Proof of the Riemann Hypothesis

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Abstract

In this article we will prove the problem equivalent to the Riemann Hypothesis developed by Luis Báez in the article “A sequential Riesz-like criterion for the Riemann hypothesis”.

1 Introduction

The Riemann Hypothesis is a famous conjecture made by Bernhard Riemann in his article on prime numbers. Riemann, as indicated by the title of his article [1], wanted to know the number of prime numbers in a given interval of the real line, so he extended a Euler observation and defined a function called Zeta. Riemann obtained an explicit formula, which depends on the non-trivial zeros of the Zeta function, for the quantity he was looking for. Along the way, Riemann mentions that probably all non-trivial zeros of the Zeta function are, in the now called critical line, that is, when the complex argument $s = \sigma + iT$ of the Zeta function has a real part equal to one-half. - $\sigma = \frac{1}{2}$. We will prove, using the equivalent problem developed by Luis Báez-Duarte [2], the conjecture.
2 Proof

\[ q_k := \sum_{n=1}^{\infty} \left( 1 - \frac{1}{n^2} \right)^k = \sum_{n=1}^{k} \left( 1 - \frac{1}{n^2} \right)^k + \sum_{n=k+1}^{\infty} \left( 1 - \frac{1}{n^2} \right)^k \]  

(1)

We need to prove that: \( q_k = O \left( k^{-3/4} \right) \), i.e., \( q_k \leq M \cdot k^{-3/4} \) for all \( k \geq k_0 \) and \( M \) is a definite positive constant. This is equivalent to the Riemann’s hypothesis.

2.1 Treating the first sum

2.1.1 Using Hölder inequality we get

\[ \sum_{n=1}^{k} \left( 1 - \frac{1}{n^2} \right)^k \leq \left( \sum_{n=2}^{k} \frac{1}{n^{(1/4) + \Delta}} \right)^{1/p} \cdot \left( \sum_{n=2}^{k} \left( 1 - \frac{1}{n^{(1/4) - \Delta}} \right)^{k/4} \right)^{1/q} \]  

(2)

and we must determine, conveniently, \( p, q \) and \( \Delta \).

2.1.2 Finding an upper bound and changing exponent 2 of \( n \)

\[ \sum_{n=2}^{k} \left( 1 - \frac{1}{n^2} \right)^{k-q} \frac{e^{-kn}}{n^{(2 - \Delta) - q}} < \sum_{n=2}^{k} \frac{e^{-kn}}{n^{(2 - \Delta) - q}} < \sum_{n=2}^{k} e^{-\frac{kn}{n^{(2 - \Delta) - q}}} + \delta \left( \left( 2 - \frac{1}{p} - \Delta \right) \cdot q \right) \]  

(3)

where \( \delta \left( \left( 2 - \frac{1}{p} - \Delta \right) \cdot q \right) \) is an error associated with exponent change, and the error is zero if \( \left( 2 - \frac{1}{p} - \Delta \right) \cdot q > 2 \). This error will be analized later.

2.1.3 Finding an integral that is an upper bound of the sum

Let \( C = \left( 2 - \frac{1}{p} - \Delta \right) \cdot q \), where we assume for now \( C > 1 \), we have

\[ \sum_{n=2}^{k} \frac{e^{-kn}}{n^{C}} < \int_{1}^{k} \frac{e^{-xn}}{x^C} \, dx \]  

(4)
Change of variable:

\[ y = \frac{kq}{x^C} \]  
\[ x = (kq)^\frac{1}{C} \cdot y^{-\frac{1}{C}} \]  
\[ dx = (kq)^\frac{1}{C} \cdot y^{-\frac{1}{C} - 1} dy \]

\[ \int_1^k \frac{e^{-\frac{kq}{x^C}}}{x^C} dx = \int_1^k \frac{e^{-y}}{kq} \cdot y \cdot (kq)^\frac{1}{C} \cdot y^{-\frac{1}{C} - 1} dy \]

\[ \frac{(kq)^\frac{1}{C} \cdot \int_0^k e^{-y} \cdot y^{-\frac{1}{C}} dy}{(kq)^\frac{1}{C} \cdot \int_0^\infty e^{-y} \cdot y^{-\frac{1}{C}} dy} < \frac{(kq)^\frac{1}{C} \cdot \int_0^k e^{-y} \cdot y^{-\frac{1}{C}} dy}{(kq)^\frac{1}{C} \cdot \int_0^\infty e^{-y} \cdot y^{-\frac{1}{C}} dy} \]

Therefore

\[ \sum_{n=2}^k \frac{e^{-\frac{kq}{n^C}}}{n^C} < \frac{(kq)^\frac{1}{C} \cdot \Gamma \left( 1 - \frac{1}{C} \right)}{C} \]

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \frac{(kq)^\frac{1}{C} \cdot \Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{C} \]

2.1.4 Replacing sum by integral in Hölder inequality

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \left( \sum_{n=2}^k \frac{1}{n^{(\frac{1}{C} + \frac{1}{\Delta}) p}} \right)^{1/p} \cdot \left( \frac{(kq)^\frac{1}{C} \cdot \Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{C} \right)^{1/q} \]

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \left( \sum_{n=2}^k \frac{1}{n^{(\frac{1}{C} + \frac{1}{\Delta}) p}} \right)^{1/p} \cdot \left( \frac{\Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{k^{\frac{1}{C} - 1}} \right)^{1/q} \cdot k^{\frac{1}{C} - \frac{1}{2}} \]

i.e., using Hölder’s inequality,

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \left( \sum_{n=2}^k \frac{1}{n^{(\frac{1}{C} + \frac{1}{\Delta}) p}} \right)^{1/p} \cdot \left( \frac{\Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{k^{\frac{1}{C} - 1}} \right)^{1/q} \cdot k^{\frac{1}{C} - \frac{1}{2} - 1} \]

or

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \left( \sum_{n=2}^k \frac{k}{n^{(\frac{1}{C} + \frac{1}{\Delta}) p}} \right)^{1/p} \cdot \left( \frac{\Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{k^{\frac{1}{C} - 1}} \right)^{1/q} \cdot k^{\frac{1}{C} - \frac{1}{2}} \]

\[ \sum_{n=2}^k \frac{(1 - \frac{1}{n^C})^{kq}}{n^{Cq}} < \left( \sum_{n=2}^k \frac{k}{n^{(\frac{1}{C} + \frac{1}{\Delta}) p}} \right)^{1/p} \cdot \left( \frac{\Gamma \left( 1 - \frac{1}{C} \right) + \delta(C)}{k^{\frac{1}{C} - 1}} \right)^{1/q} \cdot k^{\frac{1}{C} - \frac{1}{2} - 1} \]
and finally, using the fact that arithmetic mean is greater than harmonic mean, we get

\[
\sum_{n=2}^{k} \left( \frac{1 - \frac{1}{n^p}}{n^{C+q}} \right)^{kq} < \left( \sum_{n=2}^{k} \frac{1}{n^{(\frac{1}{p} + \Delta)\cdot p}} \right)^{\frac{1}{p}} \cdot \left( \frac{q^{\frac{1}{p} - 1}}{C} \Gamma \left( 1 - \frac{1}{C} \right) + \frac{\delta(C)}{k^{\frac{1}{p} - 1}} \Gamma \left( \frac{2}{3} \right) \right)^{\frac{1}{q}} \cdot k^{-\frac{1}{q}} \cdot \frac{1}{\Gamma \left( \frac{2}{3} \right)} \cdot k^{\frac{1}{q}}
\]

(17)

2.1.5 Choosing \( q \) to obtain \(-\frac{3}{4}\) power

Therefore we need to solve

\[
\frac{1}{qC} - \frac{1}{q} = -\frac{3}{4}
\]

(18)

and solving the equations we arrive at

\[
q := \frac{4}{C}
\]

(19)

and because of Hölder condition \( \frac{1}{q} + \frac{1}{p} = 1 \) we get

\[
p = \frac{4}{4-C}.
\]

(20)

We can choose \( C = \left( 2 - \frac{1}{p} - \Delta \right) \cdot q = 3 \) which implies \( \Delta = \frac{8p - 3Cp - 4}{4p} = \frac{8 \cdot 4 - 3 \cdot 4 - 4}{16} = -\frac{1}{2} \) therefore \( 1 + \Delta \cdot p = 1 - \frac{1}{2} \cdot 4 = -1 \)

2.1.6 Final Hölder inequality

\[
\sum_{n=2}^{k} \left( \frac{1 - \frac{1}{n^p}}{n^{C+q}} \right)^{kq} < \left( \sum_{n=2}^{k} \frac{1}{n^{\frac{1}{p}}} \right)^{\frac{1}{p}} \cdot \left( \frac{\left( \frac{1}{4} \right)^{\frac{1}{3}}}{3} \Gamma \left( \frac{2}{3} \right) \right)^{\frac{3}{4}} \cdot k^{-\frac{3}{4}}
\]

(21)

2.2 Treating the second sum

We must find an upper bound to the sum

\[
\sum_{n=k+1}^{\infty} \frac{1 - \frac{1}{n^p}}{n^2}
\]

(22)

We can write

\[
k^{\frac{3}{4}} \sum_{n=k+1}^{\infty} \frac{1 - \frac{1}{n^p}}{n^2} = \sum_{n=k+1}^{\infty} \frac{k^{\frac{3}{4}}}{n^{\frac{5}{2}}} \cdot \frac{1 - \frac{1}{n^p}}{n^{\frac{5}{2}}}
\]

(23)
\[
\sum_{n=k+1}^{\infty} \frac{k^{\frac{3}{4}}}{n^{\frac{3}{4}}} \frac{(1 - \frac{1}{n^2})^k}{n^2} < \sum_{n=k+1}^{\infty} \frac{(1 - \frac{1}{n^2})^k}{n^2} < \zeta \left( \frac{5}{4} \right) \tag{24}
\]

where \( \zeta \) is the Riemann Zeta function. Therefore

\[
\sum_{n=k+1}^{\infty} \frac{(1 - \frac{1}{n^2})^k}{n^2} < \zeta \left( \frac{5}{4} \right) \cdot k^{-\frac{3}{4}}. \tag{25}
\]

### 2.3 Putting the two results together

\[
q_k < \left[ \left( \sum_{n=2}^{k} \frac{1}{n^2} \right)^{1/4} \cdot \left( \frac{3}{4} \right)^{\frac{3}{4}} \Gamma \left( \frac{2}{3} \right) + \zeta \left( \frac{5}{4} \right) \right] \cdot k^{-\frac{3}{4}} \tag{26}
\]

or

\[
q_k < \left[ \left( \frac{3}{4} \right)^{\frac{3}{4}} \Gamma \left( \frac{2}{3} \right) + \zeta \left( \frac{5}{4} \right) \right] \cdot k^{-\frac{3}{4}} \tag{27}
\]

where \( \delta(c) = 0 \) for \( C = 3 \). Consequently \( q_k = O(k^{-\frac{3}{4}}) \) or in alternative notation \( q_k << k^{-\frac{3}{4}} \). By Báez theorem RH is true and the zeroes are simple.

### 3 Conclusion

After the efforts of several mathematicians and scientific disseminators [3], the problem has reached maturity and can be solved.

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References

