Hamiltonian Flow of the Riemann ξ-Function

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Abstract

The Riemann ξ-Function can be expressed as $\xi(s) = u(x,y) + iv(x,y)$ where $s = x + iy$. The structure of a Hamiltonian flow in the critical strip, $0 \leq x \leq 1$, $0 \leq y \leq \infty$ of $\dot{x} = u(x,y)$, $\dot{y} = -v(x,y)$ is determined by its behavior near zeros of ξ(s). Phase portraits are considered and proved that all zeros of the Riemann ξ-Function on the critical line are saddle points.

1. Introduction

This paper is dependent on papers from [1-3]. The Riemann Zeta function ζ(s) is a function of the complex variable s= x + iy, defined in the half plane x > 1 by the absolutely convergent series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1)$$

ζ(s) can be extended by analytical continuation to the whole complex plane, with only a simple pole at s = 1 and trivial zeros at the negative even integers that is ,when s is one of -2, -4, -6, -8 .... ζ(s) has an infinity of zeros on the critical line, x = ½. The Riemann hypothesis is stated that all the non-trivial zeros of the Riemann Zeta function must lie on the critical line, x= ½.

In order to eliminate pole at s = 0 ,1 and all trivial zeros, the ξ-function is formulated as

$$\xi(s) = \frac{1}{2} s(s-1)\frac{\Gamma(\frac{s}{2})\zeta(s)}{\pi^{s/2}} , \qquad (2)$$

which satisfies the functional equation

$$\xi(s) = \xi(1-s), \qquad (3)$$

and has the same zeros as ζ(s) in the critical strip , 0 < x < 1.

$\xi(s)$ is an entire function with real and imaginary parts $u(x,y)$ and $v(x,y)$, thus

$$\xi(x+iy) = u(x,y) + iv(x,y)\,, \tag{4}$$

where $s = x + iy$.

From Eq. (3), relationship of $u(x,y)$, $v(x,y)$ in the critical strip can be stated as:

$$u(x,y) = u(1-x,y),$$
$$v(x,y) = -v(1-x,y). \tag{5}$$

From these symmetries , the following results applying along x= ½, such as

$$v(\tfrac{1}{2},y) = 0,$$
$$\frac{\partial u}{\partial x}(1/2,y) = 0 \tag{6}$$

Since $\xi(s)$ is an analytical function of s, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ,\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{7}$$

2. Phase Portraits of Hamiltonian Systems

The Jacobian matrix of $\dot{x} = u(x,y)$, $\dot{y} = -v(x,y)$ is defined as

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \end{bmatrix} \tag{8}$$

Let $\alpha = \frac{\partial u}{\partial x}$ and $\beta = \frac{\partial u}{\partial y}$. By using relationship from Eq. (7), J can be represented as

$$J = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \tag{9}$$

At zeros of ξ(s) on the critical line, α = 0 and $\beta \neq 0$, then Eigen values of J at zeros of ξ(s) on the critical line are $\pm\beta$ and its Eigen vectors are $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ and $\left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$,respectively. Thus zeros of ξ(s) on the critical line are saddle points as shown in Fig. 1 and Fig. 2 for the first zeros and the second zero at $\rho = \frac{1}{2} + i14.1347$ and $\rho_2 = \frac{1}{2} + i21.0220$, respectively. As shown in [2], the vorticity of Riemann zero on the critical line alternate in sign as one move along it. The first and second Riemann zero has vorticity − and $+$, respectively.

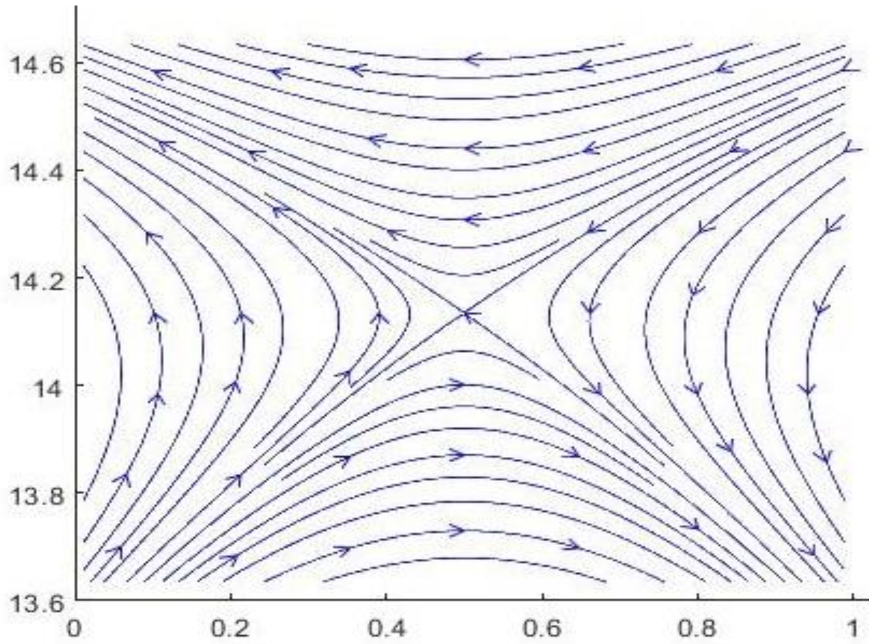

Figure 1. The phase portrait of $\dot{x}$ = u(x,y), $\dot{y}$ = -v(x,y) near $\rho_1 = \frac{1}{2} + i14.1347$

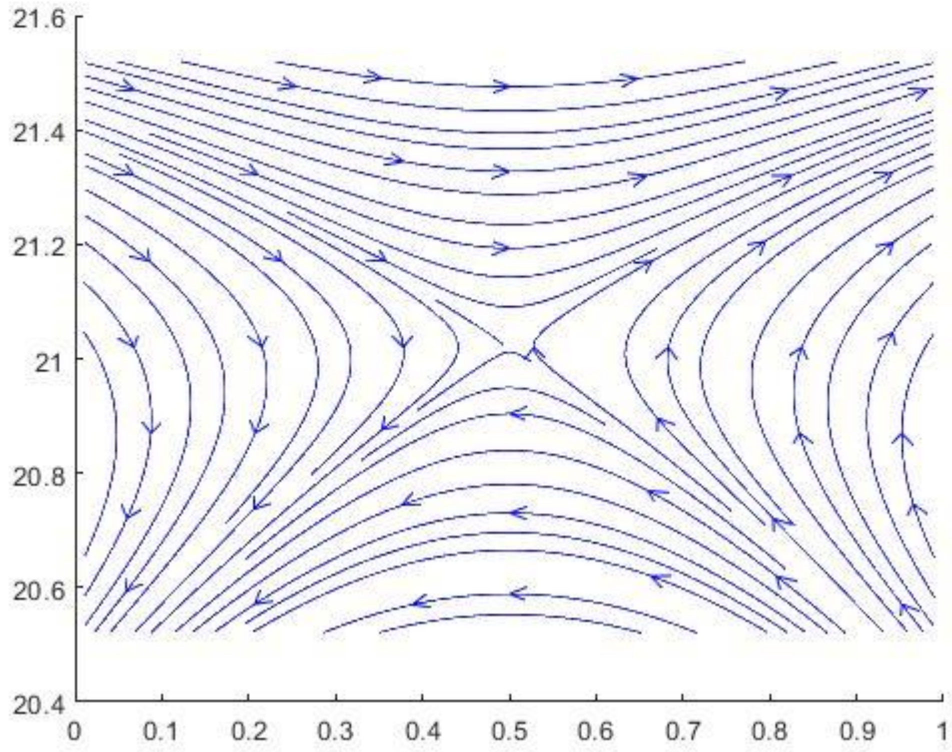

Figure 2. The phase portrait of $\dot{x}$ = u(x,y), $\dot{y}$ = -v(x,y)

near $\rho_2 = \frac{1}{2} + i21.0220$

3. Index Theory of Dynamical Systems and Application to the Critical Strip

Consider a dynamical system in the plane represented by

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y) \qquad (10)$$

Index theory provides global information as compared with local information from linearization about fixed points. To find an index of a closed curve, pick some curve C that does not have a fixed point on it. Let ∅ be the angle that the flow vector on C make w.r.t x-axis and $[\emptyset]_C$ be a net change in ∅ over one counterclockwise of C (in radians). Then the index of the closed C, I_C, defined as

$$I_C = \frac{1}{2\pi}[\emptyset]_C \qquad (11)$$

As shown in [4], the index of a closed curve C encloses a saddle point is -1. By the index theory, the index of a closed curve is additive, that is, when C is sub-divided as

$$C = C_1 + C_2,$$

then

$$I_C = I_{C_1} + I_{C_2} \tag{12}$$

Let consider the Hamiltonian system $\dot{x} = u(x,y), \dot{y} = -v(x,y)$ in the critical strip, $0 \le x \le 1$, $0 \le y \le \infty$. This critical strip can be sub-divided into $R_{i,i+1}$, i= 1,2,…….∞ that index theory can be applied to each subdivision separately.

The first region $R_{1,2}$ is defined as a rectangle with four corners at (1,0), (1,y_{12}), (0, y_{12}) and (0,0), Im(ρ_1) $< y_{12} <$ Im(ρ_2). A path from (1,y_{12}) to (0, y_{12}) does not pass through any zeros of ξ(s).

All other regions $R_{i,i+1}$, I = 2, 3, …..are defined as a rectangle with four corners at (1, $y_{i-1,i}$), (1, $y_{i,i+1}$), (0, $y_{i,i+1}$), and (0, $y_{i-1,i}$), Im(ρ_{i-1}) $< y_{i-1,i} <$ Im(ρ_i) and Im(ρ_i) $< y_{i,i+1} <$ Im(ρ_{i+1}) .

Paths from (1, $y_{i,i+1}$) to (0, $y_{i,i+1}$) and from (0, $y_{i-1,i}$) to (1, $y_{i-1,i}$) do not pass through any zeros of ξ(s).

Let $C_{i,i+1}$ be a closed path along the perimeter of $R_{i,i+1}$ in the counter clockwise direction, (1, $y_{i-1,i}$) → (1, $y_{i,i+1}$) → (0, $y_{i,i+1}$) → (0, $y_{i-1,i}$) →(1, $y_{i-1,i}$). As shown by [2], angles along $C_{i,i+1}$ from (1, $y_{i-1,i}$) to (1, $y_{i,i+1}$) and along $C_{i,i+1}$ from (0, $y_{i,i+1}$) to (0, $y_{i-1,i}$) rotate in the clockwise direction. With clockwise direction of these angles and condition from Eq. (3), the index of $C_{i,i+1}$ must be -1.

For purposes of illustration, the region $R_{2,3}$ is considered. Let $y_{1,2}$ = 16 and $y_{2,3}$ = 22 , A $C_{2,3}$ is a closed path , (1, $y_{1,2}$) → (1, $y_{2,3}$) → (0, $y_{2,3}$) → (0, $y_{1,2}$) → (1, $y_{1,2}$).

Define $\emptyset_1$, $\emptyset_2$ as angles at (1, $y_{1,2}$) and (1, $y_{2,3}$), respectively, one can find that $\emptyset_1$, $\emptyset_2$ are 3.041 radians and 0.018 radians, respectively.

A net angle changed from $(1, y_{1,2}) \rightarrow (1, y_{2,3})$ = -($\emptyset_1$ - $\emptyset_2$),

A net angle changed from $(1, y_{2,3}) \rightarrow (0, y_{2,3})$ = -2$\emptyset_2$,

A net angle changed from $(0, y_{2,3}) \rightarrow (0, y_{1,2})$ = -($\emptyset_1$ - $\emptyset_2$),

A net angle changed from $(0, y_{1,2}) \rightarrow (1, y_{1,2})$ = -2(π-$\emptyset_1$).

Thus, the angle changed = -($\emptyset_1$ - $\emptyset_2$) -2$\emptyset_2$ -($\emptyset_1$ - $\emptyset_2$) -2(π-$\emptyset_1$)= -2π.

Clearly, a net change of angle is -2π. Thus, the index of $C_{2,3}$ is -1.

Conclusions

The Hamiltonian flow of $\dot{x} = u(x,y)$, $\dot{y} = - v(x,y)$ near its critical points is analyzed. Phase portraits are considered and proved that all zeros of the Riemann ξ-Function on the critical line are saddle points. Also by sub-divide the critical strip, index theory can be applied to each subdivision separately. Results indicate that the index of a closed curve around each subdivision is -1.

References

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