On dimensionally reduced flows and anisotropic helicity polarization effects

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It is shown that the formulation and algorithm for the real Schur flow (RSF) can be generalized to general dimensionally reduced flows of, say, the Navier-Stokes equation. Substantive survey and discussions of the generalizations of our idea and method to derive various models are presented. The fine effects of helicity in two types of RSFs, as the case studies, are discussed with statistical calculations, implying consistent and interesting directivity properties determined cooperatively by the local polarization of helical motions and by the directional/global anisotropy of dimensional reduction of the dynamics, thus promising physical applications and theoretical productions in this direction.

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I. INTRODUCTION

Flows of reduced dimension(s) constitute the sub-spaces of the general system, defined, say, by the solutions of the Navier-Stokes equations. The dimensional reduction is represented by the vanishing, uniformly in ‘space’ (and, for the interest of this note, also ‘time’), specific component(s) of the velocity gradient tensor (VGT) supposed to be well defined ‘almost everywhere’ (except for some possibly singular point set of zero measure), which defines the structural relations that correspond to particular physical effect and need to be respected in the computations. A well-known mechanism leading to dimension reduction is the Taylor-Proudman effect and its analog in magnetohydrodynamics (see, e.g., Chandrasekhar for the incompressible flows and Ref. for remarks on the extensions to compressible cases).

[Dimensionally reduced models of flows also appear in different problems, including, for example, fluid flow in fractured porous media, biofilm growth and blood flow and oxygen transport, among others. Note however that, instead of taking the conventional route of physically motivated ordering and expansion of (small) parameters, we simply start from the mathematical structure of VGT to arrive at the reduced theoretical models and, if not simultaneously, then, ponder on the possible physical foundations and implications.] Various properties (e.g. the Lee Bound, i.e. the maximum amount of non-normality given the norm of the tensor, as evaluated with the Schur transform by Keylock) of flows with general reduction of dimension may help understanding the complete or full-dimensional turbulence by the comparisons with those of the latter. We will not consider the ensemble of flows with randomly (but uniformly) decimated VGT matrix elements, but the idea may be of some value in modeling turbulence: see, e.g., the ensemble of “chiral base flows (CBF)” [i.e., the helical real Schur flows (see below)] used for the “fastening” effect of helicity in Ref. following one of whose scenarios about the (giant) CBF associated to local spatial averaging or filtering in the (very) large-eddy simulations one can envisage potential applications (see, e.g., Yu, Zhao and Lu and references therein for the sub-grid models with Schur decomposition techniques).

The real Schur flow (RSF) presents a special and generic (in the sense that the VGT can always be transformed to the real Schur form by orthogonal transformations of the coordinates) dimensional reduction with two non-diagonal components of the VGT matrix vanishing. As remarked in Ref. with the relevant physical ideas (e.g., Li, Tian and
Zhang\textsuperscript{11}, it is not impossible that RSFs in $d$-space with $d > 3$ may be of relevance to applications of fluid dynamics in 3- or 2-space physics: for instance, advected two-dimensional (2D) or three-dimensional (3D) passive scalar(s) may be viewed as the $d - 2$ or $d - 3$ velocity component(s) independent of the corresponding coordinate(s). In Ref. \textsuperscript{12} for 3-space flows, two apparently complementary types of RSFs were both considered. Partly due to the two facts, respectively, that we have so far found ‘elegance’ (in the sense of topology and geometry) only in the two-component-two-dimensional-coupled-with-one-component-three-dimensional (2D2D3D) RSF and that we can indeed always choose either type as the objective for the real Schur transformation from the velocity gradient matrix in a fixed coordinate frame, only the 2D2D3D RSF was focused in the subsequent theoretical,\textsuperscript{12} analytical\textsuperscript{10} and computational\textsuperscript{13} investigations. However, some reflection leads to the following understanding: the other type with the VGT corresponding to the local adjoint map in the dual space presents an apparently different two-component-three-dimensional-coupled-with-one-component-one-dimensional (3D3D1D) velocity field, thus the consideration of the alternative form of RSFs can shed more lights.

Just as the connection of 2D2D3D RSF with the fast rotation limit of a compressible fluid,\textsuperscript{2} each of the other dimensionally reduced flows may also have its own direct relevance to some specific physical situation(s), the potential mathematical and/or physical value being nonestimatable. For systematic investigations, individually or as a whole, analytical or numerical, we need clear formulations and working algorithms, which is the purpose of this note to show that those for 2D2D3D RSF can be generalized to the general case. A particular purpose is to promote the theoretical thinking about the decomposition, specification and physical characterization of the sub-spaces of NSFs: the former two are represented in the formulations of dimensionally-reduced flows, and the latter is rich, in principle endless, thus with only a particular effort of fine helicity effect analysis being presented, emphasizing the (local) ‘polarization’\textsuperscript{14} superposed onto the (global) ‘anisotropy’ or ‘isotropy’.

\section{RSF AND BEYOND}

Consider for simplicity the isothermal 2D2D3D RSF characterized by\textsuperscript{12}

\begin{equation}
    u_{1,3} \equiv 0 \equiv u_{2,3},
\end{equation}
with the pressure \( p = c^2 \rho \) and the gradient of enthalpy \( \nabla \Pi := \rho^{-1} \nabla p \) [thus \( \Pi \) and \( c^2 \ln \rho \) is equivalent up to a spatially uniform time function \( T(t) \) irrelevant in the essential formulation], in a periodic box of dimension \( L_1 \times L_2 \times L_3 \) (but some results also apply in more general situations\(^{10}\)) with \( \langle \bullet \rangle_{123} := \int \int \int \frac{\bullet}{L_1 L_2 L_3} \), \( \langle \bullet \rangle_{12} := \int \int \frac{\bullet dx_1 dx_2}{L_1 L_2} \), \( \langle \bullet \rangle_3 := \int \bullet dx_3/L_3 \) (other averages to be used later are similarly defined) and \( \rho = u \cdot \nabla \ln \rho + \nabla \cdot u \), we have the 2D2D3D RSF iff\(^{11}\)

\[
\begin{align*}
\partial_t \ln \rho &= \langle \rho \rangle_{123} - \langle \rho \rangle_3 - \langle \rho \rangle_{12}, \\
\partial_t u + u \cdot \nabla u &= -\nabla \Pi,
\end{align*}
\]

given the 2D2D3D initial field. Here, \( \Pi \) should be decomposed into two functions, \( \mathcal{P}_h \) and \( \mathcal{P}_3 \), one of only the horizontal coordinate \( x_h := \{x_1, x_2\} \) and the other of only \( x_3 \):

\[ \Pi = \mathcal{P}_3(x_3) + \mathcal{P}_h(x_1, x_2). \quad (3) \]

**A. Still more on 2D2D3D RSFs**

Additional conditions may be further imposed on the 2D2D3D RSFs, and here we address two examples, one associated to a more accurate asymptote of the compressible Taylor-Proudman theorem with horizontal incompressibility\(^2\) and the other to a particular characterization of swirls (when exist\(^{13}\)).

1. **Horizontally incompressible 2D2D3D RSF**

We now check the horizontal velocity \( \langle \mathbf{u}_h := \{u_1, u_2\} \rangle \) equation for the 2D2D3D RSF in Ref. \(^{10}\),

\[ \partial_t \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \mathbf{u}_h = -\nabla_h \Pi. \quad (4) \]

The horizontal incompressibility \( \nabla_h \cdot \mathbf{u}_h = 0 \) then permits the introduction of such a stream function \( \Psi \) that \( \mathbf{u}_h = \nabla_h \Psi \times \bar{x}_3 \) where \( \bar{x}_3 \) is the unit vector along the vertical direction. Curling Eq. \(^{4}\) and applying the equality \( \nabla \times \mathbf{u}_h = \bar{x}_3 \nabla^2 \Psi \), we then obtain the equation of \( \Psi \) in which \( \mathcal{P}_h \) solves the associated Poisson equation, as is the classical wisdom of hydrodynamics and applied in various reduced models\(^{14}\), offering the basis of an algorithm naturally guarantee the horizontal incompressibility. Note that now the horizontal
incompressible flow pressure and the total thermodynamic pressure are connected by the vertically-averaged Eq. (2a),
\[ \partial_t (\ln \rho)_3 + \langle \varrho \rangle_3 = 0. \] (5)

2. Canonical 2D2D3D RSF and “lone” Schur flows

When there is a conjugate pair of complex eigenvalues \( \lambda_{cr} \pm \lambda_{ci} \hat{i} \), one can further rotate the coordinate frame in the \( x_1-x_2 \) plane to let
\[ u_{1,1} = u_{2,2} = \lambda_{cr}, \] (6)
and the transformation of the coordinates is uniquely determined by three rotation parameters. Li, Zhang and He\(^\text{15}\) then observed that
\[ \lambda_{ci}^2 = \psi(\psi + \gamma), \]
where \( \psi \) is the amplitude of the ‘canonical rotation’ around the \( x_3 \) axis. We may then say the 2D2D3D RSF satisfying uniformly Eq. (6) “canonical”.

Now \( u_{1,1} \equiv u_{2,2} \) is seen from Eq. (4) to be satisfied (forever from the beginning) iff
\[ \left( \frac{u_1^2}{2} \right)_{,11} - \left( \frac{u_2^2}{2} \right)_{,22} + u_2 u_{1,21} - u_1 u_{2,12} = \Pi_{22} - \Pi_{11}, \] (7)
which may be used as the condition for, say, the pseudo-time algorithm to let the internal iterations converge to the canonical 2D2D3D RSF. Such a formulation and algorithm, if indeed working, however does not guarantee \( \psi \neq 0 \) in any spatial domain, and, so far, it is not very clear what a sense it makes to consider a flow in a (sub)domain with uniformly (or ‘almost everywhere’) \( \psi = u_{2,1} \equiv 0 \) and \( u_{1,1} \equiv u_{2,2} \), i.e., a ‘canonical 2D2D3D RSF’, incongruously with 1D \( u_1 \) and without ‘canonical rotations’ (a specific identification of ‘swirls’).

While, contrary to the above somewhat disappointing “no-go” effort of canonical 2D2D3D RSF, we have been able to formulated and semi-analytically computed flows with \( \psi \equiv 0 \), thus no conjugate pair of eigenvalues (“lone”) for swirls.\(^\text{16}\) Now, with the additional
\[ u_{1,2} \equiv 0 \] (8)
besides Eq. (1), we have one-dimensional (1D) \( u_1 \) and thus 1D2D3D RSF, or the “lone Schur flow (LSF)”. For such an LSF, as analytically shown and practically implemented in the numerical simulations in Ref. \(^\text{16}\), Eq. (2a) should be further refined to be
\[ \partial_t \ln \rho = 2\langle \varrho \rangle_{123} - \langle \varrho \rangle_{23} - \langle \varrho \rangle_{13} - \langle \varrho \rangle_{12}. \] (9)
The dynamics and structures of such LSF are of course highly nontrivial and fascinating, both mathematically and physically concerning, for instance, the global behavior of 3D dynamical system and ‘vortex genesis’ of flows with even more systematic numerical and theoretical studies being initiated.

B. Then 3D3D1D RSF

The 3D3D1D RSF is characterized by

\[ u_{3,1} \equiv 0 \equiv u_{3,2}, \tag{10} \]

starting from which we see that all the analyses leading to the semi-analytical algorithm for the 2D2D3D RSF in Ref. carry over, mutatis mutandis.

It is found that the same Eq. applies also for 3D3D3D RSF, which should not be surprising for \( \Pi_{ij} = \Pi_{ji} \), the vanishing of which is the starting point of deriving the equation. In other words, the differences between the two types of RSFs rely on the initial condition and other sources/sinks. However, these 2D2D3D- and 3D3D1D-RSF sub-spaces can have an intersection of 2D2D1D (two-component-two-dimensional-and-one-component-one-dimensional) flows characterized by both Eqs. and , i.e., with

\[ u_{1,3} \equiv u_{2,3} \equiv u_{3,1} \equiv u_{3,2} \equiv 0, \tag{11} \]

which then raises the question: given the 2D2D1D initial field and acceleration, how the flow evolves? Ideally, due to Eq. , the (self)-advection nonlinear term is a 2D2D1D inertial “force”, thus nothing breaking the 2D2D1D property [integrating Eqs. with \( u_1 = \langle u_1 \rangle_3 \), \( u_2 = \langle u_2 \rangle_3 \) and \( u_3 = \langle u_3 \rangle_{12} \) then embodies the semi-analytical algorithm for simulating such a 2D2D1D flow]. However, instabilities, if indeed (depending on the Reynolds number etc.), will drive 2D2D1D flow with “disturbances” out of the sub-space of the intersection into the larger space depending on the choice of the model, here the 3D3D1D RSF: for example, the semi-analytical algorithm established for the 2D2D3D RSF, with \( u_1 = \langle u_1 \rangle_3 \) and \( u_2 = \langle u_2 \rangle_3 \) for integrating Eq., carries over straightforwardly, however only the 3D3D1D but not the 2D2D1D property is precisely perserved, allowing the numerical errors well playing the role of “disturbances”.

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C. And beyond

From the point of view of decomposition and specification of the solution space of NSF, it still makes sense to check other cases with particular uniform characteristics of the VGT. Here, we show that the idea and method of establishing the corresponding governing equations can be applied for more general situations. For example, consider again for simplicity the barotropic ideal flow with two-component-three-dimensional-and-one-component-two-dimensional (3D3D2D) velocity field, *i.e.*, with only uniformly $u_{i,j} \equiv 0$ for $i \neq j$, say, without loss of generality,

$$u_{2,1} \equiv 0,$$  \hspace{1cm} (12)

we can similarly show that Eq. (2a) of the governing equations for such 3D3D2D flow should be replaced with

$$\partial_t \ln \rho = \langle \varrho \rangle_{2} - \langle \varrho \rangle_{1} - \langle \varrho \rangle_{1}.$$  \hspace{1cm} (13)

An alternative 3D3D2D flow for $i = j$ is that, without loss of generality, of uniform

$$u_{1,1} \equiv 0$$  \hspace{1cm} (14)

which similarly indicates that $\Pi$ is either linear in $x_1$ or only a function of $x_2$ and $x_3$ and that Eq. (2a) for such 3D3D2D flow in a cyclic box should be replaced with

$$\partial_t \ln \rho + \langle \varrho \rangle_{1} = 0.$$  \hspace{1cm} (15)

Flows with other reduction(s) of dimension with, for instance, further $u_{2,3} \equiv 0$ can also be similarly treated, and there are still other cases of other specific properties, the list of which we of course can not exhaust or tabulate: the union of all the specified sub-spaces by pure dimensional reduction, far from completing the NSF solution space though and for the time being without clear definite physical relevance of all other dimensionally reduced flows though, plays a fundamental and illuminating role in understanding RSF (more remarks will be offered in Sec. [IV]) and deserves to be explored by systematically analyzing each type of the flows one by one (see, e.g., that for the LSF[16]).

D. Special remarks on the speculated “Jordan standard flow (JSF)”

As briefly remarked in Ref. [10], a JSF with the VGT matrix uniformly of the (real, or course) Jordan standard form (which can be of specific physical value) is not as generic as
the RSF, thus not of the fundamental value as the latter, in the sense that only that every complex (or other algebraically-closed-field), but not real, square matrix is similar to the Jordan standard form. Nevertheless, constructing the JSF is a mathematically well-defined problem with which most audiences would come up when confronted with the other flows such as the RSFs for the first time, thus it is worthwhile to offer more discussions.

As remarked in Sec. (II B), due to the fact that $\Pi_{ij} = \Pi_{ji}$, the flow with elements of the VGT matrix below the diagonal being uniformly vanishing, as should be the case of JSF, should ideally also have vanishing elements above the diagonal, if initially set to be so. But, either the initial initial field or the acceleration, if exists, or, the disturbances allowed can lead to nonvanishing elements above the diagonal of the VGT matrix. Thus, it may seem that JSF should be dynamically working in our model with the only additional condition of $u_{3,1} \equiv 0$ imposed on the LSF mentioned in the last section (II C), which could practically be realized by adding an operation of setting $u_3 = \langle u_3 \rangle_1$ (to analytically prohibit any disturbances, from whatever numerical errors, say, beyond the JSF model) to the algorithm of LSF computation.

However, first, $u_{2,1}$ and $u_{3,2}$ of the Jordan standard form of the VGT matrix acquire the values of either 1 or 0, thus not even nontrivial (in the sense that $u_{2,1}$ and $u_{3,2}$ do change with time) flows in the space $C^1$ with first order continuous derivatives; and, second, when $u_{2,1}$ or $u_{3,2}$ is of value 1, the two diagonal elements beside it should be of the same value: these two discontinuous conditions obviously means that, even if a JSF existed, it would be not “conventional” (associated particularly to the behavior in $C^1$ referred to in the above) and is not within the scope of this note, though could be of some other value to pursue.

III. ON THE “FINE” EFFECT OF HELICITY

Turbulent flows with dimension reduction are anisotropic by definition, which enriches the dynamical structures. Ref. [2] starts from compressible 2D2D3D RSF to examine the Taylor-Proudman effect (TPE) on reducing the horizontal compressibility. The helicity “fastening” effect on such RSF was proposed to be the TPE by boosting to a rotating frame in which the helicity vanishes: see below for the defining formula of helicity a flow is helical when the helicity is not zero. Helical isotropic compressible turbulence then was treated as the replicas of such “chiral base flows” of the same helicity with disorder in the rotation directions. Note
that the “fastening” effect of helicity was initially noticed from the comparison of the formal statistical absolute equilibrium spectra for the helical and nonhelical cases (see also Ref. [3]).

Absolute equilibria are of course not turbulence statistics and the relation with the latter depends highly on the time scales of turbulent dynamical processes which are in general unclear. Now, for the anisotropic RSF turbulence, using still the absolute equilibrium might sound stretching the methodology and argument a bit too far. Nevertheless, for lack of reliable theory, it still makes sense to check the results, which could be useful for helping understanding dimensionally reduced flows. Thus, we here present the results for insights of the “fine” effects of helicity on the full-dimensional NSF and RSFs as case studies.

The canonical absolute equilibria is determined by the distribution \( \sim \exp\{-C\} \), where the ‘constant of motion’ \( C = \alpha E + \beta H \). In terms of the Fourier coefficients \( \hat{u}_k \) of wavevector \( k \) of the variable \( v \), the (approximate) energy \( E = \sum_k (|\hat{u}_k|^2 + c^2|\hat{\zeta}_k|^2)/2 \), where \( \zeta = \ln \rho - \ln \rho_0 \) with \( \rho_0 \) being the equilibrium density (see Ref. [3] and references therein), and the helicity \( H = \sum_k \hat{k} \times \hat{u}_k \cdot \hat{u}^*_k/2 \), where \( \hat{i}^2 = -1 \) and the complex conjugate \( \hat{u}^*_k = \hat{u}_{-k} \) comes from the reality of \( u \). Writing \( \hat{u}_i = R_i + iI_i \) \((i = 1, 2, 3)\), we have the covariance matrix of \( (R_1, R_2, R_3, I_1, I_2, I_3) \),

\[
\begin{pmatrix}
\alpha^2 - \beta^2 k_1^2 & \beta k_1 & 0 & 0 & 0 & 0 \\
\beta k_1 & \alpha^2 - \beta^2 k_2^2 & \beta k_2 & 0 & 0 & 0 \\
0 & \beta k_2 & \alpha^2 - \beta^2 k_3^2 & \beta k_3 & 0 & 0 \\
0 & 0 & \beta k_3 & \alpha^2 - \beta^2 k_3^2 & \beta k_3 & 0 \\
0 & 0 & 0 & \beta k_3 & \alpha^2 - \beta^2 k_3^2 & \beta k_3 \\
0 & 0 & 0 & 0 & \beta k_3 & \alpha^2 - \beta^2 k_3^2 \\
\end{pmatrix}
\]

(16)

for the full NSF; while those of the 2D2D3D and 3D3D1D RSFs are, respectively,

\[
\begin{pmatrix}
\delta_{0,k_1} (\alpha^2 - \beta^2 k_1^2) & -\delta_{0,k_1} \beta k_1 k_2 & 0 & 0 & 0 & \delta_{0,k_1} \beta k_1 \\
\delta_{0,k_1} \beta k_1 k_2 & \delta_{0,k_1} (\alpha^2 - \beta^2 k_2^2) & 0 & 0 & 0 & \delta_{0,k_1} \beta k_2 \\
0 & 0 & \delta_{0,k_1} (\alpha^2 - \beta^2 k_3^2) & 0 & 0 & \delta_{0,k_1} \beta k_3 \\
0 & 0 & 0 & \delta_{0,k_1} (\alpha^2 - \beta^2 k_3^2) & 0 & \delta_{0,k_1} \beta k_3 \\
0 & 0 & 0 & 0 & \delta_{0,k_1} (\alpha^2 - \beta^2 k_3^2) & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_{0,k_1} (\alpha - \beta k_1) \\
\end{pmatrix}
\]

(17)
where the Kronecker deltas, \( \delta_{0,k_3}, \delta_{0,k_1}, \) and \( \delta_{0,k_2}, \) come with the definition of RSFs. Note that in all cases, the density mode \( \hat{\zeta} \) is independent of other variables in the absolute equilibrium distribution with the spectrum \( \langle c^2|\hat{\zeta}|^2 \rangle = \frac{2}{\alpha} \) with the numerator 2 coming from the equal contributions of the real and imaginary parts of \( \hat{\zeta} \). A systematic comparative analysis of the fine anisotropic effects of helicity, as preliminarily presented for the numerical results of 2D2D3D RSFs in Ref. [13], on various models (including the LSF, RSF and others) against the corresponding numerical results should be of specific value (say, for the directivity of the aeroacoustic noise problem), calling however for smart ideas for, say, a unified and neat treatment also from the audiences.

A lot of qualitative physical indications about turbulence may already be inferred from the above results, especially with comparisons. For example, most obviously, each “0” element indicates decorrelation, and the many more vanishing elements in Eqs. (17 and 18) then at least implies correspondingly weaker correlations in the turbulence of such RSFs than that of full NSF, with reasonably fair setups for comparison. The 3D3D1D RSF convariance matrix (18) appear simplest, and also obvious is that related to \( \langle R^2_3 \rangle \) and \( \langle I^2_3 \rangle \), which are of value \( \frac{1}{\alpha} \) as that of the density mode or 0 for either \( k_1 \neq 0 \) or \( k_2 \neq 0 \), indicating that much more (fraction of) velocity “energy” (power of fluctuations) will be partitioned to the \( u_1 \) and \( u_2 \) components. The helicity fastening effect can obviously also be inferred with the similar reasoning in Ref. 2, but now anisotropically, with the particular property of \( \langle |u_3|^2 \rangle = \langle R^2_3 \rangle + \langle I^2_3 \rangle \). Comparing \( \langle R^2_i \rangle \) or \( \langle I^2_i \rangle \) for \( i = 1, 2 \) and 3 among Eqs. (16, 17 and 18), we also see, in the interesting local and global directivity properties caused by nonvanishing helicity with \( \beta \neq 0 \) on all the correlations (especially the diagonal self-correlation components for the “energy”), marked differences which may persist to some degree in turbulence and aeroacoustic noises: the ‘local polarization’ in the local \( k \)-frame already presents in the isotropic full-dimensional NSF absolute equilibria of Eq. (16), due to nonvanishing helicity \( (\beta \neq 0) \), and the global or
‘directional’ anisotropy characterized most clearly by the Kronecker deltas presents in the absolute equilibria of RSFs in Eqs. (17 and 18) with additional local polarization different to that of NSF (for discussions of directional and local anisotropy in incompressible rotating turbulence, see, e.g., Cambon and Jacquin14 and references therein.) We remark that the results appear nicely ‘self-consistent’: for example, \( \langle R_1^2 \rangle = \langle I_1^2 \rangle \) of the 2D2D3D RSF is just that of the full-dimensional NSF with the local polarization associated to \( k_3 \) becoming so strong as to be the extremal case of the directional anisotropy characterized by \( \delta_{0,k_3} \); and, other results are of similarly comparative fashion.

Even finer results and implications can be obtained by calculating the longitudinal- and transversal-mode spectra of the velocity, which however will lead to even more complicated formulae and which is not presented here. Such very detailed analysis would be better be accompanied with systematic numerical checks (see, e.g., Ref. 13) which is beyond the scope of this note. Actually, the discussions in Ref. 13 already paid some attention to the anisotropic properties, but with Eq. (17) and even more detailed (but more complex) longitudinal- and transversal-mode spectra the analysis can be guided to be more complete, which should be performed in the future together with other cases systematically.

### IV. DISCUSSION

Since the solution space of flows of uniform dimensional reduction is only a very special sub-space of that of NSF, the already very rich results that we have shown could deliver the pessimistic message on our efforts to understand the latter. However, as partially indicated by the success of constructing the general relativity theory from the uniformly-flat-space-time special relativity, among others, through the “equivalence principle”, we may actually be reasonably optimistic by making a loose analogy of the two situations, particularly with now even finer specifications of the subsets such as the LSF16 and others within RSF and beyond. Our results still hold with additional appropriate de-/ac-celerations and can be reasonably extended to more complex situations as already remarked earlier10 and we believe that with the current semi-analytical algorithm and other brute-force ones for more general cases, they can be powerfully used in uncovering the complexity of NSF turbulence as already indicated in the preliminary demonstrations in Ref. 13 (and also 16).

Physically, turbulence is notoriously known for the difficulties associated with the (se-
vere deviation from statistical absolute equilibrium, with few analytically precise quantities available beyond the phenomenological arguments. However, qualitative implications from the absolute equilibria may be inferred, concerning various aspects including the energy partitions, spectral transfer directions and courageously some details in the solar dynamo with the idea of broken ergodicity and broken symmetry, among others. We have also presented some relevant results for use the idea of which, as said, might appear to be stretching the argument a bit too far on the first sight but which actually already have turned out to be illuminating as our preliminary analysis shows. On the other hand, our analysis emphasizes the importance of anisotropic flows in understanding isotropic turbulence, in which sense, probably somewhat surprisingly on the first sight from the conventional view of traditional turbulence studies, the anisotropic turbulence turns out to be actually more fundamental. And, finally, we iterate one of the introductory remarks that each of such “prototypical” flows can be of direct connection with remarkable physical realities, by pointing out further that the 3D3D1D RSF with $u_3$ depending only on $x_3$ admits the discs in the $x_1$-$x_2$ plane with vanishing $u_3$ (depending of course on the choice of co-moving frame which may be put at the places of slow-varying $u_3$, say, with $u_{3,3} = 0$) and spirals of “galaxies”.

In conclusion, we remark that the dimensionally reduced flows are in general unstable (under the condition of large enough Reynolds number, say) subject to the full- or additional-dimensional perturbations in larger-space dynamics, which then also indicates the potential value in the stability, transition and control issues of turbulence of full-dimensional NSFs.

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