Classical Source-Free Electromagnetism is Equivalent to Free-Photon Quantum Mechanics

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Abstract The classical source-free electric field is a polar transverse-vector field, but the classical magnetic field is an axial transverse-vector field. A derivative-related linear transformation (which is its own inverse) of the classical axial magnetic field in fact produces an alternate polar transverse-vector representation of the classical magnetic field. The classical source-free complex-valued electromagnetic polar transverse-vector field whose real part is the classical source-free polar electric field, and whose imaginary part is the alternate polar representation of the classical source-free magnetic field, turns out to satisfy the time-dependent Schrödinger equation whose Hamiltonian operator is that of the free photon. That classical source-free complex-valued electromagnetic polar transverse-vector field can, moreover, be slightly linearly modified to become the normalized wave function of the free photon.

1. The laws of classical source-free electromagnetism

The classical gauge-invariant Heaviside-Maxwell equations which apply to source-free electromagnetism are,

$$\text{the source-free version of Coulomb's Law, } \nabla\cdot\mathbf{E}=0, \tag{1.1a}$$

$$\text{Faraday's Law, } \nabla\times\mathbf{E}=-\dot{\mathbf{B}}/c, \tag{1.1b}$$

$$\text{Gauss' Law, } \nabla\cdot\mathbf{B}=0, \tag{1.1c}$$

$$\text{the source-free version of the Biot-Savart/Maxwell Law, } \nabla\times\mathbf{B}=\dot{\mathbf{E}}/c. \tag{1.1d}$$

Eqs. (1.1a) and (1.1c) reveal that both $\mathbf{E}$ and $\mathbf{B}$ are transverse-vector fields, and since $\mathbf{E}$ is assumed to be a *polar* vector field whose dimension is the square root of energy over volume, Eqs. (1.1b) and (1.1d) reveal that $\mathbf{B}$ is an *axial* vector field which has the same dimension as $\mathbf{E}$.

The axial/polar vector field *dichotomy* which *isolates* $\mathbf{B}$ from $\mathbf{E}$ obviously tends to frustrate the attainment of a comprehensive understanding of *the consequences* of the classical source-free electromagnetic Heaviside-Maxwell equations given in Eqs. (1.1a) through (1.1d).

Fortunately, however, there exists a derivative-related linear transformation (which is *its own inverse)* of the *axial* transverse-vector magnetic field $\mathbf{B}$ into *an alternate polar* transverse-vector magnetic-field representation $\mathbf{M}$ which has *the same dimension as that of* $\mathbf{B}$ *and* $\mathbf{E}$,

$$\mathbf{M}\equiv(-\nabla^2)^{-1/2}(\nabla\times\mathbf{B}), \tag{1.2a}$$

where for arbitrary real p the linear operator $(-\nabla^2)^p$ is given by,

$$(-\nabla^2)^p f(\mathbf{r})\equiv(2\pi)^{-3}\int|\mathbf{k}|^{2p}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}f(\mathbf{r}')\,d^3\mathbf{k}\,d^3\mathbf{r}'. \tag{1.2b}$$

Two important properties of the linear operators $(-\nabla^2)^p$ are that,

$$(-\nabla^2)^p(\nabla f(\mathbf{r}))=\nabla((-\nabla^2)^p f(\mathbf{r}))=(2\pi)^{-3}\int(i\mathbf{k})|\mathbf{k}|^{2p}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}f(\mathbf{r}')\,d^3\mathbf{k}\,d^3\mathbf{r}', \tag{1.2c}$$

and that,

$$(-\nabla^2)^{p_2}((-\nabla^2)^{p_1}f(\mathbf{r}))=(-\nabla^2)^{p_1+p_2}f(\mathbf{r}). \tag{1.2d}$$

Application of the properties given by Eqs. (1.2c) and (1.2d) yield that,

$$\nabla\cdot\mathbf{M}=\nabla\cdot((-\nabla^2)^{-1/2}(\nabla\times\mathbf{B}))=(-\nabla^2)^{-1/2}(\nabla\cdot(\nabla\times\mathbf{B}))=0, \tag{1.3a}$$

so $\mathbf{M}$ is a transverse-vector field, just as $\mathbf{B}$ is (Gauss' Law). In addition,

$$\begin{aligned}(-\nabla^2)^{-1/2}(\nabla\times\mathbf{M})&=(-\nabla^2)^{-1/2}(\nabla\times(-\nabla^2)^{-1/2}(\nabla\times\mathbf{B}))=(-\nabla^2)^{-1}(\nabla\times(\nabla\times\mathbf{B}))=\\&(-\nabla^2)^{-1}(\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B})=(-\nabla^2)^{-1}(-\nabla^2\mathbf{B})=\mathbf{B},\end{aligned} \tag{1.3b}$$

so the Eq. (1.2a) linear transformation is *its own inverse.*

We next replace Eq. (1.1c), namely $\nabla\cdot\mathbf{B}=0$, by Eq. (1.3a), namely $\nabla\cdot\mathbf{M}=0$, and furthermore insert Eq. (1.3b), namely $\mathbf{B}=(-\nabla^2)^{-1/2}(\nabla\times\mathbf{M})$ into Eqs. (1.1b) and (1.1d) in order to eliminate the axial magnetic-field vector $\mathbf{B}$ from the Heaviside-Maxwell equations for source-free electromagnetism presented in Eqs. (1.1a) through (1.1d) in favor of the alternative polar magnetic-field vector $\mathbf{M}$. Thus Faraday's Law

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of Eq. (1.1b), namely $\nabla\times\mathbf{E} = -\dot{\mathbf{B}}/c$, is replaced by $-c\nabla\times\mathbf{E} = (-\nabla^2)^{-1/2}(\nabla\times\dot{\mathbf{M}})$. To solve this last equation for $\dot{\mathbf{M}}$, we apply the curl operator $\nabla\times$ to both its left and right sides, bearing in mind that $\nabla\times$ commutes with the operator $(-\nabla^2)^{-1/2}$ (see Eq. (1.2c)), and also that $\nabla\times(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{E}$ since $\nabla\cdot\mathbf{E} = 0$, and likewise that $\nabla\times(\nabla\times\dot{\mathbf{M}}) = -\nabla^2\dot{\mathbf{M}}$ since $\nabla\cdot\dot{\mathbf{M}} = 0$. The result for $\dot{\mathbf{M}}$ in terms of $\mathbf{E}$ is,

$$\dot{\mathbf{M}} = -c(-\nabla^2)^{1/2}\mathbf{E}. \tag{1.4a}$$

The source-free Biot-Savart/Maxwell Law of Eq. (1.1d), namely $\nabla\times\mathbf{B} = \dot{\mathbf{E}}/c$, is replaced by $\dot{\mathbf{E}} = c\nabla\times((-\nabla^2)^{-1/2}(\nabla\times\mathbf{M}))$. Again, since the curl operator $\nabla\times$ commutes with the operator $(-\nabla^2)^{-1/2}$, and $\nabla\times(\nabla\times\mathbf{M}) = -\nabla^2\mathbf{M}$ because $\nabla\cdot\mathbf{M} = 0$, the simplified result for $\dot{\mathbf{E}}$ in terms of $\mathbf{M}$ is,

$$\dot{\mathbf{E}} = c(-\nabla^2)^{1/2}\mathbf{M}. \tag{1.4b}$$

We note in passing that Eq. (1.4b) implies that $\mathbf{M}$ can *also* be expressed in the form,

$$\mathbf{M} = (-\nabla^2)^{-1/2}\dot{\mathbf{E}}/c, \tag{1.4c}$$

a representation of $\mathbf{M}$ which follows as well from Eq. (1.2a) and the source-free Biot-Savart/Maxwell Law given by Eq. (1.1d).

In addition to the time-dependent Eqs. (1.4a) and (1.4b), which are clearly consistent with both $\mathbf{E}$ and $\mathbf{M}$ being polar vector fields, we of course know as well that both $\mathbf{E}$ and $\mathbf{M}$ are transverse-vector fields,

$$\nabla\cdot\mathbf{E} = 0, \tag{1.4d}$$

and,

$$\nabla\cdot\mathbf{M} = 0. \tag{1.4e}$$

The *two* real-valued vector equations given by Eqs. (1.4a) and (1.4b) can be *combined* into the *single* complex-valued vector equation,

$$id(\mathbf{E} + i\mathbf{M})/dt = c(-\nabla^2)^{1/2}(\mathbf{E} + i\mathbf{M}). \tag{1.5a}$$

Similarly, the *two* real-valued equations given by Eqs. (1.4d) and (1.4e) can be *combined* into the *single* complex-valued equation,

$$\nabla\cdot(\mathbf{E} + i\mathbf{M}) = 0. \tag{1.5b}$$

Upon multiplying Eq. (1.5a) through by the constant $\hbar$ it becomes,

$$i\hbar d(\mathbf{E} + i\mathbf{M})/dt = \hbar c(-\nabla^2)^{1/2}(\mathbf{E} + i\mathbf{M}), \tag{1.6a}$$

whose form is that of a time-dependent Schrödinger equation with the Hamiltonian operator,

$$\widehat{h} = \hbar c(-\nabla^2)^{1/2} = (|-i\hbar\nabla|^2)^{1/2}c = (|\widehat{\mathbf{p}}|^2)^{1/2}c = |\widehat{\mathbf{p}}|c, \tag{1.6b}$$

which is the relativistic energy of a *zero-mass* free particle, i.e., the relativistic energy *of the free photon.*

Notwithstanding that the *form* of Eq. (1.6a) is that of the time-dependent Schrödinger equation of a free photon, the entity $(\mathbf{E} + i\mathbf{M})$ has *shortcomings* for the role of *a normalized quantum wave function.* For example, the *dimension* of $(\mathbf{E} + i\mathbf{M})$ is the square root of energy over volume, whereas *the dimension of a normalized quantum wave function is the square root of inverse volume.* We now work out the *linear modification* of $(\mathbf{E} + i\mathbf{M})$ which renders it *a satisfactory normalized free-photon wave function.*

2. The normalized free-photon wave function in terms of classical electromagnetic fields

To *remove* the inappropriate (for a normalized quantum wave function) dimension of the square root of energy from $(\mathbf{E} + i\mathbf{M})$ we linearly transform it by the inverse of the square root of a pure number N times the free-photon Hamiltonian $\widehat{h}$,

$$\mathbf{\Psi} = (N\widehat{h})^{-1/2}(\mathbf{E} + i\mathbf{M}), \text{ where } \widehat{h} = \hbar c(-\nabla^2)^{1/2}. \tag{2.1}$$

Note that since $(\mathbf{E} + i\mathbf{M})$ satisfies the time-dependent free-photon Schrödinger equation whose Hamiltonian operator is $\widehat{h}$, the $\mathbf{\Psi}$ of Eq. (2.1) satisfies that time-dependent Schrödinger equation as well since the additional linear transformation factor $(N\widehat{h})^{-1/2}$ it features is time-independent and commutes with the Hamiltonian operator $\widehat{h}$.

We next work out the appropriate value of the pure number N which occurs in the Eq. (2.1) expression for the complex vector wave function $\mathbf{\Psi}$ by considering the expectation value of the Hamiltonian operator $\widehat{h}$ in the state $\mathbf{\Psi}$,

$$\langle\, \widehat{h}\,\rangle_{\mathbf{\Psi}} = \int \mathbf{\Psi}^* \cdot (\,\widehat{h}\,\mathbf{\Psi})\, d^3\mathbf{r} = N^{-1} \int (\mathbf{E} - i\mathbf{M}) \cdot (\mathbf{E} + i\mathbf{M})\, d^3\mathbf{r} = N^{-1} \int (|\mathbf{E}|^2 + |\mathbf{M}|^2)\, d^3\mathbf{r}. \tag{2.2}$$

Since from Eq. (1.2a), $\mathbf{M} = (-\nabla^2)^{-1/2}(\nabla \times \mathbf{B})$, and from Eq. (1.1c), $\nabla \cdot \mathbf{B} = 0$, it can be demonstrated that,

$$\int |\mathbf{M}|^2 d^3\mathbf{r} = \int |\mathbf{B}|^2 d^3\mathbf{r}, \tag{2.3}$$

by applying the lemma,

$$\int ((\nabla \times \mathbf{U}) \cdot \mathbf{V})\, d^3\mathbf{r} = \int (\mathbf{U} \cdot (\nabla \times \mathbf{V}))\, d^3\mathbf{r}, \tag{2.4}$$

which follows from integration by parts.

Therefore, from Eqs. (2.2) and (2.3),

$$\langle\, \widehat{h}\,\rangle_{\mathbf{\Psi}} = N^{-1} \int (|\mathbf{E}|^2 + |\mathbf{B}|^2)\, d^3\mathbf{r}, \tag{2.5}$$

and since $(1/2) \int (|\mathbf{E}|^2 + |\mathbf{B}|^2)\, d^3\mathbf{r}$ is known to be the energy of the source-free electromagnetic field, the value of N is 2. Upon putting this result for N into Eq. (2.1), we obtain the free-photon wave function $\mathbf{\Psi}$ in terms of the source-free electric and magnetic fields,

$$\begin{aligned}\mathbf{\Psi} = (2\hbar c)^{-1/2}\, (-\nabla^2)^{-1/4}\, (\mathbf{E} + i\mathbf{M}) = (2\hbar c)^{-1/2}\, ((-\nabla^2)^{-1/4}\, \mathbf{E} + i(-\nabla^2)^{-3/4}\, (\nabla \times \mathbf{B})) = \\ (2\hbar c)^{-1/2}\, ((-\nabla^2)^{-1/4}\, \mathbf{E} + i(-\nabla^2)^{-3/4}\, (\dot{\mathbf{E}}/c)).\end{aligned} \tag{2.6}$$

Since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{M} = 0$, it is apparent that $\nabla \cdot \mathbf{\Psi} = 0$, namely that the free-photon wave function $\mathbf{\Psi}$ is a transverse complex-valued vector field.