain $\mathcal{V}$. Therefore, there is also a form of component vectors with the ratios of norms determined by Eq. 10 spanning various boxes partitioned by k. When the 3-dimensional component vectors (spanning various boxes partitioned by k) of the 3-dimensional vector $\boldsymbol{\mathcal{X}}$ are mapped to the 2-dimensional component vectors (spanning various boxes partitioned by k) of the plane vector $\boldsymbol{\mathcal{M}}$, it is obvious that there is also a corresponding 2-dimensional form of component vectors with the ratios of norms determined by Eq. 10 (namely, the ratios of norms follows the Poisson distribution corresponding to the number of particles), but the direction is not determined. From the abovementioned decomposition method of scalar $\mathcal{M}$ (Eq. 11), it can be seen that if the ratios of norms of the component vectors of $\boldsymbol{\mathcal{M}}$ follow the Poisson distribution corresponding to the number of particles, when the norms of component vectors are expressed by the norm of $\boldsymbol{\mathcal{M}}$ itself, it is also necessary to use a unique and specified form of containing a power series (this is one of the necessary conditions. If the ratios between the norms of the component vectors are required to be directly proportional to the numbers of particles forming them at the same time, the other necessary condition is required, i.e., $u = 1$. See Part 2 of the Supplementary Information for details), that is, the method for calculating the norms of the component vectors determined by Eq. 12. At this time, the direction of each component vector is uniquely determined. Therefore, the plane mapping of the sum of all the vectors in the boxes containing the same number k of particles is the component vector determined by k in Eq. 12. When k takes all values in $\mathbb{N}_+$, the termwise sum of these terms is the unique decomposition of $\boldsymbol{\mathcal{M}}$ (spanning various boxes partitioned by k), namely,

$$\boldsymbol{\mathcal{M}} = \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-\boldsymbol{\mathcal{M}}} \boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{13}$$

According to the conclusion in Part 2 of the Supplementary Information, the norm (mathematical expectation) of each component vector is the product of the number of particles forming it and the speed of the system it located. The average velocity of the particles in each S-domain $\mathcal{V}$ is regarded as 1, so the number of particles is numerically equal to the norm of the momentum. Accordingly, the numbers of vectors distributed in various boxes are directly proportional to the norms of component vectors, and it is also comparable (computable) between S-domains. As mentioned above, scalar $\mathcal{M}$ represents the relative density of particles in the S-domain $\mathcal{V}$, which is a concept of multiples. It is obvious that $\boldsymbol{\mathcal{M}}$ should also be a relative vector. For vector $\boldsymbol{\mathcal{M}}$, it should be not only a multiple of the number of reference boxes but also a multiple of the speed of the system (that is, the norm of the average velocity of particles counted. $u = 1$ can be satisfied only if u is regarded as a relative value). Therefore, the reference value of vector $\boldsymbol{\mathcal{M}}$ is nu^* (where u^* is the absolute speed of the target domain in the background domain). Accordingly, $\boldsymbol{\mathcal{M}}$ in Section **3.3** should be exactly the relative vector sum density, and the direction of which is the same as that of the absolute sum of vectors located at that place. As mentioned above, the sum and difference operations between two spatial vectors are performed in their shared plane. In this plane, they can be decomposed respectively into a sum of plane vectors, as described in Eq. 13. Therefore, the two sets of plane component vectors can also serve as their respective spatial component vectors to correspondingly perform sum, difference or derivative operations.

3.4.2. Description of Diffusion

Suppose that the standard deviation of the projection (treated as a random variable; the same is done below) of the velocities of the k equivalent particles forming a k-particle (that is the k-generalized-particle; the same is done below) onto each equivalent coordinate axis is σ. As mentioned earlier, the speeds of k-particles follow the Maxwell distribution with scale parameter $\dfrac{\sigma}{\sqrt{k}}$ (When it is in the constrained state of $\mathrm{I}u$ not considering the Γ effect or in a completely free state, the speed of particle diffusion to uniform mixing in Fig. 3a is determined by the statistical average of the particle velocities, which is the inherent property of the system. Here, the particles in the target domain is regarded as a system with uniform distribution in the velocity direction, that is, the speeds of generalized particles follow the Maxwell distribution, and the average speed can be obtained according to the Maxwell distribution). Then, the average speed of k-particles is

$$\overline{v} = 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{\sqrt{k}}. \tag{14}$$

For k_1- and k_2-particles, the ratio of their average speeds is

$$\frac{\overline{v_1}}{\overline{v_2}} = \frac{\sqrt{k_2}}{\sqrt{k_1}}. \tag{15}$$

Because the sizes, or masses, of all 1-particles (forming k-particles) are the same, if the masses of a k_1-particle and a k_2-particle are m_1 and m_2, respectively ($m \propto k$), then according to the relationship shown in Eq. 15, the ratio of their average speeds can also be written as

$$\frac{\overline{v_1}}{\overline{v_2}} = \frac{\sqrt{m_2}}{\sqrt{m_1}}. \tag{16}$$

See Part 1 of the Supplementary Information for the detailed calculation and derivation process. According to Eq. 16, for any-particles, the product of the square root of mass and the average speed is a constant (suppose it is κ_1). Then, when the mass of a k-particle is m, its average speed is

$$\overline{v} = \frac{\kappa_1}{\sqrt{m}}. \tag{17}$$

The diffusion coefficient can be defined as follows: it is the mass or mole number of a substance that diffuses vertically through a unit of area along the diffusion direction per unit time and per unit concentration gradient. Therefore, it is believed that classical real diffusion is consistent with the essence of vector diffusion described here (the two diffusions that are achieved both require the random displacement of k-particles). According to the Einstein-Brown displacement equation, the diffusion coefficient is

$$D = \frac{\overline{x}^2}{2t}, \tag{18}$$

where $\overline{x}$ is the average displacement of k-particles along the direction of the x-axis. To replace the average displacement $\overline{x}$ in Eq. 18 with the average velocity (namely, $\overline{\boldsymbol{V}}$) of k-particles along the direction of the x-axis, this diffusion coefficient can be transformed into

$$D = \frac{\left|\overline{\boldsymbol{V}}\right|^2}{2} t^1. \tag{19}$$

The unit of the diffusion coefficient D is $\mathrm{m}^2 \cdot \mathrm{s}^{-1}$. By combining Eq. 18 and Eq. 19 (where t^1 and the t implied in $\left|\overline{\boldsymbol{V}}\right|^2$ are consistent, so $t^1 = 1$ s), the abovementioned diffusion coefficient can also be regarded as follows: it is the average area over which k-particles spread out on a plane per unit time. This average area is related to the speed of a single k-particle. If the (average) speed of a single k-particle is $\overline{v}$, then the statistical average speed of these particles in one direction is

$$\left|\overline{\boldsymbol{V}}\right| = \frac{\overline{v}}{2}. \tag{20}$$

The k-particle swarm spreads in the plane at this rate. By substituting Eq. 20 into Eq. 19 and combining $t^1 = 1$ s into the coefficient, which we then denote by κ_2, we can obtain

$$D = \kappa_2 \overline{v}^2, \tag{21}$$

where κ_2 is a constant coefficient with units of seconds (s).

By substituting Eq. 17 into Eq. 21, the diffusion coefficient of a (k-)particle swarm of (average) mass m is obtained:

$$D = \kappa_2 \left(\frac{\kappa_1}{\sqrt{m}}\right)^2 = \frac{{\kappa_1}^2 \kappa_2}{m}. \tag{22}$$

In view of the diffusion coefficient D only affecting the diffusion rate, the above equation (Eq. 22) can also be thought of as the apparent diffusion coefficient of particle(s) with mass m described by the 1-particle swarm (which forms a particle of mass m after collapse) in the constrained state of $\mathrm{I}u$. Here, we suppose that

$${\kappa_1}^2 \kappa_2 = \frac{\hbar}{2}. \tag{23}$$

As the situation in $\mathcal{R}_u$ observed from $\mathcal{R}_0$, D should also be affected by the $\Gamma[\cdot]$ effect, which is abbreviated as

$$D = \frac{\hbar \Gamma^2}{2\,m}. \tag{24}$$

3.4.3. Construction of the Generalized Diffusion Equation

Previously, we adopted the assumption that there is no interaction between point particles. Accordingly, in a time slice of a microdomain, the decomposition of the vector given by Eq. 13 must be exhibited, and all boxes containing the same number of particles in different microdomains containing different densities of vectors are equivalent. This is because there should be no differences between boxes of the same type (i.e., containing the same number of particles) when (the whole target domain is expressed as a system with a relative average speed of 1 and) the Poisson distribution determines the numbers of boxes of different types in different microdomains of different vector densities. Although the moving particles in the second or third constrained state can be distributed in a time slice of the microdomains with the same probability, when the overall behavior of k particles is counted, their average speed will inevitably slow down. At this time, there will be more or fewer particles in the unit volume of domain they located (or each box in the microdomain of domain they located), and the "slow down" effect will be retained according to the location characteristics, in other words, the degrees of freedom of particles will be reduced or affected by the second or third kind of constraint effect. The particles in various boxes partitioned by k move at their average relative speed, and the centroids of boxes containing k particles each are, on average, located at the center of each box. Among all boxes of the same type (i.e., containing k particles), the average relative speed of each k-particle is the same and must conform to the diffusion form of the Schrödinger equation (Eq. 8) determined by the diffusion coefficient for particles of this type. Therefore, according to the particle numbers k in the previously partitioned boxes, from 1 to ∞, we study the corresponding term $R(\boldsymbol{\mathcal{M}}, k)$, which is the component vector of $\boldsymbol{\mathcal{M}}$. First, we investigate the diffusion of individual terms, and then, we add them together to characterize the overall slowing behavior of diffusion.

Here, all the particles in each box containing k particles are regarded as forming a k-particle of a larger mass level, and together, all k-particles in all boxes containing k particles in microdomain $\mathcal{V}$ are called the k-particle swarm in that microdomain. Based on the above discussion, it can be considered that the average relative speed of each (k-)particle in the k-particle swarm is the same, and all of them have the same diffusion coefficient. According to the relationship given in Eq. 22 (the diffusion coefficient is inversely proportional to the mass of a k-particle, or the number of 1-particles forming a k-particle), if the diffusion coefficient of a 1-particle swarm is D_1, then the diffusion coefficient of a k-particle swarm is

$$D_k = D_1 \cdot \frac{1}{k}, \tag{25}$$

where $\frac{1}{k}$ is called the diffusion coefficient factor.

When the particles are in the constrained state of Iu or in a completely random state, the diffusion behavior of interest is that of a 1-particle swarm. It is consistent with the Schrödinger equation when the target particle swarm moves along the average speed of u. Therefore, the diffusion coefficient is

$$D_1 = -\frac{\hbar\,\Gamma^2}{2\,m}. \tag{26}$$

The diffusion equation determined by this coefficient describes the kinetics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m on the basis of the apparent diffusion rate (after deceleration) determined by the 1-particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the 1-particles in the background field. When the particles are in the constrained state of IIIu, according to the above discussion, the case of $k > 1$ must be considered. Then, the diffusion coefficient of a k-particle swarm can be obtained by substituting Eq. 26 into Eq. 25, namely,

$$D_k = -\frac{\hbar\,\Gamma^2}{2\,m} \cdot \frac{1}{k}. \tag{27}$$

This is equivalent to the proportional decline in the apparent diffusion rate of a target object (or the aggregation after collapse) of mass m due to the slowdown in the speed of the k-particles forming the target object. The meaning of the diffusion equation determined by this diffusion coefficient is similar to the case for 1-particles as considered above, that is, the kinetics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m are described on the basis of the apparent diffusion rate (after deceleration) determined by the k-particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the k-particles in the background field.

By taking the second partial derivative of $R(\boldsymbol{\mathcal{M}}, k)$ (this is the plane vector sum in the boxes containing k moving particles, namely, the k-particle swarm, which is one of the component vectors in the whole microdomain $\mathcal{V}$) with respect to location (x, y, z), $\Delta R(\boldsymbol{\mathcal{M}}, k)$ can be obtained. It should be emphasized that the absolute sizes of the two (infinitesimal) microdomains $\mathcal{V}_1$ and $\mathcal{V}_2$, which are selected to compare their differences, are equal when calculating the derivative of the vector $\boldsymbol{\mathcal{M}}$. After multiplying $\Delta R(\boldsymbol{\mathcal{M}}, k)$ by the diffusion coefficient for the k-particle swarm (Eq. 27) and then adding the products together from $k = 1$ to ∞, the complete generalized diffusion expression (including coefficients) can be obtained as follows:

$$-\frac{\hbar\,\Gamma^2}{2\,m}\sum_{k=1}^{\infty}\left[\frac{1}{k}\cdot\Delta R\,(\boldsymbol{\mathcal{M}}, k)\right]. \tag{28}$$

The diffusion calculated in this way is the generalized diffusion from the whole (infinitesimal) microdomain $\mathcal{V}_1$ to $\mathcal{V}_2$. Eq. 28 can be simplified as follows:

$$-\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}}\left[\Delta\boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}})\right], \tag{29}$$

where $T^2(\boldsymbol{\mathcal{M}}) = \left(\frac{\partial\boldsymbol{\mathcal{M}}}{\partial x}\right)^2 + \left(\frac{\partial\boldsymbol{\mathcal{M}}}{\partial y}\right)^2 + \left(\frac{\partial\boldsymbol{\mathcal{M}}}{\partial z}\right)^2$. By combining the left-hand side of Eq. 8 with Eq. 29, a complete expression for the generalized diffusion equation for vectors is obtained:

$$\mathbf{i}\,\frac{\partial\boldsymbol{\mathcal{M}}}{\partial t} = -\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}}\left[\Delta\boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}})\right]. \tag{30}$$

Therefore, the expression for the generalized diffusion coefficient with the two kinds of special constrained effects is

$$Đ = -\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}}. \tag{31}$$

The diffusion coefficient here is not a constant but rather a natural exponential function that varies with the relative vector density of moving particles. Hence, the generalized diffusion equation and the generalized diffusion coefficient $Đ$ for vectors in the constrained state of $\mathrm{III}u$ have been determined. In this constrained state, the ratios of norms of the spatial equivalent vectors in a microdomain can be determined in accordance with the Poisson distribution, while the norms and directions of the spatial equivalent vectors in the complex plane can be determined in accordance with Eq. 30. Thus, the basic effective information for a spatial (moving) particle swarm in the constrained state of $\mathrm{III}u$ has been derived.

The slowing down of diffusion based on spatial location is the only manifestation of the statistical effect of location aggregation (the second kind of constrained state) in diffusion. Obviously, the second kind of special constrained state effect of particles can be reflected according to the treatment method in Eq. 28. As mentioned above, the statistical effects include the location and direction aggregation. For the case of velocity direction aggregation, because the particles are in the system with a speed of

u, the diffusion coefficient will be affected by the Γ effect, and the statistical effect of this case is also added to the equation. In summary, all of the statistical (constrained) effects have be incorporated into Eq. 28.

3.5. Further Study of Eq. 30

Eq. 30 is the equation describing the generalized diffusion of a randomly moving particle swarm in the constrained state of IIIu. When

$$\Delta \boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}}) = 0. \tag{32}$$

$\boldsymbol{\mathcal{M}}$ does not vary with time t, and a particle swarm that meets this condition is a nondispersive particle swarm. Such a particle swarm can also be regarded as a particle of a higher mass level, which is composed of a set of particles of a lower mass level that are in the constrained state of IIIu.

To investigate the shape of a nondispersive particle swarm in detail, it is assumed that $\boldsymbol{\mathcal{M}}$ is a function only of location (x, y, z) in the constrained state of IIIu not considering the Γ effect. In 3-dimensional space, the following initial conditions are specified for Eq. 32:

$$\begin{cases} \boldsymbol{\mathcal{M}}(x, y, z) = \mathcal{M}_0, & x^2 + y^2 + z^2 = {r_1}^2, \\ \boldsymbol{\mathcal{M}}(x, y, z) = 0, & x^2 + y^2 + z^2 = {r_2}^2, \end{cases} \tag{33}$$

where r_1, r_2 and $\mathcal{M}_0$ are constants and $r_1 < r_2$. When the system is spherically symmetric (only this situation is studied in this article), the analytical solution (Eq. 34) can be obtained by solving the simultaneous equations given in Eq. 32 and Eq. 33.

$$\boldsymbol{\mathcal{M}}(x, y, z) = -\ln\left[\frac{r_1 - r_2\, \mathrm{e}^{\mathcal{M}_0}}{\mathrm{e}^{\mathcal{M}_0}(r_1 - r_2)} + \frac{r_1 r_2\left(\mathrm{e}^{\mathcal{M}_0} - 1\right)}{\mathrm{e}^{\mathcal{M}_0}(r_1 - r_2)\sqrt{x^2 + y^2 + z^2}}\right]. \tag{34}$$

See Part 3 of the Supplementary Information for the detailed Mathematica code for the solution process. Given $r_1 = 0.04$, $r_2 = 4$ and $\mathcal{M}_0 = 2 + 2\,\mathbf{i}$, the distribution of mass density ($|\boldsymbol{\mathcal{M}}|$) can be obtained, as illustrated in Fig. 5.

It can be seen from the figure that the mass of such a stable particle is almost entirely concentrated in a small spherical area near the center of a larger spherical region and that the rest of this region is very sparse (with a very low mass density), similar to the structure of an atom. Moreover, the equations for the 2-dimensional case under the same conditions are also solved in this article; see Part 3 of the Supplementary Information for details.

4. Conclusions

In this article, the generalized diffusion behavior of randomly moving particles in the constrained state of IIIu is discussed, and a kinetic equation (Eq. 30) for these particle is given. Notably an existence condition for nondiffusion particles (Eq. 32) is also included in this equation. In the more general case, that is, in the third kind of general constrained state, we can divide the whole system into countless fragments according to time and domain. Each fragment can be approximated to the case in IIIu. We use Eq. 30 to find the results for each segment and then splice them together. Thus, the whole problem of the third kind of general constraint can be solved.

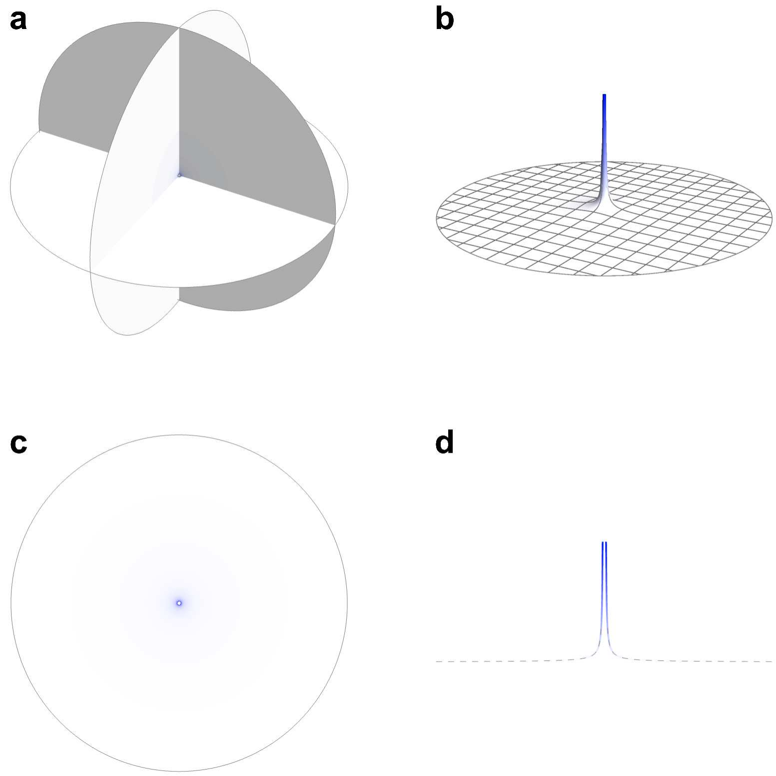

Figure 5. Distribution of mass density for a particle swarm meeting the conditions given by Eq. 32 and Eq. 33 (shown from various perspectives): **a**, 3-dimensional density distribution; **b**, 2-dimensional density distribution at $z = 0$; **c**, 2-dimensional density distribution on the plane at $z = 0$; **d**, 1-dimensional density distribution at $y = 0$ and $z = 0$. For convenience of comparison, the three (two) coordinate axes in each figure are displayed at a scale of $1 : 1$.

Acknowledgements

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Supplementary Material

Supplementary Information

See the **Supplementary Information** for detailed description of the models, derivations, additional figures, and computational method.

References

[1] Norris, J. R. (**1998**). *Markov chains*. UK: Cambridge University Press.
[2] Mrörters Y., Peres Y. (**2010**). *Brownian Motion*. UK: Cambridge University Press Cambridge Series in Statistical and Probabilistic Mathematics.

[3] Guo, T. Study on the average speed of particles from a particle swarm derived from a stationary particle swarm. *Sci Rep* 11, 13290 (**2021**). https://doi.org/10.1038/s41598-021-92402-w

Appendix:

Supplementary Information
(Mathematica v12.3.1.0 code of TraditionalForm)

Title:
Study on the Kinetics of a Special Particle Swarm

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NOTE:
1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "*In*[∘]:=" and is bold by default according to Mathematica's rules.

Part 1. The Square of the Norm of the Average Velocity is Proportional to the Number of Vectors

As described in the main text, the k-particle is a general particle composed of k 1-particles. Each 1-particle is moving at the same speed c and in a random direction in the 3-dimensional Cartesian coordinate system (they are in a completely free state or in the constrained state of Iu not considering the Γ effect). Suppose that the standard deviation of the projection of the velocity of any one of the k equivalent 1-particles forming a k-particle onto each equivalent coordinate axis is σ. According to the my previous study[1], the speed of k-particles (or k particles in a certain domain) follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$.

Then, the average velocity of the k-particles (or k particles in a certain domain) is

In[∘]:= $\overline{\boldsymbol{v}} = \mathbf{Mean}\left[\mathbf{MaxwellDistribution}\left[\frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{k}}}\right]\right]$

Out[∘]= $\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k}}$

For k_1- and k_2-particles, the ratio of their average velocity $\overline{v}_1 / \overline{v}_2$ =

In[∘]:= $\frac{2\sqrt{\frac{2}{\pi}}\,\boldsymbol{\sigma}}{\sqrt{\boldsymbol{k_1}}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\boldsymbol{\sigma}}{\sqrt{\boldsymbol{k_2}}}$

Out[∘]= $\frac{\sqrt{k_2}}{\sqrt{k_1}}$

And because: $m_1 = \mu\, k_1$ and $m_2 = \mu\, k_2$, where μ is the scale factor or the mass of 1-particle. $\overline{v}_1 / \overline{v}_2$ is also equal to

In[∘]:= $\mathbf{Simplify}\left[\frac{\sqrt{\frac{\boldsymbol{m_2}}{\boldsymbol{\mu}}}}{\sqrt{\frac{\boldsymbol{m_1}}{\boldsymbol{\mu}}}}, \mathbf{Assumptions} \to \boldsymbol{\mu} > \mathbf{0}\right]$

Out[∘]= $\frac{\sqrt{m_2}}{\sqrt{m_1}}$

Therefore, the square of the average velocity of particles is directly proportional to the mass of particles or the number of 1-particles forming it.

References

[1] Guo, T. Study on the average speed of particles from a particle swarm derived from a stationary particle swarm. Sci Rep 11, 13290 (**2021**). https://doi.org/10.1038/s41598-021-92402-w

Part 2. The Norm of the Component Vector is Proportional to the Number of Vectors Forming It

When the total vector value of a specified vector swarm is determined, the mean norms between different component vectors should be proportional to the number forming them in the constrained state of IIIu. The following proves this viewpoint in detail.

According to my previous study[1], let $\mathcal{M}$k being the norm of momentum of k particles observed from $\mathcal{R}_u$, the probability density of momentum norm formed by k particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ can be expressed as (This code takes approximately 70 seconds):

In[•]:= **Clear["Global`*"];**

$$\mathcal{D} = \mathbf{TransformedDistribution}\left[\sqrt{(k\,u)^2 + \mathcal{M}\mathrm{k}^2 - 2\,k\,u\,\mathcal{M}\mathrm{k}\,\mathrm{Cos}[\mathrm{ArcCos}[\eta]]}\,,\right.$$

$$\left.\left\{\mathcal{M}\mathrm{k} \approx \mathbf{MaxwellDistribution}\left[\frac{\sqrt{k}\,\sqrt{c^2-u^2}}{\sqrt{3}}\right], \eta \approx \mathbf{UniformDistribution}[\{-1, 1\}]\right\}\right];$$

$$\mathbf{FullSimplify}[\mathbf{PDF}[\mathcal{D}, x], \mathbf{Assumptions} \to c > 0 \land 0 < u < c]$$

Out[•]=

$$\begin{cases} \dfrac{\sqrt{3}\,x\left(e^{\frac{6\,u\,x}{c^2-u^2}}-1\right)e^{-\frac{3\,(k\,u+x)^2}{2\,k\,(c^2-u^2)}}}{k\,u\,\sqrt{2\,\pi\,c^2\,k-2\,\pi\,k\,u^2}} & k > 0 \land ((x > 0 \land k\,u > x) \lor k\,u < x) \\ -\dfrac{\sqrt{6\,\pi}\,\sqrt{c^2\,k-u\,x}\,(5\,u\,x-2\,c^2\,k)\,\mathrm{erf}\left(\frac{\sqrt{6}\,x}{\sqrt{c^2\,k-u\,x}}\right)+4\,x\,e^{\frac{6\,x^2}{u\,x-c^2\,k}}\left(c^2\,(6\,k+2)-u\,(2\,u+3\,x)\right)-8\,x\,(c-u)\,(c+u)}{4\,\sqrt{6\,\pi}\,k^{5/2}\,u\,((c-u)\,(c+u))^{3/2}} & k\,u = x \land k > 0 \end{cases}$$

The meaningful part (first branch) is selected as valid.

In view of the above conclusions, we find the mean value of this distribution (This code takes approximately 50 seconds).

In[•]:=

$$\overline{\mathcal{Y}}_k = \mathbf{FullSimplify}\left[\mathbf{Mean}\left[\mathbf{ProbabilityDistribution}\left[\frac{\sqrt{3}\,x\left(e^{\frac{6\,u\,x}{c^2-u^2}}-1\right)e^{-\frac{3\,(k\,u+x)^2}{2\,k\,(c^2-u^2)}}}{k\,u\,\sqrt{2\,\pi\,c^2\,k-2\,\pi\,k\,u^2}}, \{x, 0, +\infty\}\right]\right], \mathbf{Assumptions} \to c > u > 0 \land k > 0\right]$$

Out[•]=

$$\frac{(c^2+(3\,k-1)\,u^2)\,\mathrm{erf}\left(\frac{\sqrt{\frac{3}{2}}\,k\,u}{\sqrt{k\,(c-u)\,(c+u)}}\right)+\sqrt{\frac{6}{\pi}}\,u\,e^{\frac{3\,k\,u^2}{2\,(u^2-c^2)}}\,\sqrt{k\,(c-u)\,(c+u)}}{3\,u}$$

We find the limit of the ratio of this mean value $\overline{\mathcal{Y}}_k$ and k when k approaches $+\infty$.

In[•]:=

$$\mathbf{Simplify}\left[\mathbf{Limit}\left[\frac{\overline{\mathcal{Y}}_k}{k}, k \to +\infty\right], \mathbf{Assumptions} \to u > 0\right]$$

Out[•]=

$$\begin{cases} -u & \arg(c^2-u^2) \geq \pi \\ u & \text{True} \end{cases}$$

The second brunch is meaningful. Therefore, when k is a large number, the norm of the mean value $\overline{\mathcal{Y}}_k$ is directly proportional to the number k forming $\overline{\mathcal{Y}}_k$, namely, $\overline{\mathcal{Y}}_k = k \cdot u$.

Eq. 11 in the main text determines the proportion of particle number distributed in various boxes partitioned by k, and these particles are distributed in each box of $\mathcal{V}$ with equal probability. That is, the particles are randomly extracted from the microdomain $\mathcal{V}$ to be distributed in each box. When the number of extractions is large enough, the norm of each component vector partitioned by k should be directly proportional to the number of particles according to the probability and the scale factor is u.

The unique expansion of scalar $\mathcal{M}$ in the form of including power series is

$$\mathcal{M} = \sum_{k=1}^{\infty} \frac{e^{-\mathcal{M}} \mathcal{M}^k}{(k-1)!}$$

If the corresponding terms marked by k are directly proportional between the expansion of the norm $|\boldsymbol{\mathcal{M}}|$ of vector $\boldsymbol{\mathcal{M}}$ and the expansion of the scalar $\mathcal{M}$ representing the number of particles, or the numbers of particles are allowed to be proportional to the norms of vectors they form, the number $\mathcal{M}$ of particles must be equal to the norm $|\boldsymbol{\mathcal{M}}|$ of the vector $\boldsymbol{\mathcal{M}}$ they form besides they are required to obey Poisson distribution. According to the above conclusion $\overline{\mathcal{Y}}_k = k \cdot u$, the average speed $u = 1$ is needed in the system.

References

[1] Guo, T. Study on the average speed of particles from a particle swarm derived from a stationary particle swarm. Sci Rep 11, 13290 (**2021**). https://doi.org/10.1038/s41598-021-92402-w

Part 3. Solving Process of Eq. 32 in the Main Text

To solve the partial differential equation Eq. 32 in the main text, it is assumed that the system is spherically symmetric because it is isotropic at a huge scale. Therefore, we make the conversion from rectangular to spherical coordinates (note that φ is used to denote the azimuthal angle, whereas θ is used to denote the polar angle), namely, $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = \cos\theta$.

In the case of spherical symmetry, the change of function $\boldsymbol{\mathcal{M}}(r)$ does not depend on θ and φ, but is related to r. Therefore, after the coordinate transformation, and the first and the second derivatives are obtained, to omit the terms that depends on angles θ and φ, we can obtain:

In[∘]:= **Simplify**$\left[\frac{2}{r} D[\mathcal{M}[r], \{r, 1\}] + D[\mathcal{M}[r], \{r, 2\}] - (D[\mathcal{M}[r], \{r, 1\}])^2 \left((\mathrm{Sin}[\theta]\, \mathrm{Cos}[\varphi])^2 + (\mathrm{Sin}[\theta]\, \mathrm{Sin}[\varphi])^2 + (\mathrm{Cos}[\theta])^2\right)\right]$

Out[∘]= $\mathcal{M}''(r) - \mathcal{M}'(r)^2 + \frac{2\,\mathcal{M}'(r)}{r}$

To solve the abovementioned differential equation under the boundary condition $\boldsymbol{\mathcal{M}}(r_2) = 0$.

In[∘]:= **DSolve**$\left[\left\{\mathcal{M}''[r] - (\mathcal{M}'[r])^2 + \frac{2}{r} \mathcal{M}'[r] == 0, \mathcal{M}[\mathrm{r2}] == 0\right\}, \mathcal{M}[r], r\right]$

Out[∘]= $\{\{\mathcal{M}(r) \to \log(r) - \log(1 + c_1 r) - \log(\mathrm{r2}) + \log(1 + c_1\, \mathrm{r2})\}\}$

Suppose another boundary condition is $\boldsymbol{\mathcal{M}}(r_1) = \mathcal{M}_0$, then

In[∘]:= $r = \mathrm{r1}$;
Solve$[\mathrm{Log}[r] - \mathrm{Log}[1 + \mathrm{c1}\, r] - \mathrm{Log}[\mathrm{r2}] + \mathrm{Log}[1 + \mathrm{c1}\, \mathrm{r2}] == \mathcal{M}0, \mathrm{c1}]$

Out[∘]= $\left\{\left\{\mathrm{c1} \to \frac{\mathrm{r1} - \mathrm{r2}\, e^{\mathcal{M}0}}{\mathrm{r1}\, \mathrm{r2} \left(e^{\mathcal{M}0} - 1\right)}\right\}\right\}$

Therefore, the solution of the above differential equation is as follows:

In[∘]:= **Clear["Global`*"];**

$$\mathbf{c1 = \frac{r1 - r2\, e^{\mathcal{M}0}}{r1\, r2\, (e^{\mathcal{M}0} - 1)};}$$

Simplify[Log[*r*] − Log[1 + c1 *r*] − Log[r2] + Log[1 + c1 r2]]

Out[∘]= $$-\log\left(\frac{r\,(\mathrm{r1} - \mathrm{r2}\, e^{\mathcal{M}0})}{\mathrm{r1}\,\mathrm{r2}\,(e^{\mathcal{M}0} - 1)} + 1\right) + \log(r) + \log\left(\frac{e^{\mathcal{M}0}\,(\mathrm{r1} - \mathrm{r2})}{\mathrm{r1}\,(e^{\mathcal{M}0} - 1)}\right) - \log(\mathrm{r2})$$

To restore the above solution in spherical to the solution in 3-dimensional rectangular coordinates, then

In[∘]:= $$\boldsymbol{r = \sqrt{x^2 + y^2 + z^2};}$$

$$\mathbf{FullSimplify}\left[-\mathbf{Log}\left[\frac{r\,(\mathrm{r1} - \mathrm{r2}\, e^{\mathcal{M}0})}{\mathrm{r1}\,\mathrm{r2}\,(e^{\mathcal{M}0} - 1)} + 1\right] + \mathbf{Log}[r] + \mathbf{Log}\left[\frac{e^{\mathcal{M}0}\,(\mathrm{r1} - \mathrm{r2})}{\mathrm{r1}\,(e^{\mathcal{M}0} - 1)}\right] - \mathbf{Log}[\mathrm{r2}],\ \mathbf{Assumptions} \to \mathrm{r2} > 0 \land \mathrm{r1} > 0\right]$$

Out[∘]= $$-\log\left(\frac{(\mathrm{r1} - \mathrm{r2}\, e^{\mathcal{M}0})\,\sqrt{x^2 + y^2 + z^2}}{e^{\mathcal{M}0} - 1} + \mathrm{r1}\,\mathrm{r2}\right) + \log\left(\frac{e^{\mathcal{M}0}\,(\mathrm{r1} - \mathrm{r2})}{e^{\mathcal{M}0} - 1}\right) + \frac{1}{2}\log(x^2 + y^2 + z^2)$$

To verify the above results:

In[∘]:= $$\mathcal{M}[x, y, z] := -\mathbf{Log}\left[\frac{(\mathrm{r1} - \mathrm{r2}\, e^{\mathcal{M}0})\,\sqrt{x^2 + y^2 + z^2}}{e^{\mathcal{M}0} - 1} + \mathrm{r1}\,\mathrm{r2}\right] + \mathbf{Log}\left[\frac{e^{\mathcal{M}0}\,(\mathrm{r1} - \mathrm{r2})}{e^{\mathcal{M}0} - 1}\right] + \frac{1}{2}\mathbf{Log}[x^2 + y^2 + z^2];$$

$$\mathbf{FullSimplify}\left[\frac{\partial^2 \mathcal{M}(x, y, z)}{\partial x^2} + \frac{\partial^2 \mathcal{M}(x, y, z)}{\partial y^2} + \frac{\partial^2 \mathcal{M}(x, y, z)}{\partial z^2} - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial x}\right)^2 - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial y}\right)^2 - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial z}\right)^2\right]$$

Out[∘]= 0

Therefore, the above equation is the solution of Eq. 43 in the main text.

Similarly, the 2-dimensional case can also be solved.

In[∘]:= **Clear["Global`*"];**

$$\mathbf{Simplify}\left[D[\mathcal{M}[r], \{r, 2\}] + \frac{1}{r} D[\mathcal{M}[r], \{r, 1\}] - (D[\mathcal{M}[r], \{r, 1\}])^2\right]$$

Out[∘]= $$\mathcal{M}''(r) - \mathcal{M}'(r)^2 + \frac{\mathcal{M}'(r)}{r}$$

In[∘]:= $$\mathbf{DSolve}\left[\left\{\mathcal{M}''[r] - \mathcal{M}'[r]^2 + \frac{\mathcal{M}'[r]}{r} == 0,\ \mathcal{M}[\mathrm{r2}] == 0\right\},\ \mathcal{M}[r],\ r\right]$$

Out[∘]= $\{\{\mathcal{M}(r) \to \log(-\log(\mathrm{r2}) + c_1) - \log(-\log(r) + c_1)\}\}$

In[∘]:= ***r* = r1;**

Solve[Log[−Log[r2] + c1] − Log[−Log[*r*] + c1] == *M*0, c1]

Out[∘]= $$\left\{\left\{\mathrm{c1} \to \frac{e^{\mathcal{M}0}\log(\mathrm{r1}) - \log(\mathrm{r2})}{e^{\mathcal{M}0} - 1}\right\}\right\}$$

In[•]:= Clear["Global`*"];

$$c1=\frac{e^{\mathcal{M}0}\,\mathrm{Log[r1]}-\mathrm{Log[r2]}}{e^{\mathcal{M}0}-1};$$

Simplify[Log[−Log[r2] + c1] − Log[−Log[r] + c1]]

$$\text{Out[•]=}\quad \log\left(\frac{e^{\mathcal{M}0}(\log(\mathrm{r1})-\log(\mathrm{r2}))}{e^{\mathcal{M}0}-1}\right)-\log\left(\frac{e^{\mathcal{M}0}\log(\mathrm{r1})-\log(\mathrm{r2})}{e^{\mathcal{M}0}-1}-\log(r)\right)$$

In[•]:= $r=\sqrt{x^2+y^2}$;

$$\mathrm{FullSimplify}\Big[\mathrm{Log}\Big[\frac{e^{\mathcal{M}0}(\mathrm{Log[r1]}-\mathrm{Log[r2]})}{e^{\mathcal{M}0}-1}\Big]-\mathrm{Log}\Big[\frac{e^{\mathcal{M}0}\,\mathrm{Log[r1]}-\mathrm{Log[r2]}}{e^{\mathcal{M}0}-1}-\mathrm{Log}[r]\Big],$$

$$\mathrm{Assumptions}\to \mathrm{r2}>0\land \mathrm{r1}>0\Big]$$

$$\text{Out[•]=}\quad \log\left(\frac{e^{\mathcal{M}0}\log\left(\frac{\mathrm{r1}}{\mathrm{r2}}\right)}{e^{\mathcal{M}0}-1}\right)-\log\left(\frac{\log\left(\frac{\mathrm{r1}}{\mathrm{r2}}\right)}{e^{\mathcal{M}0}-1}+\log(\mathrm{r1})-\frac{1}{2}\log\left(x^2+y^2\right)\right)$$

$$\text{In[•]:=}\quad \mathcal{M}[x,y]:=\mathrm{Log}\Big[\frac{e^{\mathcal{M}0}\,\mathrm{Log}\left[\frac{\mathrm{r1}}{\mathrm{r2}}\right]}{e^{\mathcal{M}0}-1}\Big]-\mathrm{Log}\Big[\frac{\mathrm{Log}\left[\frac{\mathrm{r1}}{\mathrm{r2}}\right]}{e^{\mathcal{M}0}-1}+\mathrm{Log[r1]}-\frac{1}{2}\mathrm{Log}\left[x^2+y^2\right]\Big];$$

$$\mathrm{FullSimplify}\Big[\frac{\partial^2\mathcal{M}(x,y)}{\partial x^2}+\frac{\partial^2\mathcal{M}(x,y)}{\partial y^2}-\left(\frac{\partial\mathcal{M}(x,y)}{\partial x}\right)^2-\left(\frac{\partial\mathcal{M}(x,y)}{\partial y}\right)^2\Big]$$

Out[•]= 0

To verify the above conclusion, the results of analytical solution and the numerical solution under the same conditions are plotted (This code takes approximately 38 seconds):

In[•]:= Clear["Global`*"];

$$\mathcal{M}[\mathrm{x\_},\mathrm{y\_}]:=\mathrm{Log}\Big[\frac{e^{\mathcal{M}0}\,\mathrm{Log}\left[\frac{\mathrm{r1}}{\mathrm{r2}}\right]}{e^{\mathcal{M}0}-1}\Big]-\mathrm{Log}\Big[\frac{\mathrm{Log}\left[\frac{\mathrm{r1}}{\mathrm{r2}}\right]}{e^{\mathcal{M}0}-1}+\mathrm{Log[r1]}-\frac{1}{2}\mathrm{Log}\left[x^2+y^2\right]\Big];$$

$$\mathrm{r1}=\frac{4}{100};$$

r2 = 4;

$\mathcal{M}0 = 1 + 2\,i$;

Ω = ImplicitRegion[$\mathrm{r1}^2 \le x^2 + y^2 \le \mathrm{r2}^2$, {$x$, y}];

G1 = Show[Plot3D[Norm[$\mathcal{M}$[x, y]], {x, y} ∈ Ω, PlotRange → {0, $\sqrt{8}$},

ColorFunction → (Hue[0.65, #3] &), MeshStyle → None, BoundaryStyle → None, PlotPoints → 300,

AxesLabel → {Style["x ", Italic], Style["y", Italic], Rotate["Density   ", $\frac{\pi}{2}$]},

AxesStyle → Directive[Black, FontFamily → "Arial", FontSize → 15], TicksStyle → Black,

BoxStyle → Directive[Black, Thickness → 0.0018], BoxRatios → Automatic, ViewPoint → {15, −26, 16},

Epilog → Text[Style["a", 15, FontFamily → "Arial", Bold, Black], {−0.07, 0.92}, {−1, 1}]],

$$\mathrm{Table}\Big[\Omega1=\mathrm{ImplicitRegion}\Big[\frac{9}{100}\le x^2+i^2\le 16,\{x\}\Big];\ \mathrm{If}\Big[i^2\le\frac{9}{100},\ \mathrm{xx}=\sqrt{\frac{9}{100}-i^2},\ \mathrm{xx}=0\Big];$$

ParametricPlot3D[{x, i, Norm[$\mathcal{M}$[x, i]]}, x ∈ Ω1, PlotStyle → Thickness[0.0018], PlotPoints → 300,

$$\mathrm{ColorFunction}\to\left(\mathrm{GrayLevel}\Big[0.4,\ 1-\#3*\frac{\mathrm{Norm}[\mathcal{M}[\mathrm{xx},i]]}{\mathrm{Norm}\left[\mathcal{M}\left[0,\frac{3}{10}\right]\right]}\Big]\ \&\right)\Big],\ \{i,-3.5,3.5,0.5\}\Big],$$

```
Table[Ω1 = ImplicitRegion[9/100 ≤ j^2 + y^2 ≤ 16, {y}]; If[j^2 ≤ 9/100, yy = Sqrt[9/100 - j^2], yy = 0];
    ParametricPlot3D[{j, y, Norm[ℳ[j, y]]}, y ∈ Ω1, PlotStyle → Thickness[0.0018],
      PlotPoints → 300, ColorFunction → (GrayLevel[0.4, 1 - #3 * Norm[ℳ[j, yy]]/Norm[ℳ[0, 3/10]]] &)],
    {j, -3.5, 3.5, 0.5}], ParametricPlot3D[{4 Cos[ϕ], 4 Sin[ϕ], 0}, {ϕ, 0, 2 π},
    PlotStyle → Directive[Gray, Thickness[0.0018]], PlotPoints → 300]];
Needs["NDSolve`FEM`"];
mesh = ToElementMesh[Ω, MeshRefinementFunction →
    Function[{vertices, area}, area > 3/100000 × (1/10 + 80 Norm[Mean[vertices]])]];
uif = NDSolveValue[{∂²u(x, y)/∂x² + ∂²u(x, y)/∂y² - (∂u(x, y)/∂x)^2 - (∂u(x, y)/∂y)^2 == 0, DirichletCondition[
     u[x, y] == ℳ0, x^2 + y^2 == r1^2], DirichletCondition[u[x, y] == 0, x^2 + y^2 == r2^2]}, u, {x, y} ∈ mesh];
G2 = Show[Plot3D[Norm[uif[x, y]], {x, y} ∈ mesh, PlotRange → {0, Sqrt[8]},
    ColorFunction → (Hue[0.65, #3] &), MeshStyle → None, BoundaryStyle → None,
    AxesLabel → {Style["x  ", Italic], Style["y", Italic], Rotate["Density      ", π/2]},
    AxesStyle → Directive[Black, FontFamily → "Arial", FontSize → 15], TicksStyle → Black,
    BoxStyle → Directive[Black, Thickness → 0.002], BoxRatios → Automatic, ViewPoint → {15, -26, 16},
    Epilog → Text[Style["b", 15, FontFamily → "Arial", Bold, Black], {-0.07, 0.92}, {-1, 1}]],
  Table[Ω2 = ImplicitRegion[9/100 ≤ x^2 + i^2 ≤ 16, {x}]; If[i^2 ≤ 9/100, xx = Sqrt[9/100 - i^2], xx = 0];
    ParametricPlot3D[{x, i, Norm[uif[x, i]]}, x ∈ Ω2, PlotStyle → Thickness[0.0018], PlotPoints → 300,
      ColorFunction → (GrayLevel[0.4, 1 - #3 × Norm[uif[xx, i]]/Norm[uif[0, 3/10]]] &)], {i, -3.5, 3.5, 0.5}],
  Table[Ω2 = ImplicitRegion[9/100 ≤ j^2 + y^2 ≤ 16, {y}]; If[j^2 ≤ 9/100, yy = Sqrt[9/100 - j^2], yy = 0];
    ParametricPlot3D[{j, y, Norm[uif[j, y]]}, y ∈ Ω2, PlotStyle → Thickness[0.0018],
      PlotPoints → 300, ColorFunction → (GrayLevel[0.4, 1 - #3 × Norm[uif[j, yy]]/Norm[uif[0, 3/10]]] &)],
    {j, -3.5, 3.5, 0.5}], ParametricPlot3D[{4 Cos[ϕ], 4 Sin[ϕ], 0}, {ϕ, 0, 2 π},
    PlotStyle → Directive[Gray, Thickness[0.0018]], PlotPoints → 300]];
s1 = GraphicsRow[{G1, G2}, ImageSize → 500, Spacings → Scaled[-0.06]];
Pane[s1, {500, 200}, ImageMargins → {{-22, -30}, {-18, -26}}]
```

Out[∘]=

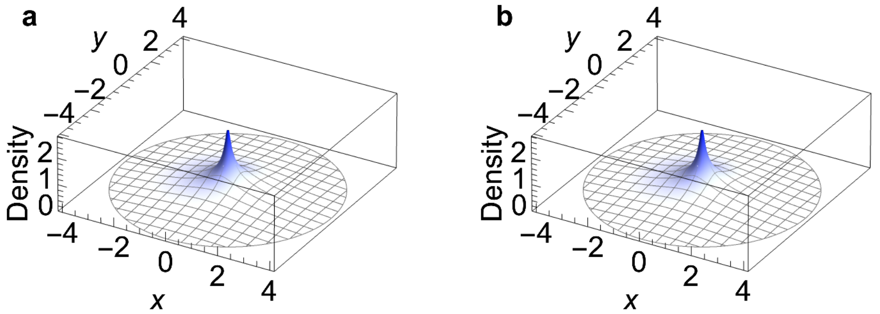

Figure S1 | Distribution of the mass density of a particle swarm meeting conditions $\mathcal{M}(0, 0) = 1 + 2i \wedge (\mathcal{M}(x, y) = 0 \wedge x^2 + y^2 = 4^2)$. **a**, The analytical solution. **b**, The numerical solution.

It can be seen from Fig. S1b that the numerical solution and the analytical solution achieve a perfect agreement.

Part 4. Figures Used in the Main Text

NOTE: To run these codes normally, the contents behind "MyDirection=" in each cell should be modified. It is similar to MyDirection="/Users/yourdirection/"

Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ##

In[∘]:=

```
Clear["Global`*"];
MyDirection = "/Users/gotall/Library/Mobile
    Documents/com~apple~CloudDocs/SPaper/Normal Paper/Reduce3/Latex/";
aa = Graphics[{{Blue, Thickness[0.003], Circle[{0, 1/2}, 1.04]}, {Red, Thickness[0.003], Circle[{0, 0}, 2]},
    {RGBColor[0, 0, 1, 1], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, 0.5}, {1.4, 0.5}}]}},
    {Green, Point[{0, 1/5}]}, Text[Style["R", 18, FontFamily → "Euclid Math One", Blue], {0, 1.07}],
    Text[Style["u", Italic, 12, FontFamily → "Arial", Blue], {0.132, 1.03}],
    Text[Style["Target (Sub-) Domain", 18, FontFamily → "Arial", Blue], {0.01, 2/3}],
    Text[Style["Total (Parent/Background) Domain", 18, FontFamily → "Arial", Red], {0, -4/5}],
    Text[Style["R", 18, FontFamily → "Euclid Math One", Red], {0, −1.25}],
    Text[Style["0", 12, FontFamily → "Arial", Red], {0.132, −1.3}],
    Text[Style["Microdomain", 18, FontFamily → "Arial", Green], {0, 0}],
    Text[Style["u", Italic, 18, FontFamily → "Arial", Blue], {1.53, 0.51}]}];
Export[MyDirection <> "figure1.eps", aa, Background → None];
```

Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ##

Figure2 ## ## ## ## ## ## ## ## ## ## ## ## ## ##

```
In[∘]:= Clear["Global`*"];
MyDirection = "/Users/gotall/Library/Mobile
      Documents/com~apple~CloudDocs/SPaper/Normal Paper/Reduce3/Latex/";
{rr1, bb1} = Last@Reap@Scan[If[#[[1]]^2 + #[[2]]^2 < 1, Sow[#, "Red"], Sow[#, "Blue"]] &,
       RandomReal[{-2, 2}, {2000, 2}]];
𝓡1 = ImplicitRegion[x^2 + y^2 > 1, {{x, -2, 2}, {y, -2, 2}}];
𝓡2 = ImplicitRegion[x^2 + y^2 < 1, {{x, -2, 2}, {y, -2, 2}}];
{rr2, bb2} = {RandomPoint[𝓡1, 1000], RandomPoint[𝓡2, 600]};
bb = Graphics[{{Blue, Dashed, Thickness[0.0016], Circle[{0, 0}, 1]}, {Red, Point[rr1]},
     {Blue, Point[bb1]}, {Blue, Dashed, Thickness[0.0016], Circle[{4.5, 0}, 1]},
     {RGBColor[0, 0, 1, 1], Arrowheads[0.03], {Thickness[0.004], Arrow[{{0, 0}, {1.4, 0}}]}},
     Text[Style["u", 18, Italic, FontFamily → "Arial", Blue], {1.53, 0.01}],
     Text[Style["(a)", 18, FontFamily → "Arial", Black], {-2, 2}], {Red,
       Point[rr2 + Table[{4.5, 0}, {i, Length[rr2]}]]}, {Blue, Point[bb2 + Table[{4.5, 0}, {i, Length[bb2]}]]},
     {Blue, Arrowheads[0.03], {Thickness[0.004], Arrow[{{4.5, 0}, {5.9, 0}}]}},
     Text[Style["u", 18, Italic, FontFamily → "Arial", Blue], {6.03, 0.01}],
     Text[Style["(b)", 18, FontFamily → "Arial", Black], {2.5, 2}]},
   Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.004]], Directive[Red, Thickness[0.004]]},
        {Style["Particles included in statistics", FontFamily → "Arial", FontSize → 18],
         Style["Particles not included in statistics", FontFamily → "Arial", FontSize → 18]},
        Joined → {False, False}, LegendLayout → "Row", LegendFunction →
         (Framed[#, RoundingRadius → 4, Background → White, FrameStyle → GrayLevel[0.58]] &)],
      Scaled[{1/2, 0.11}]], ImageSize → 700];
Export[MyDirection <> "figure2.eps", bb, Background → None];
## ## ## ## ## ## ## ## ## ## ## Figure2 ## ## ## ## ## ## ## ## ## ## ## ## ## ##
## ## ## ## ## ## ## ## ## ## ## Figure3 ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= Clear["Global`*"];
MyDirection = "/Users/gotall/Library/Mobile
    Documents/com~apple~CloudDocs/SPaper/Normal Paper/Reduce3/Latex/";
cc = Graphics[{{RGBColor[178/255, 252/255, 61/255], Rectangle[{0, 0}, {1, 1}]},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Rectangle[{1, 0}, {2, 1}]},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0.7, 0.54}, {1.3, 0.54}}]}},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.3, 0.46}, {0.7, 0.46}}]}},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Arrowheads[0.06],
      {Thickness[0.006], Arrow[{{0, -1.3}, {1.2, -1.3}}]}},
    {RGBColor[178/255, 252/255, 61/255], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {0.8, -0.3}}]}},
    {Orange, {Thickness[0.002], DotDashed, Line[{{1, -0.05}, {1, 1.05}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{0.8, -0.3}, {2, -0.3}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{1.2, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.2, -1.3}, {0.8, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {1, -0.8}}]}},
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {0.45, 0.5}],
    Text[Style["A", 17, FontFamily → "Arial", White], {0.513, 0.456}],
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {1.55, 0.5}],
    Text[Style["B", 17, FontFamily → "Arial", White], {1.616, 0.455}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.63}],
    Text[Style["A", 17, FontFamily → "Arial", Orange], {1.063, 0.59}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.38}],
    Text[Style["B", 17, FontFamily → "Arial", Orange], {1.065, 0.34}],
    Text[Style["Φ", 24, FontFamily → "Arial", Orange], {1.06, 1.08}],
    Text[Style["O", 24, FontFamily → "Arial", Orange], {0, -1.39}],
    Text[Style["B", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255, 0.5]], {1.2, -1.39}],
    Text[Style["A", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255]], {0.7, -0.28}],
    Text[Style["C", 24, FontFamily → "Arial", Orange], {2.02, -0.4}],
    Text[Style["M", 24, FontFamily → "Arial", Orange], {0.972, -0.93}],
    Inset[Style["a", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, 1.12}],
    Inset[Style["b", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, -0.2}]}];
Export[MyDirection <> "figure3.eps", cc, Background → None];
## ## ## ## ## ## ## ## ## ## ## Figure3 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
## ## ## ## ## ## ## ## ## ## ## Figure4 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= Clear["Global`*"];
MyDirection = "/Users/gotall/Library/Mobile
    Documents/com~apple~CloudDocs/SPaper/Normal Paper/Reduce3/Latex/";
text = Graphics[{Gray, Line[{{1, 0}, {1, 10}}], Line[{{2, 0}, {2, 10}}], Line[{{3, 0}, {3, 10}}],
    Line[{{4, 0}, {4, 10}}], Line[{{5, 0}, {5, 10}}], Line[{{6, 0}, {6, 10}}],
    Line[{{7, 0}, {7, 10}}], Line[{{8, 0}, {8, 10}}], Line[{{9, 0}, {9, 10}}],
    Line[{{0, 1}, {10, 1}}], Line[{{0, 2}, {10, 2}}], Line[{{0, 3}, {10, 3}}], Line[{{0, 4}, {10, 4}}],
    Line[{{0, 5}, {10, 5}}], Line[{{0, 6}, {10, 6}}], Line[{{0, 7}, {10, 7}}], Line[{{0, 8}, {10, 8}}],
    Line[{{0, 9}, {10, 9}}], Orange, Rectangle[{6, 4}, {7, 5}]}, PlotRangePadding → 1/1000];
dd = Show[{Plot3D[Sin[x + Cos[y]], {x, −3, 3}, {y, −3, 3}, PlotPoints → 60, MaxRecursion → 3,
     PlotStyle → Texture[text], Mesh → None, Lighting → "Neutral", PlotLabels → Placed["", {0, 0}],
     BoundaryStyle → None, Boxed → False, Axes → None, ViewPoint → {1, −1.9, 1.4}],
    Graphics3D[{{Thickness[0.007], Black,
       Arrow[{{0, 0, 0}, {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
            −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1}} +
          {{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {0.88, −0.3, Sin[0.88 + Cos[−0.3]]}}]},
      {Text[Style["N", 14, FontFamily → "Arial", Bold, Italic, Black],
        {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
          −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1} +
         {0.88, −0.3, Sin[0.88 + Cos[−0.3]]} + {0.02, 0.03, 0.23}]},
      {Thickness[0.007], Blue, Arrow[{{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {1.88, −0.5, 2}}]},
      {Text[Style["X", 14, FontFamily → "Euclid Math One", Bold, Blue], {2.01, −0.5, 2.01}]},
      {Text[Style["Σ", 14, FontFamily → "Arial", Italic, Gray], {−2.14, −1.5, 0.7}]},
      {Text[Style["dS", 14, FontFamily → "Arial", Orange], {0.55, −0.8, 1.39}]}}]}];
dd = Pane[dd, {400, 300}, ImageMargins → {{−10, −54}, {−75, −40}}];
Export[MyDirection <> "figure4.eps", dd, Background → None];
```

Figure4 ## ## ## ## ## ## ## ## ## ## ## ## ##

Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ##

This code takes approximately 47 seconds.

```
In[∘]:= Clear["Global`*"];
MyDirection = "/Users/gotall/Library/Mobile
    Documents/com~apple~CloudDocs/SPaper/Normal Paper/Reduce3/Latex/";
M[x_, y_, z_] := −Log[(r1 − r2 e^M0)/(e^M0 (r1 − r2)) + (r1 r2 (e^M0 − 1))/(e^M0 (r1 − r2) Sqrt[x^2 + y^2 + z^2])];
r1 = 4/100;
r2 = 4;
M0 = 2 + 2 i;
Ω1 = ImplicitRegion[r1^2 ≤ x^2 + y^2 + z^2 ≤ r2^2, {x, y, z}];
G1 = SliceDensityPlot3D[Norm[M[x, y, z]], "CenterPlanes", {x, y, z} ∈ Ω1,
   PlotRange → {0, Sqrt[8]}, ColorFunction → (Hue[0.65, #1] &), Boxed → False, Axes → None,
   BoundaryStyle → Directive[Thickness[0.0005], Gray], ClippingStyle → Transparent, PlotPoints → 200];
Ω2 = ImplicitRegion[r1^2 ≤ x^2 + y^2 ≤ r2^2, {x, y}];
G2 = Show[Plot3D[Norm[M[x, y, 0]], {x, y} ∈ Ω2,
    PlotRange → {{−4.52, 4.52}, {−4.52, 4.52}, {−1.7, 4}}, ColorFunction → (Hue[0.65, #3] &),
    MeshStyle → None, BoundaryStyle → None, Boxed → False, Axes → None,
    BoxRatios → Automatic, ImageSize → {392, 392}, PlotPoints → 200, ViewPoint → {1.2, −2, 0.7}],
```

```
Table[Ω3 = ImplicitRegion[81/10000 ≤ x^2 + i^2 ≤ 16, {x}]; If[i^2 ≤ 81/10000, xx = Sqrt[81/10000 - i^2], xx = 0];
  ParametricPlot3D[{x, i, Norm[M[x, i, 0]]}, x ∈ Ω3, PlotStyle → Thickness[0.0014], PlotPoints → 200,
   ColorFunction → (GrayLevel[0.4, 1 - #3 × Norm[M[xx, i, 0]]/Norm[M[0, 9/100, 0]]] &)], {i, -3.5, 3.5, 0.5}],
 Table[Ω3 = ImplicitRegion[81/10000 ≤ j^2 + y^2 ≤ 16, {y}]; If[j^2 ≤ 81/10000, yy = Sqrt[81/10000 - j^2], yy = 0];
  ParametricPlot3D[{j, y, Norm[M[j, y, 0]]}, y ∈ Ω3, PlotStyle → Thickness[0.0014],
   PlotPoints → 200, ColorFunction → (GrayLevel[0.4, 1 - #3 × Norm[M[j, yy, 0]]/Norm[M[0, 9/100, 0]]] &)],
  {j, -3.5, 3.5, 0.5}], ParametricPlot3D[{4 Cos[θ], 4 Sin[θ], 0}, {θ, 0, 2 π},
  PlotStyle → Directive[Gray, Thickness[0.0014]], PlotPoints → 200]];
G3 = DensityPlot[Norm[M[x, y, 0]], {x, y} ∈ Ω2, PlotRange → {{-4.07, 4.07}, {-4.07, 4.07}, {0, Sqrt[8]}},
   ColorFunction → (Hue[0.65, #1] &), Frame → False, PlotPoints → 200,
   Epilog → {Directive[Thickness[0.0014], Gray], Circle[{0, 0}, 4]}];
Ω4 = ImplicitRegion[r1^2 ≤ x^2 ≤ r2^2, {x}];
G4 = Plot[Norm[M[x, 0, 0]], {x, -4, 4}, PlotRange → {{-4.07, 4.07}, {-0.05, Sqrt[8]}},
   ColorFunction → (Hue[0.65, #2] &), PlotPoints → 1500, PlotStyle → {Thickness → 0.005},
   Frame → False, Axes → None, AspectRatio → Automatic];
G4 = Show[{G4, Plot[Norm[M[x, 0, 0]], {x, -4, 4}, PlotRange → {{-4.07, 4.07}, {-0.05, 0.8}},
     PlotPoints → 1500, PlotStyle → Directive[GrayLevel[0.5], Thickness → 0.0053, Dashed],
     Frame → False, Axes → None, AspectRatio → Automatic]}];
ee = GraphicsGrid[{{G1, Pane[G2, {400, 400}, ImageMargins → {{-70, -78}, {-80, -90}}]}, {G3, G4}},
   Spacings → {109, 109}, ImageSize → 512,
   Epilog → {Text[Style["a", 22, FontFamily → "Arial", Black, Bold], Scaled[{-0.3587, 1.3533}]],
     Text[Style["b", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.6222, 1.3533}]],
     Text[Style["c", 22, FontFamily → "Arial", Black, Bold], Scaled[{-0.3587, 0.3797}]],
     Text[Style["d", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.6222, 0.3797}]]}];
Export[MyDirection <> "figure5.eps", ee, Background → None];

## ## ## ## ## ## ## ## ## ## ## Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```