QUANTUM THEORY OF GRAVITY:

A NEW FORMULATION OF THE GUPTA-FEYNMAN BASED QUANTUM FIELD THEORY OF EINSTEIN GRAVITY

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Abstract

In this manuscript we do the Quantum Field Theory (QFT) of Einstein's Gravity (EG) based on the developments previously made by Suraj N. Gupta and Richard P. Feynman, using a new and more general mathematical theory based on Ultrahyperfunctions [1]
Ultrahyperfunctions (UHF) are the generalization and extension to the complex plane of Schwartz 'tempered distributions. This manuscript is an **application** to Einstein's Gravity (EG) of the mathematical theory developed by Bollini et al [2, 3, 4, 5] and continued for more than 25 years by one of the authors of this paper. A simplified version of these results was given in [6] and, based on them, (restricted to Lorentz Invariant distributions) QFT of EG [7] was obtained. We will quantize EG using the **most general quantization approach**, the Schwinger-Feynman variational principle [8], which is more appropriate and rigorous than the popular functional integral method (FIM). FIM is not applicable here because our Lagrangian contains derivative couplings.
We use the Einstein Lagrangian as obtained by Gupta [9, 10, 11], but we added a new constraint to the theory. Thus the problem of lack of unitarity for the S matrix that appears in the procedures of Gupta and Feynman.
Furthermore, we considerably simplify the handling of constraints, eliminating the need to appeal to ghosts for guarantying unitarity of the theory.
Our theory is obviously non-renormalizable. However, this inconvenience is solved by resorting to the theory developed by Bollini et al. [2, 3, 4, 5, 6]
This theory is based on the thesis of Alexander Grothendieck [12] and on the theory of Ultrahyperfunctions of Jose Sebastiao e Silva [1]
Based on these papers, a complete theory has been constructed for 25 years that is able to quantize non-renormalizable Field Theories (FT).
Because we are using a Gupta-Feynman based EG Lagrangian and to the new mathematical theory we have avoided the use of ghosts, as we have already mentioned, to obtain a unitary QFT of EG

KEYWORDS: Quantum Field Theory; Einstein gravity; Non-renormalizable theories, Unitarity. Ultrahyperfunctions

Contents

1 Introduction

The problem of infinities that appear in a QFT is one of the most important problems that are present in it These infinities emerge when defining the Lagrangians of the QFT's, since the products of fields that arise in them are products of Vector Distributions (VD), or more generally, Vector Ultrahyperfunctions (VUHF) in the quantum case and products of Ultrahyperfunctions (UHF) in the case of the classic QFt's. This was rigorously established by L. Schwartz in two extensive papers published in the Annales del Institut Fourier, [13, 14] In them Schwartz makes an extensive and detailed description of the DVs and shows that the product of two of them is not defined, just like the usual distributions. A VD is a continuous linear functional defined on a space of test functions and that takes values in a Locally Convex Topological Vector Space (LCTVS) The appearance of those product it is what produces the appearance of the infinities in the Lagrangeans of the QFT's and these infinities are propagated throughout the resulting theory. In particular in the product of propagators in the phase space, or in its convolution in the momentum space.

More than 25 years ago one of the authors of this manuscript, together with C. G. Bollini, worked to solve this problem using a new mathematical theory: the theory of Ultrahyperfunctions [1]. It was resolved in 4 extensive papers published in IJTP, [2, 3, 4, 5] through the development of a new mathematical theory: the Ultrahyperfunctions convolution theory. The explanation for the use of Ultrahyperfunctions instead of VU is based on the fact that L. Schwartz proved in [13, 14] that the products of VD are completely determined if the product of the corresponding distributions over the same test function space is known To construct this theory Schwartz was based on the theory developed by A. Grothendieck in his thesis [12]

Ultrahyperfunctions are the generalization and extension to the complex plane of the usual distributions defined by L. Schwartz and I. M. Guelfand and are originally known as Ultradistributions of J. Sebastiao e Silva, since they were defined and studied by this extraordinary Portuguese mathematician in an extensive paper published in Mathematische Annalen [1] Once the convolution of Ultrahyperfunctions is known, the convolution of distributions is immediately known. Having managed to define a convolution of Ultrahyperfunctions, the infinities of the QFT's do not appear, and thus they are now finite, it is not necessary to regularize the integrals that appear in them, and, furthermore, it is not necessary to renormalize said theories.

To quantize a non-renormalizable QFT is to find an appropriate product of distributions (a product in a ring with zero divisors in the configuration space) an old problem of functional analysis successfully solved in [2, 3, 4, 5, 6].

At the same time, we keep all existing solutions in the problem of running coupling constants and the renormalization group. With that convolution the UHF space is transformed into a ring with zero-divisors. In it, one has now defined a product between the ring-elements. Thus, any unitary-causal-Lorentz invariant theory quantified in such a manner becomes predictive. **The distinction between renormalizable on non-renormalizable QFT's becomes unnecessary now**.

In our work we do not use counter-terms to remove infinities from the theory bf because our convolutions are always finite. it Also we don't need to use counter terms, since a non-renormalizable theory involves an infinite number of them.

With our convolution, that uses Laurent's expansions (LE) in the parameter employed to define the LE, all finite constants of the convolutions become completely determined, eliminating arbitrary choices of finite constants. The independent term in the Laurent expansion yield the convolution value.

Until now, the attempts to do a QFT of Einstein's Gravity, failed because the quantization of the theory was carried out in: 1) In a Hilbert space with undefined metric; 2) The theory obtained was not unitary; 3) It was not known how to treat non-renormalizable QFTs.

The only problem with the Ultrahyperfunctions theory is that it turns out to be extremely complex mathematically. In a first attempt to apply our theory we achieved a QFT of EG just considering Lorentz Invariant tempered distributions [7] through a simplified version of the UHF convolution [6]. In this manuscript we have managed to make a general QFT of EG, using the theory of UHF to full.

To achieve this we have resorted to the QFT of EG developed by Suraj N. Gupta [9, 10, 11] with a choice of an additional constraint, making a theory similar to that of Quantum Electrodynamics. As a result, we obtain a QFT of EG that is finite and unitary to all perturbative order This was attempted without success first by Gupta and then by Feynman, in his Acta Physica Polonica work [15].
The manuscript is organized as follows:

- In Section 2 we present the preliminary material needed in this paper.
- In Section 3 we present Einstein's Lagrangean used in this theory
- In Section 4 we quantize the theory
- In Section 5 the graviton's self-energy is evaluated up to second order.
- In Section 6 we introduce axiones into our theory and deal with the axions-gravitons interaction.
- In Section 7 we calculate the graviton's self-energy in the presence of axions.
- In Section 8 we evaluate, up to second order, the axion's self-energy.
- Section 9 is dedicated to the conclusions of this work
- In Appendix A we discuss the convolution of Ultrahyperfunctions
- In Appendix B we obtain a mathematical formula used in this paper.

2 Preliminary Materials

In this paper we will not use the functional integral to quantify the gravitational field for two reasons: 1) It does not serve to treat Ultrahyperfunctions, since it cannot take into account the singularities that said Ultrahyperfunctions have in a strip that surrounds the real axis. 2) The interacting Lagrangean has derivative couplings of the graviton field. Instead we will use the most general method of quantization known, which is the Variational Schwinger-Feynman Method [8] which is able to deal even with high order supersymmetric theories, as exemplified by [16, 17]. Such theories can not be quantized with the usual Dirac-brackets technique.

For that purpose, we write the action for a set of fields in the form:

$$\boldsymbol{\mathcal{S}}[\sigma(x),\sigma_0,\phi_A(x)]=\int\limits_{\sigma_0}^{\sigma(x)}\mathcal{L}[\phi_A(\xi),\partial_\mu\phi_A(\xi),\xi]d\xi, \tag{2.1}$$

where $\sigma(x)$ if a space-like surface passing through the point x. σ_0 is a surface at the remote past, at which all field variations vanish. The Schwinger-Feynman variational principle establishes that:

"Any Hermitian infinitesimal variation $\delta\boldsymbol{\mathcal{S}}$ of the action induces a canonical transformation of the vector space in which the quantum system is defined, and the generator of this transformation is this same operator $\delta\boldsymbol{\mathcal{S}}$".

As a consequence of this statement we obtain [8]:

$$\delta\phi_A = i[\delta\boldsymbol{\mathcal{S}}, \phi_A]. \tag{2.2}$$

Thus, for a Poincare transformation we have

$$\delta\boldsymbol{\mathcal{S}} = a^\mu \boldsymbol{\mathcal{P}}_\mu + \frac{1}{2}a^{\mu v}\boldsymbol{\mathcal{M}}_{\mu v}, \tag{2.3}$$

Therefore, the variation of the field is given by:

$$\delta\phi_a = a^\mu \hat{P}_\mu \phi_A + \frac{1}{2}a^{\mu v}\hat{M}_{\mu v}\phi_A. \tag{2.4}$$

From (2.2),(2.3) and (2.4) we obtain

$$\partial_\mu \phi_A = i[\boldsymbol{\mathcal{P}}_\mu, \phi_A]. \tag{2.5}$$

In particular $\mu = 0$ we have:

$$\partial_0 \phi_A = i[\boldsymbol{\mathcal{P}}_0, \phi_A]. \tag{2.6}$$

This result is used to quantize the QFT's. In particular we will use it to quantize the EG. .

3 The Gupta-Feynman based Lagrangian of Einstein's QFT

According to Gupta, the Lagrangian of EG is given by [9, 10, 11]:

$$\mathcal{L}_G = \frac{1}{\kappa^2}\boldsymbol{R}\sqrt{|g|} - \frac{1}{2}\eta_{\mu v}\partial_\alpha h^{\mu\alpha}\partial_\beta h^{v\beta}, \tag{3.1}$$

where $\eta^{\mu\nu} = diag(1,1,1,-1)$, $h^{\mu\nu} = \sqrt{|g|}g^{\mu\nu}$ The effect of the second term in (3.1) is to fix the gauge. We effect now the linear approximation

$$h^{\mu v} = \eta^{\mu v} + \kappa\phi^{\mu v}, \tag{3.2}$$

where κ^2 is the gravitation's constant and $\phi^{\mu v}$ the graviton field. We write then:

$$\mathcal{L}_G = \mathcal{L}_L + \mathcal{L}_I, \tag{3.3}$$

where

$$\mathcal{L}_L = -\frac{1}{4}\left[\partial_\lambda \phi_{\mu v}\partial^\lambda \phi^{\mu v} - 2\partial_\alpha \phi_{\mu\beta}\partial^\beta \phi^{\mu\alpha} + 2\partial^\alpha \phi_{\mu\alpha}\partial_\beta \phi^{\mu\beta}\right] \tag{3.4}$$

and, up to 2nd order, one has [9, 10, 11]:

$$\mathcal{L}_I = -\frac{1}{2}\kappa\phi^{\mu v}\left[\frac{1}{2}\partial_\mu \phi^{\lambda\rho}\partial_v \phi_{\lambda\rho} + \partial_\lambda \phi_{\mu\beta}\partial^\beta \phi^\lambda_v - \partial_\lambda \phi_{\mu\rho}\partial^\lambda \phi^\rho_v\right] \tag{3.5}$$

where we have made use of the constraint:

$$\phi^\mu_\mu = 0. \tag{3.6}$$

This constraint is required in order to satisfy gauge invariance [18] As a consequence, the equation of motion of the graviton is given by:

$$\Box\phi_{\mu v} = 0, \tag{3.7}$$

The solution of the previous equation is given by:

$$\phi_{\mu v} = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \left[\frac{a_{\mu v}(\vec{k})}{\sqrt{2k_0}}e^{ik_\mu x^\mu} + \frac{a^+_{\mu v}(\vec{k})}{\sqrt{2k_0}}e^{-ik_\mu x^\mu}\right] d^3k, \tag{3.8}$$

with $k_0 = |\vec{k}|$.

4 The correct quantization of the theory

We need remember some usual definitions. The energy-momentum tensor is given by

$$T_\rho^\lambda = \frac{\partial \mathcal{L}}{\partial \partial^\rho \phi^{\mu v}} \partial^\lambda \phi^{\mu v} - \delta_\rho^\lambda \mathcal{L}, \tag{4.1}$$

From this definition we obtain the time-component of the four-momentum vector

$$\boldsymbol{\mathcal{P}}_0 = \int T_0^0 \; d^3x. \tag{4.2}$$

Using the expression (3.4) of the Lagrangian of the free fields we obtain:

$$T_0^0 = \frac{1}{4}\left[\partial_0 \phi_{\mu v} \partial^0 \phi^{\mu v} + \partial_j \phi_{\mu v} \partial^j \phi^{\mu v} - 2\partial_\alpha \phi_{\mu 0} \partial^0 \phi^{\mu\alpha} - 2\partial_\alpha \phi_{\mu j} \partial^j \phi^{\mu\alpha} + \right.$$

$$\left. 2\partial_\alpha \phi^{\mu\alpha} \partial_0 \phi_\mu^0 + 2\partial_\alpha \phi^{\mu\alpha} \partial_j \phi_\mu^j \right]. \tag{4.3}$$

Consequently, from this last equation we arrive at:

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{4} \int |\vec{k}| \left[a_{\mu v}(\vec{k}) a^{+\mu v}(\vec{k}) + a^{+\mu v}(\vec{k}) a_{\mu v}(\vec{k})\right] d^3k. \tag{4.4}$$

We now use the equation (2.6) and we have

$$[\boldsymbol{\mathcal{P}}_0, a_{\mu v}(\vec{k})] = -k_0 a_{\mu v}(\vec{k})$$

$$[\boldsymbol{\mathcal{P}}_0, a^{+\mu v}(\vec{k})] = k_0 a^{+\mu v}(\vec{k}). \tag{4.5}$$

Replacing (4.4) im (4.5) we obtain at the integral equation:

$$|\vec{k}| a^{+\rho\lambda}(\vec{k}') = \frac{1}{2} \int |\vec{k}| [a_{\mu v}(\vec{k}), a^{+\rho\lambda}(\vec{k}')] a^{+\mu v}(\vec{k}) \; d^3k. \tag{4.6}$$

The solution of this equation is

$$\left[a_{\mu v}(\vec{k}), a^{+\rho\lambda}(\vec{k}')\right] = \left[\delta_\mu^\rho \delta_v^\lambda + \delta_v^\rho \delta_\mu^\lambda\right] \delta(\vec{k} - \vec{k}'). \tag{4.7}$$

As customary, in the Gupta quantization for the graviton, the physical state $|\psi>$ of the theory is defined via the equation

$$\phi_\mu^\mu |\psi >= 0. \tag{4.8}$$

We use now the the usual definition for the graviton's propagator

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) = < 0|T[\phi_{\mu\nu}(x)\phi^{\rho\lambda}(y)]|0> . \tag{4.9}$$

Thus the propagator then turns out to be

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) = \frac{i}{(2\pi)^4}(\delta_\mu^\rho \delta_v^\lambda + \delta_v^\rho \delta_\mu^\lambda) \int \frac{e^{ik_\mu(x^\mu - y^\mu)}}{k^2 - i0} d^4k. \tag{4.10}$$

Usando (4.4) we can write:

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{4} \int |\vec{k}| \left[a_{\mu v}(\vec{k}) a^{+\mu v}(\vec{k}') + a^{+\mu v}(\vec{k}') a_{\mu v}(\vec{k})\right] \delta(\vec{k} - \vec{k}') d^3k d^3k', \tag{4.11}$$

According (4.7) we get:

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{4} \int |\vec{k}| \left[2a^{+\mu v}(\vec{k}') a_{\mu v}(\vec{k}) + \delta(\vec{k} - \vec{k}')\right] \delta(\vec{k} - \vec{k}') d^3k d^3k'. \tag{4.12}$$

We then obtain:

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{2}\int |\vec{k}| a^{+\mu v}(\vec{k}) a_{\mu v}(\vec{k}) d^3k, \tag{4.13}$$

Here where we have used the fact that the product of two deltas with the same argument vanishes [2], i.e., $\delta(\vec{k}-\vec{k}')\delta(\vec{k}-\vec{k}')=0$. This proves that using Ultrahyperfunctions is here equivalent to adopting the normal order in the definition of the time-component of the four-momentum

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{4}\int |\vec{k}| : \left[a_{\mu v}(\vec{k}) a^{+\mu v}(\vec{k}) + a^{+\mu v}(\vec{k}) a_{\mu v}(\vec{k})\right] : d^3k. \tag{4.14}$$

Now, we must insist on the fact that the physical state should satisfy not only Eq. (4.8) but also the relation (see [9, 10, 11])

$$\partial_\mu \phi^{\mu v} |\psi >= 0. \tag{4.15}$$

The resulting theory is similar to that obtained for QED, using the Guppta-Bleuler quantization method. This show that the obtained heory is unitary for any finite perturbative order. If we take into account the degrees of freedom of the theory, we conclude that we have only one type of free graviton ϕ^{12}.Thus we have only one type of graviton with two possible transverse polarizations. Obviously, this happens for a non-interacting theory, as remarked by Gupta.

4.1 Loss of unitarity if our constraint is not used

If we do NOT use the new constraint (4.8), we have

$$\boldsymbol{\mathcal{P}}_0 = \frac{1}{2}\int |\vec{k}| \left[a^{+\mu v}(\vec{k}) a_{\mu v}(\vec{k}) - \frac{1}{2} a^{+\mu}_{\mu}(\vec{k}) a^{v}_{v}(\vec{k})\right] d^3k, \tag{4.16}$$

The Feynman-Schwinger variational principle [8] now leads us to:

$$|\vec{k}| a^{+}_{\rho\lambda}(\vec{k}') =$$

$$\frac{1}{2}\int |\vec{k}| \left\{ a^{+\mu v}(\vec{k})[a_{\mu v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')] - \frac{1}{2} a^{+\mu}_{\mu}(\vec{k})[a^{v}_{v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')]\right\} d^3k, \tag{4.17}$$

The solution of this integral equation is now given by:

$$\left[a_{\mu v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')\right] = \left[\eta_{\mu\rho}\eta_{v\lambda} + \eta_{v\rho}\eta_{\mu\lambda} - \eta_{\mu v}\eta_{\rho\lambda}\right]\delta(\vec{k}-\vec{k}'). \tag{4.18}$$

The above is the usual graviton's quantification.The resulting theory leads to a S matrix that is not unitary [9, 10, 11, 15].

5 The exact self energy of the graviton

To evaluate the self-energy (SE) of the graviton, we make use of the generalized Feynman parameters. This is:

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta))}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^{\alpha+\beta}} dx \tag{5.1}$$

We now make use of the interaction Hamiltonian $\mathcal{H}_I$. Note that the Lagrangian contains derivative interaction terms.

$$\mathcal{H}_I = \frac{\partial \mathcal{L}_I}{\partial \partial^0 \phi^{\mu\nu}} \partial^0 \phi^{\mu\nu} - \mathcal{L}_I. \tag{5.2}$$

A typical SE term has the form:

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1}. \tag{5.3}$$

where $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$

$$A = (p-k)^2 - i0 \quad ; \quad \alpha = 1-\lambda$$

$$B = \rho - i0 \quad ; \quad \beta = 1-\lambda$$

n As we already said, to evaluate the integral, we use the Feynman parameters. After a Wick rotation we obtain

$$k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{1-\lambda} * k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{1-\lambda} =$$

$$i\int_0^1 x^{-\lambda}(1-x)^{-\lambda}dx\times$$

$$\frac{\Gamma(2-2\lambda)}{\Gamma^2(1-\lambda)}\int \frac{(p_{\alpha_1}-k_{\alpha_1})(p_{\alpha_2}-k_{\alpha_2})p_{\alpha_3}p_{\alpha_4}}{[(p-kx)^2+a]^{2-2\lambda}}d^4p \tag{5.4}$$

Here we have:

$$a = k^2x(1-x) \tag{5.5}$$

After the variables-change $u = p - kx$ we find

$$k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{\lambda-1} * k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{\lambda-1}. =$$

$$i\int_0^1 x^{-\lambda}(1-x)^{-\lambda}dx\times$$

$$\frac{\Gamma(2-2\lambda)}{\Gamma^2(1-\lambda)}\int \frac{f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,x,u)}{(u^2+a)^{2-2\lambda}}d^4p \tag{5.6}$$

where f has the form:

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,x,u) = \frac{1}{24}[\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}]u^4 + \frac{1}{4}[\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}(1-x)^2+$$

$$\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}x(x-1)+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}x(x-1)+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}x(x-1)+$$

$$\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3}x(x-1)+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}(1-x)^2]u^2 + k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}x^2(x-1)^2 \tag{5.7}$$

5.1 Self-Energy evaluation for $\lambda = 0$

To evaluate SE we must do the Laurent expansion of the preceding result around $\lambda = 0$, according to (A.4) of Appendix A. We obtain like this:

$$k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{\lambda-1} * k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{\lambda-1} =$$

$$-i\frac{\pi^2}{5!\lambda}\{6(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\rho^2 -$$

$$[6(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}) -$$

$$4(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+$$

$$\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})]\,\rho + 2k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\} -$$

$$\frac{i6\pi^2}{5!}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\left[\ln\rho^2 - \frac{137}{30}\right]\rho^2+$$

$$i\frac{\pi^2}{5!}\left\{\frac{3}{2}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln\rho^2 - \frac{56}{15}\right] -\right.$$

$$(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln\rho^2 - \frac{97}{30}\right]\right\}\rho-$$

$$i\frac{2\pi^2}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln\rho^2-\frac{47}{30}\right]+\sum_{n=1}^{\infty}a_n\lambda^n\Bigg\}. \tag{5.8}$$

The exact value of the convolution is the independent term of λ in (5.8). So we get

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)=k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{-1}=-$$

$$\frac{i6\pi^2}{5!}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\left[\ln\rho^2-\frac{137}{30}\right]\rho^2+$$

$$i\frac{\pi^2}{5!}\left\{\frac{3}{2}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln\rho^2-\frac{56}{15}\right]-\right.$$

$$(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+$$

$$\left.\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln\rho^2-\frac{97}{30}\right]\right\}\rho-$$

$$i\frac{2\pi^2}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln\rho^2-\frac{47}{30}\right]\Bigg\}. \tag{5.9}$$

We have to deal with 1296 diagrams of this kind.

6 Including Axions into the theory

In 1977 Peccei and Quinn postulated a hypothetical elementary particle to solve the strong CP problem in quantum chromodynamics. [19] They called that particle axion. It should have a low enough mass (within a certain range). As the Dark Matter theory evolved, several experts concluded that the axion could be a candidate for a component of dark matter. It is for this reason that we have included axions in our theory. Thus we have now axions interacting with the graviton. The Lagrangian becomes

$$\mathcal{L}_{GM}=\frac{1}{\kappa^2}\boldsymbol{R}\sqrt{|g|}-\frac{1}{2}\eta_{\mu v}\partial_\alpha h^{\mu\alpha}\partial_\beta h^{v\beta}-\frac{1}{2}\left[h^{\mu v}\partial_\mu\phi\partial_v\phi+m^2\phi^2\right]. \tag{6.1}$$

The complete Lagrangian now has the form::

$$\mathcal{L}_{GM}=\mathcal{L}_L+\mathcal{L}_I+\mathcal{L}_{LM}+\mathcal{L}_{IM}, \tag{6.2}$$

where

$$\mathcal{L}_{LM}=-\frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi+m^2\phi^2\right], \tag{6.3}$$

so that $\mathcal{L}_{IM}$ becomes the interaction Lagrangian for the axion-graviton action

$$\mathcal{L}_{IM}=-\frac{1}{2}\kappa\phi^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{6.4}$$

The new term in the interaction Hamiltonian is

$$\mathcal{H}_{IM}=\frac{\partial\mathcal{L}_{IM}}{\partial\partial^0\phi}\partial^0\phi-\mathcal{L}_{IM}. \tag{6.5}$$

7 The complete Self Energy of the Graviton

To evaluate the complete SE, we again resort to generalized Feynman parameters, only in this case the calculation is more complex.

$$\frac{1}{A^\alpha B^\beta C^\gamma D^\delta}=\frac{\Gamma(\alpha+\beta+\gamma+\delta))}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}\times$$

$$\int_0^1\int_0^1\int_0^1\frac{x^{\alpha-1}(1-x)^{\beta-1}x_1^{\alpha+\beta-1}(1-x_1)^{\gamma-1}x_2^{\alpha+\beta+\gamma-1}(1-x_2)^{\delta-1}}{\{\{[Ax+B(1-x)]x_1+C(1-x_1)\}x_2+D(1-x_2)\}^{\alpha+\beta+\gamma+\delta}}dx\;dx_1\;dx_2 \tag{7.1}$$

donde:

$$A=(p-k)^2+m^2-i0 \quad ; \quad \alpha=1$$
$$B=(p-k)^2-i0 \quad ; \quad \beta=-\lambda$$
$$C=\rho+m^2-i0 \quad ; \quad \gamma=1$$
$$D=\rho-i0 \quad ; \quad \delta=-\lambda$$

The new contribution to the SE of the graviton due to the presence of the axions is given by:

$$\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k)=k_{\alpha_1}k_{\alpha_2}(\rho+m^2-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho+m^2-i0)^{-1}. \tag{7.2}$$

After a Wick rotation we obtain

$$k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{\lambda}(\rho+m^2-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{\lambda}(\rho+m^2-i0)^{-1}.=$$
$$i\int_0^1\int_0^1\int_0^1(1-x)^{-\lambda-1}x_1^{-\lambda}x_2^{1-\lambda}(1-x_2)^{-\lambda-1}dx\,dx_1\,dx_2\times$$
$$\frac{\Gamma(2-2\lambda)}{\Gamma^2(-\lambda)}\int\frac{p_\mu p_r(k_v-p_v)(k_s-p_s)}{[(p-kx_1x_2)^2+a]^{2-2\lambda}}d^4p \tag{7.3}$$

where

$$a=k^2x_1x_2(1-x_1x_2)+m^2.(xx_1x_2+x_2-x_1x_2) \tag{7.4}$$

After the variables-change $u=p-kx_1x_2$ we find

$$k_{\alpha_1}k_{\alpha_2}\rho^{\lambda}(\rho+m^2-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}\rho^{\lambda}(\rho+m^2-i0)^{-1}.=$$
$$i\int_0^1\int_0^1\int_0^1(1-x)^{-\lambda-1}x_1^{-\lambda}x_2^{1-\lambda}(1-x_2)^{-\lambda-1}dx\,dx_1\,dx_2\times$$
$$\frac{\Gamma(2-2\lambda)}{\Gamma^2(-\lambda)}\int\frac{f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,x_1,x_2,u)}{(u^2+a)^{2-2\lambda}}d^4p \tag{7.5}$$

where f is given by:

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4,x_1,x_2,u)=\frac{1}{24}[\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}]u^4+\frac{1}{4}[\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}(1-x_1x_2)^2+$$
$$\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}x_1x_2(x_1x_2-1)+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}x_1x_2(x_1x_2-1)+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}x_1x_2(x_1x_2-1)+$$
$$\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3}x_1x_2(x_1x_2-1)+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}(1-x_1x_2)^2]u^2+k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}(x_1x_2)^2(x_1x_2-1)^2 \tag{7.6}$$

Evaluating the first integral in p and x we obtain for example:

$$\frac{i\pi^2}{4}[\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}]\times$$
$$\frac{\Gamma(-2-2\lambda)}{\Gamma(1-\lambda)\Gamma(-\lambda)}\int_0^1\int_0^{x_1}x_1^{-3-\lambda}y^{3+\lambda}(x-y)^{-1-\lambda}[k^2x_1(1-y)+m^2]^{2+2\lambda}$$
$$F\left(-2-2\lambda,-\lambda;1-\lambda;\frac{m^2x_1}{k^2x_1(1-y)+m^2}\right)dx_1dy \tag{7.7}$$

Since the integral is convergent at $\lambda=0$ using our theory, which partly uses Guelfand's regularization, we obtain:

$$\frac{i\pi^2}{64}[\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}]\times$$

$$\int_0^1 \int_0^{x_1} x_1^{-3} y^3 (x-y)^{-1} [k^2 x_1(1-y) + m^2]^{2+2\lambda} dx_1 dy \tag{7.8}$$

When evaluating this last integral we have:

$$\frac{-i\pi^2}{64} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \left(\frac{5}{2}\rho^2 + 4m^2\rho + \frac{9}{4}m^4\right) \tag{7.9}$$

The other integrals are calculated in a similar way. The end result is:

$$k_\mu k_r(\rho + m^2 - i0)^{-1} * k_v k_s(\rho + m^2 - i0)^{-1} =$$
$$\frac{-i\pi^2}{64} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \left(\frac{5}{2}\rho^2 + 4m^2\rho + \frac{9}{4}m^4\right) +$$
$$\frac{1\pi^2}{8} [\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2}] \left(\frac{41}{400}\rho - \frac{3}{2}m^2\right) +$$
$$\frac{i\pi^2}{8} [\eta_{\alpha_1\alpha_3} k_{\alpha_2} k_{\alpha_4} + \eta_{\alpha_1\alpha_4} k_{\alpha_2} k_{\alpha_3} + \eta_{\alpha_2\alpha_3} k_{\alpha_1} k_{\alpha_4} + \eta_{\alpha_2\alpha_4} k_{\alpha_1} k_{\alpha_3}] \left(\frac{103}{900}\rho + \frac{35}{144}m^2\right) \tag{7.10}$$

We have to deal with 9 diagrams of this kind.
Accordingly, our desired self-energy total is a combination of $\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)$ and $\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k)$.

8 Self Energy of the Axion

We now proceed to evaluate the SE of the axion. A typical term of the self-energy is:

$$\Sigma_{vr}(k) = k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}. \tag{8.1}$$

In four dimensions one has

$$p_{\alpha_1} p_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}] = \int \frac{p_{\alpha_1} p_{\alpha_2}}{(p^2 + m^2 - i0)[(p-k)^2 - i0]} d^4p. \tag{8.2}$$

With the Feynman generalized parameters used above we obtain

$$k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} (\rho - i0)^{\lambda} * (\rho - i0)^{\lambda - 1} =$$
$$i \frac{\Gamma(2-2\lambda)}{\Gamma(-\lambda)\Gamma(1-\lambda)} \int_0^1 (1-x)^{-1-\lambda} x_1^{-\lambda} (1-x)^{-\lambda} \int \frac{p_{\alpha_1} p_{\alpha_2}}{[(p - kx_1)^2 + a]^{2-2\lambda}} d^4k dx, \tag{8.3}$$

where

$$a = m^2 x(1 - x_1) + k^2 x_1 (1 - x_1) \tag{8.4}$$

We evaluate the integral (8.3) and find

$$k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1} = \frac{i\eta_{\alpha_1\alpha_2} \pi^2 m^2}{8} \tag{8.5}$$

9 Discussion

In this paper we have performed the Quantum Field Theory of Einstein's gravity using a very advanced mathematical theory: the Lorentz Invariant Ultrahyperfunctions convolution theory.. [2, 3, 4, 5] It is nothing more than having defined a product in a ring with divisors of zero in the configuration space. **This theory is not a regularization method**. It is a theory apt to quantize non-renormalizable QFT's

Since the functional integral is not a suitable mathematical tool to perform the quantization of a theory that contains Ultrahyperfunctions, we have resorted to the more general quantization method for the QFT 's known until now. The variational principle of Feynman and Schwinger

The resulting QFT is finite, unitary, and Lorentz Invariant. As an example of the power of the theory used, we have calculated the SE of the graviton, adding to it the presence of dark matter, represented in this case by axions. It should also be noted that we have added to the QFT of the Gupta-Feynman EG, an additional constraint The addition of this new constraint allows us to make a QFT of the unit EG.

We must clarify that the scarcity of bibliography in this paper is due to the fact that the theory developed in it is completely new.

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A The Convolution of two Lorentz Invariant Ultrahyperfunctions

We clarify that the content of this appendix has been taken from the references [4, 5] in order to simplify the reading of the paper.

In [4] formula (7.34) we have obtained a conceptually simple but rather lengthy expression for the convolution of two Lorentz invariant tempered ultradistributions:

$$H_\lambda(\rho,\Lambda) = \frac{1}{8\pi^2\rho}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2} F(\rho_1)G(\rho_2)\rho_1^\lambda\rho_2^\lambda\left\{\Theta[\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times\right.\right.$$

$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\Bigg\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\Bigg\}\Bigg\}-\frac{i}{2}\times$$

$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-\right.$$

$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)\right]\}$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$

$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_2\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)\right]\}\}\}\; d\rho_1\, d\rho_2 \tag{A.1}$$

This defines an ultradistribution in the variables ρ and Λ for
$|\Im(\rho)| > \Im(\Lambda) > |\Im(\rho_1)| + |\Im(\rho_2)|$
Let $\mathfrak{B}$ be a vertical band contained in the complex λ-plane $\mathfrak{P}$. Integral (A.1) is an analytic function of λ defined in the domain $\mathfrak{B}$. Moreover, it is bounded by a power of $|\rho\Lambda|$. Then, $H_\lambda(\rho,\Lambda)$ can be analytically continued to other parts of $\mathfrak{P}$. Thus, we define

$$H(\rho) = H^{(0)}(\rho, i0^+) \tag{A.2}$$

$$H_\lambda(\rho, i0^+) = \sum_{-m}^{\infty} H^{(n)}(\rho, i0^+)\lambda^n \tag{A.3}$$

As in the other cases, we define now

$$\{F * G\}(\rho) = H(\rho) \tag{A.4}$$

as the convolution of two Lorentz invariant tempered ultradistributions.

Alternatively we can use the formula obtained in [5], formula (10.1) for Ultrahyperfunctions of exponential type:

$$H_{\gamma\lambda}(\rho,\Lambda)=\frac{1}{8\pi^2\rho}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}[2\cosh(\gamma\rho_1)]^{-\lambda}F(\rho_1)[2\cosh(\gamma\rho_2)]^{-\lambda}G(\rho_2)$$

$$\{\Theta[\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times\right.$$

$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$
$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}\right\}-\frac{i}{2}\times$$
$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$
$$\frac{(\rho_1-\rho_2)}{2}[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-$$
$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2[\ln(-\rho_1-\rho_2+\Lambda)-$$
$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+\right.$$
$$\frac{(\rho_1-\rho_2)}{2}[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-$$
$$\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(-\rho_1-\rho_2+\Lambda)-$$
$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_2[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}\}\}\; d\rho_1\, d\rho_2 \tag{A.5}$$
$$|\Im(\rho)|>\Im(\Lambda)>|\Im(\rho_1)|+|\Im(\rho_2)|\;;\;\gamma<\min\left(\frac{\pi}{2\mid\Im(\rho_1)\mid};\frac{\pi}{2\mid\Im(\rho_2)\mid}\right)$$

We define

$$H(\rho)=H^{(0)}(\rho,i0^+)=H^{(0)}_{\gamma}(\rho,i0^+) \tag{A.6}$$

$$H_{\gamma\lambda}(\rho,i0^+)=\sum_{-m}^{\infty}H^{(n)}_{\gamma}(\rho,i0^+)\lambda^n \tag{A.7}$$

If we take into account that singularities (in the variable Λ) are contained in a horizontal band of width $|\sigma_0|$ we have:

$$H_{\gamma\lambda}(\rho,i0^+)=\sum_{-m}^{\infty}H^{(n)}_{\gamma\lambda}(\rho,i\sigma)\frac{(-i\sigma)^n}{n!}\qquad \sigma>|\sigma_0| \tag{A.8}$$

As in the other cases we define now

$$\{F*G\}(\rho)=H(\rho) \tag{A.9}$$

as the convolution of two Lorentz invariant ultradistributions of exponential type.
Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier antitransform of $H_{\gamma\lambda}(\rho,i0^+)$:

$$\hat{H}_{\gamma\lambda}(x)=\sum_{n=-m}^{\infty}\hat{H}^{(n)}_{\gamma}(x)\lambda^n \tag{A.10}$$

If we define:

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{F_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}F(\rho)\}$$
$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{G_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}G(\rho)\} \tag{A.11}$$

then

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4 \hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x) \tag{A.12}$$

and taking into account the Laurent's developments of $\hat{f}$ and $\hat{g}$:

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}_{\gamma}^{(n)}(x)\lambda^n$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}_{\gamma}^{(n)}(x)\lambda^n \tag{A.13}$$

we can write:

$$\sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \right) \lambda^n \tag{A.14}$$

$(m = m_f + m_g)$
and as a consequence:

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \tag{A.15}$$

The Feynman propagators corresponding to a massless particle F and a massive particle G are, respectively, the following ultrahyperfunctions:

$$F(\rho) = -\Theta[-\Im(\rho)]\rho^{-1}$$
$$G(\rho) = -\Theta[-\Im(\rho)](\rho + m^2)^{-1} \tag{A.16}$$

where ρ is the complex variable, such that on the real axis one has $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$. On the real axis, the previously defined propagators are given by:

$$f(\rho) = F(\rho + i0) - F(\rho - i0) = (\rho - i0)^{-1}$$

$$g(\rho) = G(\rho + i0) - G(\rho - i0) = (\rho + m^2 - i0)^{-1} \tag{A.17}$$

These are the usual expressions for Feynman propagators.

Consider first the convolution of two massless propagators . We use (A.17), since here the corresponding ultrahyperfunctions do not have singularities in the complex plane. We obtain from (A.1) a simplified expression for the convolution:

$$h_\lambda(\rho) = \frac{\pi}{2\rho} \iint_{-\infty}^{\infty} (\rho_1 - i0)^{\lambda-1}(\rho_2 - i0)^{\lambda-1} \left[(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2\right]_+^{\frac{1}{2}} \, d\rho_1 \, d\rho_2 \tag{A.18}$$

This expression is nothing other than the usual convolution:

$$h_\lambda(\rho) = (\rho - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1} \tag{A.19}$$

B A Mathematical Proof

According to the Ultrahyperfunctions theory we can write:

$$\oint_{\Gamma} \ln(a - z)\phi(z)dz = \int_{-\infty}^{\infty} [\ln(a - x - i0) - \ln(a - x + i0)]\phi(x)dx = -2i\pi \int_{-\infty}^{\infty} H(x - a)\phi(x)dx \tag{B.1}$$

So we have the correspondence:

$$-\frac{1}{2\pi i}\ln(a-z) \longleftarrow H(x-a) \tag{B.2}$$

Using now the Dirac formula for Ultrahyperfunctions we obtain:

$$-\frac{1}{2\pi i}\ln(a-z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{H(x-a)}{x-z}dx = \frac{1}{2\pi i}\int\limits_{a}^{\infty}\frac{1}{x-z}dx \tag{B.3}$$

Thus:

$$\ln(a-z) = -\int\limits_{a}^{\infty}\frac{1}{x-z}dx \tag{B.4}$$

We then have for $a > 0$

$$\ln a = -\int\limits_{a}^{\infty}\frac{1}{x}dx \tag{B.5}$$

According to the result obtained by Guelfand in [20]

$$\int\limits_{0}^{\infty}\frac{1}{x}dx = 0 \tag{B.6}$$

And therefore:

$$\ln a = \int\limits_{0}^{a}\frac{1}{x}dx \tag{B.7}$$