Cauchy's logico-linguistic slip and the "Heisenberg" uncertainty principle

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If Cauchy would have been more careful about his own language, then he could have written down the "Heisenberg" uncertainty principle a century earlier than Heisenberg. Such logico-linguistic self-criticism can provide new insights to physicists in the pursuit of truth and reality.

The laws of motion and therefore, classical mechanics[1], are founded on elementary calculus which begins with the definition of "derivative". Such literature is so well established today that anybody educated in high school physics can understand the language without having a hint of doubt. However, if one studies carefully how Cauchy defined the notion of derivative in his book[2], it is radically different from the standard practice in physics[4–6]. Quite remarkably, as I find, if Cauchy would have been careful enough with his own language, then he could have written down the "Heisenberg" uncertainty principle as a condition for violation of the laws of motion and therefore, for the invalidity of classical mechanics. This fact brings to light the question that how physics could have evolved differently if the uncertainty principle would have been theoretically written down by Cauchy around one century earlier than the time of Heisenberg[10]. More importantly, it raises the concern regarding how truthfully the theories of physics reflect the physicist's experience of experiment which is the only accepted mean to pursue truth and reality through science. With an aim to elucidate such concerns as my personal opinion, I intend to analyze Cauchy's definition of derivative as he put it in his own words in his book[2]. While there is a much broader virtue associated with the present discussion from the philosopher's and the logician's perspective, which I have elaborated in ref.[8], I believe that the central issue is worth the attention of the mainstream physics community irrespective of the philosopher's and the logician's assessment of such an opinion. Before proceeding, for the ease of understanding of the reader, I may specify the meanings of the following symbols to be used: ":=", "$\ni$" and "$\Leftrightarrow$" stand for "defined as", "such that" and "equivalent to", respectively.

Cauchy defined "derivative of a function", on page no. 11 of ref.[2], as follows:

"$\cdots$ *function* $y = f(x) \cdots$ *variable* $x \cdots$ *an infinitely small increment attributed to the variable produces an infinitely small increment of the function itself.* $\cdots$ *set* $\Delta x = i$*, the two terms of the ratio of differences*

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}. \tag{1}$$

will be infinitely small quantities. $\cdots$ *these two terms indefinitely and simultaneously will approach the limit of zero, the ratio itself may be able to converge toward another limit,...*"

Cauchy's "infinitely small quantities", in modern mathematical notation, appears as "$\Delta x \to 0, \Delta y \to 0$" in the above scenario. Therefore, according to Cauchy's prescription, *both "$\Delta y \to 0$" and "$\Delta x \to 0$" need to hold so that the derivative is definable.* Thus, from expression (1), considering a truthful conversion of verbal statements into mathematical notations, **I assert** the following:

$$\boxed{\textbf{Cauchy's definition:} \quad \frac{dy}{dx} := \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \ni \ y = f(x).} \tag{2}$$

I have skipped the unnecessary step of setting $\Delta x = i$. In the **standard practice**[4, 5], one finds the following:

$$\boxed{\textbf{Cauchy's definition:} \quad \frac{dy}{dx} := \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \ni \ y = f(x).} \tag{3}$$

Obviously one can note what the difference is in the two scenarios.

$$\boxed{\textbf{The difference is the omission of "}\Delta y \to 0\textbf{".}} \tag{4}$$

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Since I have not found any instance in the literature where Cauchy explicated the importance of such an issue, I consider (4) as **Cauchy's logico-linguistic slip**. I have used the adjective "logico-linguistic" because I am scrutinizing *how reasonably the mathematical expressions convey the sense carried by the verbal expressions* i.e. *reasoning* and *language* are the matters of concern. In any case, the essence of this adjective will appear to be more justified as I proceed to explain the significance of "$\Delta y \to 0$" which can be elucidated through the following conversation.

Question: *Why does it matter if "$\Delta y \to 0$" is not written in the mathematical expression and how does it affect our understanding of physics?*

Answer: *Writing "$\Delta y \to 0$" matters because, if it is not written then the following three problem arises.*

1. *The ratio $\Delta y/\Delta x$ diverges as $\Delta x \to 0$, rather than converging to some limit.*
2. *The mathematical expression is not a truthful conversion of the verbal statements of Cauchy.*
3. *The physicist does not realize why in some situations classical mechanics fails.*

While the first two of the above three assertions can be immediately verified, *the third assertion needs more elaboration which I provide as follows.* The physicist deals with *quantities* which are expressed in terms of some standard quantity, called *unit*, of the same *physical dimension* e.g. length is expressed in terms of length unit like meter, kilometer, etc.; time is expressed in terms of time unit like second, microsecond, etc.[3]. Let me denote the chosen units of length and time as λ_0 and T_0 respectively i.e. λ_0 stands for meter, kilometer, etc. and T_0 stands for second, microsecond, etc. I write the displacement (of an object as a whole) as $\Delta x = \Delta n_x \lambda_0$ and the time lapse during this displacement as $\Delta t = \Delta n_t T_0$. Further, I call such expressions as "physico-mathematical" due to the involvement of physical dimensions alongside the numbers $\Delta n_x, \Delta n_t$. Now, I note the following inter-conversion between verbal statements and physico-mathematical expressions as follows:

$$\text{"}\Delta x \text{ is an infinitesimally small length compared to the length unit } \lambda_0\text{"} \Leftrightarrow \Delta x \lll \lambda_0 \Leftrightarrow \Delta n_x \lll 1, \tag{5}$$

$$\text{"}\Delta t \text{ is an infinitesimally small time compared to the time unit } T_0\text{"} \Leftrightarrow \Delta t \lll T_0 \Leftrightarrow \Delta n_t \lll 1. \tag{6}$$

I must write "$\Delta n_x \to 0, \Delta n_t \to 0$" instead of "$\Delta n_x \lll 1, \Delta n_t \lll 1$" in accord with the currently accepted standard notation (however, see ref.[7]). So, I adopt the following convention of writing:

$$\Delta n_x \lll 1 \quad \Leftrightarrow \quad \Delta n_x \to 0, \tag{7}$$

$$\Delta n_x \lll 1 \quad \Leftrightarrow \quad \Delta n_t \to 0. \tag{8}$$

Therefore, under the validity of the conditions (5) and (6), and using the notation adopted in (7) and (8), instantaneous velocity can be defined, according to Cauchy's definition of derivative as follows:

$$\boxed{\frac{dx}{dt} := \lim_{\substack{\Delta n_x \to 0 \\ \Delta n_t \to 0}} \frac{\Delta x}{\Delta t} = \left(\lim_{\substack{\Delta n_x \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_x}{\Delta n_t} \right) \frac{\lambda_0}{T_0} = n_v v_0 \ \ni n_v := \lim_{\substack{\Delta n_x \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_x}{\Delta n_t} \ \& \ v_0 := \frac{\lambda_0}{T_0}.} \tag{9}$$

Now, to write down the laws of motion, the derivative of momentum needs to be defined according to Cauchy's prescription. So, let me consider momentum (p) to be a quantity in its own right (which is sufficient for the present purpose). I denote a change in p as $\Delta p = \Delta n_p p_0$, where p_0 is the momentum unit in terms of which Δp is expressed. Then the instantaneous rate of change of momentum can only be defined when the following condition is also fulfilled alongside condition (6):

$$\text{"}\Delta p \text{ is an infinitesimally small momentum compared to the momentum unit } p_0\text{"} \Leftrightarrow \Delta p \lll p_0 \Leftrightarrow \Delta n_p \lll 1. \tag{10}$$

Then, adopting the usual convention of notation, I may write $\Delta n_p \to 0 \quad \Leftrightarrow \quad \Delta p \lll p_0$, from which the definition of instantaneous rate of change of momentum can be written from Cauchy's prescription:

$$\boxed{\frac{dp}{dt} := \lim_{\substack{\Delta n_p \to 0 \\ \Delta n_t \to 0}} \frac{\Delta p}{\Delta t} = \left(\lim_{\substack{\Delta n_p \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_p}{\Delta n_t} \right) \frac{p_0}{T_0} = n_F F_0 \ \ni n_F := \lim_{\substack{\Delta n_p \to 0 \\ \Delta n_t \to 0}} \frac{\Delta n_p}{\Delta n_t} \ \& \ F_0 := \frac{p_0}{T_0}.} \tag{11}$$

Now, I may assert that the verbal statements of the laws of motion can only be expressed in physico-mathematical terms to do further calculations if and only if instantaneous velocity and instantaneous rate of change of momentum can be defined according to Cauchy's prescription. In view of this, I may conclude that the conditions (5), (6) and (10) need to remain valid so that classical mechanics is applicable. Consequently, I can assert that the following derived condition needs to hold for the laws of motion and hence, classical mechanics to be applicable:

$$\Delta x.\Delta p \lll L_0 \quad \ni \; L_0 := \lambda_0 p_0, \tag{12}$$

which is obtained from (5) and (10) and L_0 is an angular momentum unit. Then, I may now conclude that the laws of motion, and hence classical mechanics, fail when either or both of (5) and (10) do not hold. This failure can be written as

$$\boxed{\Delta x.\Delta p \gtrsim L_0.} \tag{13}$$

Here, I have assumed that the condition (6) remains valid and did not bring it into discussion so that the focus remains only on Δx and Δp because these quantities appear as the numerators in the definitions of instantaneous velocity and instantaneous rate of change of momentum, respectively. In view of this, I hope, I have been able to explain the significance of "$\Delta y \to 0$" in the context of physics and, in the process, I have justified my third assertion regarding how the physicist fails to realize the limitation of classical mechanics owing to Cauchy's logico-linguistic slip. It seems to me very appropriate now to wonder why Poincare claimed, on page no. 6 of ref.[9], that "*it is precisely in the proofs of the most elementary theorems that the authors of classic treatises have displayed the least precision and rigour.*"

From the modern standpoint, the identification of "L_0" with the Planck constant "h"[11] can be investigated upon by extending such inquiry further as follows. The Schroedinger equation for a massive free particle in one spatial dimension is written as

$$\frac{i\lambda_c^2}{4\pi}\frac{\partial^2\psi}{\partial x^2} = \tau_c\frac{\partial\psi}{\partial t} \quad \ni \lambda_c = \frac{h}{mc}, c\tau_c = \lambda_c. \tag{14}$$

$m = n_m m_0$ is the mass of the particle where m_0 is the mass unit in terms of which m is expressed. $c = n_c v_0$ is the velocity of light in vacuum expressed in terms of the velocity unit $v_0 = \lambda_0/T_0$ and $h = n_h L_0$ is the Planck constant expressed in terms of the angular momentum unit L_0. ψ is called wave function, which is a function of the space and time variables, denoted by "x" and "t", respectively. Identification of "L_0" and "h" means $n_h = 1$ i.e. the units must be chosen accordingly. However, such a choice is associated with doubts. To define "$\partial^2\psi/\partial x^2$", at first **I need to define** "$\partial\psi/\partial x$" and while doing so, the following two options arise:

$$\textbf{Option 1: } \text{"}\Delta x \text{ is infinitesimally small compared to } \lambda_0\text{"} \;\Leftrightarrow\; \Delta x \lll \lambda_0, \tag{15}$$

$$\textbf{Option 2: } \text{"}\Delta x \text{ is infinitesimally small compared to } \lambda_c\text{"} \;\Leftrightarrow\; \Delta x \lll \lambda_c. \tag{16}$$

Given the choice of units for $n_h = 1$ to hold and considering the condition (13) as the flag bearer of quantum mechanics, it now becomes an intricate play of reasoning because, along with the possibilities (15) and (16), I need to take care of the relations among $\Delta p, p_0, mc$. Such reasoning needs to be done from *direct experience* and "*intuition*" in the laboratory while dealing with measuring units and measured quantities, rather than based on "*formalism*" supplied by the theorist detached from experiment[13]. An investigation is necessary for each and every step of the theory which now looks like a language that the experimenter speaks in the laboratory while making measurements in the pursuit of truth and reality. After all, as Peres boldly asserted from his "*pragmatic and strictly instrumentalist*" viewpoint, on page no. (xi) of ref.[12], "*quantum phenomena do not occur in a Hilbert space, they occur in a laboratory.*"

Science could have developed in a different way without Cauchy's logico-linguistic slip. Nevertheless, history of science can not be changed, but lessons can be learned from such logico-linguistic self-criticism (or self-inquiry[7]), which Brouwer might have called "*inner inquiry*"[14], to do science with better understanding and more refined reasoning. Noting contemporary views, in the context of my opinion, Gisin's emphasis on language of physics stands more justified now[15], albeit from a different standpoint[7, 8].

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