An Algorithm for Determining The Irrationality of a Series

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Abstract

We give a way to determine the irrationality of a certain type of series.

Irrationality of z_n

Given that z_n is complete and $s_n^k \notin \Xi_k^n$, we create a nested sequence of intervals using the following algorithm. We will illustrate it first with z_2. Let

$$t_4 = (s_2^2, \overline{[4]_2})$$

where $\overline{[4]})$ indicates the least upper bound single digit in base 4. In this case $t_4 = (1/4, 2/4)$. Repeat with the next term and this same number base until this upper bound decimal becomes fixed. So as $s_9 = 1/4 + 1/9 < .2_4, s_{16} < .2_4, \ldots, s_{36}^2 < .2_4$, but $s_{49} > .2_4$, we have

$$t_{49} = (s_7^2, \overline{[4]_3})$$

When this upper bound in a fixed base becomes fixed, go to the next term and repeat. For this first term, define the final set with

$$T_1 = (s_7^2, \overline{[4]_3}).$$

Using the next denominator in the series as a new number base and iterating to the next partial sum, we have

$$t_{64} = (s_8^2, \overline{[9]_5})$$

and $t_{9^2} = (s_9^2, \overline{[9]_5})$, etc.... Eventually

$$T_2 = (s_{k_9}^2, \overline{[9]_6})$$

and we notice

$$T_2 \subset T_1,$$

where the subscript gives the number of the term.

If the next term does not move in from the right skip it and go to the next term until a term does move in from the right. Such a term will have to exist as increasing decimal bases have greater and greater accuracy. As they never are as accurate as the lower bound, an inequality between the two is certain to eventually exist.

This skipping occurs with the next term, $1/16$, as $11/16 = .6875 > 6/9 = .\overline{6}$, but $32/49 = .6530$ and this moves in from the base 9 upper bound. As this series is complete, all rational numbers in $(0,1)$ are excluded and z_2 is irrational.

Other series

Notice if one considers the decimal, base 10 series for an irrational, like $\sqrt{2}$, the denominators are powers of ten, giving partials that are condensed to a fration in a power of ten base – that doesn't repeat. A rational number in the form of a repeating decimal does repeat. The algorithm will show $e - 2$ is irrational, and the telescoping series is rational.