Notes on the Prouhet–Tarry–Escott problem

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Abstract: A summary of selected solutions to the ideal Prouhet–Tarry–Escott problem up to size 7

Prouhet–Tarry–Escott

The Prouhet-Tarry-Escott (PTE) problem seeks two n-tuples of integers x_i and y_i such that the sums of like powers up to k are equal

$$\sum_{i=1}^{n} {x_i}^p = \sum_{i=1}^{n} {y_i}^p \,, 1 \le p \le k$$

k is called the degree, and n the size of the problem. If x_i and y_i are permutations of the same numbers then the solution is trivial. The system of equations is written with the notation $[x_i] =_k [y_i]$. For non-trivial solutions $k < n$. If $k = n - 1$, the solution is called ideal. Chains of n-tuples which are PTE solutions in pairs $[x_i] =_k [y_i] =_k [z_i] \ldots$, are also of interest. The problem is equivalent to seeking pairs of polynomials which fully factorise over the integers and which differ by a polynomial of degree $n - k - 1$, a non-zero integer constant in the ideal case,

$$\prod_{i=1}^{n}(x - x_i) - \prod_{i=1}^{n}(x - y_i) = C$$

Versions of the PTE problem date back to Euler, but it was introduced in its current form by Prouhet in 1851 [1] and was studied in detail by Tarry who found solutions on size 6 and 8 [2] and Escott who found solution of size 7 [3] in the early twentieth century. Solutions of size 9 were found by Letac in 1942 [7]

Solutions are said to be equivalent if they differ only by permutations of the elements, addition of an integer constant, multiplication by a rational constant, or any combination of these operations.

In particular, negating each element on both sides leads to an equivalent solution. If these are the same up to permutations then the solution is called symmetric. The nature of symmetric solutions differs for odd vs even size n. If n is odd then the elements on the left side are the negatives of the elements on the right and the even power equations are automatically satisfied. If n is even, the elements on either side fall into pairs differing in sign, and the odd power equations are automatically satisfied.

In the current state of art of the PTE problem, ideal solutions are known for all sizes up to 10 [5] and also for size 12 [4,6].

size 2 degree 1

The ideal PTE problem of size two only requires that $x_1 + x_2 = y_1 + y_2$. Any solution is therefore equivalent to the symmetric form $[a, -a] =_1 [b, -b]$ which can be extended to an infinite chain of solutions.

size 3 degree 2

For size three, a linear and quadratic equation in three variables must be satisfied. For the symmetric case $[x_1, x_2, x_3] =_2 [-x_1, -x_2, -x_3]$ the quadratic is satisified and the linear is

$$x_1 + x_2 + x_3 = 0$$

So in general for the symmetric case

$$\boxed{[u, v, -u - v] =_2 [-u, -v, u + v]}$$

For the general case consider a linear substitution

$$[u, p + q, r - s,] =_2 [-u, p - q, r + s]$$

Five variables are sufficient up to equivalence. The necessary equations then reduce to

$$u + q = s$$

$$pq = rs$$

By the factorisation principle on the second equation take

$$p = ab, q = cd, r = ad, s = cb$$

And then solve the first equation for u to provide the general solution up to equivalence parameterised by four variables

$$\boxed{[cb - cd, ab + cd, ad - cb] =_2 [cd - cb, ab - cd, ad + cb]}$$

The general solution can also be expressed in a symmetric form parameterised by six variables [13,14]

$$x_1 = ap + bq + cr$$
$$x_2 = aq + br + cp$$
$$x_3 = ar + bp + cq$$
$$y_1 = ap + br + cq$$
$$y_2 = aq + bp + cr$$
$$y_3 = ar + bq + cp$$

This solution can be understood in terms of factorisations over the commutative ring generated by an element ω subject to $\omega^3 = 1$ so that a general element takes the form $= a + b\omega + c\omega^2$. The ring has a conjugation $\bar{u} = a + c\omega + b\omega^2$. With $v = p + q\omega + r\omega^2$, $x = x_1 + x_2\omega + x_3\omega^2$ and $y = y_1 + y_2\omega + y_3\omega^2$ the solution reduces $x = \bar{u}v$, $y = uv$.

If the additional relation $1 + \omega + \omega^2 = 0$ is imposed then the ring reduces to the Eisenstein integers and the general solution can be simplified to the four parameter form, but with less symmetry.

Solution chains of length 2^{l-1} can be generated using products of l Eisenstein integers with arbitrary selections of conjugates taken.

In polynomial form, the general solution to size three case above gives

$$\begin{aligned}(x - x_1)(x - x_2)(x - x_3) - (x - y_1)(x - y_2)(x - y_3)\\ = (a - b)(b - c)(c - a)(p - q)(q - r)(r - p)\end{aligned}$$

size 4 Degree 3

The general size 4 ideal case requires three equations, but the symmetric case reduces to one

$$[a, -a, b, -b] =_3 [c, -c, d, -d]$$

$$a^2 + b^2 = c^2 + d^2$$

This has the well-known general solution from products of Gaussian integers and their conjugates

$$a = pq - rs, b = pr + qs, c = pq + rs, d = pr - qs$$

Chains of symmetric solutions of size 4 can be formed using products of more Gaussian integers and their conjugates.

The general ideal case of size 4 was solved by “Crussol” in 1913 [15].

The linear equation is resolved by making use of an additive constant to set both sides to zero

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 0$$

The remaining quadratic and cubic power equations are transformed by a linear substitution

$$x_1 = X_1 + X_2 + X_3$$
$$x_2 = X_1 - X_2 - X_3$$
$$x_3 = -X_1 - X_2 + X_3$$
$$x_4 = -X_1 + X_2 - X_3$$
$$y_1 = Y_1 + Y_2 + Y_3$$
$$y_2 = Y_1 - Y_2 - Y_3$$
$$y_3 = -Y_1 - Y_2 + Y_3$$
$$y_4 = -Y_1 + Y_2 - Y_3$$

This reduces the system of equations to

$$X_1{}^2 + X_2{}^2 + X_3{}^2 = Y_1{}^2 + Y_2{}^2 + Y_3{}^2$$

$$X_1X_2X_3 = Y_1Y_2Y_3$$

Taking account of the common factors in the second equation, a general solution parameterised by 9 integers $t_{ij}, i = 1,2,3, j = 1,2,3$ is given by

$$X_i = t_{i1}t_{i2}t_{i3} \quad i = 1,2,3$$

$$Y_j = t_{1j}t_{2j}t_{3j} \quad j = 1,2,3$$

Leaving just one equation to be resolved

$$t_{11}{}^2t_{12}{}^2t_{13}{}^2 + t_{21}{}^2t_{22}{}^2t_{23}{}^2 + t_{31}{}^2t_{32}{}^2t_{33}{}^2 = t_{11}{}^2t_{21}{}^2t_{31}{}^2 + t_{12}{}^2t_{22}{}^2t_{32}{}^2 + t_{13}{}^2t_{23}{}^2t_{33}{}^2$$

Crussol and other authors since have completed the solution by treating it as a quadratic in the three variables t_{ii} with the remaining variables taken as givens. Standard methods can be used to parameterise all solutions over the rational numbers, which can then be transformed to integers by multiplying through by all denominators. This provides a solution which is complete, but opaque and unsymmetrical. An alternative approach is to recognise the equation as a determinant expression

$$\begin{vmatrix} t_{11}{}^2 & t_{33}{}^2 & t_{22}{}^2 \\ t_{23}{}^2 & t_{12}{}^2 & t_{31}{}^2 \\ t_{32}{}^2 & t_{21}{}^2 & t_{13}{}^2 \end{vmatrix} = 0$$

The ideal PTE problem of size 4 is therefore equivalent to seeking 3 by 3 singular matrices with all square elements. Note that a solution being symmetric is equivalent to one of the elements being zero.

The singularity of a matrix is equivalent to there being a linear relationship between the rows or columns. I.e there exist integers a, b, c such that

$$at_{11}{}^2 + bt_{33}{}^2 + ct_{22}{}^2 = 0$$
$$at_{23}{}^2 + bt_{12}{}^2 + ct_{31}{}^2 = 0$$
$$at_{32}{}^2 + bt_{21}{}^2 + ct_{13}{}^2 = 0$$

For example, in the specific case of $a = b = 1, c = -1$ the problem requires three Pythagorean triples.

Multiplication or division of the elements in any row or column to an integer only affects the overall solution by a constant multiplier. If we are interested in constructing solutions up to equivalence then we can freely apply such factors. It can be arranged that no element in the bottom row is zero. By this means it is possible to reduce the last row of the matrix to all unit elements while keeping the other two rows in integer form.

$$\begin{vmatrix} t_{11}{}^2 & t_{33}{}^2 & t_{22}{}^2 \\ t_{23}{}^2 & t_{12}{}^2 & t_{31}{}^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

The linear relationship is then subject to the condition $a + b + e = 0$ and it can be assumed that the three coefficients are relatively prime in pairs. In this case the general solution in integers to $ax^2 + by^2 + ec = 0$ can be parameterised up to a common factor by

$$x = bu^2 + cv^2, \qquad y = ew^2 + au^2, \qquad z = av^2 + bw^2$$

$$u + v + w = 0$$

A general solution up to equivalence is therefore given by

$$X_1 = (bu^2 + cv^2)(cr^2 + ap^2)$$

$$X_2 = (av^2 + bw^2)(bp^2 + cq^2)$$

$$X_3 = (cw^2 + au^2)(ar^2 + bp^2)$$

$$Y_1 = (bu^2 + cv^2)(ar^2 + bp^2)$$

$$Y_2 = (av^2 + bw^2)(cr^2 + ap^2)$$

$$Y_3 = (cw^2 + au^2)(bp^2 + cq^2)$$

$$a + b + c = u + v + w = p + q + r = 0$$

An alternative general solution is to substitute with

$$t_{11} = R - 2Sg$$
$$t_{33} = R - 2Sh$$
$$t_{22} = R - 2Sk$$
$$t_{23} = d$$
$$t_{12} = e$$
$$t_{31} = f$$

Then the determinant equation becomes

$$4RS(ge^2 - gf^2 + hf^2 - hd^2 + kd^2 - ke^2) \\ - 4S^2(g^2e^2 - g^2f^2 + h^2f^2 - h^2d^2 + k^2d^2 - k^2e^2)$$

A general fifth degree polynomial solution up to equivalence parameterised by d, e, f, g, h, k is therefore given by taking

$$R = g^2e^2 - g^2f^2 + h^2f^2 - h^2d^2 + k^2d^2 - k^2e^2$$
$$S = ge^2 - gf^2 + hf^2 - hd^2 + kd^2 - ke^2$$
$$x_1 = (e + d + f)R - 2(eg + dk + fh)S$$
$$x_2 = (e - d - f)R - 2(eg - dk - fh)S$$
$$x_3 = (-e - d + f)R - 2(-eg - dk + fh)S$$
$$x_4 = (-e + d - f)R - 2(-eg + dk - fh)S$$
$$y_1 = (f + e + d)R - 2(fg + ek + dh)S$$
$$y_2 = (f - e - d)R - 2(fg - ek - dh)S$$
$$y_3 = (-f - e + d)R - 2(-fg - ek + gh)S$$
$$y_4 = (-f + e - d)R - 2(-fg + ek - gh)S$$

An alternative way to analyse the size 4 case is by polynomial splitting. Write a solution as

$$p(x)r(x) - q(x)s(x) = C$$

Where

$$p(x) = (x - x_1)(x - x_2)$$
$$r(x) = (x - x_3)(x - x_4)$$
$$q(x) = (x - y_1)(x - y_2)$$
$$s(x) = (x - y_3)(x - y_4)$$

Then $p(x) - q(x) = Al(x)$ where $l(x) = x - z$ is linear with z a rational number.

Also $s(x) - r(x) = Ak(x)$ where $k(x) = x - w$ is linear with w a rational number.

Furthermore $\big(p(x) - q(x)\big)r(x) - q(x)\big(s(x) - r(x)\big) = C$

$$l(x)r(x) - q(x)k(x) = C/A$$

Which is an ideal solution of size 3

In addition $\big(p(x) - q(x)\big)s(x) - p(x)\big(s(x) - r(x)\big) = C$

$$l(x)s(x) - p(x)k(x) = C/A$$

In summary, an ideal solution of size 4 can be split to produce two ideal solutions of size 3 that share a number on either side and have the same constant. There are 18 ways to make such a split from one solution.

size 5 degree 4

For the symmetric case of size 5 only the linear and cubic equation need to be resolved. Solutions can found with suitable non-general substitutions. For the symmetric cases that most commonly arise for low numbers it is found that one element is often the sum of two others on the same side. This can be written as,

$$[-x-y-z, -x-y+z, x, y, x+y] =_4 [-x-y, -y, -x, x+y-z, x+y+z]$$

In this case the linear equation is satisfied and the cubic reduces to

$$(2zz + xy)(x + y) = 0$$

No element can be zero so the general solution of this case is given by

$$z = ab, x = -aa, y = 2bb$$

A general parameterisation for the ideal symmetric case of size 5 up to equivalence can be derived by a method of linear composition [10]. Suppose two solutions are known $[a_i]$ and $[b_i]$

$$\sum_{j=1}^{5} a_j = 0, \qquad \sum_{j=1}^{5} {a_j}^3 = 0$$

$$\sum_{j=1}^{5} b_j = 0, \qquad \sum_{j=1}^{5} {b_j}^3 = 0$$

Seek another solution $[x_i]$ that is a linear combination

$$x_i = Aa_i - Bb_i$$

The linear equation $\sum_{i=1}^{5} x_i = 0$ is immediately satisfied while the cubic $\sum_{i=1}^{5} {x_i}^3 = 0$ reduces to

$$AB\left(A\sum_{j=1}^{5} {a_j}^2 b_j - B\sum_{j=1}^{5} {b_j}^2 a_j\right) = 0$$

The solutions given by $A = 0$ and $B = 0$ are equivalent to the originals, but a new solution is obtained from

$$A = \sum_{j=1}^{5} {b_j}^2 a_j$$

$$B = \sum_{j=1}^{5} {a_j}^2 b_j$$

$$x_i = a_i \sum_{j=1}^{5} {b_j}^2 a_j - b_i \sum_{j=1}^{5} {a_j}^2 b_j$$

To get a general solution it is then sufficient to choose trivial solutions for $[a_i]$ and $[b_i]$ such that any potential non-trivial general solution is a linear combination up to some factor, e.g.

$$[a_i] = [m, 0, r, -m, -r]$$

$$[b_i] = [0, m, -m, s, -s]$$

Which gives a general solution to the symmetric case of size 5 up to equivalence as follows

$$\boxed{\begin{gathered} A = m^2 r - s^2 m - s^2 r \\ B = -r^2 m + m^2 s - r^2 s \\ [x_i] = [mA, mB, rA - mB, sB - mA, -rA - sB] \end{gathered}}$$

The non-symmetric solutions of size 5 have not been fully resolved. However, there is a splitting theorem that can be applied when the following sum relations apply

$$x_1 + x_2 = y_1 + y_2, \qquad x_3 + x_4 + x_5 = y_3 + y_4 + y_5$$

One relation implies the other. This splitting condition may seem like a very special case, but computational searches show that such a relation usually holds for some choice of the elements, at least for small solutions. Indeed it usually holds in 2, 3 or even 4 different ways. There are exceptions.

Given this relation, a split can be made using a similar procedure as was done for the size four case

$$p(x)r(x) - q(x)s(x) = C$$

Where

$$p(x) = (x - x_1)(x - x_2)$$
$$r(x) = (x - x_3)(x - x_4)(x - x_5)$$
$$q(x) = (x - y_1)(x - y_2)$$
$$s(x) = (x - \ y_3)(x - y_4)(x - x_5)$$

Then $p(x) - q(x) = A$.

Also $s(x) - r(x) = Ak(x)$ where $k(x) = x - w$ is linear with w a rational number.

Furthermore $\big(p(x) - q(x)\big)r(x) - q(x)\big(s(x) - r(x)\big) = C$

$$r(x) - q(x)k(x) = C/A$$

Which is an ideal solution of size 3

In addition $\big(p(x) - q(x)\big)s(x) - p(x)\big(s(x) - r(x)\big) = C$

$$s(x) - p(x)k(x) = C/A$$

In summary, an ideal solution of size 5 with a splitting condition can be split into two ideal solutions of size three with the same constant and an element shared on one side when the linear sums in each are made equal.

The complete solution to the general ideal case of size 5 is not known, but non-symmetric parametric solutions have been given by Choudhry [11]

size 6 degree 5

Consider an ideal solution of size 3 in zeroed form

$$[x_1, x_2, x_3] =_2 [y_1, y_2, y_3]$$

$$x_1 + x_2 + x_3 = 0$$

$$p(x) - q(x) = C = y_1y_2y_3 - x_1x_2x_3$$

The negatives of each side can be used to provide a chain of solutions.

$$[x_1, x_2, x_3] =_2 [-x_1, -x_2, -x_3] =_2 [y_1, y_2, y_3] =_2 [-y_1, -y_2, -y_3]$$
$$p(x) + q(-x) = D = -y_1y_2y_3 - x_1x_2x_3$$

It follows that

$$p(x)p(-x) - q(x)q(-x)$$
$$= p(x)(p(-x) - q(-x)) \ + \ q(-x)(p(x) - q(x))$$
$$= p(x)C + q(-x)C = CD$$

So for any ideal solution of size 3 there is a symmetric ideal solution of size 6

$$[x_1, x_2, x_3, -x_1, -x_2, -x_3] =_5 [y_1, y_2, y_3, -y_1, -y_2, -y_3]$$

This relation implies in particular that ${x_1}^4 + {x_2}^4 + {x_3}^4 = {y_1}^4 + y^4 + {y_3}^4$, a result that could have been derived by direct algebraic means from the original ideal solution. A more general result is that if $[x_i] =_{n-1} [y_i]$ is an ideal solution of size n and $\sum_{i=1}^{n} x_i = 0$ then $\sum_{i=1}^{n} {x_i}^{n+1} = \sum_{i=1}^{n} {y_i}^{n+1}$ [9], which can be derived most simply using a polynomial method as above.

There are symmetric ideal solutions of size 6 that do not take the above form. There are also non-symmetric ideal solutions. Again Choudhry provides parametric solutions [11]

size 7 degree 6

A parameterisation for a class of symmetric ideal solutions is known [5,11] Choudhry has already provided an elegant derivation but an alternative method using a relation with the general case of size 4 may provide some new light.

Start by assuming a splitting condition $x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 = 0$.

$$p(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$q(x) == (x - x_5)(x - x_6)(x - x_7)$$

$$p'(x) = p(-x)$$

$$q'(x) = -q(-x)$$

Then

$$p(x)q(x) - p'(x)q'(x) = C$$

$$p(x) - p'(x) = Ax$$

$$q'(x) - q(x) = A$$

$$\big(p(x) - p'(x)\big)q(x) - p'(x)\big(q'(x) - q(x)\big) = C$$

$$q(x)x - p'(x) = C/A$$

Therefore finding a symmetric ideal solution of size 7 with this splitting condition is equivalent to finding a general ideal solution of size 4 which includes a zero element when zeroed. I.e. when $x_5 + x_6 + x_7 = 0$ then

$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7] =_6 [-x_1, -x_2, -x_3, -x_4, -x_5, -x_6, -x_7]$$

Is equivalent to

$$[0, x_5, x_6, x_7] =_3 [-x_1, -x_2, -x_3, -x_4]$$

Using the analysis of the size 4 case above, this is equivalent to finding a 3 by 3 singular matrix with square elements

$$\begin{vmatrix} {A_1}^2 & {A_2}^2 & {A_3}^2 \\ {a_1}^2 & {a_2}^2 & {a_3}^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

With the extra condition that

$$A_1 a_2 + A_2 a_3 + A_3 a_1 = 0$$

This is more tractable if the further ansatz $a_1 + a_2 + a_3 = 0$ is imposed. A general solution giving the zero element is then

$$A_1 = P - 2(a_3 - a_1)Q$$

$$A_2 = P - 2(a_1 - a_2)Q$$

$$A_3 = P - 2(a_2 - a_3)Q$$

And the determinant equation is resolved up to common factors by

$$Q = (a_3 - a_1)({a_2}^2 - {a_3}^2) + (a_1 - a_2)({a_3}^2 - {a_1}^2) + (a_2 - a_3)({a_1}^2 - {a_2}^2)$$

$$P = (a_3 - a_1)^2({a_2}^2 - {a_3}^2) + (a_1 - a_2)^2({a_3}^2 - {a_1}^2) + (a_2 - a_3)^2({a_1}^2 - {a_2}^2)$$

$$\begin{aligned} x_5 &= A_1 a_2 - A_2 a_3 - A_3 a_1 \\ x_6 &= -A_1 a_2 - A_2 a_3 + A_3 a_1 \\ x_7 &= -A_1 a_2 + A_2 a_3 - A_3 a_1 \\ -x_1 &= A_1 a_3 + A_2 a_1 + A_3 a_2 \\ -x_2 &= A_1 a_3 - A_2 a_1 - A_3 a_2 \\ -x_3 &= -A_1 a_3 - A_2 a_1 + A_3 a_2 \\ -x_4 &= -A_1 a_3 + A_2 a_1 - A_3 a_2 \end{aligned}$$

Setting $a_1 = g, a_2 = h, a_3 = -(g + h)$, these equations can be combined to give a parameterised (partial) solution for the ideal symmetric case of size 7 in terms of degree 5 polynomials in two integer parameters.

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