# A Proof For 3X+1 Guess

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## Abstract

Build a special odd tree model according with $(\times 3+1)\div 2^k$ algorithm, to depart odd numbers in different groups.Carry out research into the tree,found that counts of elements in the tree reduce and converge downward one by one layer,values of elements converge downward to 1,and all odd numbers can appear in the tree only one time.Then prove the "3x+1"guess indirectly.

**Key Words:**3x+1 odd tree

## I Prove Some Lemmas

**Lemma 1:**All odd numbers can be written in one of three forms:6a+1,6a+3,6a+5(a>=0).

**Lemma 2:**Transform odd 6a+1: $\left((6a+1)\times 2^{2k}-1\right)\div 3$ ,(a>=1,k>=1),generate an infinite data sequence,which each element is odd,and any element in the sequence is 4 times of the previous element(if exists) plus 1;The No. k odd make $(\times 3+1)\div 2^{2k}$ ,(k>=1)transformation get a result 6a+1.

Prove: k=1: $\left((6a+1)\times 2^2-1\right)\div 3=(24a+3)\div 3=8a+1$, is odd

k=2: $\left((6a+1)\times 2^4-1\right)\div 3=(96a+15)\div 3=32a+5$, is odd

Use method of induction,suppose $\left((6a+1)\times 2^{2k}-1\right)\div 3=2\text{m}+1$ m>0, k>=1

Then $\left((6a+1)\times 2^{2(k+1)}-1\right)\div 3=((6a+1)\times 2^{2k}-1)\div 3+((6a+1)\times 2^{2k}\times 3\div 3$

$=2m+1+((6a+1)\times 2^{2k}$ is odd

Hence each element in the sequence is odd.

$$\left(\left(\left((6a+1)\times 2^{2k}-1\right)\div 3\right)\right)\times 4+1=\left((6a+1)\times 2^{2(k+1)}-4\right)\div 3+1=\left((6a+1)\times 2^{2(k+1)}-1\right)\div 3$$

Hence No. k+1 element is 4 times No. k element Plus 1.

With $\left((6a+1)\times 2^{2k}-1\right)\div 3$ make transformation of $(\times 3+1)\div 2^{2k}$ obviously get 6a+1.

**Lemma 3:** Transform odd 6a+5: $\left((6a+5)\times 2^{2k-1}-1\right)\div 3$ ,(a>=0,k>=1),generate an infinite data sequence,which each element is odd,and any element in the sequence is 4 times of the previous element(if exists) plus 1;The No. k odd make $(\times 3+1)\div 2^{2k-1}$,(k>=1)transformation get a result 6a+5.

Prove: k=1: $\left((6a+5)\times 2\ -1\right)\div 3=(12a+9)\div 3=4a+3$ is odd

k=2: $\left((6a+5)\times 2^3-1\right)\div 3=(48a+39)\div 3=16a+13$ is odd

Use method of induction,suppose $\left(\left(6a+5\right)\times 2^{2k-1}-1\right)\div 3=2\mathrm{m}+1$ m>0,k>=1

Then $\left(\left(6a+5\right)\times 2^{2k+1}-1\right)\div 3=((6a+5)\times 2^{2k-1}-1)\div 3+((6a+5)\times 2^{2k-1}\times 3\div 3$

$$=2m+1+((6a+5)\times 2^{2k-1}$$ is odd

Hence each element in the sequence is odd.

$$\left(\left(\left(\left(6a+5\right)\times 2^{2k-1}-1\right)\div 3\right)\right)\times 4+1=\left(\left(6a+5\right)\times 2^{2(k+1)-1}-4\right)\div 3+1=\left(\left(6a+5\right)\times 2^{2(k+1)-1}-1\right)\div 3$$

Hence No. k+1 element is 4 times No. k element Plus 1.

With $\left(\left(6a+5\right)\times 2^{2k-1}-1\right)\div 3$ make transformation of $(\times 3+1)\div 2^{2k-1}$ obviously get 6a+5.

**Lemma 4:** Transform odd 6a+3: $\left(6a+3\right)\times 2^{k}-1$,(a>=0,k>=1),data generated can not be divided exactly by 3.

## Ⅱ Build A Tree Model

Below we build digit tree using Lemma 1--4.

Use 1 as root place in layer 0,data sequence generated by transformation of $(1\times 2^{2k}-1)\div 3$,(k>1) place in layer 1,as up node of 1 in layer 0.Here because case k=1, $(1\times 4-1)\div 3=1$,is duplication of 1 in layer 0,remove it from layer 1.

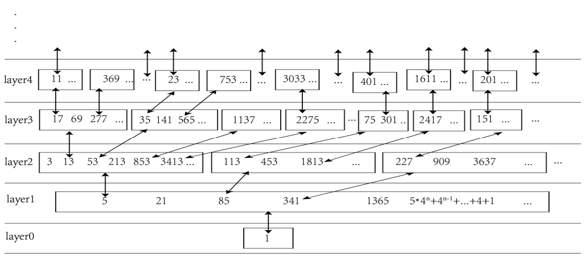
Build layer 2 using layer 1,for every element in layer 1,make transformation in order as follow:

Case element is 6a+1: do $\left(\left(6a+1\right)\times 2^{2k}-1\right)\div 3$,(a>0,k>=1),generate an odd sequence place behind of the sequence last generated in layer 2,as up node of 6a+1 in layer 1.

Case 6a+5: do $\left(\left(6a+5\right)\times 2^{2k-1}-1\right)\div 3$,(a>=0,k>=1) ,generate an odd sequence place behind of the sequence last generated in layer 2,as up node of 6a+5 in layer 1.

Case 6a+3:do nothing.

Use same method do transformation in order for every element of every sequence in one layer ,and generate data and node of next layer.Finally we can build a determined unique digit tree,as shown in graph 1.



Graph 1: 3x+1 odd tree

## Ⅲ Prove Some Properties Of The Tree

Below prove some mathematical properties of the tree.

**Property 1:**All elements in the tree are odd.

According to Lemma 1--4 can get it.And all elements downward calculation accord to $(\times 3+1)\div 2^{2k}$,(k>=1)algorithm.I call it “3X+1 odd tree”.

**Property 2:**If a data is an element of the tree,use the data as root build a tree using same method,then all elements in the new tree are elements in the tree.

**Property 3:**Elemens in random one sequence are arrayed by the order…6a+1,6b+5,6c+3…,first element is one of three forms.

Prove:We can easily know random one form of 3 forms can appear in the first position of a sequence as regulation of graph 1.To random layer i sequence j,if no. k element in the sequence is 6a+1,then no. k+1 element is $(6a+1)\times 4+1=24a+5=6b+5$,is form of 6b+5;No. k+2 element is $((6a+1)\times 4+1)\times 4+1=96a+21=6c+3$, is form of 6c+3.

Similarly we can prove case of first element is 6b+5 or 6c+3.

We can also prove that the first element of the sequence generated by odd 6b+5 upward calculation is 2 times of the first element generated by odd 6a+1(if exists) plus 1.

**Property 4:**Any element(except 1)in the tree do downward calculation $(\times 3+1)\div 2^k$ ,(k>=1),get an unique data,drop and only drop one layer.

**Property 5:**Numbers of elements in the tree diverge upward,reduce and converge downward one by one layer.

Prove:According to Lemma 2--4 and Property 3,each element(except 6a+3) in one layer build an infinite sequence in the up layer,on the contrary,one sequence in a layer can only build one element in the down layer.Hence,numbers of elements in the tree reduce downward one by one layer.The lowest layer have only one element 1,then numbers of elements converge downward.Because all elements are generated from 1 upward calculation one by one layer,all elements do downward calculation converge to 1.This is to say,although each element does downward calculation,data generated per time is sometimes big,sometimes small varying,the numbers of data it can build reduce per time,data value finally converge to 1.

**Property 6:**If the position of a data is determined in the tree,then the downward calculation route of the data is unique.

**Property 7:**The first element of random one sequence(except sequences in layer 0,1) can not be built from an odd $(\times 4+1)$ calculation.

Prove:With random layer i sequence j(i>1,j>=1),if no. k element is 6a+1,the first element of the corresponding sequence in layer i+1 is $((6a+1)\times 2^2-1)\div 3=(24a+3)\div 3=8a+1$, $(8a+1-1)\div 4=2a$ ,is an even,hence first element can not be gotten from odd;If no. k element is 6a+5,the first element of the corresponding sequence in layer i+1 is $((6a+5)\times 2\ \ -1)\div 3=(12a+9)\div 3=4a+3$ , $(4a+3-1)=4a+2$ ,can not be divided exactly by 4.Sequences in layer i+1 can only be built from 6a+1 or 6a+5,hence we prove it.This is to say,the first element can not be written in binary form x01…01(odd x>1).

**Property 8:**If two elements in two sequences are equal,then the first elements in the two sequences are equal,the two sequences are complete equivalence.

Prove:According to Lemma 2--3 and Property 7,if the first element of layer i sequence j $A_{ij}$ is $a_{ij1}$,the sequence can be written in binary form $a_{ij1}$, $a_{ij1}$01…01, layer l sequence m can be written in binary form $a_{lm1}$, $a_{lm1}$01…01, $a_{ij1}$ and $a_{lm1}$ itself can not be written in binary form x01…01(odd x>1).If $a_{ij1}$ is equal to $a_{lm1}$,then two sequences are complete equivalence.If a certain $a_{ij1}01\ldots01$ is equal to a certain $a_{lm1}01\ldots01$,do $\left(-1\right)\div 4=\mathrm{odd}$ calculation separately to them,datas gotten must be equal.Continue do until can not perform this calculation,should get data $a_{ij1}$ and $a_{lm1}$, $a_{ij1}=a_{lm1}$, two sequences are complete equivalence.

**Property 9:**Elements in same layer are not equal to each other.

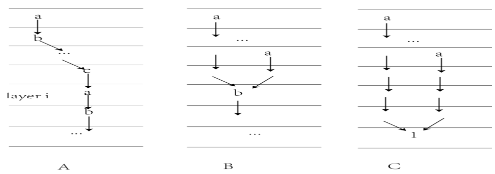
Prove:Obviously,elements are not equal to each other in layer 0 and layer 1.To prove elements in other layers are not equal to each other in same layer,we only need to prove first elements of sequences in same layer are not equal to each other.With random 6a+1 and 6b+5,do $\left(\left(6a+1\right)\times 2^2-1\right)\div 3=(24a+3)\div 3=8a+1$,

$\left(\left(6b+5\right)\times 2\ \ -1\right)\div 3=(12b+9)\div 3=4b+3$ ,if 8a+1=4b+3,then 4a-1=2b,odd=even,contradict.Hence,first elements of random sequences built from 6a+1 and 6b+5 are not equal to each other.If elements are not equal to each other in layer i,with all 6a+1 and 6b+5 elements in layer i,do calculations above,get sequences in layer i+1,which first elements are not equal to each other.

**Property 10:** Elements in different layers are not equal to each other.

Prove:If data a exists in two different layers,there are 3 cases,as shown in graph 2.

Case A:downward calculation route of a include a.From layer i downward,each layer should have a sequence which is complete equivalence to one sequence in some one up layer,but obviously sequence in layer 1 is not complete equivalence to any sequence in up layers,because sequence in layer 1 has the binary form 101,101…01, sequences in up layers have the form x,x01…01(odd x>1),x itself can not be written in binary form y01…01(odd y>1).Hence,case A is not established.In fact,in case A we even can not use data a to build a tree(upward,downward,horizontal expand) which counts of elements reduce downdward.



Graph 2: Same Data In Different Layers

Case B and C: downward calculation routes of a in two different layers intersect in some one down layer or layer 0.Since each downward calculation get a unique result,to let two different routes intersect, one of the two routes must do cross-layer calculation,do not accord to Property 4.

Attention,in the proof of Property 9 and Property 10,we use special characters of layer 0 and layer 1,is just to make proof easier,during the procedure of tree model building,we only use Lemma 1--4,this is to say, use any odd number to build a tree,it should also accord to Property 1--10.

## Ⅳ Prove 3X+1 Guess

Below prove all odds do $(\times 3+1)\div 2^k$ calculation must converge to 1.

Prove:If the data is in “3X+1 odd tree” built above,it must converge to 1.If exists an odd a do not appear in the tree,use a build a tree with same methods above,then all elements in the new tree do not appear in “3X+1 odd tree”,but the new tree also accord to Property 1--10.

Since $\left(2x+1\right)\times 3+1=6x+4$ ,(x>0),to random x>0,k>0, $\left(6x+4\right)\div 2^k\neq 2x+1$ ,hence a do not converge to itself.Suppose with a do downward calculation converge to odd b,and b>1,this case b can continue do $(\times 3+1)\div 2^{k}$ calculation,and layer number reduce for every calculation,possible datas can be generated are fewer and fewer,until build 1,can not continue do calculation.Hence a must converge to 1,and must exist in “3X+1 odd tree”.

Come here,we have proved “3X+1”guess.In fact, “3X+1 odd tree”include all odd numbers,and each odd number appear only one time.