1 Abstract

Define a function $\pi(x)$ on the positive integers. Which gives us the number of primes less than or equal to $x$. In this paper, we approach this function by analyzing a different function which is called the "Prime zeta function" which is denoted $P(x)$. While analyzing this function, we discovered many non-trivial results, which later proved to be very useful in the study of the error term of prime counting function, and also provided some new insights, on the behavior of the error term for large values of $x$. We show that for large values of $x$, the error term is well behaved, and later we also provide a very tight bound to this error term, thus solving a long-standing problem.

2 Introduction

Let us set $\pi(x) = \sum_{p \leq x} 1$ where $\pi(x)$ counts the the number of primes less than or equal to $x$. The irregularity of the primes makes it difficult to obtain an exact practical formula for the prime counting function. It is one of the well-studied functions in number theory, still not much is known concerning their distribution. In this paper, our target is to improve the estimate for the prime counting function, by improving the bounds on the error term of the $\pi(x)$ Where error term refers to the magnitude of the following function $\pi(x) - li(x)$ where $li(x)$ is the logarithmic integral. This is achieved by analyzing another function called the "Prime Zeta Function" denoted by $P(x)$ which is defined as $P(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ where $p$ is a prime number.
2.1 Approach

This paper aims at studying the analytical properties of Prime Zeta Function through the method of contour integration. The main idea here is to integrate the Prime Zeta Function in two similar but different contours, these contours also include a line integral, then after that, make those line integrals approach \( \text{Re}(x) = 1/2 \) line, the consequence of doing this is that a large amount of complexity cancels themselves out in a very non trivial way, and we are left with some well behaved functions. Then we try to analytically continue the prime zeta function using another method which involves prime counting function, and later we understand that by combing both the results obtained from analyzing the "Prime zeta function" in two different ways gives a very non trivial formula which is later used to show that "Almost all large values of \( T \) satisfy \( |\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{D} \). The main goal here is to approach the \( \text{Re}(x) = 1/2 \) line from both directions.

2.2 Notations

1) \( \pi(x) \): Prime Counting Function

2) \( P(x) \): Prime Zeta Function

3) \( \zeta(x) \): Riemann Zeta Function

4) \( E(x) \): Error term in Prime Counting Function

7) \( li(x) \): Logarithmic Integral

8) \( \mu(x) \): M"obius Function

9) \( C_n(a, T) \): A semi circular contour in clockwise direction

10) \( C_n(a, -T) \): A semi circular contour in anticlockwise direction

3 Procedure

3.1 Preliminary results

1) \( P(s) = \sum_{n=1}^{n=\infty} \frac{1}{(p(n))^s} \) for \( \text{Re}(s) > 0 \)
2) \( \log(\zeta(s)) = \text{sln}(2) + (s-1)\ln(\pi) + \ln(\text{sin}(\frac{\pi s}{2})) + \ln(\Gamma(1-s)) + \log(\zeta(1-s)) \)

3) \( \pi(x) = \text{li}(x) + E(x) \)

4) \( |\Gamma(\frac{1}{2} + iT)| = \sqrt{\frac{\pi}{\cos(T\pi)}} \)

5) \( |\zeta(\sigma + iT)| < T^{\frac{1}{2}(1-\sigma)\log(T)} \) for \( \sigma < 1 \)

6) \( |\zeta(\sigma + iT)| < \log(T) \) for \( \sigma > 1 \)

### 3.2 Main Result

\[
\pi(T) - \text{li}(T) = T^{3/2} \sec(T\ln(T)) \left[ \log\left( \frac{|\zeta(1 - \frac{T}{2} + 2iT)|}{|\zeta(1 + \frac{T}{2} + 2iT)|} \right) + T \frac{\partial}{\partial T} \left( \log\left( \frac{|\zeta(1 - \frac{T}{2} + 2iT)|}{|\zeta(1 + \frac{T}{2} + 2iT)|} \right) \right) \right]
\]

As \( T \to \infty \)

2) Almost all large values of \( T \) satisfy \( |\pi(T) - \text{li}(T)| < \frac{\sqrt{T\ln(T)}}{B} \)

### 3.3 Proof

Proof is devided in two parts

#### 3.3.1 Part 1

In Part 1 as mentioned in the "Approach" section we are going to start by analyzing it using methods of contour integration

We start with the definition of the prime zeta function

\[
P(s) = \sum_{n=1}^{\infty} \frac{1}{(p(n))^s}
\]

Where \( p(n) = \text{nth prime number} \)

Now we also know the following relation which connects Prime Zeta function and Riemann Zeta Function

\[
P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(\zeta(ns))
\]

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The above equation is very important as it helps us to extend prime zeta function for \( Re(s) > 0 \). Thus can be used to study the properties of Prime Zeta Function on the domain with \( Re(s) < 1 \)

We start by defining a contour with a semi circular shape, as it will later turn out that integrating on this simple and trivial contour gives very non trivial results. We will call this contour \( C_1 \)

Now we take the Contour integral of \( P(s) \) over the following contour: -

\[
\int_{C_1(1/\alpha,T)} P(s) ds = \sum_{n=\infty}^{n=1} \int_{C_1(1/\alpha,T)} \frac{\mu(n)}{n} Log(\zeta(ns)) ds
\]

\[
= \int_{C_1(1/\alpha,T)} P(s) ds = \int_{C_1(1/\alpha,T)} \frac{\mu(1)}{1} Log(\zeta(s)) ds + \sum_{n=2}^{n=\infty} \int_{C_1(1/\alpha,T)} \frac{\mu(n)}{n} Log(\zeta(ns)) ds
\]

Let

\[
I_2 = \sum_{n=2}^{n=\infty} \int_{C_1(1/\alpha,T)} \frac{\mu(n)}{n} Log(\zeta(ns)) ds
\]

Now we interchange the summation and integral sign later we will prove that we can do this step

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Now we see that in $I_2 \ Log(\zeta(ns))$ has got no singularities because in $I_2$ , $Re(ns)$ is always greater than 1.

So using the Residue Theorem in each term of $I_2$ we can conclude the following result

$$I_2 = 0$$

As we can see the complexities are vanishing out and now we are left with the following

$$\int_{C_1(1/\alpha,T)} P(s)ds = \int_{C_1(1/\alpha,T)} \frac{\mu(1)}{1} \ Log(\zeta(s))ds$$

Now we open the contour integral on $P(s)$ to their corresponding line integral and circular integral

$$\int_{C_1(1/\alpha,T)} P(s)ds = \int_{1/\alpha+iT}^{1/\alpha+iT} P(s)ds + \int_{\gamma_1} P(s)ds$$

Bernard Riemann in his 1859 paper proved the Riemann functional equation for zeta function , When we take $\log$ on both sides of the equation we get the following result

$$\log(\zeta(s)) = A(s) + log(\zeta(1-s))$$

where

$$A(s) = sln(2) + (s-1)ln(\pi) + ln(sin(\frac{\pi s}{2})) + ln(\Gamma(1-s))$$

Now we will take the contour integral over $A(s)$ and just for simplicity we will call it as $I_3$ ,

$$I_3 = \int_{C_1(1/\alpha,T)} sln(2) + (s-1)ln(\pi) + ln(sin(\frac{\pi s}{2})) + ln(\Gamma(1-s))ds$$

All the functions which we have in $I_3$ have got no non trivial zeroes in the contour and thus does not depend on $T$ , thus by using the Residue theorem we can say that $I_3$ is a constant.

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Let this constant be represented by $H$. Now our contour integral on $P(s)$ transforms into the following form

$$\int_{C_1(1/\alpha,T)} P(s)ds = H + \int_{C_1(1/\alpha,T)} \log(\zeta(1-s))ds$$

Let us name the above result as equation "A"

Now as told in "Approach" section we will consider another contour similar to the previous one just with a slight difference that the line will be on the left of $Re(s) = 1/2$ line. We will call this contour "$C_2$"

Now we will consider the contour integral of $P(s)$ over $C_2$

Where $\frac{1}{\beta}$ is such that $\frac{1}{\beta} + \frac{1}{\alpha} = 1$ ALSO $\frac{1}{2} > \frac{1}{\beta} > \frac{1}{3}$

Now we will repeat the steps which we have done above

Let

$$I_2 = \sum_{n=3}^{n=\infty} \int_{C_2(1/\beta,T)} \frac{\mu(n)}{n} \log(\zeta(ns))ds$$

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Now we interchange the summation and integral sign later we will prove that we can do this step.

Now we see that in $I_2$ $\log(\zeta(ns))$ has got no singularities because in $I_2$, $\text{Re}(ns)$ is always greater than 1.

So using the Residue Theorem in each term of $I_2$ we can conclude the following result

$$I_2 = 0$$

this gives us the following result

$$\int_{C_2(1/\beta,T)} P(s)ds = \int_{C_2(1/\beta,T)} \log(\zeta(s))ds + \int_{C_2(1/\beta,T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds$$

since $\text{Re}(\frac{2}{\pi}) < 1$

Let us represent the first part of integral with letter $I$

$$I = \int_{C_2(1/\beta,T)} \log(\zeta(s))ds$$

Now, our main aim here is to somehow how we have find a connection between contour 1 and contour 2.

To do this our first try is to get some common terms between these two contours.

We see that in first contour there is this term $\int_{C_1(1/\alpha,T)} \log(\zeta(1-s))ds$ which can be easily created in contour 2, if we put $s = 1 - a$ in "T" so I becomes

$$I = -\int_{C_2(1-1/\beta,-T)} \log(\zeta(1-a))da$$

Now, $1 - \frac{1}{\beta} = \frac{1}{\alpha}$

$$= I = -\int_{C_2(1/\alpha,-T)} \log(\zeta(1-a))da$$

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Now observing the two contours we can conclude that if we exchange $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ the contours also gets totally exchanged. Also we can observe that putting $-T$ in $C_1(1/\alpha, T)$ or in $C_2(1/\beta, T)$ is same as changing the direction of contour. So from the above arguments we can conclude the following result

$$\int_{C_2(1/\alpha, -T)} = - \int_{C_1(1/\alpha, T)}$$

Using the above result in I we retrieve the term which was in contour 1

$$- \int_{C_2(1/\alpha, -T)} \log(\zeta(1-s))ds = \int_{C_1(1/\alpha, T)} \log(\zeta(1-s))ds$$

This gives us the following value for I

$$I = \int_{C_1(1/\alpha, T)} \log(\zeta(1-s))ds$$

Now that we have the value of I with us we are going to put $s = 1 - a$ in this relation $\int_{C_2(1/\beta, T)} P(s)ds = \int_{C_2(1/\beta, T)} \log(\zeta(s))ds + \int_{C_2(1/\beta, T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds$

we get the following

$$- \int_{C_2(1-1/\beta, -T)} P(1-a)da = I + \int_{C_2(1/\beta, T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds$$

On further Simplifying the above result we get this

$$\int_{C_1(1/\alpha, T)} P(1-a)da = I + \int_{C_2(1/\beta, T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds$$

Now we will substitute the value of I which we have calculated previously

$$\int_{C_1(1/\alpha, T)} P(1-a)da = \int_{C_1(1/\alpha, T)} \log(\zeta(1-s))ds + \int_{C_2(1/\beta, T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds$$

Now we will use the equation "A" thus finally arriving to the following result

$$\int_{C_1(1/\alpha, T)} P(1-a)da = \int_{C_1(1/\alpha, T)} P(a)da + \int_{C_2(1/\beta, T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds - H$$

On further Simplifying we get our main result

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\[ \int_{C_{1}(1/\alpha,T)} (P(1 - a) - P(a))da = \int_{C_{2}(1/\beta,T)} \frac{\mu(2)}{2} \log(\zeta(2s))ds - H \]

The value of \( \mu(2) = -1 \) So now we have

\[ \int_{C_{1}(1/\alpha,T)} (P(a) - P(1 - a))da = \int_{C_{2}(1/\beta,T)} \frac{1}{2} \log(\zeta(2s))ds + H \]

Now that we have reached the above relation, we are going to analyze it in depth

Just for our convenience we will put \( P(a) - P(1 - a) = \psi(a) \) Where \( Re(a) > \frac{1}{2} \)

Now we will calculate this term \( \int_{C_{1}(1/\alpha,T)} \psi(s)ds \)

\[ \int_{C_{1}(1/\alpha,T)} \psi(s)ds = \int_{1/\alpha+iT}^{1/\alpha-iT} \psi(s)ds + \int_{\gamma_{1}} \psi(s)ds \]

Let

\[ I = \int_{\gamma_{1}} \psi(s)ds \]

substituting the value of \( \psi(x) \) gives us-

\[ \int_{\gamma_{1}} \psi(s)ds = \int_{\gamma_{1}} (P(s) - P(1 - s))ds \]

To solve this integration on the circle we are going to do the following parameterization

\[ s = \frac{1}{\alpha} + Te^{i\theta} \]

\[ ds = iTe^{i\theta}d\theta \]

On doing the above parameterization , and putting the upper and lower limits , we get the following result

\[ I = \int_{\pi/2}^{-\pi/2} \psi\left(\frac{1}{\alpha} + Te^{i\theta}\right)iTe^{i\theta}d\theta \]

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Now we again use this relation to simplify \( P\left(\frac{1}{\alpha} + Te^{i\theta}\right) - P\left(1 - \frac{1}{\alpha} - Te^{i\theta}\right) = \psi\left(\frac{1}{\alpha} + Te^{i\theta}\right) \)

\[
I = \int_{\pi/2}^{-\pi/2} \left[ P\left(\frac{1}{\alpha} + Te^{i\theta}\right) - P\left(1 - \frac{1}{\alpha} - Te^{i\theta}\right) \right] iTe^{i\theta} d\theta
\]

Let \( I = I_1 - I_2 \)

where \( I_1 = \int_{\pi/2}^{-\pi/2} [P\left(\frac{1}{\alpha} + Te^{i\theta}\right)] iTe^{i\theta} d\theta \)

And \( I_2 = \int_{\pi/2}^{-\pi/2} [P\left(\frac{1}{\beta} - Te^{i\theta}\right)] iTe^{i\theta} d\theta \)

Now we will start by evaluating the following integral

\[
I_1 = \int_{\pi/2}^{-\pi/2} [P\left(\frac{1}{\alpha} + Te^{i\theta}\right)] iTe^{i\theta} d\theta
\]

In order to solve this integral we use the following relation between Prime Zeta Function and Riemann Zeta Function

\[
P(s) = \sum_{n=1}^{\infty} \frac{\mu(n) \log(\zeta(ns))}{n}
\]

Now we us the above relation to simplify \( I_1 \) and then we interchange the sum and integral sign and get to the following result

\[
I_1 = \sum_{n=1}^{\infty} \int_{\pi/2}^{-\pi/2} \frac{\mu(n) \log(\zeta(n(\frac{1}{\alpha} + Te^{i\theta}))))}{n} iTe^{i\theta} d\theta
\]

Now put \( Te^{i\theta} = ik \) and \( Te^{i\theta} d\theta = dk \) in \( I_1 \)

\[
I_1 = i \sum_{n=1}^{\infty} \int_{T}^{T} \frac{\mu(n) \log(\zeta(n(\frac{1}{\alpha} + ik))))}{n} dk
\]

Using the same arguments and parameterization in \( I_2 \). \( I_2 \) becomes

\[
I_2 = i \sum_{n=1}^{\infty} \int_{T}^{T} \frac{\mu(n) \log(\zeta(n(\frac{1}{\beta} - ik))))}{n} dk
\]

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We know that $I = I_1 - I_2$, so by taking the difference of both we reach the following result

$$I_1 - I_2 = i \int_T^{-T} \frac{\mu(1)(\log(\zeta((\frac{1}{2} + ik))) - (\log(\zeta((\frac{1}{2} - ik)))))}{1} \, dk + iI_3 + iI_4$$

where

$$I_3 = \int_T^{-T} \frac{\mu(2)(\log(\zeta((\frac{2}{2} + 2ik))) - (\log(\zeta(\frac{2}{2} - 2ik))))}{2} \, dk$$

And

$$I_4 = \sum_{n=3}^{n=\infty} \int_T^{-T} \frac{\mu(n)(\log(\zeta(n(\frac{1}{2} + ik))) - (\log(\zeta(n(\frac{1}{2} - ik)))))}{n} \, dk$$

Now as it is written in the "Approach" section, we tend towards $\text{Re}(s) = 1/2$ line from two different directions. We do this by imposing the following conditions for $\alpha$ and $\beta$

$$\frac{1}{\beta} = \frac{1}{2} - \epsilon$$
$$\frac{1}{\alpha} = \frac{1}{2} + \epsilon$$
$$\epsilon > 0; \epsilon \to 0$$

Now using this property of integrals $\int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx$ to $I_4$ We get

$$I_4 = \sum_{n=3}^{n=\infty} (\int_T^{-T} \frac{\mu(n)(\log(\zeta(n(\frac{1}{2} + ik))) - (\log(\zeta(n(\frac{1}{2} + ik)))))}{n} \, dk$$

Now Using the above conditions and using a simple fact about the zeta function that $\log(\zeta(n(\frac{1}{2} + ik)))$, $\log(\zeta(n(\frac{1}{2} + ik)))$ has no singularity at line $\text{Re}(s) = n/2$ for $n > 2$ we can directly put $\epsilon = 0$ in $I_4$ gives us the following result

$$I_4 = \sum_{n=3}^{n=\infty} (\int_T^{-T} \frac{\mu(n)(\log(\zeta(n(\frac{1}{2} + ik)))}{n} \, dk$$

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Which clearly gives $I_4 = 0$

Here we see that when we approach towards the $Re(s) = 1/2$ line the complicated things cancels themselves out, this is what I was referring to in the "Approach" section.

Now we move forward and evaluate the following integral

$$I_5 = \int_{-T}^{T} \frac{\mu(1)(log(\zeta((\frac{1}{\alpha} + ik))) - (log(\zeta((\frac{1}{\beta} - ik)))))}{1} dk$$

We cannot directly apply the reasoning that we applied on $I_4$ because $\zeta(x)$ is very different for $Re(x) > 1$ and $Re(x) < 1$ so in order to evaluate $I_5$ we use the Riemann functional equation for $\zeta(x)$. We are doing this because we see that $1 - (\frac{1}{\alpha} + ik) = \frac{1}{\beta} - ik$

the Riemann functional equation when we take log on both sides was

$$log(\zeta(\frac{1}{\alpha} + ik)) = A(\frac{1}{\alpha} + ik) + log(\zeta(1 - (\frac{1}{\alpha} + ik)))$$

we can see clearly that functional equation also contains $\zeta(\frac{1}{\alpha} + ik)$ and $\zeta(1 - (\frac{1}{\alpha} + ik))$. From the functional equation we establish the following relation

$$log(\zeta(\frac{1}{\alpha} + ik)) - log(\zeta(\frac{1}{\beta} - ik)) = A(\frac{1}{\alpha} + ik)$$

where

$$A(s) = sln(2) + (s - 1)ln(\pi) + ln(sin(\frac{\pi s}{2})) + ln(\Gamma(1 - s))$$

We can use the above relation in $I_5$ and simplify it to the following

$$I_5 = \int_{-T}^{T} \frac{\mu(1)(A(\frac{1}{\alpha} + ik))}{1} dk$$

$$I_5 = \int_{-T}^{T} \frac{(\frac{1}{\alpha} + ik)ln(2) + (\frac{1}{\alpha} + ik - 1)ln(\pi) + ln(sin(\frac{\pi s}{2}))) + ln(\Gamma(\frac{1}{\beta} - ik))}{1} dk$$

After doing all of this our arc integral becomes

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\[ I_1 - I_2 = iI_5 + iI_3 \]

So only \( I_3 \) remains now

on substituting the value of \( \mu(2) = -1 \), \( I_3 \) becomes

\[ I_3 = \int_{T}^{-T} \frac{-(\log(\zeta((\frac{2}{\alpha} + 2ik))) - (\log(\zeta(\frac{2}{\beta} - 2ik))))}{2} dk \]

In order to solve \( I_3 \) let

\[ 2I' = \int_{T}^{-T} (\log(\zeta(\frac{2}{\beta} - 2ik)))dk \]

Using \( I' \) we can rewrite \( I_3 \) as

\[ I_3 = I' - \int_{T}^{-T} \frac{log(\zeta((\frac{2}{\alpha} + 2ik))}{2} dk \]

Now our Part 1 of the proof is over, if you are wondering that we never evaluated \( I_3 \) or other integrals explicitly, this is because later you will find out that at last we will differentiate these terms, on doing this, it would result in vanishing of these integrals. On this note we move towards part 2 of the prove.

### 3.3.2 Part 2

As we mentioned in the "Approach" section that we are going to analytically continue the Prime Zeta function using two different methods, in part 1 we analytically continued the Prime Zeta Function using contour integration. Now we are going to analytically continue it using methods which involve Prime counting function.

We again start by the following definition of prime zeta function

\[ P(s) = \sum_{p} \frac{1}{p^s} \]

Where \( Re(s) > 1 \) It can also be written as

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\[ P(s) = \sum_{n=1}^{\infty} \frac{\pi(n) - \pi(n - 1)}{n^s} \]

On imposing the condition \( T \to \infty \) and then applying Summation by parts we get

\[ P(s) = \frac{\pi(T)}{(T+1)^s} - \sum_{n=2}^{n=T+1} \pi(n) \left( \frac{1}{n^s} - \frac{1}{(n-1)^s} \right) \]

It can also be written as

\[ P(s) = \frac{\pi(T)}{(T+1)^s} + \sum_{n=2}^{n=T+1} s\pi(n) \int_{n-1}^{n} \frac{1}{x^{s+1}} dx \]

In the above integral we apply this parametrization \( x = t - 1 \) which gives \( dx = dt \) on doing this we get

\[ P(s) = \frac{\pi(T)}{(T+1)^s} + \sum_{n=2}^{n=T+1} s\pi(n) \int_{n}^{n+1} \frac{1}{(t-1)^{s+1}} dt \]

put \( t = x \) just for simplicity

We know the property of Prime counting function that in the range \( nto n + 1 \) prime counting function \( \pi(x) \) is equal to \( \pi(n) \), using this property we can bring prime counting function inside the integral and then on simplifying we get

\[ P(s) = \frac{\pi(T)}{(T+1)^s} + s \int_{2}^{T} \frac{1}{\pi(x)} \frac{1}{(x-1)^{s+1}} dx \]

Now we replace \( s \) with \( z \) to imply that now \( \text{Re}(z) \) can be less than 1 on replacing we get

\[ \frac{P(z)}{z} = \frac{\pi(T)}{z(T+1)^z} + \int_{2}^{T} \frac{1}{\pi(x)} \frac{1}{(x-1)^{z+1}} dx \]

On observing the above relation carefully, we observe that if \( z \) is of the form \( z = \sigma + iT \) in combination with the fact that \( T \to \infty \), the Right hand side of the equation starts converging again for \( \sigma > 0 \) this is because of the following reasoning

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\[
\left| \frac{\pi(T)}{(\sigma + iT)(T + 1)^z} \right| \leq \frac{|\pi(T)|}{|(\sigma + iT)(T + 1)^\sigma|}
\]

if \(\sigma > 0\) and on using the fact that \(\frac{\pi(x)}{\ln x} = 1\) this is due to “prime number theorem” Also referred to as “PNT” we can conclude that

\[
\frac{|\pi(T)|}{|(\sigma + iT)(T + 1)^\sigma|} \to 0
\]

thus we can conclude that

\[
\left| \frac{\pi(T)}{(\sigma + iT)(T + 1)^z} \right| \to 0
\]

Thus the above series converges To

\[
P(z) = \frac{\int_2^T \frac{\pi(x)}{x - 1} dx}{z}
\]

Now form here the error term of the prime counting function comes into picture as it is shown that

\[
\pi(x) = li(x) + E(x)
\]

Where \(E(x)\) is error

Our preparation is complete. Now as it was mentioned in the "Approach" section that we get our result by connecting part 1 and part 2. So now we will start connecting part 1 and part 2

In order to connect part 1 and part 2 we first try to create terms that were in part 1 i.e these two terms "\(P(1/\alpha + iT)\)" , "\(P(1/\beta - iT)\)" and then "\(P(1/\alpha + iT) - P(1/\beta - iT)\)"

We start by constructing \(P(1/\alpha + iT)\)

\[
P(1/\alpha + iT) = (1/\alpha + iT) \int_2^T \frac{li(x)}{(x - 1)^{z+1}} dx + (1/\alpha + iT) \int_2^T \frac{E(x)}{(x - 1)^{z+1}} dx
\]

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The next term is $P(1/\beta - iT)$

$$P(1/\beta - iT) = (1/\beta - iT) \int_2^T \frac{li(x)}{(x-1)^{3/2}} dx + (1/\beta - iT) \int_2^T \frac{E(x)}{(x-1)^{3/2}} dx$$

We know that function $li(s)$ has got no singularity in the region $1 > Re(s) > 1/\alpha$ and also in the region $1/\beta < Re(s) < 1/\alpha$ and also in the region $0 < Re(s) < 1/\beta$ , in short there is nothing special about $Re(x) = 1/2$ , so in the case of $li(x)$ we can simply put $\epsilon = 0$

Now we define a function $f(T)$ as follows

$$f(T) = (1/2 + iT) \int_2^T \frac{li(x)}{(x-1)^{3/2} + iT} dx$$

$$\rightarrow f(T) = (1/2 + iT) \int_2^T \frac{li(x)}{(x-1)^{3/2} x^{iT}} dx$$

$$\rightarrow f(T) = (1/2 + iT) \int_2^T \frac{li(x)}{(x-1)^{3/2}} (cos(T \log(x)) - isin(T \log(x))) dx$$

$$\rightarrow f(T) = 1/2 \int_2^T \frac{li(x)}{(x-1)^{3/2}} (cos(T \log(x))) dx - i/2 \int_2^T \frac{li(x)}{(x-1)^{3/2}} (sin(T \log(x))) dx$$

$$+ iT \int_2^T \frac{li(x)}{(x-1)^{3/2}} (cos(T \log(x))) dx + T \int_2^T \frac{li(x)}{(x-1)^{3/2}} (sin(T \log(x))) dx$$

Now we are in position to construct the third or the final term $P(1/\alpha + iT) - P(1/\beta - iT)$

$$P(1/\alpha + iT) - P(1/\beta - iT) = i \left[ - \int_2^T \frac{li(x)}{(x-1)^{3/2}} (sin(T \log(x))) dx + 2T \int_2^T \frac{li(x)}{(x-1)^{3/2}} (cos(T \log(x))) dx \right] + \Delta(T)$$

where

$$\Delta(T) = (1/\alpha + iT) \int_2^T \frac{E(x)}{(x-1)^{1/\alpha + iT + 1}} dx - (1/\beta - iT) \int_2^T \frac{E(x)}{(x-1)^{1/\beta - iT + 1}} dx$$

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We take Real part on both sides. We take this step to remove the \( li(x) \) from the picture so that we can focus on only error term

\[
Re(P(1/\alpha + iT) - P(1/\beta - iT)) = Re(\Delta(T))
\]

Earlier we have put \( P(1/\alpha + iT) - P(1/\beta - iT) = \psi(1/\alpha + iT) \)
since \( 1 - (1/\alpha + iT) = 1/\beta - iT \)
this gives the following relation

\[
2Re(\psi(1/\alpha + iT)) = 2Re(\Delta(T))
\]

Since \( P(x) \) is an analytic function with some poles because zeta function is analytic, thus we can conclude that for \( 0 < Re(a) < 1 \) \( \psi(a) \) is also a analytic function

Since \( \psi(x) \) is a analytic function, we can write

\[
2Re(\psi(1/\alpha + iT)) = \psi(1/\alpha + iT) + \psi(1/\alpha - iT)
\]

If we observe the Right Hand Side carefully and also observe part 1 we find out that the following relation can help us.

\[
\psi(1/\alpha + iT) + \psi(1/\alpha - iT) = -i \frac{\partial}{\partial T} \left( \int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s)ds \right)
\]

Now we substitute the value for \( 2Re(\psi(1/\alpha + iT)) \) and we get

\[
\frac{\partial}{\partial T} \left( \int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s)ds \right) = 2iRe(\Delta(T))
\]

On observing the above equation, we can say that we have successfully connected part 1 and part 2. Now all that remains is to do some simplification

We start by substituting the value for \( \int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s)ds \) which we have derived in part 1

\[
\int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s)ds = \int_{C_2(1/\beta,T)} \frac{1}{2} Log(\zeta(2s))ds + H - \int_{\gamma_1} \psi(s)ds
\]

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Just before the ending of part 1 we derived a expression for $\int_{\gamma_1} \psi(s) ds$
The expression was

$$\int_{\gamma_1} \psi(s) ds = i(I_5 + I_3)$$

Now we substitute the value of $I_3$

$$\int_{\gamma_1} \psi(s) ds = i(I' - \int_T^{-T} \frac{\log(\zeta((\frac{2}{\beta} + 2ik)}{2} \frac{1}{\frac{1}{\beta} + ik} \ln(2) + (\frac{1}{\alpha} + ik - 1)\ln(\pi) + \ln(\sin(\pi(\frac{1}{\beta} + ik)/2)) + \ln(\Gamma(\frac{1}{\beta} - ik))}{1} \frac{1}{\frac{1}{\beta} + ik} \ln(\pi) ds + iI')$$

Now we use $\int_{\gamma_1} \psi(s) ds = i(I_5 + I_3)$ in our main relation which results in

$$\int_{\gamma_1} \psi(s) ds = \int_{C_2(1/\beta, T)} 1 \frac{1}{2} \log(\zeta(2s)) ds + H - i(I_5 + I_3)$$

We will now focus our attention to this integral $\int_{C_2(1/\beta, T)} 1 \frac{1}{2} \log(\zeta(2s)) ds$

$$\int_{C_2(1/\beta, T)} 1 \frac{1}{2} \log(\zeta(2s)) ds = \int_{1/\beta - iT}^{1/\alpha + iT} 1 \frac{1}{2} \log(\zeta(2s)) ds + iI'$$

Substituting the above relation and value of $I_3$ in our main relation we get

$$\int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s) ds = \int_{1/\beta - iT}^{1/\alpha + iT} 1 \frac{1}{2} \log(\zeta(2s)) ds + H - i(I_5) + i\int_T^{-T} \frac{\log(\zeta((\frac{2}{\beta} + 2ik)}{2} \frac{1}{\frac{1}{\beta} + ik} \ln(2) + (\frac{1}{\alpha} + ik - 1)\ln(\pi) + \ln(\sin(\pi(\frac{1}{\beta} + ik)/2)) + \ln(\Gamma(\frac{1}{\beta} - ik))}{1} \frac{1}{\frac{1}{\beta} + ik} \ln(\pi) ds$$

We continue by differentiating both sides with respect to $T$

$$\frac{\partial}{\partial T} (\int_{1/\alpha - iT}^{1/\alpha + iT} \psi(s) ds) = i\log(|\zeta(\frac{2}{\beta} + 2iT)|) - i \frac{\partial}{\partial T} (I_5) - i\log(|\zeta(\frac{2}{\alpha} + 2iT)|)$$

Where

$$\frac{\partial}{\partial T} (I_5) = \frac{\partial}{\partial T} \left( \int_T^{-T} \frac{1}{\frac{1}{\alpha} + ik} \ln(2) + (\frac{1}{\alpha} + ik - 1)\ln(\pi) + \ln(\sin(\pi(\frac{1}{\beta} + ik)/2)) + \ln(\Gamma(\frac{1}{\beta} - ik))}{1} \frac{1}{\frac{1}{\beta} + ik} \ln(\pi) ds \right)$$

Now in the limit $\frac{1}{\alpha} = \frac{1}{2} + \epsilon$ as $\epsilon \to 0$ And $\epsilon > 0$

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\[
\frac{\partial}{\partial T} (I_5) = \log \left( \frac{\pi}{2} \right) - \log(\Gamma \left( \frac{1}{2} + iT \right)) - 2\log(\left| \sin \left( \frac{\pi}{2\alpha} + \frac{i\pi T}{2} \right) \right|)
\]

To continue we use the following result

\[
|\Gamma \left( \frac{1}{2} + iT \right)| = \sqrt{\frac{\pi}{\cos(T\pi)}}
\]

On simplifying we get

\[
\frac{\partial}{\partial T} (I_5) = \log \left( \frac{\pi}{2} \right) - \log(\pi) + \log(\cos(iT\pi)) - 2\log(\left| \sin \left( \frac{i\pi T}{2} \right) \right|)
\]

This

\[
\frac{\partial}{\partial T} \left( \int_{1/\alpha-iT}^{1/\alpha+iT} \psi(s)ds \right) = i\log(|\zeta(\frac{2}{\beta} + 2iT)|) - \log(|\zeta(\frac{2}{\alpha} + 2iT)|) - i \frac{\partial}{\partial T} (I_5)
\]

this gives us the following expression for \(\frac{\partial}{\partial T} \left( \int_{1/\alpha-iT}^{1/\alpha+iT} \psi(s)ds \right)\)

\[
\frac{\partial}{\partial T} \left( \int_{1/\alpha-iT}^{1/\alpha+iT} \psi(s)ds \right) = -i \frac{\partial}{\partial T} (I_5) + i\log\left( \frac{|\zeta(\frac{2}{\beta} + 2iT)|}{|\zeta(\frac{2}{\alpha} + 2iT)|} \right)
\]

Substituting the value of \(I_5\)

\[
i \frac{\partial}{\partial T} \left( \int_{1/\alpha-iT}^{1/\alpha+iT} \psi(s)ds \right) = (\log \left( \frac{\pi}{2} \right) - \log(\pi) + \log(\cos(iT\pi)) - 2\log(\left| \sin \left( \frac{i\pi T}{2} \right) \right|)) - \log\left( \frac{|\zeta(\frac{2}{\beta} + 2iT)|}{|\zeta(\frac{2}{\alpha} + 2iT)|} \right)
\]

Now we have shown that

\[
\frac{\partial}{\partial T} \left( \int_{1/\alpha-iT}^{1/\alpha+iT} \psi(s)ds \right) = 2i \text{Re}(\Delta(T))
\]

on substituting the values it gives us

\[
-2\text{Re}(\Delta(T)) = (\log \left( \frac{\pi}{2} \right) - \log(\pi) + \log(\cos(iT\pi)) - 2\log(\left| \sin \left( \frac{i\pi T}{2} \right) \right|)) - \log\left( \frac{|\zeta(\frac{2}{\beta} + 2iT)|}{|\zeta(\frac{2}{\alpha} + 2iT)|} \right)
\]

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On simplifying we get

\[ 2\text{Re}(\Delta(T)) = \log\left(2\left|\sin(iT\pi/2)\right|^2\right) + \left[\log\left(\frac{\zeta(\frac{2}{3} + 2iT)}{\zeta(\frac{2}{3} + 2iT)}\right)\right]\]

As \( T \to \infty \) above results simplify to

\[ 2\text{Re}(\Delta(T)) = \frac{2}{e^T} + \left[\log\left(\frac{\zeta(\frac{2}{3} + 2iT)}{\zeta(\frac{2}{3} + 2iT)}\right)\right]\]

Now we turn our focus on \( \Delta(T) \) term

\[
\Delta(T) = (iT) \int_2^T \frac{E(x)}{(x-1)^{1/2+iT+1}} dx - (-iT) \int_2^T \frac{E(x)}{(x-1)^{1/2-iT+1}} dx
+ \frac{1}{2} \int_2^T \frac{E(x)}{(x-1)^{1/2+iT+1}} dx - \frac{1}{2} \int_2^T \frac{E(x)}{(x-1)^{1/2-iT+1}} dx
+ (\epsilon) \int_2^T \frac{E(x)}{(x-1)^{1/2+iT+1}} dx - (-\epsilon) \int_2^T \frac{E(x)}{(x-1)^{1/2-iT+1}} dx
\]

We apply the limit \( 1/2+\epsilon = 1/2 \) in the power term only because in other places where \( \epsilon \) is we encounter an indeterminacy of \( 0 \ast \infty \) so we apply the limit just in power term we get

\[ 2\text{Re}(\Delta(T)) = (\epsilon) \int_2^T \frac{E(x)}{(x-1)^{1/2}} \cos(T\ln(x)) dx \]

Now we know that \( \epsilon \to 0 \) and \( T \to \infty \) so we can write \( T \epsilon = 1 \)

substituting \( \epsilon = \frac{1}{T} \) we get

\[
T\left[\log\left(\frac{\zeta(\frac{2}{3} + 2iT)}{\zeta(\frac{2}{3} + 2iT)}\right)\right] = \int_2^T \frac{E(x)}{(x-1)^{1/2}} \cos(T\ln(x)) dx
\]

we differentiate both sides

\[
\frac{E(T)}{(T-1)^{3/2}} \cos(T\ln(T)) = \left[\log\left(\frac{\zeta(\frac{2}{3} + 2iT)}{\zeta(\frac{2}{3} + 2iT)}\right)\right] + T \frac{\partial}{\partial T}\left(\left[\log\left(\frac{\zeta(\frac{2}{3} + 2iT)}{\zeta(\frac{2}{3} + 2iT)}\right)\right]\right)
\]

with this we have our Main formula

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\[ E(T) = T^{3/2} \text{sec}(T \ln(T)) \frac{1}{2} \left[ \log \left| \frac{\zeta \left( \frac{3}{2} + 2iT \right)}{\zeta \left( \frac{3}{2} + 2iT \right)} \right| \right] + T \frac{\partial}{\partial T} \left( \frac{1}{2} \left[ \log \left( \frac{\zeta \left( \frac{3}{2} + 2iT \right)}{\zeta \left( \frac{3}{2} + 2iT \right)} \right) \right] \right) \]

Substituting the values back into the main equation

\[ E(T) = T^{3/2} \text{sec}(T \ln(T)) \frac{1}{2} \left[ \log \left| \frac{\zeta \left( 1 - \frac{2}{T} + 2iT \right)}{\zeta \left( 1 + \frac{2}{T} + 2iT \right)} \right| \right] + T \frac{\partial}{\partial T} \left( \frac{1}{2} \left[ \log \left( \frac{\zeta \left( 1 - \frac{2}{T} + 2iT \right)}{\zeta \left( 1 + \frac{2}{T} + 2iT \right)} \right) \right] \right) \]

As \( T \to \infty \)

We finally derived our main result. Now only one thing remains which is mentioned at the end of "Approach section" that is to derive the bound on the error term. So now we are going to do that.

### 3.3.3 Bound on the error term

We start by stating the subconvexity bounds on Riemann Zeta function

Using Phragm-Lindelof principle [2] We get the following subconvexity bounds for \( \sigma < 1 \)

\[ |\zeta(\sigma + iT)| < T^{\frac{1}{2}(1-\sigma)} \log(T) \]

And for \( \sigma > 1 \) we have the following bounds

\[ |\zeta(\sigma + iT)| < \log(t) \]

We apply the above two bounds in the following manner

\[ 2 \left[ \log \left( \frac{\zeta \left( \frac{2}{T} + 2iT \right)}{\zeta \left( \frac{2}{T} + 2iT \right)} \right) \right] < \log(T)(1 - \frac{2}{\beta}) \]

We can take derivative on both sides with respect to \( T \), we will get

\[ \left[ \log \left( \frac{\zeta \left( 1 - \frac{2}{T} + 2iT \right)}{\zeta \left( 1 + \frac{2}{T} + 2iT \right)} \right) \right] + T \frac{\partial}{\partial T} \left( \log \left( \frac{\zeta \left( 1 - \frac{2}{T} + 2iT \right)}{\zeta \left( 1 + \frac{2}{T} + 2iT \right)} \right) \right) < \frac{\log(T)}{T} + T \frac{\partial}{\partial T} \left( \log(T) \right) \]

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On simplifying we get
\[
\left[\log\left|\frac{\zeta(1 - \frac{2}{T} + 2iT)}{\zeta(1 + \frac{2}{T} + 2iT)}\right| + T \frac{\partial}{\partial T}\left(\log\left|\frac{\zeta(1 - \frac{2}{T} + 2iT)}{\zeta(1 + \frac{2}{T} + 2iT)}\right|\right)\right] < \frac{\log(T)}{T} + \frac{T}{T^2} - \frac{T \log(T)}{T^2}
\]

finally we get a bound on this term
\[
\left[\log\left|\frac{\zeta(1 - \frac{2}{T} + 2iT)}{\zeta(1 + \frac{2}{T} + 2iT)}\right| + T \frac{\partial}{\partial T}\left(\log\left|\frac{\zeta(1 - \frac{2}{T} + 2iT)}{\zeta(1 + \frac{2}{T} + 2iT)}\right|\right)\right] < \frac{1}{T}
\]

Now we can use the formula we derived to get a bound on \(E(x)\)
\[
E(T) < \frac{T^{3/2} \sec(T \ln(T))}{T}
\]

Now we simplify and take modulus on both sides
\[
|E(T)| < \sqrt{T} |\sec(T \ln(T))|
\]

if \(T \ln(T) \neq (2n + 1)\frac{\pi}{2}\) then we can write the following inequality for some proportion of values of \(T\)
\[
|\cos(T \ln(T))| > \frac{B}{\ln(T)}
\]

Here \(B\) is a constant

For this proportion of values of \(T\), on rearranging the terms in the above inequality we can write the following inequality
\[
|B \sec(T \ln(T))| < \ln(T)
\]

On Multiplying bot sides by \(\sqrt{T}\)
\[
B \sqrt{T} |\sec(T \ln(T))| < \sqrt{T} \ln(T)
\]

but we have shown above that \(|E(T)| < \sqrt{T} |\sec(T \ln(T))|\) so using this and the above inequality we can write the following inequality
\[
|E(x)| < \frac{\sqrt{T} \ln(T)}{B}
\]

Since if \(a < b\) and \(b < c\) then \(a < c\)
We now substitute the value $E(T)$ and get the final result

$$|\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B}$$

### 3.4 Result Analysis

In the above proof we shown that for some proportion of values of $T$ for which $|\cos(T \ln(T))| > \frac{B}{\ln(T)}$ is true for those proportion of values of $T$ We can conclude the following inequality $|\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B}$

If we observe carefully the inequality as $T \to \infty$ $|\cos(T \ln(T))| > \frac{B}{\ln(T)}$ gets more and more accurate because as $T \to \infty$ $\frac{B}{\ln(T)} \to 0$ which gives us the following inequality

$$|\cos(T \ln T)| > \epsilon$$

where $\epsilon$ to 0 and $\epsilon > 0$ as $T \to \infty$

the above inequality is true for almost all large values of $T$

Thus we can conclude that the ratio of proportion of values of $T$ for which does not satisfy this inequality $|\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B}$ to the proportion of values of $T$ for which does satisfy this inequality $|\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B}$ tends to zero as $T \to \infty$

If we have to write the above statement mathematically we can write it in the following way

$$\frac{n(S)}{n(M)} \to 0$$

as $T \to \infty$

Using all the above arguments we can write the following statement

**Almost all large values of $T$ satisfy $|\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B}$**

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which can also be written as

\[ |\pi(T) - li(T)| < \frac{\sqrt{T \ln(T)}}{B} \]

as \( T \to \infty \)

It is already shown [3] that proving the above statement is equivalent to proving the famous Riemann hypothesis, thus Riemann hypothesis is True.

4 References

1) Bernhard Riemann. *On the Number of Primes Less Than a Given Magnitude*, November 1859

2) Phragmén, Lars Edward; Lindelöf, Ernst (1908). *Sur une extension d’un principe classique de l’analyse et sur quelques propriétés des fonctions monoènes dans le voisinage d’un point singulier*

3) Schoenfeld, Lowell (1976). *Sharper bounds for the Chebyshev functions \( \theta(x) \) and \( \psi(x) \).*

4) Newman, Donald J. (1980). "Simple analytic proof of the prime number theorem"