Problem of identity and quadratic equation

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Abstract

Given "$ab = 0$", considering the arithmetic truth "$0.0 = 0$" we conclude that one possibility is "both $a = 0$ and $b = 0$". Consequently, the roots of a quadratic equation are mutually inclusive. Therefore, the concerned variable can acquire multiple identities in the same process of reasoning or, at the same time. The law of identity gets violated, which we call the problem of identity. In current practice such a step of reasoning is ignored by choice, resulting in the subsequent denial of "$0.0 = 0$". Here, we deal with the problem of identity without making such a choice of ignorance. We demonstrate that the concept "identity of a variable" is meaningful only in a given context and does not have any significance in isolation other than the symbol, that symbolizes the variable, itself. We demonstrate visually how we actually realize multiple identities of a variable at the same time, in practice, in the context of a given quadratic equation. In this work we lay the foundations, based on which we intend to bring forth some hitherto unattended facets of reasoning that concern two basic differential equations which are pivotal to the literature of physics.

Keywords: Quadratic equation; Identity; Contextual truth; Visual mathematics

Contents

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1 Introduction

Mathematical logic has developed in its own right into a vast literature with the main motive of completely removing doubts from the processes of human reasoning and actually 'calculate' such reasons like that of mathematics e.g. see refs.[16, 8]. In fact, such was the attempt made by Boole to write down the laws of human thoughts in analogy with the laws of algebra [10]. This is what we call formal reasoning or formal logic or, simply, logic. There are certain laws, namely the fundamental laws of logic, which form the basis of such formal reasoning e.g. see ref.[1]. Now, we may note that the laws of algebra are also manifestations of human thoughts and to reach conclusions in mathematical calculations we also apply reason. In this article we intend to investigate whether those reasons are consistent with the laws of logic if we formally reason to reach the conclusions of a mathematical calculation. In particular, we investigate how we generally reach conclusions regarding each of the expressions being zero, when we are given that the product of those two expressions is zero. We demonstrate, through formal reasoning, how the laws of logic do not hold universally when humans execute such a process of reasoning to reach the conclusion, by considering simple quadratic equations as examples. Our demonstration is intended to showcase, not the violation of the laws of logic which are nevertheless extremely useful, but to point out that there is a hitherto untrodden pasture beyond formal reasoning that may enable us to have a different viewpoint towards mathematics and mathematical logic. This work as a piece of demonstration, with particular examples, of how we reason while doing simple mathematical calculations concerning product of two expressions being zero. We may call such demonstration as a process of self-inquiry[4], which Brouwer might have called "inner inquiry"[17] i.e. how consciously and truthfully we can reason while doing simple mathematical calculations so that the steps of formal reasoning and the respective limitations are all admitted in writing.

In this work we lay the foundation, based on which we plan to explore, in two companion articles[23, 22], certain hitherto unattended facets of reasoning that concern two basic differential equations, the solutions of which require solving quadratic equations as intermediate steps. These differential equations are of central importance in the physics literature, namely, the equation of motion for the simple harmonic oscillator and the second degree eikonal equation. In view of this, we intend to present our work in a way that can be understood and applied by the reader having very basic knowledge of mathematics and logic to analyze the situations where quadratic equations appear, like the above two differential equations. We believe that our work should be of considerable interest for scientific community irrespective of the categories like the mathematician, logician and physicist. The article is arranged in the following manner.

In Section (2), we begin with a discussion regarding how the truth of "$0.0 = 0$" had been indirectly denied by De Morgan while drawing conclusion about the possibilities when we are given that the product of two expressions is zero. Certainly, we point out what the possibilities are if such truth is not denied. We discuss the associated logical subtleties. Consequently, we set the stage for further discussions regarding how the truth of "$0.0 = 0$" is used in the literature and the associated concerns from the logical perspective.

In Section (3), we present a discussion regarding whether the two roots of a quadratic equation are mutually exclusive or mutually inclusive. We explain that our general habit is to consider the roots as mutually exclusive, but this consideration is based on the denial of the truth of "$0.0 = 0$". We choose to ignore such truth not due to wrong mathematics, but due a logical problem, namely, the violation of the law of identity. When such logical problem does not arise in some particular context, we do not deny the validity of "$0.0 = 0$". That is, we are in a habit of considering the contextual validity of the proposition "$0.0 = 0$".

In Section(4), we present our discussion through a formal analysis so as to give our arguments a logical form. We analyze our reasoning through truth tables and logical expressions by structuring the problem at hand in terms of propositions. We explain through the use of truth table that the polynomial form of the quadratic equation is only logically implied by the factored form, which is nevertheless a consequence of the universal validity of "$0.0 = 0$". We take responsibility of explaining how to deal with the (logical) problem of identity and consider the universal, rather than contextual, validity of "$0.0 = 0$".

In Section (5), we present a visual demonstration to support our formal reasoning. Considering "x" as the variable involved in a quadratic equation, we argue that the phrase "identity of x" is meaningless, in isolation, except that it identifies the symbol itself. Rather, it is the context that provides the meaning to the phrase "identity of x" and here the context is the quadratic equation. We clarify such issues through visual demonstration and then proceed to show in what sense we actually deal with simultaneous multiple identities of a variable while explaining the roots of the equation with a graphical plot. We argue that the real problem is not to admit that a variable can have multiple identities in a context, but to find a way to represent such multiple identities and proceed to do mathematical calculations with such representations.

In Section (6), we provide an outlook that concerns an exploration of how our discussion can have possible consequences in the context of two differential equations, namely, the equation of motion for the simple harmonic oscillator and the second degree eikonal equation, which are of central importance in the literature of physics.

2 De Morgan's "either ..... or ....." and the truth of "$0.0 = 0$"

The question that we aim to investigate is the following: If we are given that a Product of Two Expressions is equal to Zero (PTEZ) can we conclude that one possibility is that both of the expressions are equal to zero? In mathematical terms, the question can be stated as follows: If we are given $a.b = 0$, can we conclude that one possibility is $a = 0 = b$? De Morgan explicitly stated, in a footnote on page no. 110 of ref.[2], the following:

"*When a product $ab = 0$, either a or $b = 0$, for if both have a value, the product, by common rules, has a value.*"

However, he did not explain why it is not possible that "both have a value" that is equal to "zero" i.e. "both $a = 0$ and $b = 0$" or equivalently "$a = 0 = b$". Such a precise logical statement by De Morgan is noteworthy, given the fact that we do have the arithmetic truth "$0.0 = 0$" that lets us write the following:

$$a = 0 = b \qquad \therefore \quad ab = 0.0 = 0. \tag{1}$$

So, indeed the statement "both $a = 0$ and $b = 0$" is consistent with the given relation "$ab = 0$" considering the validity of the proposition"$0.0 = 0$".

In fact, if we deny the proposition "$0.0 = 0$" then we can not answer the following question: *If $x = 0$ then what is the value of x^2?* This is because, to answer this question, we have to write "x^2" as "$x.x$", then we have to write "0" in place of "x" to obtain "0.0" and finally we write "0" in place of "0.0" using "$0.0 = 0$" to conclude that "$x^2 = 0$". In view of such explanations, we believe that "$a = 0 = b$" is a valid conclusion that we can draw from "$ab = 0$" in addition to the two cases which De Morgan considered.

Now, one can provide a *counter argument* to our reasoning, and in favour of avoiding the use of "$0.0 = 0$", to justify De Morgan's choice of excluding "$a = 0 = b$" from consideration in the following way. From $ab = 0$, we can conclude two (mutually exclusive) cases:

- $a = \frac{0}{b} \ni b \neq 0$. Therefore, $a = 0, b \neq 0$.
- $b = \frac{0}{a} \ni a \neq 0$. Therefore, $b = 0, a \neq 0$.

In such a way of calculation, there is no use of "$0.0 = 0$". Furthermore, in this way of reaching the conclusion we can have "either $a = 0$ or $b = 0$", but not "$a = 0 = b$", which is in accord with De Morgan's claim.

Such a *counter argument*, we believe, is a passage around the problem and not a resolution of it. This can be understood by dealing a slightly different problem in the two different methods of reasoning. We change the above problem by writing "a" in place of "b" and then try to find the possible values of a from "$a^2 = 0$". Then from the equation $a^2 = 0$ we can conclude only one valid case, and that also by using "$0.0 = 0$", as follows:

- $a^2 = a.a = n.0 = 0 \ni n \neq 0$. Since $a \neq 0 = a$ is a logical contradiction (i.e. if $a = 0$, then $a \neq 0$), this conclusion is invalid on logical grounds.
- $a^2 = a.a = 0.n = 0 \ni n \neq 0$. Since $a = 0 \neq a$ is a logical contradiction (i.e. if $a \neq 0$, then $a = 0$), this conclusion is invalid on logical grounds.
- $a^2 = a.a = 0.0 = 0$. There is no logical problem in this case as $a = 0 = a$. This is the only valid case.

Therefore, without using "$0.0 = 0$" we can not reach any logically valid conclusion from "$a^2 = 0$" regarding the possible values of a. The one and only logically valid possibility is "$a = 0$" that can be verified if and only if we use "$0.0 = 0$". So, for the verification also we need to use "$0.0 = 0$".

Now, if we use the division method (adopted in the *counter argument*) to reach a conclusion and try to avoid using "$0.0 = 0$", we get into trouble on logical grounds. In order to use the division method, we need to write "$a = 0/a$" from "$a^2 = 0$" and then we have to draw any conclusion (like the way we provided the *counter argument*, in favour of De Morgan). If we do that then we do not arrive at a single valid conclusion. Rather, such an attempt leads us either to a contradiction or to an undecidable situation in the following way:

- If we put $a = n \neq 0$ in the denominator on the right hand side of the equation $a = 0/a$, then we obtain $a = 0/n = 0$. So, we have $a \neq 0 = a$. Thus we arrive at a logical contradiction.
- If we put $a = 0$ in the denominator on the right hand side of the equation $a = 0/a$, then we obtain $a = \frac{0}{0}$ which is an indeterminate form. This is an undecidable situation i.e. we can not decide whether it is possible to arrive at a valid conclusion or not.

Therefore, in view of this above discussion, we prefer not to draw conclusions from any PTEZ through the application of division method (as adopted in the *counter argument*) and rather we rely on the use of "$0.0 = 0$".

Further, if we carefully analyze the process by which we perform division, first and foremost it is dependent on our understanding of subtraction, which is in turn understood in terms of addition and negative numbers; secondly, we do take help of multiplication (tables) to perform division in arithmetic. In fact, this is how De Morgan explained "division" in Section IV of ref.[3]. Carrying forward such intuition to the algebraic framework, we consider that drawing a conclusion from a PTEZ using "$0.0 = 0$" comes logically prior to the method of division based on which *counter argument* can be given.

In view of the above discussion we shall draw conclusion from any PTEZ using the truth of "$0.0 = 0$" only and not by the application of division method adopted in the *counter argument*. This clarification is necessary because now we shall discuss the logical relation between the two roots of a quadratic equation by considering the truth of "$0.0 = 0$" only and not in light of the division method.

3 Contextual validity of "$0.0 = 0$" and the roots of a quadratic equation

Let us consider the following question:

> *What are the possible solutions of the equation $x^2 = 4$?*

The general answer is that:

> *There are two possible solutions (or roots) of this equation i.e. $x = 2$ and $x = -2$.*

However, if we further consider the following question:

> *Are these two solutions mutually exclusive or mutually inclusive?*

then the scenario becomes a bit tricky from the logical perspective. The question can be rephrased in a formal way:

> *In the context of the statement "$x^2 = 4$", which of the following options is true?*
>
> 1. *EITHER "$x = 2$" OR "$x = -2$",*
> 2. *"$x = 2$" OR "$x = -2$".*

"$EITHER....OR....$" means only one of the statements can be true and not both i.e. mutually exclusive. "OR" means one or both of the statements can be true i.e. mutually inclusive. In the jargon of mathematical logic the respective symbols are "$\veebar$", "$\vee$" [9]. However, we shall not introduce such symbols right now, but later, and proceed with the use of words so that the reader can understand the issue even if he has no prior acquaintance with the specific symbols of mathematical logic. Keeping this simplicity, as much as possible, is a necessity in this present discussion because the issues that we are going to investigate involve the most basic steps of mathematics which we take for granted in general.

Now, to carry out such an investigation it is necessary to analyze the process of reasoning through which we generally come to conclusion about the two roots. In terms of formal logic, we are interested in investigating the logical relationships among the statements "$x = 2$", "$x = -2$" and "$x^2 = 4$". During general discussions available in a standard, preliminary modern text, the two roots are considered as mutually inclusive through the use of the words like "*at least* one of the factors is zero" after writing "$x^2 = 4$" in the factorized form "$(x-2)(x+2) = 0$" [21]. This means, either exclusively "$x - 2 = 0$ and $x + 2 \neq 0$", or exclusively "$x - 2 \neq 0$ and $x + 2 = 0$", or exclusively "both $x - 2 = 0$ and $x + 2 = 0$". The process of reasoning through which we can come to such conclusion goes as follows:

BBOX

1. "$\underbrace{\underbrace{(x-2)}_{=0}.\underbrace{(x+2)}_{=n\neq 0}}_{\text{“}0.n=0\text{”}} = 0$" $\leftarrow$ "$(x-2)=0$" AND "$(x+2)\neq 0$" $\equiv$ "$x=2$" AND "$x\neq -2$"
2. "$\underbrace{\underbrace{(x-2)}_{=n\neq 0}.\underbrace{(x+2)}_{=0}}_{\text{“}n.0=0\text{”}} = 0$" $\leftarrow$ "$(x-2)\neq 0$" AND "$(x+2)= 0$" $\equiv$ "$x\neq 2$" AND "$x= -2$"
3. "$\underbrace{\underbrace{(x-2)}_{=0}.\underbrace{(x+2)}_{=0}}_{\text{“}0.0=0\text{”}} = 0$" $\leftarrow$ "$(x-2)= 0$" AND "$(x+2)= 0$" $\equiv$ "$x= 2$" AND "$x= -2$"

Each of the above three processes of reasoning is exclusive of the other two because each of the three processes is an individual analysis. While we do not find any unjustifiably wrong mathematics performed in the above explanation, certainly what looks unacceptable is the third case where "both $x = 2$ and $x = -2$". It is a very legitimate concern to raise that consideration of the statement "$0.0 = 0$" to be true, leads to an apparently uncomfortable situation because x attains two distinct values simultaneously, which is a violation of the law of identity on logical grounds i.e. we have $x = 2 \neq -2 = x$ which is equivalent to writing $2 = x \neq x = -2$. This results due to the fact that the equality sign ("=") can be used reversibly, which is contained in the law of identity in logic e.g. one can see ref.[5] for a detailed discussion of the law of identity. So, we may emphasize that the third case is not at all unacceptable from the mathematical point of view because we have not done any wrong calculation. However, it is objectionable from the logical point of view because in one and "the same process of reasoning", we have used the symbol "x" in two different senses i.e. once in the sense of "2" and then again in the sense of "-2". This is what remains problematic as we note that, Boole, on page number 6 of ref.[10], put emphasis on the basic principles of reasoning with symbols as follows:

"*..... first, that from the sense once conventionally established we never, in the same process of reasoning, depart; secondly, that the laws by which the process is conducted be founded exclusively upon the above fixed sense or meaning of the symbols employed.*"

Thus, what we are dealing with is a logical problem concerning the law of identity and not any wrong mathematical calculation. This becomes even more evident when such problem with the law of identity does not arise, even if we consider the validity of "$0.0 = 0$", to analyze the possible conclusions that we can draw from an equation like $(x - 2)(y + 2) = 0$. The situation can be presented as follows:

xyBOX

1. $\underbrace{“\underbrace{(x-2)}_{=0}.\underbrace{(y+2)}_{=n\neq 0}}_{“0.n=0”} = 0” \quad \leftarrow \quad “(x-2)=0” \text{ AND } “(y+2)\neq 0” \quad \equiv \quad “x=2” \text{ AND } “y\neq -2”$
2. $\underbrace{“\underbrace{(x-2)}_{=n\neq 0}.\underbrace{(y+2)}_{=0}}_{“n.0=0”} = 0” \quad \leftarrow \quad “(x-2)\neq 0” \text{ AND } “(y+2)=0” \quad \equiv \quad “x\neq 2” \text{ AND } “y=-2”$
3. $\underbrace{“\underbrace{(x-2)}_{=0}.\underbrace{(y+2)}_{=0}}_{“0.0=0”} = 0” \quad \leftarrow \quad “(x-2)=0” \text{ AND } “(y+2)=0” \quad \equiv \quad “x=2” \text{ AND } “y=-2”$

Therefore, as is stands, we arrive at an uncomfortable situation where “$0.0=0$” leads us to logical problems depending on the nature of the PTEZ. That is, if we consider the law of identity to be universally true, then we have to consider “$0.0=0$” to be contextually true. In other words, when we face trouble from the logical point of view, we *choose to ignore* the truth of “$0.0=0$” and exclude that particular step of reasoning from consideration. We may call this *choice of ignorance*, which is indeed the usual practice that we adopt but possibly remain unaware of. We may showcase such unawareness regarding our own reasoning by rewriting BBOX with the choice of ignorance (which is our general habit), as follows in BBOXC.

BBOXC

1. $\underbrace{“\underbrace{(x-2)}_{=0}.\underbrace{(x+2)}_{=n\neq 0}}_{“0.n=0”} = 0” \quad \leftarrow \quad “(x-2)=0” \text{ AND } “(x+2)\neq 0” \quad \equiv \quad “x=2” \text{ AND } “x\neq -2”$
2. $\underbrace{“\underbrace{(x-2)}_{=n\neq 0}.\underbrace{(x+2)}_{=0}}_{“n.0=0”} = 0” \quad \leftarrow \quad “(x-2)\neq 0” \text{ AND } “(x+2)=0” \quad \equiv \quad “x\neq 2” \text{ AND } “x=-2”$
3. ~~$\underbrace{“\underbrace{(x-2)}_{=0}.\underbrace{(x+2)}_{=0}}_{“0.0=0”} = 0” \quad \leftarrow \quad “(x-2)=0” \text{ AND } “(x+2)=0” \quad \equiv \quad “x=2” \text{ AND } “x=-2”$~~ (choice of ignorance)

In fact, such was the case for Boole, who used the contextual validity of “$0.0=0$” while introducing an algebra with variables which can have value “EITHER 0 OR 1”, in ref.[10]. We explain this as follows. On page number 37 of ref.[10], Boole wrote the following: “*We know that* $0^2=0$*, and that* $1^2=1$*; and the equation* $x^2=x$*, considered as algebraic, has no other roots than* 0 *and* 1.” We believe, by “0^2” Boole meant “0.0 (zero times zero)” and based on this belief we conclude with certainty that Boole considered the validity of “$0.0=0$”. However, Boole's conclusion that the variables can take value “EITHER 0 OR 1” is an outcome of his *choice of ignorance* regarding the validity of “$0.0=0$” that is equivalent to the acceptance of “$0.0\neq 0$”. Certainly Boole did not emphasize, in ref.[10], on whether he considered mutual exclusiveness or mutual inclusiveness of the two roots and this indicates how he was possibly unaware of the logical hazards when we go through such a process of self-inquiry i.e. an analysis of how truthful we are regarding our own process of reasoning. We present Boole's reasoning as follows:

Boole - BOX

1. $“\underbrace{x}_{=0}.\underbrace{(x-1)}_{=n\neq 0} = 0” \leftarrow “x = 0” \text{ AND } “(x-1) \neq 0” \equiv “x = 0” \text{ AND } “x \neq 1”$ — $“0.n = 0”$
2. $“\underbrace{x}_{=n\neq 0}.\underbrace{(x-1)}_{=0} = 0” \leftarrow “x \neq 0” \text{ AND } “(x-1) = 0” \equiv “x \neq 0” \text{ AND } “x = 1”$ — $“n.0 = 0”$
3. ~~$“\underbrace{x}_{=0}.\underbrace{(x-1)}_{=0} = 0” \leftarrow “x = 0” \text{ AND } “(x-1) = 0” \equiv “x = 0” \text{ AND } “x = 1”$~~ (choice of ignorance) — $“0.0 = 0”$

Now, Boole's case is different from what we have analyzed in BBOX regarding quadratic equation. This is because, unlike BBOX, Boole had to consider both “$0.0 = 0$” and “$0.0 \neq 0$” within the context of the same problem and this can be explained as follows. Considering the choice of ignorance i.e. “$0.0 \neq 0$”, it becomes now impossible to verify that $x = 0$ satisfies the equation $x^2 = x$ which Boole considered to begin with. Thus, Boole's choice of ignorance, while lets him conclude that x can take value “EITHER 0 OR 1”, on the other hand halts him from verifying that 0 is actually a root of the concerned quadratic equation. If we consider the totality of three cases of Boole-BOX as a whole and consider it as one context, then certainly Boole's analysis if founded on a contradiction i.e. the proposition “$0.0 = 0$” is both true and false in the same context or “the same process of reasoning”. On the other hand, if we consider that each of the three steps of Boole-BOX as an individual process of reasoning, then we can say that Boole considered “$0.0 = 0$” to be only contextually valid – Boole's reasoning is then saved from a contradiction. So, from Boole-BOX, we can conclude that the general practice is to use “$0.0 = 0$” as contextually valid while drawing conclusion regarding the factors of a PTEZ.

We however aim to investigate the logical issues which arise if we do not apply the choice of ignorance and consider “$0.0 = 0$” as a universal truth. In doing so, besides having to deal with the violation of the law of identity, we find that the polynomial form “$x^2 = 4$” and the factored form “$(x-2)(x+2) = 0$” are not logically equivalent. We explain the situation as follows. In the expression “$x^2 = 4$”, only one value of x can be used, and not both, to satisfy the equation i.e. one can write either (exclusively) an expression “$2^2 = 4$” or (exclusively) an expression “$(-2)^2 = 4$”, but not both. The situation can be understood if we try to solve the following problem:

Given “$x^2 = 4$”, verify that there are two possibilities viz. “$x = 2$”, “$x = -2$”.

The answer to this question can be given as follows:

ABOX

1. $“x = 2” \equiv “x^2 = (2)^2” \rightarrow “x^2 = 4”$
2. $“x = -2” \equiv “x^2 = (-2)^2” \rightarrow “x^2 = 4”$
3. $“x = 2” \text{ AND } “x = -2” \rightarrow “x^2 = x.x = (2).(-2) = -4 \neq 4” \equiv “x^2 \neq 4”$

There is no use of “$0.0 = 0$”. Also, there is no threat to the law of identity. Here, we have two processes of reasoning corresponding to two reachable conclusions, compared to three cases for the factorized form. Therefore, an uncomfortable situation appears yet again, where we find that the expressions “$x^2 = 4$” and “$(x-2)(x+2) = 0$” are not equivalent from the logical perspective simply because the former does not contain all the truths associated with the later. Nevertheless, this discomfort never arises in the usual practice because, like Boole, we are also in a habit of applying the choice of ignorance that renders the two expressions logically equivalent.

Before we proceed further, in view of what we have discussed until now, it is important to note a statement by Descartes. In fact, we may note that we could find only one instance in the literature where it was Descartes who admitted of the mutual inclusiveness of the two roots of a quadratic equation, in writing. On page no. 159 of ref.[25], Descartes wrote the following:

" *Multiplying together the two equations* $x-2=0$ *and* $x-3=0$*, we have* $x^2-5x+6=0$*, or* $x^2=5x-6$*. This is an equation in which x has the value* 2 *and at the same time x has the value* 3*.* "

If we *assume* that this classic text of Descartes (i.e. ref.[25]) does not contain a printing mistake (or wrong translations from French to English), then certainly the underlined portion of the above statement manifests his views regarding the mutual inclusiveness of the two roots. Certainly, here, to multiply together the two equations $x-2=0$ and $x-3=0$, Descartes needed to use "$0.0=0$" so that he could write $(x-2)(x-3)=0.0=0$. From here he could multiply the factors on the left hand side to write $(x-2)(x-3)=x^2-5x+6$ and then he could write 0 on the right hand side to complete the equation. However, Descartes did not explain how he could use $x=2$ and $x=3$ "at the same time" to show that $x^2-5x+6=0$. Had he tried to verify his own statement at each and every step as a process of self-inquiry, he would have found that such verification is not possible in the polynomial form. Further, we believe that Boole would have replaced Descartes' phrase "at the same time" with "in the same process of reasoning" to write "*... x has the value* 2 *and in the same process of reasoning x has the value* 3." in place of the underlined part of Descartes' statement. What we want to mean is that, Descartes did not discuss the logical problem associated with his assertion which leads to the conclusion "$2=3$". Nevertheless, we have mentioned Descartes' statement for precise nitpicking of words so as to highlight how the problem, which we are dealing with, is really present in the literature of mathematics but somehow has remained unattended. So, we now rejoin the main flow of our discussion to proceed further.

4 Formal reasoning with the quadratic equation

To present the discussion in formal terms, now we mark the relevant statements written in quotes (" ") with capital letters and consider them as propositions which can be either true (T) or false (F) and we denote the corresponding negations by an overhead bar ($\bar{}$), as written in LVBOX (below). Also, after explaining the issues in words, we shall now use the symbols of mathematical logic to do the formal analysis.

LVBOX

$$A:\text{“}x^2=4\text{”},\quad B:\text{“}(x-2)(x+2)=0\text{”},\quad B_-:\text{“}x=2\text{”},\quad B_+:\text{“}x=-2\text{”},$$

$$\bar{A}:\text{“}x^2\neq 4\text{”},\quad \bar{B}:\text{“}(x-2)(x+2)\neq 0\text{”},\quad \bar{B}_-:\text{“}x\neq 2\text{”},\quad \bar{B}_+:\text{“}x\neq -2\text{”}.$$

Now, we can write BBOX, taking into account the mutual exclusivity of each of the options, formally as follows:

FBBOX

1. $B \quad \leftarrow \quad B_- \wedge \bar{B}_+$
2. $B \quad \leftarrow \quad \bar{B}_- \wedge B_+$
3. $B \quad \leftarrow \quad B_- \wedge B_+$

$$\therefore B \equiv (B_- \wedge \bar{B}_+) \veebar (\bar{B}_- \wedge B_+) \veebar (B_- \wedge B_+) \equiv (B_- \vee B_+)$$

Also, we can write ABOX, taking into account the mutual exclusivity of each of the options, formally as follows:

FABOX

1. $B_- \quad \rightarrow \quad A$
2. $B_+ \quad \rightarrow \quad A$
3. $B_- \wedge B_+ \quad \rightarrow \quad \bar{A}$

$$\therefore \quad (B_- \veebar B_+) \equiv A$$

Further, to interpret the status of the law of identity in the respective situations, we write the following:

IBOX

I: "$x = x$, *for all* x" $\bar{I}$: "$x \neq x$, *for all* x"

Now, we write the truth table for A and B, with B_-, B_+ as inputs and taking into account the status of the law of identity in different contexts, as follows:

B_+	B_-	A	B	I
F	F	F	F	U
T	F	T	T	T
F	T	T	T	T
T	T	F	T	F

We have written U (for 'undecided') to mean that we can not decide whether I is true or false in a particular situation which, in the present scenario, is when the mathematical problem itself is not stated i.e. the first row of the above truth table. From the above truth table we can conclude that $B \to A$.

However, if we apply the *choice of ignorance* by simply not considering the last row of the above truth table as follows:

B_+	B_-	A	B	I
F	F	F	F	U
T	F	T	T	T
F	T	T	T	T
~~T~~	~~T~~	~~F~~	~~T~~	~~F~~

then we can conclude that $A \equiv B$, which is nevertheless the standard practice adopted by us in general. In this case, the scenario in which the law of identity is violated, does not arise because we have chosen to ignore that situation according to will. Here, we do *not* consider the *choice of ignorance* and rather choose to make some choices of ignorance at some subtler levels of reasoning. So, we work with the first truth table among the above two. Therefore, we shall proceed with B, alongside the acknowledgment of $B \to A$ which is a consequence of the universal validity of "$0.0 = 0$". So, now our responsibility is to investigate how to deal with the problem regarding the law of identity in the context of quadratic equations. If we follow the literature, as far as the law of identity is concerned, the concepts of "variable" (which we have denoted by "x", "y", etc.) and "equality($=$)" themselves get questioned, as one can understand from refs.[14, 12, 6, 26, 18] and the references therein. Nevertheless, we do not want to indulge in a review of what the respective authors of these references have debated on. Rather, we shall stick to the particular context which we are dealing with and adhere to our attitude of demonstrating facts through explicit means so that our process of reasoning can be directly realized and put into use for further requirement. In what follows, we adopt a visual way to demonstrate the situation so as to support what we have explained through formal reasoning regarding the equation $(x-2)(x+2) = 0$.

5 Visual demonstration in support of the formal reasoning

When logic gets in trouble, and especially when becomes a challenge to accept certain truth that goes against a long standing belief, it is hard to rely on such reasoning however rational it may look. In such cases, if possible, any directly realizable argument becomes of immense value to support such belief defying process of reasoning. Certainly, when the question of direct realization is concerned, our reliance on our own perceptions becomes of utmost value and among all our modes of perception, visual perception comes to the foremost if the ease of understanding of a situation is of concern. The value of a visual demonstration in mathematics is undeniable[20]. Therefore, to explain the process of reasoning by which we come to conclusion about the two roots of the equation $(x-2)(x+2) = 0$, we plot the function $f(x) = (x-2)(x+2)$ on a graph paper along the y-axis or ordinate, and then point out the cutting points of the plot, with the x-axis or abscissa. If one considers $(2-x)$ instead of $(x-2)$, that will not affect our arguments. Now, in this method of demonstration, the meaning of "x" is the line along with the cuts which mark the numbers $\cdots, -2, -1, 0, 1, 2, \cdots$ – see fig.(1).

If we want to really express what "x" means other than the symbol itself, then it has no true identity that is meaningful. Rather, it is a symbolic representation of the collection of operations that we do to draw fig.(1). However, given the context, namely the equation $(x-2)(x+2) = 0$ whose roots need to be determined, the

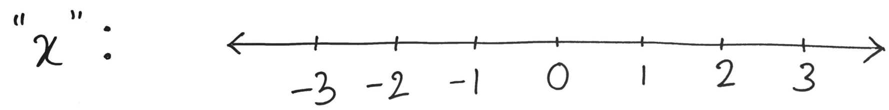


Figure 1: The symbol "x" gets its meaning through the operations of drawing a line and making cuts to demonstrate the numbers $\cdots, -2, -1, 0, 1, 2, \cdots$.

cutting points of the plot with the abscissa, namely I_1 and I_2 in fig.(2), provide some particular values which we may call the identities of "x" in this particular context. We say that "x can take such and such values". These

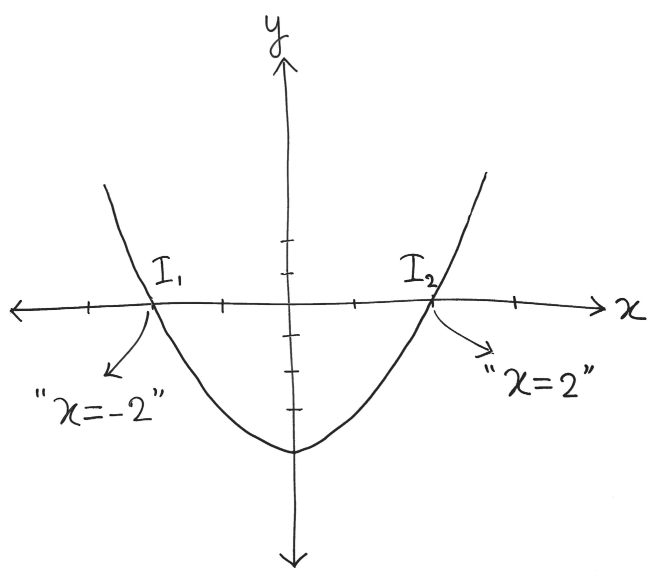
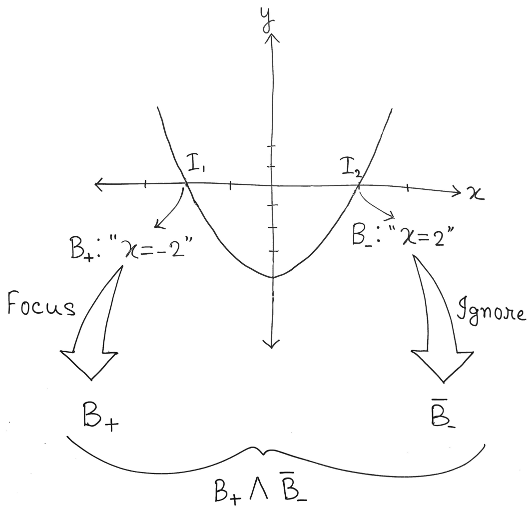


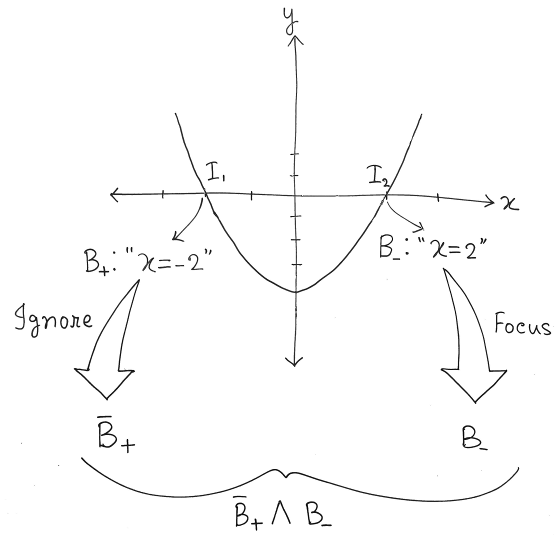
Figure 2: The curve provides the context in which "x" acquires the identities through the intersection points of the curve with the abscissa, namely, I_1 and I_2.

are the identities of "x" whose meaning goes beyond the symbol itself, nevertheless, only under this particular context. Now, the crucial question is whether "x" can have multiple identities in the same process of reasoning, or, in other words, simultaneous multiple identities. The answer depends on what we call as "the same process of reasoning".

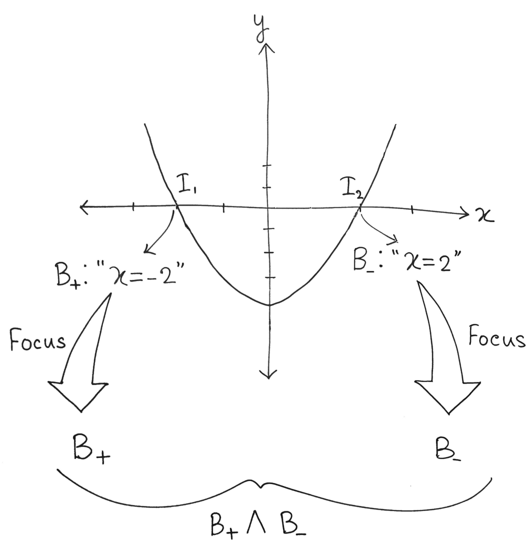
If we choose to focus at I_1 and ignore I_2, to explicate the truth $B_+ \wedge \bar{B}_-$, then in this process of reasoning, "x" has only one identity, namely, "-2". Considering this step as one individual process of reasoning the identity of "x" remains fixed to "-2" in this same process of reasoning, but at the cost of ignoring I_2 i.e. negation of B_-. This is what we have demonstrated through fig.(3(a)).



(a) This is an individual process of reasoning where we focus on I_1 and ignore I_2. This demonstrates $B_+ \wedge \bar{B}_-$. "x" has one fixed identity "-2" in this one and same process of reasoning.



(b) This is an individual process of reasoning where we focus on I_2 and ignore I_1. This demonstrates $\bar{B}_+ \wedge B_-$. "x" has one fixed identity "2" in this one and same process of reasoning.



(c) This is an individual process of reasoning where we focus on both I_1 and I_2. This demonstrates $B_+ \wedge B_-$. "x" has two fixed identities "-2" and "2" in this one and same process of reasoning.

Figure 3: The above three diagrams demonstrate a collection of three individual processes of reasoning. In this collective one and same process of reasoning, "x" has at least one and at most two identities among "-2" and "2". This diagram as a whole provides the visual demonstration of the mutual inclusiveness of the two roots, which we have formally written as $B \equiv B_+ \vee B_-$.

If we choose to focus at I_2 and ignore I_1, to explicate the truth $\bar{B}_+ \wedge B_-$, then in this process of reasoning,

"x" has only one identity, namely, "2". Considering this step as one individual process of reasoning, the identity of "x" remains fixed to "2" in this same process of reasoning, but at the cost of ignoring I_1 i.e. negation of B_+. This is what we have demonstrated through fig.(3(b)).

Nevertheless, such deliberate ignorance, we believe, is equivalent to admittance of inability to handle the whole truth. Here, the whole truth is the two simultaneous identities of x in the whole process of reasoning that lead us to the conclusion of two roots visualized as the two cutting points viz. I_1, I_2.

If the identity of "x" is fixed to "2", say, then any other identity is automatically denied. So, denial of any other specific identity becomes unnecessary. However, if we know of more than one possible identities of "x" in a specific context, then the denial of the other or the others become necessary. In our problem, due to our knowledge of both I_1, I_2 in fig.(2), we become aware of multiple identities of "x". Then, when we focus on I_1, we have to write that we ignore I_2. Similarly, when we focus on I_2, we have to write that we ignore I_1. Each of these individual acts of focusing are the individual processes of reasoning that we have just discussed in the previous paragraphs. Then, certainly the whole truth contains the cognition of both I_1 and I_2 simultaneously in one and same process of reasoning, as visualized in fig.(3(c)), which we write as $B_+ \wedge B_-$.

In a nutshell, while we consider the roots of the equation we do deal with multiple identities of "x" in the same process of reasoning and now by "same process of reasoning" we mean both the individual focuses on I_1 and I_2, and the focus on both I_1 and I_2. This is what we have demonstrated through fig.(3) which shows the collection of three individual processes of reasoning as a whole. In this one whole and same process of reasoning "x" has at least one identity and at most two identities among "-2" and "2" i.e. the identities, or roots in the present context, are mutually inclusive.

6 Outlook: Differential Equations

In view of our demonstration with quadratic equation, we believe, admitting that there can be multiple identities of "x" in the same process of reasoning is not a problem. Rather, the real problem is whether we can represent mutually inclusive multiple identities of "x" in writing and do calculations with such representation. Then the question comes – calculate what? As far as the satisfaction of the equation with simultaneous multiple identities of "x" is concerned, we have shown how to do this and that is where the role of "$0.0 = 0$" gets manifested. Therefore, we need broader contexts where such quadratic equations appear as sub-contexts – an intermediate step – and then we can check how the role of simultaneous multiple identities of a variable can be put to use. We mention two immediate applications of our discussion so that its importance can be foreseen.

First is the case of a differential equation of the following form: $\frac{d^2x}{dt^2} = -\omega^2 x$ where x is a function of t and ω is a given number. This equation is known as the equation of a simple harmonic oscillator in the literature of physics and this differential equation is possibly one of the most basic equations which any physics student has to learn e.g. see ref.[15]. One of the easiest and the most widely used method to solve this equation is to proceed with an *assumption* that the solution is of the form $x = Ae^{kt}$, where A is some non-zero constant and k needs to be determined; for example, one can see Boole's explanation of this method in ref.[11]. Proceeding with this assumption, one immediately obtains the following quadratic equation: $k^2 + \omega^2 = 0$, which certainly has two roots, namely, $k = i\omega$ and $k = -i\omega$. The question arises, in light of the present discussion, how the mutual inclusiveness of these two roots gets manifested in the solution. Interestingly, the answer is related to the principle of linear superposition which leads us to the general solution of the following form $x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$ that violates the assumption because the general solution can not be cast into the form $x = Ae^{kt}$. To mention, it is not unknown today that the problem of identity is related to the principle of superposition, which however is only discussed in the context of quantum mechanics e.g. see the relevant references cited in ref.[27]. However, the case of simple harmonic oscillator which we have just brought into discussion, appears in classical physics. So, the question arises whether the logical subtleties are associated with the way we reason on mathematical grounds or with the "classical" and "quantum" distinction of physics. We plan to elaborate on this issue in ref.[23].

Second is the case of a differential equation of the following form: $\left(\frac{\partial f}{\partial x}\right)^2 = \frac{1}{v^2}\left(\frac{\partial f}{\partial t}\right)^2$, where f is a function of x and t and v is a given number. This equation is one of the two eikonal equations which arise in the study of waves in one dimension with phase velocity v (e.g. see ref.[7]) and more specifically in the context of geometric optics approximation (e.g. see ref.[19] to find a three dimensional generalization). If we consider how Boole explained the solution of such a differential equation in ref.[11], then the first step to consider is to factorize the differential equation and obtain a pair of differential equations as follows: $\frac{\partial f}{\partial x} = \frac{1}{v}\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x} = -\frac{1}{v}\frac{\partial f}{\partial t}$. So, the question regarding the mutual inclusiveness of the two factors of the differential equation is there to deal with from the beginning. As far as the problem of identity is concerned, it is trivially manifest because of the following reasons. f is the solution of the second degree differential equation, but the factorized differential equations are a collection of two different differential equations of first degree. Since there is a difference in signature in the two first degree differential equations, it is easy to see that the solutions of the two cases differ. Then this leads to the situation $f \neq f$ and the problem of identity gets manifest at the level of the solution. To avoid this problem one needs to write the two differential equations as $\frac{\partial f_-}{\partial x} = \frac{1}{v}\frac{\partial f_-}{\partial t}$, $\frac{\partial f_+}{\partial x} = -\frac{1}{v}\frac{\partial f_+}{\partial t}$ so as to symbolically express the difference in the solutions. However, if these equations are written in such a way, then the product does not give back the original second degree equation. We plan to elaborate on such issues in ref.[22].

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