A New Theory on the Derivation of Metacentric Radius Governing the Stability of Ships

Tsutomu HORI † and Manami HORI ††

Abstract

In this paper, we develop a new theory on the derivation of the transverse metacentric radius which governs the stability of ships.

As a new development in its derivation process, it was shown that the direction of movement of the center of buoyancy due to lateral inclination of ship is the direction of the half angle of the heel angle θ.

By finding it, we were able to derive a metacentric radius worthy of its name by showing that the metacentric radius correctly represents the radius centered on the metacenter, which is the center of inclination.

Keywords : ***Metacentric Radius, Ship's Stability, Naval Architecture, Half Angle of the Heel Angle, Movement of the Center of Buoyancy***

1. Introduction

The transverse metacentric radius $\overline{BM}$, which governs the stability performance of ships, can be calculated as follows, where V is the volume of underwater portion and I_{CL} is the quadratic moment about the centerline of the water plane.

$$\overline{BM} = \frac{I_{CL}}{V} \quad \cdots\cdots (1)$$

Here, the above equation is a well-known result as a basic formula in naval architecture.

Eq. (1) for this $\overline{BM}$ was derived by Bouguer(1), and Nowacki(2) & Ferreiro(3) have introduced the historical background. It is also described by Goldberg(4) in the US " Principles of Naval Architecture ", the bible of naval architecture. More recently, it has been considered by Mégel and Kliava(5). In Japan and other countries, it has been described by Takagi(6), Nishikawa(7), Sugihara(8), Ohgushi(9), Ohta & Kuwahara *et al.*(10), and Akedo(11) in the past, and recently by Nohara & Shoji(12), Barrass & Derrett(13), Ikeda & Furukawa *et al.*(14) and Shin(15) in many textbooks of naval architecture and nautical mechanics.

† *Professor* , Naval Architecture Course, Department of Engineering, Faculty of Engineering, Nagasaki Institute of Applied Science. 536 Aba Machi, Nagasaki, 851-0193, Japan

†† *Jewel Manami HORI of Five Stars JP,* Daughter of †

viXra *Classical Physics*, Submitted : 3 November 2021,
E-mail : HORI_Tsutomu@NiAS.ac.jp
HomePage : http://www.ship.nias.ac.jp/personnel/horiken/ *(in Japanese)*

Although the result itself does not change with respect to such a basic formula for $\overline{BM}$ in Eq. (1), as a new development in its derivation process, it was shown that the direction of movement of the center of buoyancy due to the lateral inclination of ship is the direction of the half angle $\theta/2$ of the heel angle θ. By finding it, we were able to derive a metacentric radius $\overline{BM}$ suitable for its name by showing that the metacentric radius correctly represents the radius centered on the metacenter M, which is the center of inclination. The process of new derivation (16) was published in the Journal "*NAVIGATION*" of Japan Institute of Navigation in 2017, with intentions of receiving criticism from the distinguished scholars.

In this paper, a new development of the derivation process (16),(17) for metacentric radius $\overline{BM}$ is described in detail.

2. New Derivation of Metacentric Radius $\overline{BM}$

Fig. 1 shows a three-dimensional view of the ship when it is inclined laterally by heel angle θ to the starboard side from upright position. The water line is WL and the center of buoyancy is B in the upright state, and the water line is $W'L'$ and the center of buoyancy is B' (18),(19),(20),(21),(22) after inclination. The intersection point of the center line perpendicular to WL extending from B in the upright state and the action line of the buoyancy vertically upward from B' in the inclined state is the so-called "transverse metacenter", M.

Since both hull sides of the ship can be assumed to be perpendicular to the water surface near the water line generally, the exposed part $\triangle WoW'$ and the submerged part $\triangle LoL'$ are right triangles which is similar

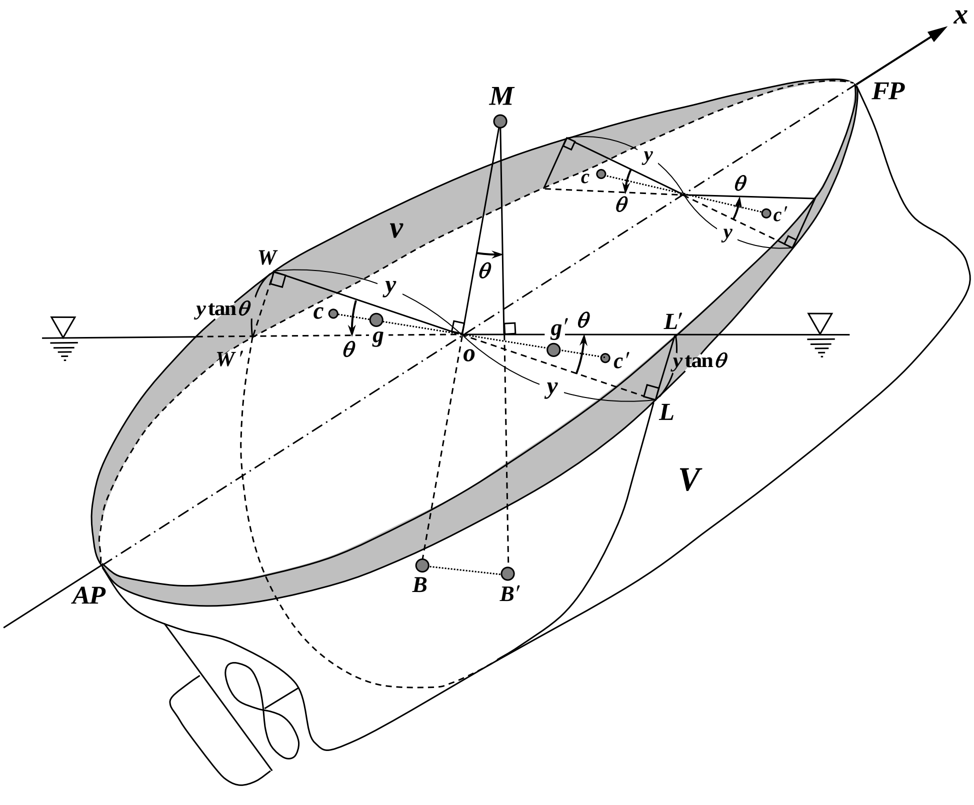

Fig. 1 Three-dimensional view of the wedge-shaped exposed and immersed portions of a laterally inclined ship.

in all cross-sections from the stern *AP* to the bow *FP*, although the waterline width $2y$ differs in the longitudinal direction *x*. Therefore, *AP-WoW'-FP* and *AP-LoL'-FP* are three-dimensionally wedge-shaped, respectively.

Since the volume *V* of ship's underwater portion remains the same after inclination, the volumes of the wedge-shaped *AP-WoW'-FP* in the exposed portion and the wedge-shaped *AP-LoL'-FP* in the immersed portion are equal. If the wedge-shaped volume is *v*, and the centroid of the exposed volume is *g* and the centroid of the immersed volume is *g'*, we can consider that a part of the underwater volume *v* has moved from *g* to *g'*.

Therefore, the direction and distance $\overline{BB'}$ when the center of buoyancy, which is the centroid of the whole underwater volume *V*, moves from *B* to *B'* are determined as follows :

$$\left.\begin{aligned} &\overline{BB'} \,/\!/\, \overline{gg'} \\ &\overline{BB'} = \frac{v}{V} \cdot \overline{gg'} \end{aligned}\right\} \cdots\cdots (2)$$

The result of Eq. (2) above is the dynamical law described in the textbooks[(4),(6)-(15)] of naval architecture and nautical mechanics, as in Eq.(1). In this paper, this law will be carefully explained in Appendix A-1. There, in Eq. (A-9) of its Appendix, *A* and *a* for area are replaced by *v* and *V* for volume.

2.1 Consideration on the direction of movement $\overline{BB'}$ *of* the center of buoyancy

Fig. 2 depicts the cross-section of the laterally inclined ship shown in Fig. 1 at a certain ship's longitudinal ordinate *x*. Since the areas of the right triangles $\triangle WoW'$ and $\triangle LoL'$ in the exposed and immersed parts of the cross-section are equal, they are written as *a*, and their centroids of area are written as *c* and *c'* respectively. Since *a* and *c*, *c'* are functions of *x*, the volumes *v* of the wedge-shaped *AP-WoW'-FP* and *AP-LoL'-FP*, their moving moments $v \cdot \overline{gg'}$, and the direction of $\overline{gg'}$ can be obtained by integrating from *AP* to *FP* in the longitudinal direction *x*, respectively, as follows :

$$\left.\begin{aligned} &v = \int_{AP}^{FP} a\,dx \\ &v \cdot \overline{gg'} = \int_{AP}^{FP} a \cdot \overline{cc'}\,dx \\ &\overline{gg'} \,/\!/\, \overline{cc'} \end{aligned}\right\} \cdots\cdots (3)$$

Here, the line segment $\overline{gg'}$ connecting *g* and *g'* is coincide with the line segment $\overline{cc'}$ connecting the centroid of are of the right triangles $\triangle WoW'$ and $\triangle LoL'$ in the cross-section, though the lengths are different, as shown in Fig. 1 and Fig. 2.

Hereafter, paying attention to the right triangle $\triangle LoL'$ of the immersive part shown in Fig. 2, let's determine the direction of $\overline{cc'}$ according to $\overline{oc'}$ on starboard side. This is the core of the argument in this paper. Here, the heel angle due to lateral inclination is $\angle LoL' = \theta$, the angle formed by $\overline{oc'}$ and the base

$\overline{oL}$ is $\angle Loc' = \varphi$, and the length of the triangular base $\overline{oL}$ corresponding to the half width of the water line WL is y. Here, the centroid c' of triangle $\triangle oLL'$ is located at two-thirds of $\overline{oL} = y$ in the base direction and one-third of $\overline{LL'} = y\tan\theta$ in the height direction, so the tangent of φ is obtained as

$$\left.\begin{aligned} \tan\varphi &= \frac{\frac{1}{3}\overline{LL'}}{\frac{2}{3}\overline{oL}} = \frac{\frac{1}{2}\overline{LL'}}{\overline{oL}} = \frac{\frac{1}{2}y\tan\theta}{y} = \frac{1}{2}\tan\theta \\ \therefore\ \varphi &= \tan^{-1}\left(\frac{1}{2}\tan\theta\right) \end{aligned}\right\} \cdots\cdots\cdots (4)$$

This result of the former in above equation means that if we extend $\overline{oc'}$ through the centroid c' of the triangle $\triangle oLL'$, it will pass through the midpoint of the opposite side $\overline{LL'} = y\tan\theta$, which confirms what geometry teaches.

Now, if we assume that $|\varphi| \ll 1$ and $|\theta| \ll 1$ in the 2nd. line of Eq. (4), the angle φ can be Taylor-expanded with respect to θ as follows :

$$\begin{aligned} \varphi &= \tan^{-1}\left(\frac{1}{2}\tan\theta\right) \\ &= \frac{1}{2}\tan\theta - \frac{1}{3}\left(\frac{1}{2}\tan\theta\right)^3 + \cdots \\ &= \frac{1}{2}\left(\theta + \frac{\theta^3}{3} + \cdots\right) - \frac{1}{24}\left(\theta + \frac{\theta^3}{3} + \cdots\right)^3 + \cdots \\ &= \frac{\theta}{2} + \frac{\theta^3}{8} + \cdots \end{aligned} \cdots\cdots\cdots (5)$$

Strictly speaking, φ is slightly larger than $\theta/2$ according to the above equation, but the following relational expression is obtained when the heel angle θ is small to some extent, actually up to about 20°, in the range where W and L' in Fig. 2 are on both hull sides. Therefore, we find that φ is a half angle of θ as follows :

$$\varphi = \angle Loc' = \frac{\theta}{2} \cdots\cdots\cdots (6)$$

By doing so, the direction of movement of $\overline{oc'}$, *i.e.* $\overline{cc'}$, could be correctly determined within the range of linear theory regarding the heel angle θ in the cross-section at longitudinal ordinate x.

Therefore, it is found from Eqs. (2), (3) and (6) that $\overline{BB'}$ in underwater volume moves in the same direction as $\overline{gg'}$ in wedge shape and $\overline{cc'}$ in cross-section, as follows :

$$\angle L''BB' \left(= \angle Log' = \angle Loc' \right) = \varphi = \frac{\theta}{2} \cdots\cdots\cdots (7)$$

The conclusion of this section is that the direction $\angle L''BB'$ of movement $\overline{BB'}$ from the upright center of buoyancy B to the inclined center of buoyancy B' is the direction of the half angle of the heel angle θ.

2.2 Metacentric radius $\overline{BM}$ in the true physical sense

Let's apply Eq. (7), which is a consequence of the previous section, to $\triangle MBB'$ in the cross-section of the inclined ship shown in Fig. 2. Since $\angle L''BM$ is a right angle, the angle $\angle MBB'$ can be obtained as

$$\angle MBB' = \angle L''BM - \angle L''BB' = \frac{\pi}{2} - \varphi = \frac{\pi}{2} - \frac{\theta}{2} \quad \cdots\cdots (8)$$

On the other hand, since the sum of the interior angles of a triangle is π, it can be written as follows :

$$\angle MBB' + \theta + \angle MB'B = \pi \quad \cdots\cdots (9)$$

Now, by using Eq. (8) in Eq. (9), the angle $\angle MB'B$ can be calculated as

$$\angle MB'B = \pi - \theta - \angle MBB' = \pi - \theta - \left(\frac{\pi}{2} - \frac{\theta}{2} \right) = \frac{\pi}{2} - \frac{\theta}{2} \quad \cdots\cdots (10)$$

Therefore, since the right-hand sides of Eqs. (8) and (10) are equal, the following equality relation is obtained.

$$\angle MBB' = \angle MB'B \left(= \frac{\pi}{2} - \frac{\theta}{2} \right) \quad \cdots\cdots (11)$$

From this relationship, we can find that $\triangle MBB'$ is an isosceles triangle with transverse metacenter M as its vertex. As a result, we were able to show the following relation.

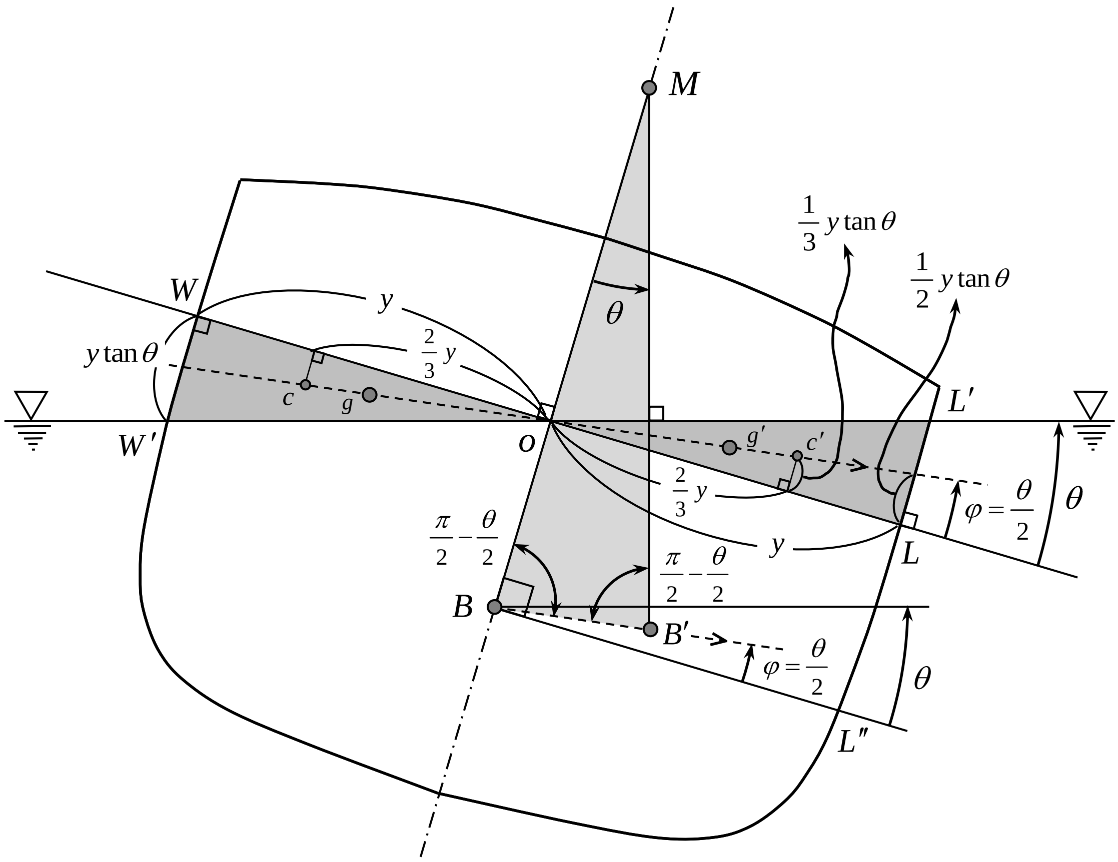

Fig. 2 Metacenter and movement of the center of buoyancy in the cross-section of a laterally inclined ship.

$$\overline{BM} = \overline{B'M} \quad \cdots\cdots (12)$$

From this equality relation, it can be seen that both $\overline{BM}$ and $\overline{B'M}$ are geometrically the radii of the circle centered on M. In this way, we have been able to derive a metacentric radius $\overline{BM}$ worthy of the name. We wouldn't like to think that it is self-righteousness of the authors to claim so.

2.3 Relationship between $\overline{BM}$ and $\overline{BB'}$

Let's find the moving distance $\overline{BB'}$ of the center of buoyancy according to the explanation in the previous section. Applying the cosine theorem to triangle $\triangle MBB'$ shown in Fig. 2, the square of $\overline{BB'}$ can be obtained by using Eq. (12) as follows :

$$\begin{aligned}\overline{BB'}^{\,2} &= \overline{BM}^{\,2} + \overline{B'M}^{\,2} - 2\,\overline{BM}\cdot\overline{B'M}\,\cos\theta = 2\,\overline{BM}^{\,2}(1-\cos\theta) \\ &= \overline{BM}^{\,2}\left(\theta^2 - \frac{\theta^4}{12} + \cdots\right) \quad \cdots\cdots (13)\end{aligned}$$

Then, by taking the square root of the above equation, $\overline{BB'}$ can be calculated as follows :

$$\begin{aligned}\overline{BB'} &= \overline{BM}\sqrt{2(1-\cos\theta)} \\ &= \overline{BM}\left(\theta - \frac{\theta^3}{24} + \cdots\right) \quad \cdots\cdots (14)\end{aligned}$$

Here, the 2nd. line of both Eqs. (13) and (14) above are the results by means of the Taylor expansion of $\cos\theta$ with respect to θ, assuming $|\theta| \ll 1$.

Therefore, when the heel angle θ is somewhat small, the moving distance $\overline{BB'}$ of the center of buoyancy can be obtained in a simple form by using only the 1st. term in the 2nd. line of Eq. (14), as follows :

$$\overline{BB'} = \overline{BM}\,\theta \;\left(= \overset{\frown}{BB'}\right) \quad \cdots\cdots (15)$$

Hence, the result of the above Eq. (15) shows that the line segment $\overline{BB'}$ is equal to the arc length $\overset{\frown}{BB'}$ with $\overline{BM}$ as its radius, when θ is small to some extent.

Therefore, the metacentric radius $\overline{BM}$ can be calculated by solving Eq. (15) as follows :

$$\overline{BM} = \frac{\overline{BB'}}{\theta} \quad \cdots\cdots (16)$$

The above Eq. (16) shows that $\overline{BM}$ can be determined by dividing the moving distance $\overline{BB'}$ of the center of buoyancy by heel angle θ .

2.4 Moving distance $\overline{BB'}$ of center of buoyancy

In this section, let us consider the determination of $\overline{BB'}$ by using the dynamical law of Eq. (2). The area a of each of the right triangles $\triangle WoW'$ and $\triangle LoL'$ in the cross-section shown in Fig. 2 and the line segment

$\overline{cc'}$ connecting their centroid can be written as follows, using the important Eq. (6), where $\varphi = \dfrac{\theta}{2}$.

$$\left.\begin{aligned} a &= \frac{1}{2} y^2 \tan\theta = \frac{1}{2} y^2 \left(\theta + \frac{\theta^3}{3} + \cdots \right) \\ \overline{cc'} &= 2\,\overline{oc'} = \frac{4}{3} y \sec\frac{\theta}{2} = \frac{4}{3} y \left(1 + \frac{\theta^2}{8} + \cdots \right) \end{aligned}\right\} \qquad \cdots\cdots (17)$$

Here, in the above equation, the Taylor-expanded form for θ is also given. The moving moment $a \cdot \overline{cc'}$ is then the product of the two in Eq. (17), and is calculated as follows :

$$\begin{aligned} a \cdot \overline{cc'} &= \frac{2}{3} y^3 \tan\theta \sec\frac{\theta}{2} \\ &= \frac{2}{3} y^3 \left(\theta + \frac{\theta^3}{3} + \cdots \right) \cdot \left(1 + \frac{\theta^2}{8} + \cdots \right) = \frac{2}{3} y^3 \left(\theta + \frac{11}{24} \theta^3 + \cdots \right) \end{aligned} \qquad \cdots\cdots (18)$$

Hence, when the heel angle θ is somewhat small, the moving moment $a \cdot \overline{cc'}$ can be obtained by using the 1st. order term with respect to θ in above Eq. (18), as follows :

$$a \cdot \overline{cc'} = \frac{2}{3} y^3 \theta \qquad \cdots\cdots (19)$$

Now, by integrating the above Eq. (19) from the stern *AP* to the bow *FP* in the longitudinal direction *x*, as shown in Eq. (3), the moving moment $v \cdot \overline{gg'}$ of the wedge-shaped volume *v* can be calculated as follows :

$$\begin{aligned} v \cdot \overline{gg'} &= \int_{AP}^{FP} a \cdot \overline{cc'}\, dx = \frac{2}{3} \theta \int_{AP}^{FP} y^3 dx \\ &= \theta \int_{AP}^{FP} \frac{(2y)^3}{12} dx = I_{CL} \cdot \theta \end{aligned} \qquad \cdots\cdots (20)$$

Here, since the integral in the above Eq. (20) corresponds to the quadratic moment of the rectangle of height $2y$ and width dx, it represents the quadratic moment I_{CL} with respect to the center line of the water plane, as shown by single-dotted line in Fig. 1. Therefore, the moving distance $\overline{BB'}$ of the center of buoyancy can be determined by the latter part of the dynamical law in Eq. (2), as follows :

$$\overline{BB'} = \frac{v \cdot \overline{gg'}}{V} = \frac{I_{CL} \cdot \theta}{V} \qquad \cdots\cdots (21)$$

The above Eq. (21) shows that the moving distance $\overline{BB'}$ can be calculated by dividing the product of the quadratic moment I_{CL} and the heel angle θ shown in Eq. (20) by the underwater volume *V* of a ship.

2.5 Calculation formula for metacentric radius $\overline{BM}$

According to the results of Sections 2.3 and 2.4, the transverse metacentric radius $\overline{BM}$ can be determined by substituting the moving distance $\overline{BB'}$ obtained in Eq. (21) for the numerator of the right-hand side in Eq.(16), as follows :

$$\overline{BM} = \frac{\overline{BB'}}{\theta} = \frac{\dfrac{I_{CL} \cdot \theta}{V}}{\theta} = \frac{I_{CL}}{V} \qquad \cdots\cdots (22)$$

The metacentric radius $\overline{BM}$ of the above Eq. (22) can be calculated only by the geometric shape of the ship under the water plane, regardless of the heel angle θ which cancels out the numerator and denominator. Therefore, $\overline{BM}$ has a meaning as a parameter which governs the stability performance of a ship. The result is a well-known formula that can be found in any textbook[(4),(6)-(15)] of naval architecture and nautical mechanics.

3. Some Considerations

In this chapter, we will consider the explanations given in the textbooks so far.

In most textbooks[(4),(6),(7),(9),(10),(11)], the moving direction of the center of buoyancy due to lateral inclination is approximated as follows, by assuming that heel angle θ in Fig. 2 is tends to zero.

$$\left.\begin{array}{l} \overline{BB'} \,/\!/\, \overline{WL} \\ \angle MBB' = \dfrac{\pi}{2} \end{array}\right\} \cdots\cdots (23)$$

As a result, the moving distance $\overline{BB'}$ of the center of buoyancy is often described as

$$\overline{BB'} = \overline{BM}\ \tan\theta \quad \cdots\cdots (24)$$

Here, Goldberg[(4)], Nishikawa[(7)], Ohgushi[(9)], and Akedo[(11)] specify the Eq. (23).

In addition, Sugihara[(8)], Nohara & Shoji[(12)], Barrass & Derrett[(13)], and Shin[(15)] do not specify the direction of movement $\overline{BB'}$, but wrote its moving distance as follws, as well as Eq. (15) in Section 2.3.

$$\overline{BB'} = \overline{BM}\ \theta \quad \cdots\cdots (25)$$

On the other hand, recent work by Ikeda & Furukawa *et al.*[(14)] accurately calculated the moving component parallel to $\overline{WL}$, not the moving distance $\overline{BB'}$. If we use the result of Eq. (11) and write it in the notation of this paper, it will be as follows :

$$\overline{BB'}\cos\frac{\theta}{2} = \overline{B'M}\ \sin\theta \quad \cdots\cdots (26)$$

After all, the correct direction of movement of $\overline{BB'}$ is still not mentioned, and they derive the result by avoiding it.

4. Summary of the Results Obtained

It is claimed in this paper that the direction $\angle L''BB'$ of movement $\overline{BB'}$ from the upright center of buoyancy B to the inclined center of buoyancy B' is the direction of the half angle of the heel angle θ due to lateral inclination as follows :

$$\angle L''BB' \left(= \angle Log' = \angle Loc' \right) = \varphi = \frac{\theta}{2} \quad \cdots\cdots \text{previously written (7)}$$

Here, the above equation is obtained by the moving direction $\angle Loc'$ of a partial area from the exposed to the immersed portion, as given in Eq. (6).

As a result, we obtained the following relationship using by Eq. (7) of Section 2.2.

$$\angle MBB' = \angle MB'B \left(= \frac{\pi}{2} - \frac{\theta}{2} \right) \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{ previously written (11)}$$

By doing so, since we were able to show that $\triangle MBB'$ shown in Fig. 2 is an isosceles triangle with metacenter M as its vertex, the following Eq. (12) was found as the radii centered on the metacenter M.

$$\overline{BM} = \overline{B'M} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ previously written (12)}$$

In this way, it is considered that the metacentric radius $\overline{BM}$ suitable for the name could be derived geometrically.

As mentioned above, the conclusions of this paper can be summarized in the above Eqs. (7), (11) and (12). Subsequently, in Section 2.3 onwards, the well-known formula (22) for the metacentric radius $\overline{BM}$ is described within the framework of the linear theory for the heel angle θ, according to the usual method.

5. Concluding Remarks

One of the authors[(23)] has been teaching "Hydrostatics of Floating Bodies" as a compulsory subject in the Department of Naval Architecture (currently the Naval Architecture Course[(24),(25)]) at *Nagasaki Institute of Applied Science* for more than ten years. Every year, especially in the last few years, I have been guilty of somewhat misrepresenting the moving direction of the center of buoyancy $\overline{BB'}$ due to lateral inclination when explaining the theory of metacenter, which is the title of this paper. I have been lecturing on it, telling myself that it is an approximation by a minimal angle of inclination. I was always going to the lecture with reluctant heart because I was afraid of being questioned by the excellent students.

By summarizing this paper, I felt relieved from this worry, but I thought that it should not be self-righteous, so I submitted it. I am prepared to receive criticism from the great scholars who already knew the insistence in this paper and are lecturing as such. In addition, if the contents of this paper have already been published in textbooks or papers, please forgive me for the lack of searching related literature by an illiterate author.

Acknowledgments

In closing this paper, we would like to say the following words of thanks from the 1st. author[(23)]. I would like to communicate my deepest gratitude to my late teacher, *Pr. Masato KURIHARA*[(26),(27),(28)], who cordially taught me the theory of "Hydrostatics of Ships" with detailed figures and formulas on the blackboard when I was a first-year undergraduate student and learned my first specialized subject of naval architecture in *the College of Naval Architecture of Nagasaki*. Therefore, I am following the appearance of my teacher at that time from more than 40 years ago as an exemplary example, when the author gives lectures to students on Hydrostatics of Floating Bodies and Theory of Ship Stability.

Finally, I would like to express my heartfelt gratitude to *Dr. Yoshihiro KOBAYASHI*, former professor at Sojo University and current president of Como-Techno Ltd., who always gave warm encouragement to the author's research and recommended that this new theory should be published in English. I am greatly inspired by the vigorous academic spirit with which he writes about the results of his research in books[(29),(30)].

References ‡

(1) Bouguer, P. : " Traité du Navire, de sa Construction, et de ses Mouvemens ", (Treatise of the Ship, its Construction and its Movements), Livre II, Section II, **Edited by Jombert, C.**, Paris, 1746 (January), p.249-324.

(2) Nowachi, H and Ferriro, L. D. : " Historical Roots of the Theory of Hydrostatic Stability of Ships ", 8th. International Conference on the Stability of Ships and Ocean Vehicles, ***Escuela Técnica Superior de Ingenieros Navales***, 2003, pp.1-30.

(3) Ferreiro, L. D. : " Ships and Science —The Birth of Naval Architecture in the Scientific Revolution, 1600-1800 —", Chapter 4 — Inventing the Metacenter, *The MIT Press* (Cambridge, Massachusetts), 2006 (November), Hard-Cover, p.187-257.

(4) Goldberg, L. L. : " Principles of Naval Architecture (2nd. Revision), Volume I - Stability and Strength ", Chapter 2 – Intact Stability, Section 3. 2 – Location of the Transverse Metacenter, Edited by Lewis, E. V., *The Society of Naval Architects and Marine Engineers*, Jersey City, NJ, 1988 (April) 1st. Printing, p.72.

(5) Mégel, J. and Kliava, J. : " On the Buoyancy Force and the Metacentre ", 2009 (June), ***arXiv:0906.1112***, pp.1-33.

(6) Takagi, Y. : " Nautical Mechanics and its Applications " (in Japanese), Chapter 6 – Stabilitys, Section 3 – Metacenter and *BM* (metacentric radius), *Seizan-dou Publishing,* 1960 (March) 1st. Printing, p.206-210.

(7) Nishikawa, H. : " Primary Hydrostatics of the ship " (in Japanese), Chapter 4 – Theory of Floating Bodies, Section 4. 10 – Transverse metacenter and *BM*, *Kaibun-dou Publishing,* 1964 (July) 1st. Printing, p.92-94.

(8) Sugihara, K. : " Nautical Theory (Ship Mechanics Division) " (in Japanese), Chapter 3 – Transverse Stability, Section 3. 3 – Calculation of $\overline{BM}$ and the approximate value, *Kaibun-dou Publishing,* 1964 (July) 1st. Printing, p.54-56.

(9) Ohgushi, M. : " Theoretical Naval Architecture (Upper Volume) —New Revision— " (in Japanese), Chapter 4 – Equilibrium of Ships, Transverse and Longitudinal Metacenter, Change of Trim, Section 4.4 – Transverse Metacenter and $\overline{BM}$, *Kaibun-dou Publishing,* 1971 (June) 1st. Printing, p.81-83.

(10) Ohta, T., Kuwahara, K., Kotani, T., Tamaki, K., Nishikawa, H., Baba, K., Masui, S. and Mitamura, T. : " Naval Architectural Engineering " (in Japanese), Chapter III – Theory and Design of the Ships, Section 2. 4. 2 – Transverse Metacenter and the Transverse Metacentric Radius, Edited by National Naval Architectural Education Research Association, *Kaibun-dou Publishing,* 1975 (June) 1st. Printing, p.164-165.

‡ Bold text in the list means that there is a HyperLink.

(11) Akedo, N. : "Basic Nautical Mechanics" (in Japanese), Chapter 3 – Stability of the Ships, Section 3.1.3 – Metacentric Radius, *Kaibun-dou Publishing,* 1983 (June) 1st. Printing, p.121-125.

(12) Nohara, T. (Original) and Shoji K. (Author) : "Nautical Naval Architecture (2nd. Edition)" (in Japanese), Chapter 9 – Stability, Section 9. 4 – Metacenter, *Kaibun-dou Publishing,* 2005 (April) 1st. Printing of 2nd. Edition, p.174-175.

(13) Barrass, C. B. and Derrett, D.R. : " Ship Stability for Masters and Mates", Part 1 – Linking Ship Stability and Ship Motions, Chapter 12 – Calculating *KB*, *BM a*nd metacentric diagrams, To find transverse *BM*, ***Elsevier Ltd.,*** 2006 (February), 6th. Edition, p.106-108.

(14) Ikeda, Y., Furukawa, Y., Katayama, T., Fujii, T., Murai, M. and Yamaguchi S : "Hydrostatics of Ships and Stability —Series(i) for the Naval Architects and the Ocean Engineers—", (in Japanese), Chapter 4 – Basic of the Stability, Section 4. 2. 1 – Transverse Stability**,** Edited by the Textbook Compilation Committee of the Ability Development Center, *Seizan-dou Publishing,* 2012 (April) 1st. Printing, p.67-68.

(15) Shin, C. I. : "Capsizing and Stability of the Ships" (in Japanese), Chapter 4 – Transverse Stability of the Ships, Section 4. 2 – How to Find the Metacentric Radius *BM* and the Position of the Metacenter, *Seizan-dou Publishing,* 2021 (July) 1st. Printing, p.41-43.

(16) Hori, T. : "A Consideration on Derivation of Ship's Transverse Metacentric Radius $\overline{BM}$ " (in Japanese), ***NAVIGATION*** (Journal of Japan Institute of Navigation), 2017 (April), ***No.200*** (First 200th Anniversary Issue), pp.75-79.

(17) Hori, T. : "New Developments in the Fundamental Theory for Hydrostatics of Floating Bodies — Part 2 : A Consideration on Derivation of Ship's Transverse Metacentric Radius $\overline{BM}$ — " (in Japanese), *Boat Engineering*, 2018 (December), No.136, pp.1-5.

(18) Hori, T. : " A Positioning on Ship's Centre of Buoyancy Derived by Surface Integral of Hydrostatic Pressure — Proof that Centre of Buoyancy is Equal to Centre of Pressure — " (in Japanese), ***NAVIGATION*** (Journal of Japan Institute of Navigation), 2018 (January), ***No.203***, pp.88-92.

(19) Hori, T. : " A Positioning on Ship's Centre of Buoyancy Derived by Pressure Integral of Hydrostatic Pressure — Part 2 : In the Case of Arbitrary Sectional Form — " (in Japanese), ***NAVIGATION*** (Journal of Japan Institute of Navigation), 2018 (July), ***No.205***, pp.28-34.

(20) Hori, T. : "New Developments in the Fundamental Theory for Hydrostatic of Floating Bodies — Part 1 : The Proof that Centre of Buoyancy is Equal to Centre of Pressure — " (in Japanese), *Boat Engineering*, 2018 (September), No.135, pp.1-10.

(21) Hori, T. : " A Positioning on Ship's Center of Buoyancy Derived by Surface Integral of Hydrostatic Pressure — Part 6 : The Proof for Submerged and Floating Body of Arbitrary Form — " (in Japanese), *NAVIGATION* (Journal of Japan Institute of Navigation), 2021 (January), No.215, pp.69-77.

(22) Hori, T. : "Proof that the Center of Buoyancy is Equal to the Center of Pressure by means of the Surface Integral of Hydrostatic Pressure Acting on the Inclined Ship", 2021 (September), ***viXra:2109.0008*** [Ver.2], Classical Physics, pp.1-20.

(23) Hori, T. : "Ship Waves Laboratory's Web Page" (in Japanese), Naval Architecture Course in *Nagasaki Institute of Applied Science*, ***http://www2.cncm.ne.jp/~milky-jun_0267.h/HORI-Lab/*** **.**

(24) Hori, T. : "Naval Architecture Course, Department of Engineering , Faculty of Engineering, Nagasaki Institute of Applied Science" (in Japanese), Introduction of Educational and Research Institutes, *NAVIGATION* (Journal of Japan Institute of Navigation), 2021 (January), No.215, pp.38-45.

(25) "Naval Architecture Course's Web Site" (in Japanese), Faculty of Engineering in *Nagasaki Institute of Applied Science*, administrated by Hori, T. , ***http://www.ship.nias.ac.jp/*** .

(26) Kurihara, M. : "On the Rolling Motion of a Buoy in Regular Waves" (in Japanese), *The Bulletin of the College of Naval Architecture of Nagasaki*, 1974 (June), No.15, No.1, pp.1-4.

(27) Kurihara, M. : "On the Motions of a Ringed Buoy in Regular Waves" (in Japanese), *The Bulletin of Nagasaki Institute of Applied Science*, 1978 (October), No.19 (Commemorative issue of the name change from former the College of Naval Architecture of Nagasaki), pp.11-15.

(28) Kurihara, M. : "On the Motions of a Floating Vertical Cylinder in Regular Waves" (in Japanese), *The Bulletin of Nagasaki Institute of Applied Science* (former the College of Naval Architecture of Nagasaki), 1979 (June), No.20, No.1, pp.1-5.

(29) Kobayashi, Y. : "Tank System of LNG-LH2 —Physical Model and Thermal Flow Analysis by Using CFD —" (in Japanese), *Seizan-dou Publishing,* 2016 (December), p.1-375.

(30) Kobayashi, Y. : "Utilization System of LNG-LH2 at Ultra-Low Temperature and Cold Heat —No Waste Energy System—" (in Japanese), *Shouwa-dou Publishing,* 2019 (April), p.1-226.

(31) Hori, T. : "A New Derivation of Metacentric Radius (i) Positioning the metacenter (63 minutes in the 1st. half) " (in Japanese), 2021 (14 October), Regular Lecture No.3 of "Stability of the Ship" (Specialized Subject of *Nagasaki Institute of Applied Science*), ***https://youtu.be/lUWbQ92zJQQ*** .

(32) Hori, T. : "A New Derivation of Metacentric Radius (ii) Calculation formula for metacentric radius (82 minutes in the 2nd. half) " (in Japanese), 2021 (21 October), Regular Lecture No.4 of "Stability of the Ship" (Specialized Subject of *Nagasaki Institute of Applied Science*), ***https://youtu.be/qAIzLKXSY4U*** .

(33) Hori, T. : "A Typical Example on Ship's Stability Theorem" (in Japanese), *NAVIGATION* (Journal of Japan Institute of Navigation), 2021 (July), No.217, pp.39-46.

(34) Hori, T. : "Example on the Stability Theory of Ships (i) Stable condition of a columnar ship with rectangular cross-section of different widths (85 minutes) " (in Japanese), 2020 (19 November), Regular Lecture No.8 of "Stability of the Ship" (Specialized Subject of *Nagasaki Institute of Applied Science*), ***https://youtu.be/PNVuRuZWYBM*** .

(35) Hori, T. : " Seminar on the Stability Theory of Ships (Outline explanation of the theory and model experiment in a small water tank) (15 minutes) " (in Japanese), 2020 (7 August), Inquiry learning for high school students (Nagasaki Prefectural Seiryo High School) conducted online, ***https://youtu.be/4T6znj1iKPI*** .

(36) Hori, T. : "Example on the Stability Theory of Ships (ii) Stable condition of a columnar ship with square cross-section of different materials (92 minutes) " (in Japanese), 2020 (26 November), Regular Lecture No.9 of "Stability of the Ship" (Specialized Subject of *Nagasaki Institute of Applied Science*), ***https://youtu.be/eeVg9ThjPd0*** .

(37) Hori, T. : "An Advanced Example on Ship's Stability Theorem —Solution for Stable Attitude of an Inclined Ship—" (in Japanese), *NAVIGATION* (Journal of Japan Institute of Navigation), 2021 (October), No.218, pp.58-65

Appendix

A-1. Movement of the centroid of whole area when a partial area moves

Fig. A-1 shows the case that a square □*ABDC* (area *A*, centroid *G*) transforms into an isosceles triangle △*CBE* (area *A*, centroid *G'*), when a right triangle △*ABC* (gray-filled area *a*, centroid *g*) is rotated 90° counterclockwise around point *C* and moved to a right triangle △*CDE* (gray-filled area *a*, centroid *g'*).

In this Appendix A-1, let's consider the distance and direction of movement of the centroid of the whole area, *i.e.*, from *G* of the square □*ABDC* to *G'* of the isosceles triangle △*CBE*. The right triangle △*CBD* (white-filled area *A*-*a*, centroid *o*) in Fig. A-1 is a fixed and common area before and after the movement. Here, the centroid *G* of the whole area is located geometrically on the line segment $\overline{og}$ connecting the respective centroids *o* and *g*, and *G'* is located on the line segment $\overline{og'}$ connecting *o* and *g'*.

§ 1. General theory

Firstly, we will develop the general theory without setting a specific area *etc.*.

For the square □*ABDC* before the move, the following equation holds from the equilibrium of the area moments of *a* and *A* around point *o*, which is the centroid of a fixed triangle △*CBD*.

$$\left.\begin{aligned} a\cdot\overline{og} &= A\cdot\overline{oG} \\ \rightarrow a\cdot\ell_g &= A\cdot\ell_G \end{aligned}\right\} \quad \cdots\cdots \text{(A-1)}$$

Here, for simplicity's sake, we have written $\overline{og}=\ell_g$, $\overline{oG}=\ell_G$. By the above equation, the following relation is obtained as :

$$\frac{\ell_G}{\ell_g}=\frac{a}{A} \quad \cdots\cdots \text{(A-2)}$$

Next, for the isosceles triangle △*CBE* after the move, the following equation holds from the equilibrium of the area moments of *a* and *A* around the point *o* as well.

$$\left.\begin{aligned} a\cdot\overline{og'} &= A\cdot\overline{oG'} \\ \rightarrow a\cdot\ell'_g &= A\cdot\ell'_G \end{aligned}\right\} \quad \cdots\cdots \text{(A-3)}$$

Here, we have abbreviated $\overline{og'} = \ell'_g$, $\overline{oG'} = \ell'_G$ in the same way. By the above equation, the following relation is obtained as well.

$$\frac{\ell'_G}{\ell'_g} = \frac{a}{A} \quad \cdots\cdots \text{(A-4)}$$

Let us now consider the trapezoid $\triangle ABEC$, which combines three right triangles, two before and after the move and one fixed. By Eqs. (A-2) and (A-4), the following relationship can be easily derived as :

$$\frac{\ell_G}{\ell_g} = \frac{\ell'_G}{\ell'_g} \left(= \frac{a}{A} \right) \quad \cdots\cdots \text{(A-5)}$$

This indicates that the scale ratio on the left side of the two small $\triangle GoG'$ and large $\triangle gog'$ triangles is equal to that on the right side. By transforming the above equation, we can obtain the relational equation as follows :

$$\frac{\ell_G}{\ell'_G} = \frac{\ell_g}{\ell'_g} \quad \cdots\cdots \text{(A-6)}$$

It shows that the ratio of the left side to the right side is the same in the two small $\triangle GoG'$ and large $\triangle gog'$ triangles. Furthermore, the apex angles of both small and large triangles are clearly common as follows :

$$\angle GoG' = \angle gog' \quad \cdots\cdots \text{(A-7)}$$

Therefore, according to Eqs. (A-6) and (A-7) above, we can see that both small and large triangles are similar as follows :

$$\triangle GoG' \backsim \triangle gog' \quad \cdots\cdots \text{(A-8)}$$

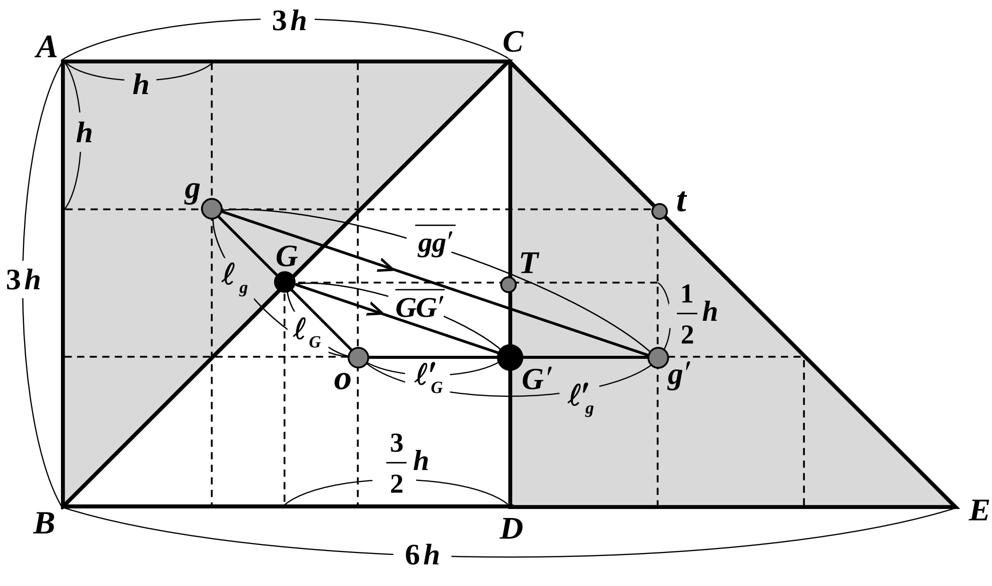

Fig. A-1 Movement of the centroid of whole area when a partial area moves.

As a result of the above discussion, it can be seen that the ratio of $\overline{GG'}$ to $\overline{gg'}$, which corresponds to the base of both triangles, is also the same as that in Eq. (A-5), and the two are parallel. It can be written as follows :

$$\left.\begin{array}{l} \dfrac{\overline{GG'}}{\overline{gg'}} = \dfrac{a}{A} \ (<1) \ \rightarrow \ \therefore \overline{GG'} = \dfrac{a}{A} \cdot \overline{gg'} \\ \overline{GG'} \ /\!/ \ \overline{gg'} \end{array}\right\} \cdots\cdots (A\text{-}9)$$

The above equation is the law of dynamics as described in textbooks[(4),(6)-(15)] on naval architecture and nautical mechanics. There is no restriction on the size of the area ratio a/A in the 1st. equation above, except that it is less than one. In this appendix, we have discussed the case where the area moves, which is the easiest to understand, but it can be applied by replacing a and A in the above Eq. (A-9) with v and V for volume and w and W for weight.

§ 2. Numerical calculations for the verification of §1

In this section, let's set numerical values for the area *etc.* and do some calculations. In that sense, the state of Fig. A-1 can be verified by the theory of §1, because the position of the centroid G and G' before and after the move is geometrically known.

As shown in Fig. A-1, the square □$ABDC$ has a side of $3h$ before the move and the isosceles triangle $\triangle CBE$ has a base of $6h$ and a height of $3h$ after the move, the two moving right triangles $\triangle ABC$ and $\triangle CDE$ have a base and a height of $3h$. Therefore, the whole area A, the moving area a and their ratio are written as follows :

$$\left.\begin{array}{l} A = 9h^2 \\ a = \dfrac{9}{2}h^2 \end{array}\right\} \rightarrow \dfrac{a}{A} = \dfrac{1}{2} \cdots\cdots (A\text{-}10)$$

Now, since the distance and direction of the movement of centroid of the whole area A due to the movement of a partial area a are shown in Eq. (A-9), we will consider the moving distance by breaking it down into its horizontal and vertical components.

Here, as shown in Fig. A-1, each component in the moving distance of centroid of a partial area a is geometrically as follows :

$$\left.\begin{array}{ll} \text{Horizontal} : & \overline{gt} = 3h \\ \text{Vertical} : & \overline{tg'} = h \end{array}\right\} \cdots\cdots (A\text{-}11)$$

Therefore, the moving distance of centroid of the whole area A can be determined for horizontal and vertical direction respectively by adopting the value of Eqs. (A-10) and (A-11) into Eq. (A-9) as follows :

$$\left.\begin{array}{ll} \text{Horizontal} : & \overline{GT} = \dfrac{1}{2}\overline{gt} = \dfrac{3}{2}h \\ \text{Vertical} : & \overline{TG'} = \dfrac{1}{2}\overline{tg'} = \dfrac{1}{2}h \end{array}\right\} \cdots\cdots (A\text{-}12)$$

Then, the result of the above equation places the point G' at one-third of the height $\overline{DC}$ of the isosceles triangle $\triangle CBE$, just above the midpoint D of the base $\overline{BE}$. This point G' is correctly the centroid of the isosceles triangle $\triangle CBE$. Since this fact is consistent with what geometry teaches, we were able to verify that Eq. (A-9), which is derived in the general theory of §1, is correct.

A-2. Lecture videos on a new derivation of metacentric radius $\overline{BM}$

The content of this paper, in which a new derivation process for metacentric radius $\overline{BM}$ is developed, is lectured to second-year students of the naval architecture course (24),(25) as a subject of "Stability of the Ship" at the university where the one of the authors (23) works.

With the recent trend of remote lectures, the situation above is shot in two parts, the 1st. half (31) and the 2nd. half (32), and on-demand teaching materials are created and uploaded as YouTube videos. The explanation is in Japanese, but if you are interested, please have a look.

A-3. Introducing examples and videos on the stability theory of ships

One of the authors (23) has published two typical examples and three videos on the stability theory of ships based on the metacentric radius, which is the main topic of this paper. We will introduce them in this Appendix A-3.

First, the basic example (33) of determining the conditions, under which a columnar ship with a rectangular cross-section (in simple terms, a homogeneous squared timber of arbitrary width and material) floats stably in an upright position, was explained in the journal "*NAVIGATION*" of *JIN* (Japan Institute of Navigation).

As a concrete example, when the specific weight of a ship is about half that of water, the theory of finding a stable width condition in an upright state is lectured at the university (24),(25) where the one of the authors (23) works. The lecture video (34) and the shot of the confirmation experiment (35) in the small water tank have been uploaded to YouTube as on-demand teaching materials.

In addition, we have also uploaded a lecture video (36) explaining about materials (*i.e.*, light or heavy in specific weight of a ship) which are stable in the upright state for the shape with a square cross-section.

Next, the solution to the problem of finding a stable attitude of a similar columnar ship which does not satisfy the above conditions and floats in an inclined position is also explained in the same journal of *JIN* as an advanced example (37).

Both the explanatory treatises and the lecture videos are explained in Japanese, but if you are interested, please take a look and read them.