The application of $f(R) = R^2$ gravity in Gravitational Waves

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Abstract

In this paper I study the applications of the $f(R)=R^2$ gravity in gravitational waves. The choice of this action emerges naturally from the field equations of a general action $f(R)$ in order to get a wave equation of the scalar curvature *R* and the Ricci tensor $R_{\mu\nu}$.

The new field equations (*NFE*) seem to be applicable to non-conservative systems (as a collapsing binary). They are derived from the action of $f(R)=R^2$ assuming that the Lagrangian matter depends on $g_{\mu\nu}$ and x^λ: $L_m\left(g_{\mu\nu},x^\lambda\right)$. The radiated energy causes a non-static space-time.

In the case of gravitational waves these equations turn out to be more general than the existing ones that are being produced from the action of $f(R)=R$. They predict both the transverse as well as the longitudinal and time oscillations.

Keywords: gravitational waves, f(R) theories, general relativity.

1. Introduction

In the 21st century $f(R)$ theories are being examined as alternative to General Relativity in order to give an explanation to the accelerated expansion of the universe and other solar-system observations. Further more in *GR* the graviton has zero mass, while in $f(R)$ models it has a non-zero mass. The LIGO observations gave a limit of $m_g c^2 < 1.2\times10^{-22}eV$ [1]. In [2] J.F. Nash had proved that the field equations of the action $I=\int\sqrt{-g}\left(2R_{\mu\nu}R^{\mu\nu}-R^2\right)d^4x$ in 4 dimensions lead to the $g^{\mu\nu}R_{;\mu\nu}=0$.

In this paper we are going to study the form of the gravitational waves that are being produced by the action of $f(R)=R^2$. With this choice the field equations of a general action $f(R)$ become the wave equation of the scalar curvature. For the non-homogeneous field equation we consider the Lagrangian matter $L_m\left(g_{\mu\nu},x^\lambda\right)$ since our system is non-conservative. While the T_{00} component of the energy-momentum tensor is the source of the

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gravitational field, the T_{ij} are the sources of the gravitational waves. The produced waves are polarized in the transverse plane as well as in propagating and time direction.

The application of the *NFE* to conservative systems such as a single mass and a keplerian binary, indicates that they are not valid in these cases.

In paragraph 7 we study the case of a single mass located at the origin of the coordinate system considering the Schwarzschild metric $g_{\mu\nu}$ as the background.

In paragraph 8 we will see that in case of a Keplerian binary it must be $R_{\mu\nu}=0$ which implies $h_{\mu\nu}=0$. This result was to be expected since the system is conservative.

2. The production of Gravitational Waves

A general form of an action in empty space may be

$$I=\int_V \sqrt{-g}\,f(R)\,d^4x \tag{1}$$

The equations of motion of (1) are

$$-\frac{1}{2}f(R)g_{\mu\nu}+f'(R)R_{\mu\nu}-f'(R)_{;\mu\nu}+f'(R)_{;\rho\beta}\,g^{\rho\beta}g_{\mu\nu}=0 \tag{2}$$

Note that the scalar curvature R is a function of the metric and its first and second derivatives while the metric is a function of the coordinates x^μ $(\mu=0,1,2,3)$, so it is $R=R(x^\mu)$. Equations (2) can then be written in the form

$$\begin{aligned}-\frac{1}{2}f(R)g_{\mu\nu}+f'(R)R_{\mu\nu}-f''(R)\left(R_{;\mu\nu}-R_{;\rho\beta}g^{\rho\beta}g_{\mu\nu}\right)&\\ -f'''(R)\left(R_{;\mu}R_{;\nu}-R_{;\rho}R_{;\beta}g^{\rho\beta}g_{\mu\nu}\right)=0&\end{aligned} \tag{3}$$

By taking the trace, eq. (3) becomes

$$-2f(R)+f'(R)R+3f''(R)R_{;\rho\beta}g^{\rho\beta}+3f'''(R)R_{;\rho}R_{;\beta}g^{\rho\beta}=0 \tag{4}$$

For $f(R)=R^2$ we finally get

$$R_{;\rho\beta}g^{\rho\beta}=0 \tag{5}$$

In the case of a non-static space-time, eq. (5) is a wave equation of the Ricci scalar. (For the metric we consider the sign (-, +, +, +)).

Let us assume now small perturbations $h_{\mu\nu}$ of a spherical diagonal([1]) static metric $\hat{g}_{\mu\nu}$. This may be the flat or Schwarzschild([2]) one. The total metric is $g_{\mu\nu} = \hat{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$, where $0 < \varepsilon \ll 1$. We also assume that $h_{\mu\nu} = h_{\nu\mu}$. The inverse metric is $g^{\mu\nu} = \hat{g}^{\mu\nu} - \varepsilon h^{\mu\nu} + O(\varepsilon^2)$. In order to be $g^{\mu\nu} g_{\nu\kappa} = \delta^{\mu}_{\kappa}$ at first order of ε, there must be $\hat{g}^{\mu\nu} h_{\nu\kappa} - h^{\mu\nu} \hat{g}_{\nu\kappa} = 0 \Rightarrow \quad h^{\mu\lambda} = \hat{g}^{\mu\nu} \hat{g}^{\kappa\lambda} h_{\nu\kappa}$.

The Schwarzschild metric at sufficiently large distances is approximately $\hat{g}_{\mu\nu} - 2\phi A_{\mu\nu}$ where $\hat{g}_{\mu\nu}$ is flat, ϕ ([3]) is the Newtonian potential and $A_{00} = A_{11} = 1$ and the rest components $A_{\mu\nu}$ are zero (in spherical coordinates). So we have $g_{\mu\nu} = \hat{g}_{\mu\nu} - 2\phi A_{\mu\nu} + \varepsilon h_{\mu\nu}$, $g^{\mu\nu} = \hat{g}^{\mu\nu} + 2\phi A^{\mu\nu} - \varepsilon h^{\mu\nu}$. Because it is $\hat{g}_{\mu\lambda} A^{\nu\lambda} - \hat{g}^{\lambda\nu} A_{\lambda\nu} = 0$ we may consider the flat metric as the background and use it to raise or lower indices: $h^{\mu\lambda} = \hat{g}^{\mu\nu} \hat{g}^{\kappa\lambda} h_{\nu\kappa}$. So we may assume the perturbations $h_{\nu\kappa}$ as being superimposed on the flat space in our next calculations.

At large distances from the source the metric "behaves" like flat so the Christoffel symbols, the Ricci tensor and the Ricci scalar are

$$
\begin{aligned}
\Gamma^{\kappa}_{\mu\nu} = \hat{\Gamma}^{\kappa}_{\mu\nu} + \frac{\varepsilon}{2}\Big[h^{\kappa\lambda} \left(\hat{g}_{\lambda\mu,\nu} + \hat{g}_{\lambda\nu,\mu} - \hat{g}_{\mu\nu,\lambda} \right) \\
+ \hat{g}^{\kappa\lambda} \left(h_{\lambda\mu,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda} \right) \Big] + O\left(\varepsilon^2\right) \\
R_{\mu\nu} = \hat{R}_{\mu\nu} + R^{(per)}_{\mu\nu}, \quad R = \hat{R} + R^{(per)}
\end{aligned}
\tag{6}
$$

The $\hat{\Gamma}^{\kappa}_{\mu\nu}$, $\hat{R}_{\mu\nu}$, and $\hat{R}$ denote the flat metric. $R^{(per)}_{\mu\nu}$, $R^{(per)}$ are the total perturbed parts (including terms of all orders of ε). Because $\hat{R} = 0$ eq. (5) becomes

$$
g^{\rho\beta} R^{(per)}_{,\rho\beta} - g^{\rho\beta} R^{(per)}_{,\kappa} \Gamma^{\kappa}_{\rho\beta} = \hat{g}^{\rho\beta} R^{(per)}_{,\rho\beta} - \hat{g}^{\rho\beta} R^{(per)}_{,\kappa} \hat{\Gamma}^{\kappa}_{\rho\beta} + O(\varepsilon) = 0 \tag{7}
$$

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[1] We can always choose the coordinate system such as the metric becomes diagonal.

[2] From the solution-set of the field equations of the action of $f(\hat{R}) = \hat{R}^2$, we choose the Schwarzschild metric because it becomes flat at infinity and also has $\hat{R} = 0$ and $\hat{R}_{\mu\nu} = 0$ like Minkowski.

[3] According to Newton's law a mass distribution $\rho(\vec{r}', t)$ within a volume V at large distances generates a field of the form $\phi(r) = \int_V \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \simeq -\frac{M_{total}}{r}$ with M_{total} located at the central mass of the system. Because we are too far we may ignore the higher terms of the Taylor expansion.

We have to solve the

$$\widehat{g}^{\rho\beta} R^{(per)}_{,\rho\beta} - \widehat{g}^{\rho\beta} R^{(per)}_{,\kappa} \widehat{\Gamma}^{\kappa}_{\rho\beta} = 0 \tag{8}$$

In spherical coordinates eq.(8) is

$$\begin{aligned} &\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 R^{(per)}}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^2 R^{(per)}}{\partial\theta^2} + \frac{\partial^2 R^{(per)}}{\partial r^2} \\ &+ \frac{1}{r^2}\left[\cot\theta \frac{\partial R^{(per)}}{\partial\theta} + 2r\frac{\partial R^{(per)}}{\partial r}\right] - \frac{\partial^2 R^{(per)}}{\partial t^2} = 0 \end{aligned} \tag{9}$$

If we approach the source so that the Newtonian potential becomes detectable, we may still use the flat metric as background. But the wave equation (9) needs the following additional terms:

$$2\phi(r)\left(\frac{2}{r}\frac{\partial R^{(per)}}{\partial r} + \frac{\partial^2 R^{(per)}}{\partial r^2} + \frac{\partial^2 R^{(per)}}{\partial t^2}\right) + 2\frac{\partial R^{(per)}}{\partial r}\phi'(r)$$

This result comes from eq.(8) for $\widehat{g}_{\mu\nu}$ being the Schwarzschild metric and taking the Taylor expansion for $\phi(r)$ and $\phi'(r)$.

3. The solution of the differential equation

The solution of eq.(9) at very long distances is

$$R^{(per)}(t,r,\theta,\varphi) = \sum_{l,m} c_{lm} Y_l^m(\theta,\varphi) \int_{-\infty}^{\infty} c(\omega) \frac{e^{i\omega(r-t)}}{r} d\omega \tag{10}$$

where $c_{lm}, c(\omega) \in \mathbb{R}$ and $Y_l^m(\theta,\varphi)$ the spherical harmonics. As can be seen the energy propagates radially outward.

4. Discussion of the solution

We may now assume that our location relative to the source has constant radial values, say (θ_o,φ_o). Then $\sum_{l,m} c_{lm} Y_l^m(\theta_o,\varphi_o) = C$, $C \in \mathbb{R}$. If we use Minkowski coordinates and assume that *r* direction coincides with *z* we have

$$R^{(per)}(t,z) = C\int_{-\infty}^{\infty} c(\omega)\frac{e^{i\omega(z-t)}}{z}d\omega$$
$$R^{(per)}(t,z;\omega) = C\left[c^{*}(\omega)\frac{e^{-i\omega(z-t)}}{z} + c(\omega)\frac{e^{i\omega(z-t)}}{z}\right] \tag{11}$$

where c, c^{*} are conjugate complex constants. Let $(x^{\mu}) = (t,x,y,z)$ so the wave vector is

$$(k_{\mu}) = (-\omega, 0, 0, \omega) \tag{12}$$

x^{μ} and k_{μ} denote the components of the four-vectors (x^{μ}) and (k_{μ}) respectively.

From eq.(6) the Ricci tensor and scalar become

$$R_{\mu\nu}^{(per)} = \frac{\varepsilon}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} + h^{\alpha}{}_{\mu,\nu\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right)$$
$$R^{(per)} = \varepsilon\left(h^{\alpha}{}_{\mu,\alpha}{}^{,\mu} - h^{\mu}{}_{\mu,\alpha}{}^{,\alpha}\right) \tag{13}$$

5. The field equations

In order to get the non-homogeneous eq. (3) for the case where $f(R) = R^{2}$ we consider the action

$$I = \int_{V}\sqrt{-g}\left(R^{2} + \bar{\kappa}L_{m}\right)d^{4}x \tag{14}$$

where L_{m} is the Lagrangian matter-density while V is a volume that contains the source. $\bar{\kappa}$ is a coupling constant which has dimensions of m^{-2}, still undefined. As a binary system collapses it loses angular momentum. So it must be $L_{m}(g_{\mu\nu}, x^{\lambda})$.

Since it is

$$T^{\mu\nu} \equiv -g^{\mu\nu}L_{m} + 2\frac{\partial L_{m}}{\partial g_{\mu\nu}} \tag{15}$$

the non-homogeneous field equations are $(c = G = 1)$

$$-\frac{1}{2}R^{2}g_{\mu\nu} + 2RR_{\mu\nu} - 2\left(R_{;\mu\nu} - R_{;\rho\beta}g^{\rho\beta}g_{\mu\nu}\right) = -\bar{\kappa}\left(T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}L_{m}\right) \tag{16}$$

Since we have $\widehat{R} = \widehat{R}_{\mu\nu} = 0$ from now on for simplicity's sake we are going to use the symbols R and $R_{\mu\nu}$, instead of $R^{(per)}$ and $R^{(per)}_{\mu\nu}$. For the reasons discussed in p.3 we are going to assume a flat background in spherical coordinates.

Eq. (16) at first order of the perturbed terms of R and $R_{\mu\nu}$ become

$$R_{;\mu\nu} - R_{;\rho\beta} g^{\rho\beta} g_{\mu\nu} = \frac{\overline{\kappa}}{2}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} L_m\right) \tag{16a}$$

By taking the trace

$$g^{\mu\nu} R_{;\mu\nu} - g^{\mu\nu} R_{;\rho\beta} g^{\rho\beta} g_{\mu\nu} = \frac{\overline{\kappa}}{2} g^{\mu\nu}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} L_m\right) \tag{16b}$$

we may finally get the

$$g^{\mu\nu}\left(R_{\rho\beta} - g_{\rho\beta} R\right)_{;\mu\nu} = \frac{\overline{\kappa}}{2}\left(T_{\rho\beta} + \frac{1}{2} g_{\rho\beta} L_m\right) \tag{16c}$$

Let us set $\tilde{G}_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu} R$. We have

$$\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\overline{\kappa}}{2}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} L_m\right) \tag{17}$$

Since $\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha\nu} = -\frac{1}{2} g_{\mu\nu} R_{;\alpha}^{;\alpha\nu} = \frac{\overline{\kappa}}{4} g_{\mu\nu} L_m{}^{,\nu}$ we may have

$$\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\overline{\kappa}}{2}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} L_m\right) \rightarrow \quad G_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\overline{\kappa}}{2} T_{\mu\nu}\,, \quad R_{;\alpha}^{;\alpha} = -\frac{\overline{\kappa}}{2} L_m \tag{18}$$

where the Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. If we take the trace $G^{\mu}{}_{\mu}$ from (18) we get

$$R_{;\alpha}^{;\alpha} = -\frac{\overline{\kappa}}{2} T^{\lambda}{}_{\lambda} \tag{19a}$$

$$R_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\overline{\kappa}}{2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\right) \tag{19b}$$

$$\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\overline{\kappa}}{2}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\right) \tag{19c}$$

Let us now assume the simplest model of two equal masses M at distance $2r_o$ orbiting around their common central mass at distances r_o with angular velocity ω_o. The orbital level may be vertical to the (θ_o, φ_o) direction. We also assume that the process of convergence has just started. So the velocities are small and for some cycles we may regard r_o being constant([4]). For non relativistic velocities and small accelerations we may assume that $\ddot{r} \simeq 0$, $\int T^n{}_n d^3\vec{r}' \sim \dot{r}^2 \simeq 0$ and $\int T^\lambda{}_\lambda d^3\vec{r}' \simeq -\int T_{00} d^3\vec{r}'$. The solutions of (19a,b) are

$$R = \frac{\bar{\kappa}}{8\pi} \int \frac{T_{00}(t_r, \vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \tag{20a}$$

$$R_{\mu\nu} = \frac{\bar{\kappa}}{8\pi} \left(\int \frac{T_{\mu\nu}(t_r, \vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' + \frac{1}{2} n_{\mu\nu} \int \frac{T_{00}(t_r, \vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' \right) \tag{20b}$$

where $\vec{r}'$ is the radius within the source, $|\vec{r}'| \ll |\vec{r}|$ and $t_r = t - |\vec{r} - \vec{r}'|$ is the retarded time. We now take the Fourier transform and then the Taylor series up to second order of $\vec{r}'$. We are also going to use Minkowski coordinates where the direction of r coincides with z $(\hat{r} \equiv \hat{z})$

$$\begin{aligned} \int \frac{T_{\mu\nu}(t_r, \vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' &= \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\sqrt{2\pi}} \int \frac{e^{i\omega|\vec{z} - \vec{r}'|}}{|\vec{z} - \vec{r}'|} \tilde{T}_{\mu\nu}(\omega, \vec{r}') d^3\vec{r}' \\ &\simeq \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(z-t)}}{\sqrt{2\pi} z} \int \tilde{T}_{\mu\nu} d^3\vec{r}' - \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(z-t)}}{\sqrt{2\pi}} \left(\frac{i\omega}{z} - \frac{1}{z^2} \right) \int z' \tilde{T}_{\mu\nu} d^3\vec{r}' \\ &\quad + \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(z-t)}}{2\sqrt{2\pi}} \left(-\frac{\omega^2}{z} - i\omega \frac{2}{z^2} + \frac{2}{z^3} \right) \int z'^2 \tilde{T}_{\mu\nu} d^3\vec{r}' \end{aligned} \tag{21}$$

In the following calculations it is useful to consider for T_{00} the above series up to second order of $\vec{r}'$, for T_{0i} up to first order and for T_{ij} only the first term of zero order. We are also going to keep the terms that are proportional to $1/z$. The term $A \equiv 1/z$ can be considered as constant because we are sufficiently far away from the source. In vacuum the solution (20a) must satisfy the $R_{,\alpha}^{,\alpha} = 0$ so from (21) it must be

$$R = \frac{\bar{\kappa} A}{8\pi\sqrt{2\pi}} \sum_{\omega = \pm 2\omega_o} e^{i\omega(z-t)} \int \tilde{T}_{33}(\omega, \vec{r}') d^3\vec{r}' \tag{22}$$

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[4] The model considered here is extremely simple and it just outlines the application of $f(R) = R^2$ in gravitational waves. It is not being used in research.

From equations (20b) combining them with (13) we have

$$R_{11}=\frac{\bar{\kappa}A}{8\pi}\left(\int T_{11}\left(t_r,\vec{r}'\right)d^3\vec{r}'+\frac{1}{2}\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'\right)=-\frac{\varepsilon}{2}h_{11,\ \alpha}^{\ \ \ ,\alpha}$$
$$R_{22}=\frac{\bar{\kappa}A}{8\pi}\left(\int T_{22}\left(t_r,\vec{r}'\right)d^3\vec{r}'+\frac{1}{2}\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'\right)=-\frac{\varepsilon}{2}h_{22,\ \alpha}^{\ \ \ ,\alpha} \tag{23a}$$

$$R_{00}=\frac{\bar{\kappa}A}{8\pi}\left(\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'-\frac{1}{2}\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'\right)$$
$$=\frac{\varepsilon}{2}\left(-h_{00,33}+2h_{03,03}-h_{33,00}-\left(h_{11}+h_{22}\right)_{,00}\right)$$
$$R_{33}=\frac{\bar{\kappa}A}{8\pi}\left(\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'+\frac{1}{2}\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'\right) \tag{23b}$$
$$=\frac{\varepsilon}{2}\left(h_{00,33}-2h_{03,03}+h_{33,00}-\left(h_{11}+h_{22}\right)_{,33}\right)$$

From (23a) we get

$$\left(h_{11}+h_{22}\right)_{,\alpha}^{,\alpha}=0 \tag{24}$$

If we set $h_{03}=0,\quad h_{00}=-h_{33}$ eq. (22) then becomes

$$\frac{\bar{\kappa}A}{8\pi}\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'=-\varepsilon h_{33,\alpha}^{\ \ ,\alpha} \tag{25a}$$

From (23b) and (24) we also have

$$\frac{\bar{\kappa}}{4\pi}A\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}'=-\varepsilon\left(h_{11}+h_{22}\right)_{,00} \tag{25b}$$

If the position vector on the orbiting plane is $\vec{\rho}=r_o\left(\cos\omega_o t_r,\quad \sin\omega_o t_r,\quad 0\right)$ the one relative to us is

$$\vec{r}_o=r_o\begin{pmatrix}\cos\theta_o\cos\varphi_o\cos\omega_o t_r-\sin\varphi_o\sin\omega_o t_r\\ \cos\theta_o\sin\varphi_o\cos\omega_o t_r+\cos\varphi_o\sin\omega_o t_r\\ -\sin\theta_o\cos\omega_o t_r\end{pmatrix}=r_o\hat{r}_o$$

Note that it is

$$T_{00} = M\left[\delta\left(\vec{r}' - \vec{r}_o\right) + \delta\left(\vec{r}' + \vec{r}_o\right)\right]$$

$$\int T_{nm} d^3\vec{r}' = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int r'_m r'_n T_{00} d^3\vec{r}' = M\frac{\partial^2}{\partial t^2} r_{(o)m} r_{(o)n} \simeq M r_o^2 \frac{\partial^2}{\partial t^2}\hat{r}_{(o)m}\hat{r}_{(o)n} \tag{26a}$$

So

$$\int T_{mn}\left(t_r, \vec{r}'\right) d^3\vec{r}' = 4\omega_o^2 M r_o^2 \left(a_{mn}\cos 2\omega_o\left(z - t\right) + b_{mn}\sin 2\omega_o\left(z - t\right)\right) \tag{26b}$$

where coefficients $a_{mn}, b_{mn} \in \mathbb{R}$ depend from $\left(\theta_o, \varphi_o\right)$. For $n = 1,2$ we have

$$\begin{gathered}\frac{\bar{\kappa}}{8\pi}\int\frac{T_{0n}\left(t_r, \vec{r}'\right)}{\left|\vec{r} - \vec{r}'\right|} d^3\vec{r}' \simeq \frac{\bar{\kappa} A}{8\pi\sqrt{2\pi}}\sum_{\omega = \pm 2\omega_o} e^{i\omega(z-t)}\int\tilde{T}_{0n}\left(\omega, \vec{r}'\right) d^3\vec{r}' \\ -\frac{i\bar{\kappa} A}{8\pi\sqrt{2\pi}}\sum_{\omega = \pm 2\omega_o}\omega e^{i\omega(z-t)}\int z'\tilde{T}_{0n}\left(\omega, \vec{r}'\right) d^3\vec{r}' = \frac{\varepsilon}{2}\left(-h_{0n,33} + h_{n3,03}\right)\end{gathered} \tag{27a}$$

If we differentiate by *t* and integrate by *z* we get

$$\frac{\bar{\kappa} A}{8\pi\sqrt{2\pi}}\sum_{\omega = \pm 2\omega_o} e^{i\omega(z-t)}\int\tilde{T}_{3n}\left(\omega, \vec{r}'\right) d^3\vec{r}' = \frac{\varepsilon}{2}\left(-h_{0n,30} + h_{n3,00}\right) \tag{27b}$$

For $h_{0n,0} = h_{n3,3}$ we have

$$\begin{gathered}\frac{\bar{\kappa} A}{8\pi\sqrt{2\pi}}\sum_{\omega = \pm 2\omega_o} e^{i\omega(z-t)}\int\tilde{T}_{3n}\left(\omega, \vec{r}'\right) d^3\vec{r}' = -\frac{\varepsilon}{2} h_{n3,\alpha}^{\ ,\alpha} \\ h_{0n,0} = h_{n3,3} \Rightarrow h_{0n} = \int h_{n3,3}\partial t\end{gathered} \tag{27c}$$

Synoptically we have

$$\begin{aligned}&\frac{\bar{\kappa} A}{8\pi}\int T_{nm} d^3\vec{r}' = -\frac{\varepsilon}{2} h_{nm,\alpha}^{\ ,\alpha} \quad n, m \neq 3 \\ &\frac{\bar{\kappa} A}{8\pi}\int T_{33} d^3\vec{r}' = -\varepsilon h_{33,\alpha}^{\ ,\alpha} \\ &h_{03} = 0, \quad h_{0n} = \int h_{n3,3}\partial t, \quad n = 1,2\end{aligned} \tag{27d}$$

From (26b), (27d) we have to solve the wave equations of the form

$$h_{mn,\alpha}^{\ ,\alpha} \sim 4\omega_o^2\left(\alpha_{mn}\cos 2\omega_o\left(z - t\right) + \beta_{mn}\sin 2\omega_o\left(z - t\right)\right) \tag{28a}$$

where $\alpha_{mn}, \beta_{mn} \in \mathbb{R}$ depend from a_{mn}, b_{mn}.

The solutions are

$$h_{mn} \sim c^{(1)}_{mn}\cos 2\omega_o\left(z-t\right)+c^{(2)}_{mn}\sin 2\omega_o\left(z-t\right)$$
$$+\omega_o t\left(\beta_{mn}\cos 2\omega_o\left(z-t\right)-\alpha_{mn}\sin 2\omega_o\left(z-t\right)\right) \tag{28b}$$

where $c^{(1)}_{mn}, c^{(2)}_{mn} \in \mathbb{R}$.

For the functions h_{11}, h_{22} from (24) and (28b) it must be $\alpha_{22}=-\alpha_{11},\quad \beta_{22}=-\beta_{11}$. From (25b) and (26b) we have

$$\frac{\bar{\kappa}}{4\pi}A\int T_{33}\left(t_r,\vec{r}'\right)d^3\vec{r}' =$$
$$-\frac{\bar{\kappa}}{2\pi}AMr_o^2\omega_o^2\sin^2\theta_o\cos 2\omega_o\left(z-t\right)=-\varepsilon\left(h_{11}+h_{22}\right)_{,00} \tag{29}$$

So it must be $c^{(2)}_{11}=c^{(2)}_{22}=c^{(1)}_{11}=0$ and $c^{(1)}_{22}=-\bar{\kappa}AMr_o^2\sin^2\theta_o/2\varepsilon$.

The perturbed matrix is then

$$\left(h_{\mu\nu}\right)=\begin{pmatrix} -h_{33} & \int h_{13,3}\partial t & \int h_{23,3}\partial t & 0 \\ \int h_{13,3}\partial t & h_{11} & h_{12} & h_{13} \\ \int h_{23,3}\partial t & h_{12} & -h_{11}+\dfrac{\bar{\kappa}AMr_o^2}{2\varepsilon}\sin^2\theta_o\cos 2\omega_o\left(z-t\right) & h_{23} \\ 0 & h_{13} & h_{23} & h_{33} \end{pmatrix} \tag{30a}$$

If we set all the rest coefficients $c^{(1)}_{nm}=c^{(2)}_{nm}=0$, for the case of $\theta_o=0$ and $\varphi_o=0$, matrix (30a) becomes

$$\left(h_{\mu\nu}\right)\sim 2\omega_o t\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin 2\omega_o\left(z-t\right) & -\cos 2\omega_o\left(z-t\right) & 0 \\ 0 & -\cos 2\omega_o\left(z-t\right) & -\sin 2\omega_o\left(z-t\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30b}$$

For a direction $\left(\theta_o,\varphi_o\right)$ the length between two points is

$$\delta s^2=\left(n_{\mu\nu}+\varepsilon h_{\mu\nu}\right)\delta x^\mu\delta x^\nu \rightarrow \quad \delta s\simeq\left.\sqrt{\delta s^2}\right|_{\varepsilon=0}+\varepsilon\left.\frac{\partial\sqrt{\delta s^2}}{\partial\varepsilon}\right|_{\varepsilon=0} \tag{31a}$$

If the points are on the x-axis with a distance x_o, as the wave passes through them we have

$$\delta s_x = x_o\left[1+\frac{\bar{\kappa} AMr_o^2\omega_o t}{16\pi}\left(3+\cos 2\theta_o\right)\cos 2\varphi_o \sin 2\omega_o\left(z-t\right)\right.$$
$$\left.-\frac{\bar{\kappa} AMr_o^2}{32\pi}\left(\sin^2\theta_o - 8\omega_o t\cos\theta_o \sin 2\varphi_o\right)\cos 2\omega_o\left(z-t\right)\right] \tag{31b}$$

If the points are on the y-axis or on the z-axis we respectively have

$$\delta s_y = y_o\left[1-\frac{\bar{\kappa} AMr_o^2\omega_o t}{16\pi}\left(3+\cos 2\theta_o\right)\cos 2\varphi_o \sin 2\omega_o\left(z-t\right)\right.$$
$$\left.-\frac{\bar{\kappa} AMr_o^2}{32\pi}\left(\sin^2\theta_o + 8\omega_o t\cos\theta_o \sin 2\varphi_o\right)\cos 2\omega_o\left(z-t\right)\right] \tag{31c}$$

$$\delta s_z = z_o\left[1+\frac{\bar{\kappa} AMr_o^2\omega_o t}{8\pi}\sin^2\theta_o \sin 2\omega_o\left(z-t\right)\right] \tag{31d}$$

On the t-axis we get

$$\mathrm{Im}\left(\delta s_t\right) = t_o\left[1+\frac{\bar{\kappa} AMr_o^2\omega_o t}{8\pi}\sin^2\theta_o \sin 2\omega_o\left(z-t\right)\right] \tag{31e}$$

where t_o could be the duration of the time unit (second).

6. The radiated energy

The energy of the two-bodies system considered here is $E = M\dot{\vec{\rho}}^2 - \frac{GM^2}{2r_o}$ where $\vec{\rho} = r_o\hat{\rho}$ is the position-vector on the orbital plane. Its energy-loss is $\dot{E} = 2M\dot{\vec{\rho}}\cdot\ddot{\vec{\rho}} + M\frac{GM}{2r_o^2}\dot{r}_o$. In the equivalent one-body problem the centripetal acceleration of the reduced mass is $-\frac{GM}{2r_o^2} = 2\left(\ddot{r}_o - r_o\omega_o^2\right)$. So assuming $\dot{r}_o, \omega_o$ as constants the rate of energy change is $\dot{E} \simeq 4Mr_o\dot{r}_o\omega_o^2$. The energy is transmitted via $R_{\mu\nu}$ to $h_{\mu\nu}$ according to equations (27d). In *SI* units the constant $\bar{\kappa}$ is $\frac{16\pi G}{c^4\left(m^2\right)}$, where the dimension m^{-2} is denoted in parenthesis.

We are going to rewrite (eq.27d) in *SI* $\left(\varepsilon h_{ij} \equiv h_{ij}\right)$:

$$n,m \neq 3: \quad \frac{4GMA}{c^4\left(m^2\right)}\frac{\partial^2}{\partial t^2}r_{(o)m}r_{(o)n} = -h_{nm,\alpha}^{\ \ \ ,\alpha}, \quad \frac{2GMA}{c^4\left(m^2\right)}\frac{\partial^2}{\partial t^2}r_{(o)3}r_{(o)3} = -h_{33,\alpha}^{\ \ \ ,\alpha} \qquad (32)$$

Both sides have dimensions m^{-2} because $h_{mn} \sim \frac{GMA}{c^2}$ (dimensionless). In (26a) we considered r_o as constant but here we are going to use the relation

$$\frac{\partial^2}{\partial t^2}r_{(o)m}r_{(o)n} = \frac{\partial^2}{\partial t^2}r_o^2\left(t\right)\hat{r}_{(o)m}\hat{r}_{(o)n}$$

We will also set $\cos\omega_o\left(z-t\right) \to \cos\omega_o\left(\frac{z}{c}-t\right)$, $\sin\omega_o\left(z-t\right) \to \sin\omega_o\left(\frac{z}{c}-t\right)$ where *c* is the speed of light while $\omega_o / c = k_o$ is the wave number.

At a fixed point *z* on the *z*-axis eq.32 become the equations of harmonic oscillator: $-h_{nm,\alpha}^{\ \ \ ,\alpha} = c^{-2}\left(\ddot{h}_{nm} + \omega_o^2 h_{nm}\right)$. Remember that $A = 1/z$ so the solutions[5] of eq.(32) (at a fixed point z) are of the form $h_{nm} = \frac{GM}{c^2 z}\tilde{f}\left(t,z,\theta_o,\varphi_o,\omega_o,r_o,\dot{r}_o,\ddot{r}_o\right)$ where function $\tilde{f}$ is dimensionless. The product $F = \sum_{\mu,\nu}\ddot{h}_{\mu\nu}\sum_{\kappa,\lambda}\dot{h}_{\kappa\lambda} \sim \frac{1}{z^2}$ has the dimension flux per unit mass $\frac{J}{kgr \cdot m \cdot s}$ and it is proportional to z^{-2}. We assumed very slow variations $\left(\dot{r}_o, \omega_o \simeq constant\right)$. The surface-integral $P = \int_0^{2\pi}\int_0^{\pi} z^2 F \sin\theta_o d\theta_o d\varphi_o$ of *F* at a given distance *z* has dimension of power per unit mass $J \cdot kgr^{-1}s^{-1}$. Its mean value at time $T = 2\pi / \omega_o$ is

$$P \simeq \frac{26624 G^2 M^2 \pi r_o^2}{135 c^4\left(m^4\right)}\omega_o^2 r_o \dot{r}_o \qquad (33)$$

P is related to the energy-loss of the system $\dot{E}$:

$$P = \frac{6656 G^2 M \pi r_o^2}{135 c^4\left(m^4\right)}\dot{E} \qquad (34)$$

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[5] In order to solve the wave equations (32) we may consider r_o and its time derivatives as constants for sufficiently small time-intervals, since all the variations are very small.

7. Comments on the Newtonian Limit and the Einstein Field Equations

The most general form of a static spherical symmetric metric is

$$\left(g_{\mu\nu}\right)=\begin{pmatrix} -f_0(r) & 0 & 0 & 0 \\ 0 & f_1(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{35}$$

Since we are interested in spaces which become flat at infinity -so they have a physical meaning, it is easy to see that it must be $R=0$. If we set $R=q(r)\xrightarrow{r\to\infty}0$ and solve the $R=q(r)$ for $f_0''(r)$ and replace to R_μ^ν, the last tensor is being separated into two parts: $R_\mu^\nu=R_\mu^{(a)\nu}+R_\mu^{(b)\nu}$. The first corresponds to the case $R^{(a)}=0$ while the second has $R_0^{(b)0}=R_1^{(b)1}=\dfrac{q(r)}{2}$ and the rest components are zero. In order to be $R=q(r)$ the solutions $f_0(r), f_1(r)$ must satisfy the $R^{(a)}=0$. But in this case it is obvious that it is also $R=0$.[4]

There are two solutions which become flat at infinity $\left(f_0(r), f_1(r)\xrightarrow{r\to\infty}1\right)$:

$$(i)\quad R_0^0=R_1^1=R_2^2=R_3^3=0\to\quad \begin{cases} f_0(r)=\dfrac{r+c}{r} \\ f_1(r)=\dfrac{r}{c+r} \end{cases} \tag{36a}$$

$$(ii)\quad R_0^0=R_1^1=-R_2^2=-R_3^3\to\quad \begin{cases} f_0(r)=1+\dfrac{c_1}{r}+\dfrac{c_2}{r^2} \\ f_1(r)=\dfrac{1}{f_0(r)} \end{cases},\quad R_0^0=-\frac{c_2}{r^4}\to 0 \tag{36b}$$

Let us now assume a single (uncharged) mass *M* located at the origin of the coordinate system. Our space is being described by the Schwarzschild([6]) metric (36a). At sufficiently large distances this becomes

$$f_0(r)=1+h(r),\quad f_1(r)=1-h(r),\quad h(r)=\frac{c}{r}\ll 1 \tag{37}$$

[6] Since there is a mass generating the field, the Schwarzschild metric is the background. The flat space is an ideal case where there are no sources at all.

Remember that in the field equations (19) the computed R and $R_{\mu\nu}$ were the perturbed parts of the background flat or Schwarzschild metric. The spacial components of the energy- momentum tensor T_{ij} act as a source for $R^{(per)}$ and $R^{(per)}_{\mu\nu}$ which then act as a source for the metric' s perturbations $h_{\mu\nu}$ ($h_{ij,\alpha}^{,\alpha} \sim \int T_{ij} d^3\vec{r}'$, $h_{00} = -h_{33}$ and $h_{0n} = \int h_{n3,3} \partial t$, $n = 1,2$). But since in our case it is $T_{ij} = 0$, there are no perturbations.

In an unperturbed space the Ricci tensor and scalar are zero so they need no source. According to the discussion on page 3, the term $h(r)$ on (37) looks like a perturbation on the flat metric but it is just an approximation of the background Schwarzschild metric. For this reason eq.(19) does not give solutions for it.

For the metric (34) the Ricci scalar at first order approximation is

$$R = -2\left(\frac{h(r)}{r^2} + \frac{h'(r)}{r}\right) - \left(\frac{2h'(r)}{r} + h''(r)\right) \tag{38}$$

This must be also equal to zero. So does $R_{\mu\nu}$. The homogeneous differential equations $\tilde{G}_{\mu\nu} = 0$ (at first order approximation of *h*) have as a common solution the $h = c/r$. Eq.(38) then becomes

$$\frac{2}{r^2}\partial_r\left(r^2\frac{h(r)}{r}\right) + \frac{1}{r}\partial_{rr}\left(rh(r)\right) = 0 \tag{39}$$

Generally for a $f(r)$ function it is $\vec{\nabla}^2 f(r) = \frac{1}{r}\partial_{rr} rf(r) = \frac{1}{r^2}\partial_r r^2 f'(r)$. So for $h(r) = c/r$ eq.(39) may be written as

$$-\vec{\nabla}^2 h(r) = 4\pi c\delta(\vec{r}) \tag{40}$$

By setting $c = 2M$ in eq. (40), we get

$$-\vec{\nabla}^2 h(r) = 8\pi M\delta(\vec{r}) \tag{41}$$

As it is known from the equations of motion $\frac{dp^i}{dt} = -m\Gamma^i_{00} = \frac{m}{2} n^{ij} h_{,j} = -m\vec{\nabla}\phi$, it must be $h(r) = -2\phi$ where $\phi = -\frac{M}{r}$. So from (41) we get the Newton's law

$$\vec{\nabla}^2 \phi = 4\pi M \delta(\vec{r}) \tag{42}$$

Eqs.(19) are also valid for the total R and $R_{\mu\nu}$. From eq.(19a) we have

$$R^{;\alpha}_{;\alpha} = -\frac{\bar{\kappa}}{2} T^{\lambda}{}_{\lambda} \rightarrow \left\{ R_{\mu\nu;\alpha}{}^{;\alpha} = -\frac{\bar{\kappa}}{2} T_{\mu\nu},\ \ \frac{1}{2} g_{\mu\nu} R^{;\alpha}_{;\alpha} = -\bar{\kappa} T_{\mu\nu} \right\} \tag{43a}$$

Eq.(19c) takes the form([7])

$$\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha} = \frac{3\bar{\kappa}}{2} T_{\mu\nu} \tag{43b}$$

We finally have $G_{\mu\nu;\alpha}{}^{;\alpha} = \frac{\bar{\kappa}}{2} T_{\mu\nu}$. The *NFE* and the *EFE* $G_{\mu\nu} = \frac{\kappa}{2} T_{\mu\nu}$ can not be transformed directly into each other because they are derived form different actions. We set $\kappa = 16\pi$.

As we will see below the *NFE* apply only in the case of a non-conservative system. For that reason the Newtonian limit $\frac{c}{r}$ is not produced directly but it is the solution of $\tilde{G}_{\mu\nu} = 0$ since in our special case we have $R = R_{\mu\nu} = 0$.

Note: For $T_{\mu\nu} \sim \delta(\vec{r})$ if we take the trace of $G_{\mu\nu} = \frac{\kappa}{2} T_{\mu\nu}$, for $\vec{r} \neq 0$ -or generally outside the region in which the source lies, we get $R = R_{\mu\nu} = 0$. According to Birkhoff's theorem the equations $G_{\mu\nu} = 0$ require the Schwarzschild solution. So the *EFE* apply in static space.

8. The *NFE* for a Keplerian binary.

In page 14 we had discussed that eqs.(19) are not valid in the case of a single mass located at the origin. What if we have two masses surrounding each other in a Keplerian orbit? In the case of our previous model the distance r_o of the two equal masses is constant. According to equation (33) no radiation is emitted from the source.

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[7] If we consider $L_m(g_{\mu\nu})$ the field equations are $\tilde{G}_{\mu\nu;\alpha}{}^{;\alpha} \sim T_{\mu\nu}$.

The system is conservative so it is $L_m\left(g_{\mu\nu}\right)$ and the field equations (17) are $\tilde{G}^{;\alpha}_{\mu\nu;\alpha} = \frac{\overline{\kappa}}{2}T_{\mu\nu}$. We set $\overline{\kappa} \to 3\overline{\kappa}, \quad \overline{\kappa} = 16\pi m^{-2}$ in order to get eq.(43b). Far away form the source we may set $g_{\mu\nu} \equiv n_{\mu\nu}$.

We consider the $R_{\mu\nu,\alpha}^{\ \ ,\alpha} = -\frac{\overline{\kappa}}{2}T_{\mu\nu}$ (eq.(43a)). For the non-diagonal elements the solutions for h_{nm} are given from eq.(27d)

$$\frac{\overline{\kappa}A}{8\pi}\int T_{nm}d^3\vec{r}' = -\frac{\varepsilon}{2}h_{nm,\alpha}^{\ \ ,\alpha}, \quad n \neq m, \quad h_{0n} = \int h_{n3,3}\partial t, \quad n,m = 1,2 \tag{44}$$

We also have

$$\begin{aligned}
&(a) \quad R = \frac{\overline{\kappa}A}{8\pi}\int T_{33}d^3\vec{r}' = \varepsilon\left(h_{00,33} - 2h_{03,30} + h_{33,00} - \left(h_{11} + h_{22}\right)_{,\alpha}^{,\alpha}\right) \\
&(b) \quad R_{00} = -\frac{\overline{\kappa}A}{8\pi}\int T_{33}d^3\vec{r}' = \frac{\varepsilon}{2}\left(-h_{00,33} + 2h_{30,03} - h_{33,00} - \left(h_{11} + h_{22}\right)_{,00}\right) \\
&(c) \quad R_{11} = -\frac{\overline{\kappa}A}{8\pi}\int T_{11}d^3\vec{r}' = -\frac{\varepsilon}{2}h_{11,\alpha}^{\ \ ,\alpha}, \quad R_{22} = -\frac{\overline{\kappa}}{2}\int T_{22}d^3\vec{r}' = -\frac{\varepsilon}{2}h_{22,\alpha}^{\ \ ,\alpha} \\
&(d) \quad R_{33} = -\frac{\overline{\kappa}A}{8\pi}\int T_{33}d^3\vec{r}' = \frac{\varepsilon}{2}\left(h_{00,33} - 2h_{03,30} + h_{33,00} - \left(h_{11} + h_{22}\right)_{,33}\right)
\end{aligned} \tag{45}$$

From (45b,d) we have

$$R_{00} + R_{33} \to \quad -\frac{\overline{\kappa}A}{4\pi}\int T_{33}d^3\vec{r}' = -\frac{\varepsilon}{2}\left(h_{11} + h_{22}\right)_{,00} - \frac{\varepsilon}{2}\left(h_{11} + h_{22}\right)_{,33} \tag{46a}$$

From (45c) since $T^{i}_{\ i} \simeq 0$

$$R_{11} + R_{22} \to \quad -\frac{\varepsilon}{2}\left(h_{11} + h_{22}\right)_{,\alpha}^{,\alpha} = \frac{\overline{\kappa}A}{8\pi}\int T_{33}d^3\vec{r}' \tag{46b}$$

Form (46a,b) we get

$$\left(h_{11} + h_{22}\right)_{,00} = 3\left(h_{11} + h_{22}\right)_{,33} \tag{46c}$$

Eq. 46c may be valid only if $h_{11} + h_{22} = 0$ or $h_{11} = h_{22} = 0$.

From (45b,d) is $R_{00}=R_{33}$ so $h_{00}=h_{03}=h_{33}=0$ ([8]). Equation (45a) suggests $R=\frac{\kappa A}{8\pi}\int T_{33}d^3\vec{r}'=0$ so (26a) lead us to $\theta_o=0$ and hence $T_{03}=T_{13}=T_{23}=T_{33}=0$. From eq.(44) we get $h_{23}=h_{13}=h_{02}=h_{01}=0$ and from eq.(13) we have $R_{00}=R_{01}=R_{02}=R_{03}=R_{13}=R_{23}=R_{33}=0$. (Remember that $R_{0n}=\frac{\varepsilon}{2}\left(-h_{0n,33}+h_{n3,03}\right)$ for $n=1,2$.) This "moved" us to a vertical direction relative to the orbital plane which was not our initial position since we had assumed $\theta_o\neq 0$.

Because $R_{01}=R_{02}=0$ from $R_{\mu\nu,\alpha}^{;\alpha}=-\frac{\kappa}{2}T_{\mu\nu}$ we get $T_{01}=T_{02}=0$ and therefor $T_{11}=T_{12}=T_{22}=0\Rightarrow R_{11}=R_{12}=R_{22}=0$ and from eq.(44) $h_{11}=h_{12}=h_{22}=0$.

So in the case of a Keplerian binary it is $R_{\mu\nu}=0$ and therefore no gravitational waves are produced. The space is static.

Actually a good simulation of a Keplerian binary is a system of two masses with low velocities (so it is $T_{ij}\simeq 0$) in a large enough distance from other gravitational sources so that it may be considered as isolated. The above result $T_{ij}=0$ just means that the equations $\tilde{G}_{\mu\nu;\alpha}^{;\alpha}=\frac{\kappa}{2}T_{\mu\nu}$ are not valid in case of a conservative system.

Note: For a metric $g_{tt}=-\left(1-2\phi\right),\quad g_{rr}=-\left(1-2\phi\right)^{-1}$ where $\phi\simeq-\frac{2M}{r}-Mr_o^{\,2}\frac{3\sin^2\theta_o\cos^2\omega_o\left(r-t\right)-1}{r^3}$ the Ricci tensor and scalar are of order $R_{00}\sim r^{-5},\quad R_{11}\sim r^{-5},\quad R_{22}\sim r^{-3},\quad R_{33}\sim r^{-3}$ and $R\sim r^{-5}$. Sufficiently far from the source we may consider them as zero.

9. Conclusions

For $T_{\mu\nu}\sim\delta\left(\vec{r}\right)$ the Einstein Field Equations in vacuum require $R=R_{\mu\nu}=0$ and the Schwarzschild solution. That is to say that we have a static space regardless of the passing of the wave. This holds either for a single mass located at the origin of the coordinate system or for two masses orbiting around each other. But with the passing of a gravitational wave why should the

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[8] If we set $h_{03}=0,\quad h_{00}=-h_{33}$ from (45b,d) we get $-h_{33,\alpha}^{;\alpha}=\frac{\kappa A}{8\pi}\int T_{33}d^3\vec{r}'=h_{33,\alpha}^{;\alpha}$ which implies $h_{33}=0$ and $\int T_{33}d^3\vec{r}'=0$.

Ricci tensor and scalar still remain invariable? The *NFE* (3) imply that the action $f(R)=R^2$ produces a wave equation for $R_{\mu\nu}$ and *R*. If T_{ij} is their source then they may vary.

The T_{00} is the source of the gravitational field. According to the *NFE* the spatial components of the energy-momentum tensor T_{ij} act as a source for the Ricci tensor and scalar which then act as a source for the metric' s perturbations $h_{\mu\nu}(t,r)$. Since $R^{(per)} \sim \int T_{33} d^3\vec{r}'$ we finally obtain the set of the non-homogeneous wave equations (28a). The perturbed metric is the (30a) with the selection of the coefficients c_{nm} as mentioned in the text. This becomes in the vertical direction the metric of (30b) satisfying also the transverse-traceless condition. The longitudinal and time oscillations appear to be interconnected and are observable in non-vertical directions. The time oscillations could result in a periodical red and blue-shift of a photon's frequency.

The amplitude given from the *NFE*'s waves is linear for time *t* while from the *EFE*'s this is time-independent. This is due to the fact that the eq.(27d) describe forced oscillations. But since the passing of the wave does not last for too long there is not enough time for the amplitude to become infinitely large.

If we apply the *NFE* to a Keplerian binary we get $R = R_{\mu\nu} = 0$. So we have a static space. This is the expected result since the system does not lose energy. Furthermore eq.(33) indicates that no energy is radiated in case of a constant r_o.

In the case of a single mass located at the origin we have $R_{\mu\nu} = R = 0$ either for the flat or the Schwarzschild metric. The Newtonian limit is not produced directly by the *NFE* but the solution $\frac{c}{r}$ is acceptable because it satisfies the equations $\tilde{G}_{\mu\nu} = 0$. The term $h(r) = 2\phi(r)$ at weak fields ($g_{00} = -1 - h(r)$, $g_{11} = 1 - h(r)$) is not a perturbation on the background flat metric but an approximation of the background Schwarzschild metric at large distances. This is the difference between $h(r)$ and $h_{\mu\nu}(t,r)$. For that reason eq.(19) does not give solutions for the static case.

From the above analysis it follows that the *NFE* apply in the case of non-conservative systems (by considering a Lagrangian matter $L_m(g_{\mu\nu}, x^\lambda)$) which cause a non-static space (via the emitted radiation) and therefore are more suitable to describe gravitational waves.

Strictly speaking the Newtonian limit is not produced directly because these equations are not valid for a static space. But the solution of $\tilde{G}_{\mu\nu} = 0$ is $\frac{c}{r}$.

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