Series for Particular Values of the Gamma function

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Abstract. We give some series for the number \( \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} \), where \( \Gamma(x) \) is the Gamma function.

Introduction

Recall that

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \int_0^{4\sqrt{7}} \frac{1}{x(112 + 21x + x^2)} \, dx = \frac{2}{4\sqrt{7}} \int_0^1 \frac{1}{\sqrt{4 + 3\sqrt{7}x^2 + 4x^4}} \, dx
\]

\[
= \frac{2}{\sqrt{7}} \int_1^{\infty} \frac{1}{\sqrt{4 + 3\sqrt{7}x^2 + 4x^4}} \, dx = \frac{1}{\sqrt{7}} \int_0^\infty \frac{1}{\sqrt{4 + 3\sqrt{7}x^2 + 4x^4}} \, dx
\]

\[
= \frac{1}{4\sqrt{2}} \sqrt{7} \int_0^1 \frac{\sqrt{64 - x^2 - 3\sqrt{7}x}}{x} \, dx = \frac{\sqrt{3}}{2} \int_0^\infty \frac{1}{\sqrt{x(252 + 63x + 4x^2)}} \, dx
\]

\[
= \frac{\sqrt{3}}{2} \int_0^\infty \frac{1}{\sqrt{x(4 + 63x + 252x^2)}} \, dx = \frac{1}{\sqrt{7}} \int_1^\infty \frac{1}{\sqrt{3\sqrt{7} + 8x}(x^2 - 1)} \, dx
\]

\[
= \frac{1}{\sqrt{7}} \int_0^1 \frac{1}{\sqrt{x(8 + 3\sqrt{7}x)(1 - x^2)}} \, dx = \frac{1}{\sqrt{3\sqrt{7} + 8\cosh x}} \int_0^\infty \frac{1}{\cosh x} \, dx
\]

\[
= \frac{1}{\sqrt{7}} \int_0^{(3 - \sqrt{7})/\sqrt{2}} \cosh^{-1} \left( \frac{1 - 3\sqrt{7}x^2}{8x^2} \right) \, dx = 2 \int_0^{2\sqrt{7}x} \frac{1}{\sqrt{112 + 21x^2 + x^4}} \, dx
\]

\[
= \frac{2}{\sqrt{7}} \int_0^\infty \frac{1}{\sqrt{8 + 3\sqrt{7} + 16(\sinh x)^2}} \, dx = \frac{2}{\sqrt{7}} \int_0^\infty \frac{1}{\sqrt{16(\cosh x)^2 - 8 + 3\sqrt{7}}} \, dx
\]

\[
= \frac{2}{\sqrt{7}} \int_0^\infty \frac{1}{\sqrt{(8 + 3\sqrt{7} + 16x^2)(x^2 + 1)}} \, dx = \frac{2}{\sqrt{7}} \int_1^\infty \frac{1}{\sqrt{16x^2 - 8 + 3\sqrt{7})(x^2 - 1)}} \, dx
\]

\[
= \frac{1}{\sqrt{7}} \int_0^1 \frac{1}{\sqrt{x(1 - x)(2 - x)(8 + 3\sqrt{7} - 3\sqrt{7}x)}} \, dx
\]
Notations: $i = \sqrt{-1}$,

The number Pi is defined by

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}$$

The Gauss hypergeometric function is defined by

$$F(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \quad |x| < 1$$

where $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$, $(a)_0 = 1$

The Appell hypergeometric function is defined by

$$F_1(a, b, c, d, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (c)_{n}}{(d)_{m+n} m! n!} x^m y^n, \quad |x| < 1, |y| < 1$$

The Gamma function is defined by

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^z}{1 + \frac{z}{n}}, \text{Re} \: z > 0$$

Other integrals

$$\frac{\Gamma(1/7) \Gamma(2/7) \Gamma(4/7)}{8 \pi \sqrt{7}} = 2\sqrt{6} \int_{0}^{\tan^{-1}\sqrt{8/(3\sqrt{7})}} \frac{1}{\sqrt{63 + \cos^4 x}} dx = 2\sqrt{6} \int_{\tan^{-1}\sqrt{3\sqrt{7}/8}}^{\pi/2} \frac{1}{\sqrt{63 + \sin^4 x}} dx$$

$$= 2\sqrt{6} \int_{0}^{\sinh^{-1}\sqrt{8/(3\sqrt{7})}} \frac{\cosh x}{\sqrt{1 + 63 \cosh^4 x}} dx = \frac{1}{4\sqrt{7}} \int_{0}^{\pi/2} \frac{1}{x(2 + x)(8 + 3\sqrt{7} + 8x)} dx$$

$$= \frac{1}{2\sqrt{7}} \int_{0}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \left(\frac{1}{2} - \frac{3\sqrt{7}}{16}\right) \sin^2 x}} dx = \sqrt{2(3 - \sqrt{7})} \int_{0}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 + (8 - 3\sqrt{7})^2 \cos^2 x}} dx$$

$$= 2\sqrt{2(3 - \sqrt{7})} \int_{0}^{1} \frac{1}{\sqrt{1 + (510 - 192\sqrt{7}) x^2 + x^4}} dx = \frac{1}{4\sqrt{7}} \int_{0}^{1/2} \frac{1}{\sqrt{x(1 - x) \left(4 - (8 - 3\sqrt{7})x(1 - x)\right)}} dx$$
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{2\sqrt{21}} F_1\left(\frac{1}{2}, \frac{1}{7} ; \frac{2}{7} , \frac{3}{7} ; \frac{8}{63 + 3\sqrt{7} i} , \frac{8}{63 - 3\sqrt{7} i} \right) + \frac{\sqrt{3}}{3} \int_0^1 \sqrt{\frac{1}{4 + 63x^2 + 252x^4}} \, dx
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{4\sqrt{7}} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} P_n\left(\frac{3\sqrt{7}}{8}\right) + \frac{2}{4\sqrt{7}} \int_a^1 \sqrt{\frac{1}{4 + 3\sqrt{7}x^2 + 4x^4}} \, dx
\]

where \(0 \leq a \leq 1\) and \(P_n(x)\) is the Legendre polynomials.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi(3 - \sqrt{7})}{\sqrt{2} \sqrt[4]{7}} - \frac{1}{2\sqrt{7}} \int_1^{6\sqrt{2} - 2\sqrt{14}} \sin^{-1}\left(\frac{6\sqrt{2} + 2\sqrt{14}}{\sqrt{1 - \frac{1}{x^2}}}\right) \, dx
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = -\frac{\sqrt{2}(3 - \sqrt{7})(-1 + \ln(6 - 2\sqrt{7}))}{4\sqrt{7}} + \frac{1}{\sqrt{7}} \int_0^{(3-\sqrt{7})/\sqrt{2}} \ln\left(1 - 3\sqrt{7} x^2 + \sqrt{1 - 6\sqrt{7} x^2 - x^4}\right) \, dx
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{\sqrt{21} + 8\sqrt{7}} + \int_s^{\infty} -7 + 35\sqrt{2} \left(2646 + \frac{\sqrt{-4630500 + \left(2646 + \frac{27}{x^2}\right)^2 + \frac{27}{x^2}}}{\sqrt{3}}\right)^{-1/3} \, dx
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}(3 - \sqrt{7})}{4\sqrt{7}} \int_0^s \frac{(\text{sech } x)^2}{\sqrt{1 - (\text{sech } x)^2 + 4(8 - 3\sqrt{7})(\text{sech } x)^4}} \, dx
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = 2\sqrt{2}(3 - \sqrt{7}) \int_0^s \frac{e^{-x}}{\sqrt{1 + (510 - 192\sqrt{7})e^{-2x} + e^{-4x}}} \, dx
\]

where \(s = 1/\sqrt{2352 + 896\sqrt{7}}\).
Series for \( \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} \)

**Entry 1.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\frac{4\sqrt{7}}{\sqrt{168 + 42\sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} 2^{-2n} \left( \frac{4 + 3\sqrt{7}}{12 + 3\sqrt{7}} \right)^n \sum_{k=0}^{n} \frac{(n)}{(k)} (-1)^k \left( \frac{6\sqrt{7}}{4 + 3\sqrt{7}} \right)^k \sum_{m=0}^{k} \frac{(k)}{(m)} \left( \frac{4\sqrt{7}/21}{2k + 2m + 1} \right)^m}{n}
\]

**Entry 2.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{\sqrt{7}} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} 2^{-2n} \frac{2n+1}{2n+1} \left( 2 - \frac{3\sqrt{7}}{4} \right)^n F\left( 2n+1, n+\frac{1}{2}, n+\frac{3}{2}, -1 \right)
\]

**Entry 3.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} (-1)^n 2^{-2n} \left( \frac{\sqrt{3}}{5 + 2\sqrt{7}} \right)^{2n+1} \sum_{k=0}^{n} \frac{(n)}{(k)} \left( \frac{\sqrt{7}/3}{2k + 1} \right)^k F\left( 2n+1, 1, k + \frac{3}{2}, \frac{2\sqrt{7}}{5 + 2\sqrt{7}} \right)
\]

**Entry 4.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{8}{\sqrt{7}} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} (-1)^n 2^{-2n} \left( 8 - 3\sqrt{7} \right)^{2n+1} F\left( 1, 2n+1, \frac{3}{2}, \frac{8}{8 + 3\sqrt{7}} \right)
\]

**Entry 5.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt{21 + 4\sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} (-1)^n 2^{-2n} \left( \frac{4}{4 + 3\sqrt{7}} \right)^n F\left( 1, n + \frac{1}{2}, 2n + \frac{3}{2}, \frac{3\sqrt{7}}{4 + 3\sqrt{7}} \right)
\]

**Entry 6.**

\[
\frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}{8\pi\sqrt{7}} = \frac{2\sqrt{2} \sqrt{7} \sqrt{14\sqrt{7} - 37}}{3} \times \sum_{n=0}^{\infty} \frac{(2n)}{(n)} (-1)^n 2^{-2n} \left( \frac{14\sqrt{7} - 37}{9} \right)^n F_1\left( 1, n + \frac{1}{2}, n + \frac{1}{2}, \frac{3}{2}, \frac{2\sqrt{7}}{5 + 2\sqrt{7}}, \frac{4\sqrt{7}}{11 + 4\sqrt{7}} \right)
\]

**Entry 7.**

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt{7} (3 + \sqrt{7})} \sum_{n=0}^{\infty} \frac{(2n)}{(n)} 2^{-4n} \left( 24\sqrt{7} - 63 \right)^n F\left( \frac{1}{2}, n + \frac{1}{2}, n + 1, \frac{1}{2} \right)
\]
Entry 8.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt{7}(3 + \sqrt{7})} F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{3 \sqrt{7}}{8 + 3 \sqrt{7}}\right)
\]

Entry 9.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt{2\sqrt{3}\sqrt{7}}} \sum_{n=0}^{\infty} \binom{2n}{n} (\frac{4n}{2n}) (-1)^n 2^{-6n} 63^{-n}
\]

Entry 10.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{4\sqrt{7}} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-4n} \left(\frac{8 - 3\sqrt{7}}{16}\right)^n
\]

Entry 11.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = 2 \sqrt{\frac{2}{21}} \sum_{n=0}^{\infty} \binom{2n}{n} (-1)^n 2^{-2n} \left(\frac{8}{3\sqrt{7}} - 1\right) \sum_{k=0}^{n} \binom{n}{k} \frac{1}{4k + 1} F\left(2n + 1, 2k + \frac{1}{2}, 2k + \frac{3}{2}, -1\right)
\]

Entry 12.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = 2 \sqrt{\frac{2}{21}} \left[ F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{3 \sqrt{7}}{63 + 3 \sqrt{7} i}, -\frac{8}{63 - 3 \sqrt{7} i}\right) + \frac{4\sqrt{3}}{71} \sum_{n=0}^{\infty} \binom{2n}{n} (-1)^n \left(\frac{63}{40164}\right)^n F\left(2n + 1, 1, 2n + \frac{3}{2}, 63, 71\right)\right]
\]

Entry 13.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = 2 \sqrt{\frac{2}{7}} (3 - \sqrt{7}) \sum_{n=0}^{\infty} \frac{1}{\sqrt{7}} \binom{2n+2}{n+1} (n+1) \sum_{k=0}^{[n/2]} (-1)^k \left(\frac{2n - 2k}{n - k}\right) \binom{n - k}{k} \left(16(8 - 3\sqrt{7})^k\right)
\]

Entry 14.
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4(3 - \sqrt{7})}{\sqrt{7}} \sum_{n=0}^{\infty} \frac{2^{-3n}}{(n+1)(2n+2)(n+1)} \sum_{k=0}^{[n/2]} (-1)^k \binom{2n - 4k}{n - 2k} \binom{2k}{k} \binom{n}{n - 2k}^{-1} \left(16(8 - 3\sqrt{7})^{2k}\right)
\]

Entry 15. For \(a = 255 + 180\sqrt{2} - 96\sqrt{7} - 68\sqrt{14}\), we have
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2} \sqrt{a}}{\sqrt{7}(3 + \sqrt{7})(1 + a)} F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1, a^2 \frac{1}{1 + a} \frac{1}{1 + a}\right)
\]
\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}}{\sqrt[4]{7} (3 + \sqrt{7})} \frac{a}{\sqrt{1 + a}} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1 + a, 1 - a \right)
\]

Entry 16.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}}{\sqrt[4]{7} (3 + \sqrt{7})} \frac{a}{\sqrt{1 + a}} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1 - \frac{1}{a}, 1 + \frac{1}{a} \right)
\]

Entry 17.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}}{\sqrt[4]{7}} \frac{(3 - \sqrt{7})}{\sqrt[4]{7}} \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \frac{(-1)^n 2^{-n} 3^{-2n} 7^{-n}}{4n + 1} \frac{(8 - 3\sqrt{7})^{2n}}{2} F \left( 1, 2n + 1, n + \frac{3}{2}, 2, \frac{3\sqrt{7}}{8 + 3\sqrt{7}} \right)
\]

Entry 18.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{3}}{\sqrt[9]{1}} \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \frac{(-1)^n 2^{-n} 3^{-n}}{4n + 1} \left( \frac{36}{91} \right)^n F \left( 1, n + \frac{1}{2}, 2n + \frac{3}{2}, \frac{9}{13} \right)
\]

Entry 19. For \( a = 1 - \sqrt[3]{3\sqrt{7}(8 - 3\sqrt{7})} \), we have

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{a}}{\sqrt[21]{2}} \sum_{n=0}^{\infty} (-1)^n 2^{-2n} 63^{-n} \sum_{k=0}^{n} (-1)^k \binom{2n - 2k}{n - k} \binom{2k}{k} \frac{(63a)^{4n-4k}}{2} \sum_{m=0}^{4n-4k} \binom{4n-4k}{m} \frac{(-1)^m a^m}{2k + 2m + 1}
\]

Entry 20.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{3^{3/7}} \sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} 2^{-2k} 3^{-k} (8 - 3\sqrt{7})^k \binom{n + k}{n - k} \binom{2k}{k} \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{(-2)^m}{2k + 2m + 1}
\]
Entry 21.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt{3} \sqrt[4]{7} \sqrt{1 + \sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n) (-1)^n 12^{-n}}{(1 + \sqrt{7})^n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{2^{2k}}{4k + 1} \text{F} \left( n + \frac{1}{2}, 1, 2k + \frac{3}{2}, \frac{7 - \sqrt{7}}{6} \right)
\]

Entry 22.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2\sqrt{2}}{\sqrt[4]{7} \sqrt{10 + 3\sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n) 2^{2n}}{(3 + 3\sqrt{7})^n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k \frac{2}{4 + 3\sqrt{7}} \sum_{m=0}^{n+k} \left( \frac{n+k}{m} \right) (-2)^m \text{F} \left( n + \frac{1}{2}, 1, n + k + \frac{3}{2}, 3\sqrt{7} - 1 \right)
\]

Entry 23.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt[4]{7} \sqrt{3 + 3\sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n) (-1)^n 2^{-2n}}{(3 + 3\sqrt{7})^n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{2^{2k}}{2n + 2k + 1} \text{F} \left( n + \frac{1}{2}, 1, n + k + \frac{3}{2}, 3\sqrt{7} - 1 \right)
\]

Entry 24.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt[4]{7} \sqrt{4 + 3\sqrt{7}}} \sum_{n=0}^{\infty} \frac{(2n) (-1)^n}{(4 + 3\sqrt{7})^n} \text{F} \left( n + \frac{1}{2}, 1, 2n + \frac{3}{2}, \frac{3\sqrt{7}}{4 + 3\sqrt{7}} \right)
\]

Entry 25.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}}{\sqrt[4]{7}} \sum_{n=0}^{\infty} \frac{1}{2n + 1} \text{F} \left( -n, n + 1, n + \frac{3}{2}, 4(8 - 3\sqrt{7}) \right)
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}}{\sqrt[4]{7}} \sum_{n=0}^{\infty} \frac{(12\sqrt{7} - 31)^n}{2n + 1} \text{F} \left( -n, \frac{1}{2}, n + \frac{3}{2}, -\frac{12\sqrt{7} - 16}{47} \right)
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}}{\sqrt[4]{7}} \sqrt{12\sqrt{7} - 31} \sum_{n=0}^{\infty} \frac{(12\sqrt{7} - 31)^n}{2n + 1} \text{F} \left( 2n + \frac{3}{2}, \frac{1}{2}, n + \frac{3}{2}, 4(8 - 3\sqrt{7}) \right)
\]

Entry 26.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7} \sin \left( \frac{3\pi}{7} \right)} \left( 7 \cdot 2^{-1/7} \text{F} \left( \frac{5}{7}, \frac{1}{7}, \frac{8}{7}, \frac{1}{2} \right) + \frac{7}{2} \cdot 2^{-2/7} \text{F} \left( \frac{6}{7}, \frac{2}{7}, \frac{9}{7}, \frac{1}{2} \right) \right)
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7} \sin \left( \frac{5\pi}{7} \right)} \left( 7 \cdot 2^{-1/7} \text{F} \left( \frac{3}{7}, \frac{1}{7}, \frac{8}{7}, \frac{1}{2} \right) + \frac{7}{4} \cdot 2^{-4/7} \text{F} \left( \frac{6}{7}, \frac{4}{7}, \frac{11}{7}, \frac{1}{2} \right) \right)
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7} \sin \left( \frac{6\pi}{7} \right)} \left( 7 \cdot 2^{-2/7} \text{F} \left( \frac{3}{7}, \frac{2}{7}, \frac{9}{7}, \frac{1}{2} \right) + \frac{7}{4} \cdot 2^{-4/7} \text{F} \left( \frac{5}{7}, \frac{4}{7}, \frac{11}{7}, \frac{1}{2} \right) \right)
\]

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Entry 27. For \( \alpha = \sqrt{3\sqrt{7}(8 - 3\sqrt{7})} \), we have

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} - 2\sum_{n=0}^{\infty} \frac{(2n)^2}{2n+1} \sum_{k=0}^{\infty} \left( \frac{n}{k} \right) (-63)^{-k} \sum_{m=0}^{k} \left( \frac{k}{m} \right) \frac{(-1)^m a^{2n+2k+2m+1}}{2n+2k+2m+1}.
\]

Entry 28.

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}(3 - \sqrt{7})}{\sqrt{21}} \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right)^2 \sum_{k=0}^{\infty} \left( \frac{n}{k} \right) (-63)^{-k} \sum_{m=0}^{3k} \left( \frac{3k}{m} \right) \frac{(-1)^m (8(8 - 3\sqrt{7}))^{n-k+m}}{2n-2k+2m+1}.
\]
Endnote

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi^2}{7\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}
\]

\[
\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{B(1/7,2/7)}{8\sqrt{7}\sin\left(\frac{3\pi}{7}\right)} = \frac{B(1/7,4/7)}{8\sqrt{7}\sin\left(\frac{5\pi}{7}\right)} = \frac{B(2/7,4/7)}{8\sqrt{7}\sin\left(\frac{6\pi}{7}\right)}
\]

where \(B(x, y)\) is the Beta function

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x > 0, y > 0
\]

References


