ON A NEW RULE OF APPROXIMATING AREA UNDER THE CURVE

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Abstract

I am going to provide a new technique of approximating area under the curve, using the Newton-Raphson Method. I am also going to provide a formula that would help us approximate any Definite Integral or help us find the area under the curve, under certain conditions. The relative error of this formula is very small, which makes it even more interesting.

Keywords: Treanungkur's Rule for Definite Integration, Alternative of Midpoint Rule, Treanungkur's Method of Finding Area under the curve, Application of Newton Raphson Method, Treanungkur's Rule.

1 Introduction

In Numerical Analysis, we often use Newton-Raphson Method to approximate roots of a polynomial function because a polynomial with degree ≥ 5 is solvable iff it forms a solvable Galios Group. The main formula of Newton-Raphson Method is :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Proposition 1. *Let us assume, a function $f(x)$, which is an increasing continuous function on the interval $[\alpha, \beta]$. Let, a root of $f(x)$ lies in the interval $[\alpha, \beta]$, for instance let it be a, then*

$$\int_a^b f(x)\,dx \approx \frac{0.935}{2}\left\{\frac{f(x_0)}{f'(x_0)}\left(f(x_0)+f(x_1)\right) + \ldots + \frac{f(x_n)}{f'(x_n)}\left(f(x_n)+f(x_{n+1})\right)\right\}$$

where,

$$x_0 = b,\;\; x_{n+1} \approx a, \textit{and}\;\; x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

2 Newton-Raphson Method

We have often encountered various polynomials in our life, most commonly we have often seen polynomials of degree 2, and we have also solved it using factorization, using Completing the square method, and using the Quadratic Formula. But it turns out to be for polynomials with degree ≥ 5, it is not easily solvable using Radicals. For instance, this equation :

$$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 + 4x + 1$$

One of the ways to solve higher degree polynomials like this, is by approximation.

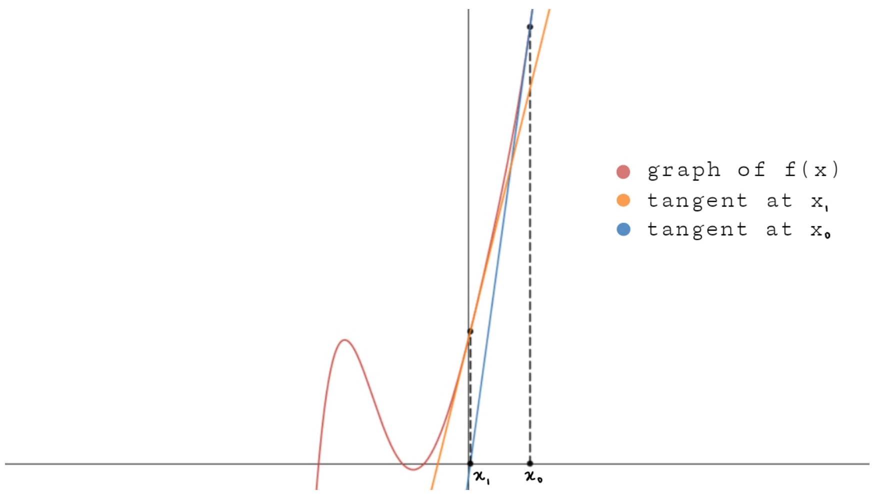

Figure 1: This is the graphical representation of the tangents drawn at x_0 and x_1

The Newton-Raphson Method gives us the formula :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

First, we have to choose a value x_0 around which the function $f(x)$ is increasing, then we have to draw a tangent passing through the point $(x_0, f(x_0))$. Then we have to find its x-intercept, and the above-mentioned formula gives the x-intercept of the tangent drawn through the point $(x_n, f(x_n))$. And by repeating the same process a couple of times we can get a value that is approximately equal to the root of $f(x)$.

3 Proof of Newton-Raphson Method

To understand my method of finding area under the curve we need to understand the proof of Newton-Raphson Method.
Let us assume, a function $f(x)$ around an initial value x_0 such that the function is increasing and x_0 is not a critical value, then it meets the curve of $f(x)$ at point (x_0,$f(x_0)$), now we have to draw a tangent from that specific point.

Now we have,

Slope of the tangent (m) = $f'(x_0)$ and,

$$y - y_{value} = m(x - x_{value})$$

Here, $(x_{value}, y_{value}) \equiv (x_0, f(x_0))$

$$\Rightarrow y - f(x_0) = m(x - x_0)$$

$$\Rightarrow y - f(x_0) = f'(x_0)(x - x_0)$$

This is the equation of the line, but we need to find its x-intercept, so let the coordinate of x-intercept be $(x_1, 0)$.

$$\Rightarrow 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow f(x_0) = f'(x_0)(x_0 - x_1)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

On generalizing this formula for 'n' we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \square$$

Newton-Raphson Method is very useful in Numerical Analysis, and it is one of the widely used methods for approximating the roots of a function. But it has some limitations like
(i) The initial value (x_0) must not be a critical value.
(ii) It's convergence is not guaranteed.
(iii) Division by zero problem can occur.
(iv) Inflection point issue might occur
(v) Symbolic derivative is required.

4 Treanungkur's Rule for Finding Area under the curve by extending Newton-Raphson Method

I, Treanungkur Mal have discovered an interesting method to find area under the curve by extending the concept of Newton-Raphson Method, so I call this rule as Treanungkur's Rule and the formula I have given as Treanungkur's Formula.

$$\int_a^b f(x)\,dx \approx \frac{0.935}{2}\left\{\frac{f(x_0)}{f'(x_0)}\left(f(x_0)+f(x_1)\right)+\ldots+\frac{f(x_n)}{f'(x_n)}\left(f(x_n)+f(x_{n+1})\right)\right\}$$

where,

$$x_0=b,\;\; x_{n+1}\approx a,\, and\;\; x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

So to find area of a function $f(x)$ we use Integration, or to be precise we use Definite Integration. Already we have many methods of approximating Definite Integrals, like Midpoint Rule, Trapezoidal Rule, Simpson's Rule, Riemann Sum, etc.
Now I want to add one more to the list, i.e Treanungkur's Rule, the formula for which has been mentioned above. Now, I will provide a proof of the Rule given above.

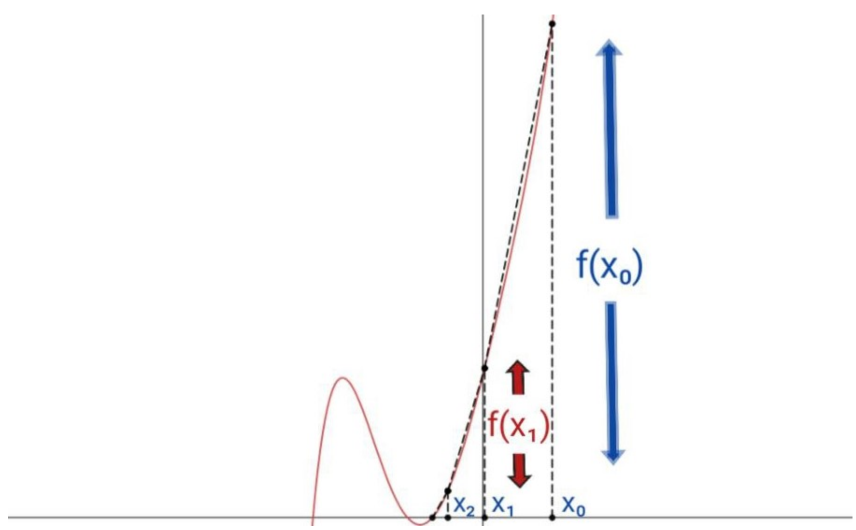

Figure 2: This is the graphical representation of the length of the sides of trapezium at x_0 and x_1

5 Proof of Treanungkur's Rule

The idea here is to form many trapeziums and then find their area, in the above picture, the black stipped lines are the trapeziums formed by using the Newton-Raphson Method. Eventually, as we progress the last figure formed will be a triangle, i.e. when $f(x_{n+1}) \approx f(a)$. Now to get into a correct result we have multiplied the answer by a factor of 0.935, the final ans reported by Treanungkur's Rule is very correct, we shall show it in some examples later. Let the area of the first trapezium be A_1 then,

$$area(A_1) = \frac{1}{2}\left(|x_0 - x_1|\right)\left(f(x_0) + f(x_1)\right)$$

Note,

$$\left(|x_0 - x_1|\right) = \left(|x_0 - x_0 + \frac{f(x_0)}{f'(x_0)}|\right) = \left(\frac{f(x_0)}{f'(x_0)}\right)$$

$$\Rightarrow area(A_1) = \frac{1}{2}\left(\frac{f(x_0)}{f'(x_0)}\right)\left(f(x_0) + f(x_1)\right)$$

where, $x_0 = b$, upper limit of the integral. Similarly,

$$area(A_2) = \frac{1}{2}\left(\frac{f(x_1)}{f'(x_1)}\right)\left(f(x_1) + f(x_2)\right)$$

$$area(A_3) = \frac{1}{2}\left(\frac{f(x_2)}{f'(x_2)}\right)\left(f(x_2) + f(x_3)\right)$$

$$\vdots$$

$$area(A_{n+1}) = \frac{1}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)\left(f(x_n) + f(x_{n+1})\right)$$

where,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Since,

$$f(x_{n+1}) \approx f(a)$$

So,

$f(x_{n+1}) \approx 0$ i.e. $area(A_{n+1}) =$ area (One and only triangle).

Therefore total Area :

$$k\,(area(A_1) + area(A_2) + ... + area(A_{n+1})) \approx \int_a^b f(x)\,dx$$

$$\Rightarrow \frac{k}{2}\left\{\frac{f(x_0)}{f'(x_0)}\,(f(x_0) + f(x_1)) + ... + \frac{f(x_n)}{f'(x_n)}\,(f(x_n) + f(x_{n+1}))\right\} \approx \int_a^b f(x)\,dx$$

By experimental data, we get a suitable value for k, i.e. 0.935

Therefore our final answer,

$$\int_a^b f(x)\,dx \approx \frac{0.935}{2}\left\{\frac{f(x_0)}{f'(x_0)}\,(f(x_0) + f(x_1)) + ... + \frac{f(x_n)}{f'(x_n)}\,(f(x_n) + f(x_{n+1}))\right\}$$

where,

$$x_0 = b, \;\; x_{n+1} \approx a, \, and \;\; x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Example 5.1 Evaluate the following Integral using Treanungkur's Rule :

$$\int_{-0.5}^{1} 2x^2 + 3x + 1\,dx$$

Solution. For solving the given Definite Integral we need to find some values for 'x' :
$x_0 = 1, f(x_0) = 6, f'(x_0) = 7$
$x_1 = 0.142, f(x_1) = 1.466, f'(x_1) = 3.568$
$x_2 = -0.26, f(x_2) = 0.3552, f'(x_2) = 1.96$
$x_3 = -0.44, f(x_3) = 0.0672, f'(x_3) = 1.24$
$x_4 = -0.49 \approx -0.5$,
So, we can stop here as $x_4 \approx -0.5$
Let,
$I_{approx} = \int_{-0.5}^{1} 2x^2+3x+1\,dx \approx \frac{0.935}{2}\left\{\frac{f(x_0)}{f'(x_0)}\,(f(x_0) + f(x_1)) + ... + \frac{f(x_n)}{f'(x_n)}\,(f(x_n) + f(x_{n+1}))\right\}$

$$\Rightarrow I_{approx} \approx \frac{0.935}{2}\left\{\frac{6}{7}\,(7.466) + \frac{1.466}{3.568}\,(1.8212) + \frac{0.3552}{1.96}\,(0.4224) + \frac{0.0672}{1.24}\,(0.0774)\right\}$$

$$\Rightarrow I_{approx} \approx 3.37767$$

By solving it using Normal Integration we get,

$$I_{original} = 3.375$$

Therefore, Relative Error = $|I_{original} - I_{approx}| = 0.00267$
and Relative Error in % = $\frac{0.00267}{3.375} \cdot 100\% = 0.0007911 \cdot 100\% = 0.0791\%$

Finally, On comparing with all the widely used methods of approximating Definite Integrals, we get the Relative Errors for the given sum in % as follows :

Relative Error for the given sum using Midpoint Rule = 1.8518 %
Relative Error for the given sum using Trapezoidal Rule = 3.7037 %
Relative Error for the given sum using Left Riemann Sum = 40.7407 %
Relative Error for the given sum using Right Riemann Sum = 48.1481 %
Relative Error for the given sum using Treanungkur's Rule = 0.0791% ◀

Example 5.2 Evaluate the following Integral using Treanungkur's Rule :

$$\int_{-0.356}^{0.5} 5x^5 + 4x^4 - 3x^3 + 2x^2 + 4x + 1 \, dx$$

Solution. For solving the given Definite Integral we need to find some values for 'x' :
$x_0 = 0.5, f(x_0) = 3.53125, f'(x_0) = 7.3125$
$x_1 = 0.017, f(x_1) = 1.06856, f'(x_1) = 4.06548$
$x_2 = -0.2457, f(x_2) = 0.192535, f'(x_2) = 2.32767$
$x_3 = -0.3284, f(x_3) = 0.035769, f'(x_3) = 1.43988$
$x_4 = -0.3533 \approx -0.356,$
So, we can stop here as $x_4 \approx -0.356$
Let,
$\Rightarrow I_{approx} = \int_{-0.356}^{0.5} 5x^5 + 4x^4 - 3x^3 + 2x^2 + 4x + 1 \, dx$
$\Rightarrow I_{approx} \approx \frac{0.935}{2} \left\{ \frac{f(x_0)}{f'(x_0)} \left(f(x_0) + f(x_1)\right) + ... + \frac{f(x_n)}{f'(x_n)} \left(f(x_n) + f(x_{n+1})\right) \right\}$

$$\Rightarrow I_{approx} \approx \frac{0.935}{2} \left\{ \frac{3.5312}{7.3125}(4.5998) + \frac{1.0685}{4.0654}(1.2610) + \frac{0.1925}{2.3276}(0.2283) + \frac{0.0357}{1.4398}(0.0393) \right\}$$

$\Rightarrow I_{approx} \approx 1.2026$
By solving it using Normal Integration we get,

$$I_{original} = 1.222010$$

Therefore, Relative Error = $|I_{original} - I_{approx}| = 0.0193$
and Relative Error in % = $\frac{0.0193}{1.222010} \cdot 100\% = 1.5793\%$

Finally, On comparing with all the widely used methods of approximating Definite Integrals, we get the Relative Errors for the given sum in % as follows :

Relative Error for the given sum using Midpoint Rule = 1.6418 %
Relative Error for the given sum using Trapezoidal Rule = 3.3510 %
Relative Error for the given sum using Simpson's 1/3 Rule = 13.3460%
Relative Error for the given sum using Simpson's 3/8 Rule = 13.3869%
Relative Error for the given sum using Left Riemann Sum = 37.8703 %
Relative Error for the given sum using Right Riemann Sum = 44.5718 %
Relative Error for the given sum using Treanungkur's Rule = 1.5793%

By the above examples, I have tried to show that my Rule for solving Indefinite Integrals is very efficient as well as it is very correct, compared to other methods or rules which are widely used today. ◀

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7 Conflict of Interest Statement

The author declare that there is no conflict of interest.

8 Ethical Statement

1. This material is the authors' own original work, which has not been previously published elsewhere.
2. The paper is not currently being considered for publication elsewhere.
3. The paper reflects the authors' own research and analysis in a truthful and complete manner.
4. The paper properly credits the meaningful contributions of co-authors

and co-researchers.
5. The results are appropriately placed in the context of prior and existing research.
6. All sources used are properly disclosed.
7. All authors have been personally and actively involved in substantial work leading to the paper, and will take public responsibility for its content.

9 Data Availability Statement

All data generated or analysed during this study are included in this published article (and its supplementary information files).

10 Author Contribution

Conception and Design of Study : Treanungkur Mal.
Acquisition of data : Treanungkur Mal.
Analysis and Interpretation of data : Treanungkur Mal.

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