Proof that the Center of Buoyancy is Equal to the Center of Pressure by means of the Surface Integral of Hydrostatic Pressure Acting on the Inclined Ship

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Summary

In this paper, we prove that "ship's center of buoyancy is equal to the center of hydrostatic pressure". This subject is an unsolved problem in physics and naval architecture, even though the buoyancy taught by Archimedes' principle (1) can be obtained clearly by the surface integral of hydrostatic pressure. Then we thought that the reason why the vertical position of the center of pressure could not be determined was that the horizontal force would be zero due to equilibrium in the upright state.

As a breakthrough, we dared to assume the left-right asymmetric pressure field by inclining the ship with heel angle θ. In that state, the force and moment due to hydrostatic pressure were calculated correctly with respect to the tilted coordinate system fixed to the floating body. By doing so, we succeeded in determining the center of pressure.

Then, by setting the heel angle θ to zero, it was proved that the center of hydrostatic pressure is equal to the well-known center of buoyancy, *i.e.*, the centroid of the cross-sectional area under the water surface.

Specifically, the above proof is first shown for a rectangular cross-section, and then for an arbitrary shape of floating body by applying Gauss's integral theorem.

Keywords : ***Center of Buoyancy, Hydrostatic Pressure, Inclined Ship, Surface Integral, Archimedes' Principle, Rectangular Section, Arbitrary Shape, Gauss's Integral Theorem***

1. Introduction

It is a well-known fact in naval architecture and physics that the position of "Center of Buoyancy" acting on a ship is equal to the center of the volume of the geometric shape under the water surface.

The buoyancy taught by Archimedes' principle (1) is clearly obtained by the surface integral of the hydrostatic pressure, but the position of the center of buoyancy is described in every textbook (on physics (2), fluid dynamics (3),(4), hydraulics (5), and naval architecture (6),(7),(8),(9), nautical mechanics (10) etc.), as the center of gravity where the volume under the water surface is replaced by water. There is no explanation as the center of pressure due to hydrostatic pressure (11),(12).

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Recently, Komatsu[13] raised the issue of "the center of buoyancy ≠ the center of pressure?" at 2007 in Japan, and it was actively discussed by Seto[14],[15], K. Suzuki[16], Yoshimura and Yasukawa[17], Komatsu[18], Yabushita and Watanabe[19] and others in the research committee and the academic meeting of the Japan Society of Naval Architects and Ocean Engineers (hereinafter abbreviated as *JASNAOE*). At the same time, in Europe, the problem was studied in detail by Mégel and Kliava[20],[21] in terms of potential energy. However, no one was able to solve this issue.

On the other hand, it is also an indisputable fact that the well-known center of buoyancy (*i.e.* the volume center of the underwater portion) is correct from the viewpoint of ship's stability[22],[23] (that is to say, positioning of the metacenter by calculating the metacentric radius[24],[25] $\overline{BM}$).

In response to this unsolved problem, we considered that the reason why the vertical center of pressure could not be determined was because the horizontal forces equilibrated to zero in the upright state. To solve this problem, Hori[26],[27] attempted in 2018 to integrate the hydrostatic pressure acting on the ship surface at the inclined state with heel angle θ. Then, the forces and moments acting on the ship were calculated with respect to a tilted coordinate system fixed to the ship. In this case, both orthogonal components of the force acting on the ship are not zero. Therefore, it was shown that the center of pressure at inclined state can be determined. By setting the heel angle θ to zero, we proved that, the center of pressure coincides with the center of area under the water surface in the upright state, *i.e.*, the well-known center of buoyancy. First, a columnar ship with the rectangular cross section[26] was proved. And then an arbitrary cross-sectional shape[27] was proved and published in the Journal "*Navigation*" of Japan Institute of Navigation (hereinafter abbreviated as *JIN*).

For this problem, Yabushita[28] showed that the center of buoyancy is the center of pressure by tilting the direction of gravity from the vertical direction in his text book. Later, Yabushita *et al.*[29] showed that the same conclusion can be obtained by tilting only the coordinate system, not by tilting the floating body or direction of gravity. Furthermore, K.Suzuki[30] gave a detailed examination of Hori's theory[26]. On the other hand, Komatsu[31] performed an analysis in which only the vertical buoyant component was extracted from the hydrostatic pressure acting on the surface of the laterally inclined floating body, as shown by Hori[26]. As a result, he claimed that the center of action of buoyancy is different from the well-known center of buoyancy. Also, Yabushita[32] et al. attempted an elaborate analysis in terms of the potential energy of buoyancy, which is adopted by Mégel and Kliava[20],[21], and showed that the height of the center of buoyancy is equal to the conventional position of the center of buoyancy. In this way, as many researchers are studying this issue with various approaches, the discussions have deepened in *JASNAOE*.

To sublate these discussions, we have illustrated that "the center of buoyancy is equal to the center of pressure" for semi-submerged cylinders[33], and submerged cylinders[34] which does not change its shape under the water even if it is inclined, and for a for triangular prisms[35], using the same method[36]. If you are interested, please read them.

In order to put an end to the above discussions, we proved that "the center of buoyancy = the center of pressure" for a submerged body with arbitrary shape[37] using Gauss's integral theorem in 2021.

Furthermore, it was published in the same journal "*Navigation*" of *JIN* that it is easier to prove a floating body of arbitrary shape[37] than our previous paper[27] by using Gauss's theorem in the same way[38].

In this paper, we summarize the proofs for the case of the rectangular cross-section[26], which is the easiest to understand, and for the floating body of arbitrary cross-sectional shape [2nd. half of 37] by applying Gauss's integral theorem.

2. Positioning of the center of hydrostatic pressure acting on the floating body with an inclined rectangular cross-section

Fig. 1 shows a two-dimensional rectangular cross-section of width $2b$ and depth $f+h$ (draft f and freeboard h) with a heel angle θ to the starboard side. The origin o is set at the center of the bottom surface, and the coordinate system fixed to the floating body is $o-\eta\zeta$ and the coordinate system fixed to the space is $o-yz$. Here, the z-axis of the latter is directed vertically upwards.

In the figure, atmospheric pressure is shown as a broken line, hydrostatic pressure as a solid line, each pressure as a thin line, and each force as a thick line. All these vectors act perpendicularly to the surface of the floating body.

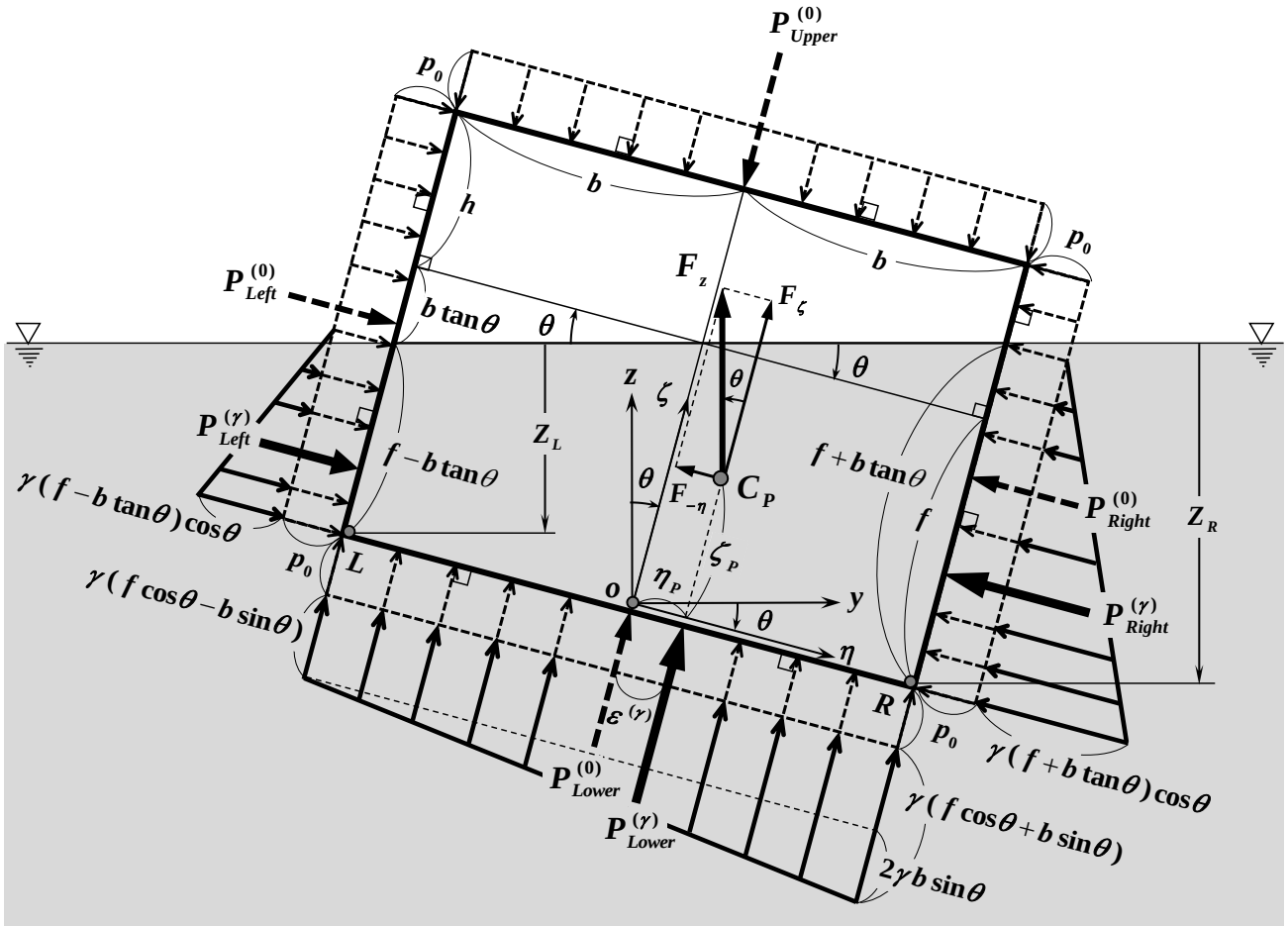

Fig. 1 Hydrostatic pressure and center of pressure acting on the inclined rectangular cross-section.

2.1 Forces due to hydrostatic pressure acting on the surface of an inclined rectangular cross-section

When the floating body is inclined laterally by heel angle θ, the left-right asymmetric pressure field is created. Then, as shown in Fig. 1, the water depths Z_L and Z_R under the still water surface at the bottom points of port L and starboard R are expressed respectively in the form

$$\left.\begin{aligned} Z_L &= (f - b\tan\theta)\cos\theta \\ Z_R &= (f + b\tan\theta)\cos\theta \end{aligned}\right\} \quad \cdots\cdots(1)$$

Let's calculate the forces P_{Left} acting on the port (indicated by the subscript "*Left*") and P_{Right} acting on the starboard (indicated by the subscript "*Right*"). P_{Left} is calculated by superimposing $P_{Left}^{(0)}$, which is obtained by the integrating the uniformly distributed atmospheric pressure acting on the port side, and $P_{Left}^{(\gamma)}$, which is obtained by the integrating the triangularly distributed hydrostatic pressure acting on the submerged area. Similarly, P_{Right} is calculated by superimposing $P_{Right}^{(0)}$ and $P_{Right}^{(\gamma)}$ on the starboard side. Therefore, if the atmospheric pressure is p_0 and the specific gravity of water is γ, the above P_{Left} and P_{Right} can be written respectively as follows :

$$\left.\begin{aligned} P_{Left} &= P_{Left}^{(0)} + P_{Left}^{(\gamma)} \\ &= p_0(f+h) + \frac{1}{2}\gamma Z_L (f - b\tan\theta) \\ &= p_0(f+h) + \frac{1}{2}\gamma (f - b\tan\theta)^2 \cos\theta \\ P_{Right} &= P_{Right}^{(0)} + P_{Right}^{(\gamma)} \\ &= p_0(f+h) + \frac{1}{2}\gamma Z_R (f + b\tan\theta) \\ &= p_0(f+h) + \frac{1}{2}\gamma (f + b\tan\theta)^2 \cos\theta \end{aligned}\right\} \quad \cdots\cdots(2)$$

The force P_{Upper} acting on the upper deck (indicated by the subscript "*Upper*") is obtained only by $P_{Upper}^{(0)}$ due to atmospheric pressure. And the force P_{Lower} acting on the bottom (indicated by the subscript "*Lower*") is obtained by superimposing $P_{Lower}^{(0)}$ due to atmospheric pressure and $P_{Lower}^{(\gamma)}$ due to hydrostatic pressure of trapezoidal distribution. Therefore, each of P_{Upper} and P_{Lower} can be written as follows :

$$\left.\begin{aligned} P_{Upper} &= P_{Upper}^{(0)} \\ &= 2p_0 b \\ P_{Lower} &= P_{Lower}^{(0)} + P_{Lower}^{(\gamma)} \\ &= 2p_0 b + \frac{\gamma Z_L + \gamma Z_R}{2} \cdot 2b \\ &= 2p_0 b + 2\gamma f b \cos\theta \end{aligned}\right\} \quad \cdots\cdots(3)$$

2.2 Forces $F_{-\eta}$ and F_{ζ} combined in the $-\eta$ and ζ direction

The combined forces $F_{-\eta}$ and F_{ζ} acting in the $-\eta$ (in the direction of the negative axis of η) and ζ directions fixed on the floating body can be obtained by using P_{Left} and P_{Right} in Eq. (2) as follows :

$$\left.\begin{aligned} F_{-\eta} &= P_{Right} - P_{Left} = P^{(\gamma)}_{Right} - P^{(\gamma)}_{Left} \\ &= 2\gamma f b \sin\theta \\ F_{\zeta} &= P_{Lower} - P_{Upper} = P^{(\gamma)}_{Lower} \\ &= 2\gamma f b \cos\theta \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

Here, it can be seen that $F_{-\eta}$ is to the leftward, and F_{ζ} is upward. And for the both forces, the atmospheric pressure p_0 is canceled out.

2.3 Forces F_y and F_z converted in the y and z direction

The horizontal component F_y and the vertical component F_z acting on the floating body can be calculated by transforming the coordinates of the both forces $F_{-\eta}$ and F_{ζ} in Eq. (4) as follows :.

$$\left.\begin{aligned} F_y &= F_{\zeta}\sin\theta - F_{-\eta}\cos\theta \\ &= 2\gamma f b(\cos\theta\sin\theta - \sin\theta\cos\theta) = 0 \\ F_z &= F_{\zeta}\cos\theta + F_{-\eta}\sin\theta \\ &= 2\gamma f b(\cos^2\theta + \sin^2\theta) = 2\gamma f b \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5)$$

Here, it can be seen that the horizontal component F_y does not act as a combined force due to pressure integration, even when the floating body is laterally inclined and the pressure field is asymmetric. On the other hand, the vertical component F_z can be written as

$$\begin{aligned} F_z &= \gamma\cdot(2b\cdot f) \\ &= \gamma\cdot(\text{Area of the rectangle under the water surface}) \\ &= \text{Buoyancy} \end{aligned} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

By the above Equation, F_z is the buoyancy exactly as taught by Archimedes' principle[1].

2.4 Moments M_{η} and M_{ζ} due to hydrostatic pressure in the η and ζ directions

First, we calculate the moment M_{η} due to the forces in the η direction. The counterclockwise moment M_{η} around the origin o due to $P^{(0)}_{Right}$, $P^{(0)}_{Left}$ and $P^{(\gamma)}_{Right}$, $P^{(\gamma)}_{Left}$ can be obtained using Eq. (2). As shown in Fig. 1, the former is multiplied by the lever up to the action point of the pressure distributed uniformly, and the latter is multiplied by the lever of the pressure distributed triangularly, so that the moment M_{η} is can be calculated as follows :

$$M_\eta = P^{(0)}_{Rigrt} \cdot \frac{f+h}{2} + P^{(\gamma)}_{Right} \cdot \frac{f+b\tan\theta}{3} - P^{(0)}_{Left} \cdot \frac{f+h}{2} - P^{(\gamma)}_{Left} \cdot \frac{f-b\tan\theta}{3}$$

$$= \gamma b \sin\theta \left(f^2 + \frac{b^2}{3}\tan^2\theta \right) \quad \cdots\cdots(7)$$

Here, the terms for atmospheric pressure p_0 is canceled out, as in the case of the forces in Eq. (2).

Next, let us consider calculating the moment M_ζ due to the forces in the ζ direction. To do this, we need to find the distance $\varepsilon^{(\gamma)}$ from origin o to the action point of $P^{(\gamma)}_{Lower}$. Here, the hydrostatic pressure of the trapezoidal distribution of acting on the bottom surface is decomposed into the uniform distribution and the triangular distribution. Since only the pressure of the triangular distribution contributes to the moment around origin o shown in Fig. 1, the distance $\varepsilon^{(\gamma)}$ can be determined by using Eq. (3) as

$$\varepsilon^{(\gamma)} = \frac{2\gamma b^2 \sin\theta \cdot \left(b - \frac{2b}{3} \right)}{P^{(\gamma)}_{Lower}} = \frac{b^2}{3f}\tan\theta \quad \cdots\cdots(8)$$

Therefore, the counterclockwise moment M_ζ around the origin o due to the forces $P^{(0)}_{Lower}$, $P^{(\gamma)}_{Lower}$ and $P^{(0)}_{Upper}$ acting in the ζ direction can also be calculated as follows :

$$M_\zeta = P^{(0)}_{Lower} \times 0 + P^{(\gamma)}_{Lower} \cdot \varepsilon^{(\gamma)} - P^{(0)}_{Upper} \times 0$$

$$= P^{(\gamma)}_{Lower} \cdot \varepsilon^{(\gamma)} = \frac{2}{3}\gamma b^3 \sin\theta \quad \cdots\cdots(9)$$

As a result, M_ζ is obtained as the numerator in Eq. (8) and, like M_η in Eq. (7), does not depend on p_0.

2.5 Positioning of center of hydrostatic pressure C_P of rectangular cross-section

Consider the determination of the position of the center of hydrostatic pressure C_P acting on the floating body with rectangular cross-section.

The counterclockwise moments M_η and M_ζ about origin o calculated in the previous section can be written by the combined forces $F_{-\eta}$ and F_ζ acting on $C_P(\xi_P, \eta_P)$, based on the hydraulic method used by Ohgushi[7], as follows :

$$\left.\begin{aligned} M_\eta &= F_{-\eta} \cdot \zeta_P \\ M_\zeta &= F_\zeta \cdot \eta_P \end{aligned}\right\} \quad \cdots\cdots(10)$$

Therefore, the distances η_P and ζ_P in the η and ζ directions from the origin o to the center of pressure C_P can be determined respectively by using $F_{-\eta}$, F_ζ of Eq. (4) in Section 2.2 and M_η, M_ζ of Eq. (7), (9) in Section 2.4, as follows :

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$$\left.\begin{aligned}
\eta_P &= \frac{M_\zeta}{F_\zeta} \\
&= \frac{P_{Lower}^{(\gamma)} \cdot \varepsilon^{(\gamma)}}{P_{Lower}^{(\gamma)}} = \varepsilon^{(\gamma)} = \frac{b^2}{3f}\tan\theta \quad \left(= \eta_G \right]_{Eq.(A-5)} \Big) \\
\zeta_P &= \frac{M_\eta}{F_{-\eta}} \\
&= \frac{\gamma b \sin\theta \left(f^2 + \frac{b^2}{3}\tan^2\theta \right)}{2\gamma f b \sin\theta} \\
&= \frac{f}{2} + \frac{b^2}{6f}\tan^2\theta \quad \left(= \zeta_G \right]_{Eq.(A-5)} \Big)
\end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (11)$$

As shown in the appendix A-1, this result (η_P, ζ_P) is coincide with the result (η_G, ζ_G) of Eq. (A-5), in which the centroid of the trapezoidal region under the water surface was geometrically determined by calculating the area moment. Hence, it is correct and equal to the well-known position of the center of buoyancy.

Then, the specific weight γ of water have been cancelled out in the denominator and numerator of Eq. (11) respectively. And, η_P is obtained as the force point $\varepsilon^{(\gamma)}$ calculated by Eq. (8), on which $P_{Lower}^{(\gamma)}$ acts.

Here, it should be noted that that the position ζ_P of the center of pressure in the ζ - direction could be determined because the zero factor $\sin\theta$ at the heel angle $\theta \to 0$ was offset in the denominator and numerator, as shown in the 2nd. part of Eq. (11). If we start and calculate as an upright state $\theta = 0$, both the denominator $F_{-\eta}$ and the numerator M_η are zero in equilibrium, so the fraction will be indeterminate forms and ζ_P cannot be determined.

To clarify this result, let's determine the pressure center in the upright state by setting the heel angle to $\theta \to 0$. Then, since the $\eta\zeta$ - coordinates tilted and fixed on the floating body coincide with the yz - coordinates fixed in space, the Eq. (11) becomes as

$$\left.\begin{aligned}
(\eta_P, \zeta_P)\big]_{\theta\to 0} &= (y_P, z_P) = \left(0, \frac{f}{2}\right) \\
\therefore\ C_P &= B
\end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (12)$$

Here, it can be obtained that the center of pressure is equal to rectangular centroid. This proves that the center of pressure C_P due to hydrostatic pressure coincides with the well-known "Center of Buoyancy, B".

3. Positioning of the center of hydrostatic pressure Acting on the inclined floating body with an arbitrary form

In this chapter, we apply the same method as used in the previous chapter, in which a rectangular shape is inclined laterally, to the floating body with the arbitrary shape. It is shown that the position of the center of pressure can be easily determined by integrating the hydrostatic pressure using Gauss's integral theorem than our previous paper[27].

Fig. 2 shows a transverse section of an arbitrarily shaped floating body with a heel angle θ to the starboard side. The origin o is placed in the center of the still water surface, and the coordinate system fixed to the floating body and tilted is $o-\eta\zeta$, and the coordinate system fixed to space is $o-yz$. Here, the

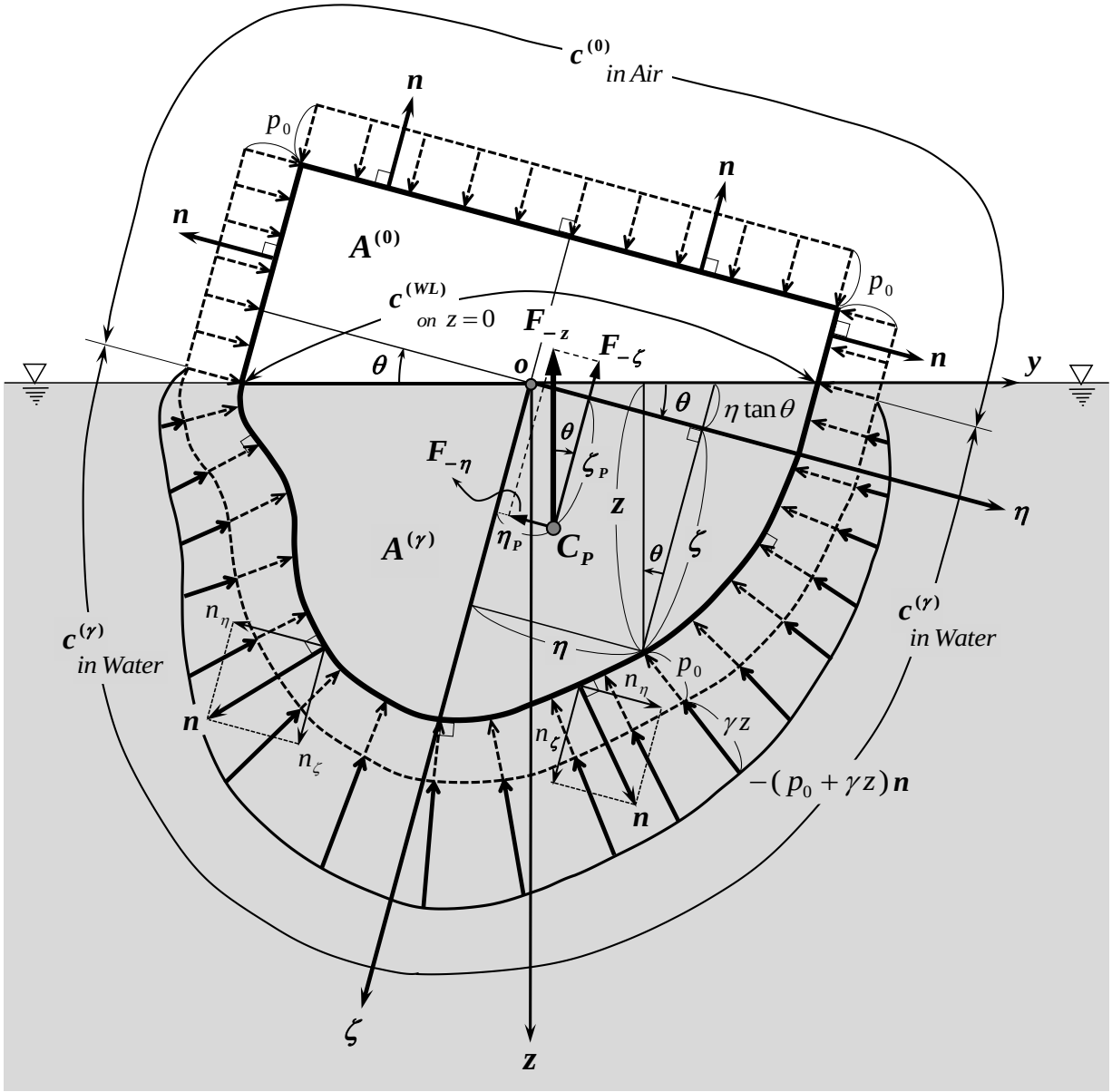

Fig. 2 Hydrostatic pressure and center of pressure acting on the inclined floating body with arbitrary form.

z - axis of the latter is vertically downwards, and the opposite direction to that of Fig. 1. Also, the outward unit normal vector standing on the surface of the floating body is $\boldsymbol{n} = n_\eta \boldsymbol{j} + n_\zeta \boldsymbol{k}$, and n_η and n_ζ are the directional cosines of the floating body fixed in the η and ζ directions, respectively.

In the figure, the atmospheric pressure is shown as a broken line, the hydrostatic pressure as a solid line, same as in Fig. 1. And all of the vectors act in the $-\boldsymbol{n}$ direction perpendicular to the floating body surface.

As shown in Fig. 2, the water depth z on the surface (η, ζ) of the floating body is written as follows :

$$\begin{aligned} z(\eta, \zeta) &= (\zeta + \eta \tan\theta)\cos\theta \\ &= \zeta \cos\theta + \eta \sin\theta \end{aligned} \quad \cdots\cdots\cdots\cdots (13)$$

Here, as in Chapter 2, if the atmospheric pressure is written as p_0 and the specific weight of water is written as γ , the hydrostatic pressure $p(\eta, \zeta)$ can be obtained as

$$p(\eta, \zeta) = p_0 + \gamma z(\eta, \zeta) \quad \cdots\cdots\cdots\cdots (14)$$

3.1 $-\eta$ directional component $F_{-\eta}$ and $-\zeta$ directional component $F_{-\zeta}$ of the total force due to hydrostatic pressure acting on the floating body

The $-\eta$ directional component $F_{-\eta}$ and the $-\zeta$ directional component $F_{-\zeta}$ of the total force acting on the floating body surface can be obtained by integrating the η and ζ components of the hydrostatic pressure p in Eq. (14). Here, the integral path is written as $c^{(0)}$ for the aerial part of the floating body and $c^{(\gamma)}$ for the underwater part, as shown in Fig. 2. Then, $F_{-\eta}$ and $F_{-\zeta}$ are calculated by the sum of the integrals respectively as follows :

$$\left.\begin{aligned} F_{-\eta} &= \int_{c^{(0)}} p_0 n_\eta \, d\ell + \int_{c^{(\gamma)}} (p_0 + \gamma z) n_\eta \, d\ell \\ &= \oint_{c^{(0)} + c^{(\gamma)}} p_0 n_\eta \, d\ell + \gamma \int_{c^{(\gamma)}} z n_\eta \, d\ell \\ F_{-\zeta} &= \int_{c^{(0)}} p_0 n_\zeta \, d\ell + \int_{c^{(\gamma)}} (p_0 + \gamma z) n_\zeta \, d\ell \\ &= \oint_{c^{(0)} + c^{(\gamma)}} p_0 n_\zeta \, d\ell + \gamma \int_{c^{(\gamma)}} z n_\zeta \, d\ell \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (15)$$

Both results are obtained by summing the line integral over the entire circumference of the floating body $c^{(0)} + c^{(\gamma)}$ for p_0 and the line integral over the underwater surface of the floating body $c^{(\gamma)}$ for z .

Here, since $z = 0$ on the still water surface (y −axis), the equality relation is not broken even if the integral term for the path $c^{(WL)}$ on the still water surface is added to the second term, as shown in Fig. 2. As a result, it can be expressed as a contour integral of $c^{(\gamma)} + c^{(WL)}$ under the water surface. Therefore, $F_{-\eta}$ and $F_{-\zeta}$ can be written as the sum of the contour integral of the two paths, respectively, as follows :

$$\left.\begin{aligned} F_{-\eta} &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 n_\eta \, d\ell \\ &\quad + \gamma \int_{c^{(\gamma)}} z n_\eta \, d\ell + \gamma \int_{c^{(WL)}} z n_\eta \, d\ell \\ &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 n_\eta \, d\ell + \gamma \oint_{c^{(\gamma)}+c^{(WL)}} z n_\eta \, d\ell \\ F_{-\zeta} &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 n_\zeta \, d\ell \\ &\quad + \gamma \int_{c^{(\gamma)}} z n_\zeta \, d\ell + \gamma \int_{c^{(WL)}} z n_\zeta \, d\ell \\ &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 n_\zeta \, d\ell + \gamma \oint_{c^{(\gamma)}+c^{(WL)}} z n_\zeta \, d\ell \end{aligned}\right\} \quad \cdots\cdots(16)$$

Therefore, the following two-dimensional (η-ζ plane) Gauss' integral theorem, in which n_η and n_ζ are the directional cosines of the outward unit normal vector in η and ζ directions, can be applied to the contour integrals of the above Eq. (16), respectively.

$$\left.\begin{aligned} \oint_c u(\eta,\zeta)\, n_\eta \, d\ell &= \iint_A \frac{\partial u}{\partial \eta} \, dA \\ \oint_c v(\eta,\zeta)\, n_\zeta \, d\ell &= \iint_A \frac{\partial v}{\partial \zeta} \, dA \end{aligned}\right\} \quad \cdots\cdots(17)$$

Then, both $F_{-\eta}$ and $F_{-\zeta}$ can be converted to the area integral, in which the area of the aerial part of the floating body is denoted as $A^{(0)}$ and the area of the underwater part as $A^{(\gamma)}$. As a result of the calculation, both forces can be expressed only in terms of the area integral of underwater $A^{(\gamma)}$, as follows :

$$\left.\begin{aligned} F_{-\eta} &= \iint_{A^{(0)}+A^{(\gamma)}} \frac{\partial p_0}{\partial \eta} \, dA + \gamma \iint_{A^{(\gamma)}} \frac{\partial z}{\partial \eta} \, dA \\ &= \gamma \iint_{A^{(\gamma)}} \frac{\partial z}{\partial \eta} \, dA \\ F_{-z} &= \iint_{A^{(0)}+A^{(\gamma)}} \frac{\partial p_0}{\partial \zeta} \, dA + \gamma \iint_{A^{(\gamma)}} \frac{\partial z}{\partial \zeta} \, dA \\ &= \gamma \iint_{A^{(\gamma)}} \frac{\partial z}{\partial \zeta} \, dA \end{aligned}\right\} \quad \cdots\cdots(18)$$

This is the result of finding that the area integral with respect to p_0 in the 1st. term of the above equation vanishes because the integrand becomes zero.

Furthermore, using Eq. (13) for water depth z, the both forces $F_{-\eta}$ and $F_{-\zeta}$ in Eq. (18) can be calculated as follows. Then, each of the 1st. term of integrand for $A^{(\gamma)}$ in the following equation will become to zero and vanish.

$$\left.\begin{aligned} F_{-\eta} &= \gamma \iint_{A^{(\gamma)}} \frac{\partial(\zeta \cos\theta + \eta \sin\theta)}{\partial\eta}\, dA \\ &= \gamma \sin\theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \sin\theta \\ F_{-\zeta} &= \gamma \iint_{A^{(\gamma)}} \frac{\partial(\eta \sin\theta + \zeta \cos\theta)}{\partial\zeta}\, dA \\ &= \gamma \cos\theta \iint_{A^{(\gamma)}} dA = \gamma A^{(\gamma)} \cos\theta \end{aligned}\right\} \cdots\cdots (19)$$

It can be seen that both are determined by the area $A^{(\gamma)}$ of the floating body under the still water surface and the heel angle θ , and do not depend on the atmospheric pressure p_0.

In addition, according to the results of Eq. (20) in the next section, $F_{-\eta}$ and $F_{-\zeta}$ are obtained as $-\eta$ and $-\zeta$ directional components of the buoyancy F_{-z} acting vertically upward, respectively.

3.2 Forces F_y and F_{-z} converted in y and $-z$ directions

The horizontal component (y direction) F_y and the vertical component ($-z$ direction) F_{-z} acting on the floating body can be obtained by transforming the coordinates of the both forces $F_{-\eta}$ and $F_{-\zeta}$ in Eq. (19) of the previous section, as follows :

$$\left.\begin{aligned} F_y &= F_{-\zeta} \sin\theta - F_{-\eta} \cos\theta \\ &= \gamma A^{(\gamma)} (\cos\theta \cdot \sin\theta - \sin\theta \cdot \cos\theta) \\ &= 0 \\ F_{-z} &= F_{-\zeta} \cos\theta + F_{-\eta} \sin\theta \\ &= \gamma A^{(\gamma)} (\cos^2\theta + \sin^2\theta) \\ &= \gamma A^{(\gamma)} = \textit{Buoyancy} \end{aligned}\right\} \cdots\cdots (20)$$

Here, it can be seen that the horizontal component F_y does not act as a combined force due to pressure integration, even when the floating body is laterally inclined and asymmetric. On the other hand, the vertical component is the product of the specific weight γ of water and the cross-sectional area $A^{(\gamma)}$ of the floating body under the water surface, and is the buoyancy itself that generates vertically upward, as taught by Archimedes' principle[1]. This situation is similar to Eq. (5) for rectangular cross sections in Chapter 2.

3.3 Moments M_η and M_ζ due to hydrostatic pressure in the η and ζ directions

In this section, we shall calculate the total counterclockwise moment M_o around the origin o due to hydrostatic pressure acting on the surface of the floating body. It can be calculated by superimposing the clockwise moment M_η due to the pressure component in the direction $-\eta$ and the counterclockwise moment M_ζ due to the pressure component in the direction $-\zeta$, based on the hydraulic method used by Ohgushi[7] for an example of the rolling gate, as follows :

$$M_o = -M_\eta + M_\zeta \quad \cdots\cdots(21)$$

M_η and M_ζ can be obtained by multiplying the integrand in Eq. (16) by ζ or η as the moment lever, respectively, in the form

$$\left.\begin{aligned} M_\eta &= \int_{c^{(0)}} p_0 \zeta n_\eta \, d\ell + \int_{c^{(\gamma)}} (p_0 + \gamma z)\, \zeta \cdot n_\eta \, d\ell \\ M_z &= \int_{c^{(0)}} p_0 \eta n_\zeta \, d\ell + \int_{c^{(\gamma)}} (p_0 + \gamma z)\, \eta \cdot n_\zeta \, d\ell \end{aligned}\right\} \quad \cdots\cdots(22)$$

Now, as in the case of forces $F_{-\eta}$ and $F_{-\zeta}$ in Eq. (16), let's connect the path $c^{(0)}$ and $c^{(\gamma)}$ with respect to p_0 and add a term for the path $c^{(WL)}$ on the still water surface with respect to z where the integral value become zero as shown in Fig. 2. Then, M_η and M_ζ can be expressed as the sum of the contour integrals of the two paths, respectively, as follows :

$$\left.\begin{aligned} M_\eta &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 \zeta n_\eta \, d\ell \\ &\quad + \gamma \int_{c^{(\gamma)}} z \zeta n_\eta \, d\ell + \gamma \int_{c^{(WL)}} z \zeta n_\eta \, d\ell \\ &= p_0 \oint_{c^{(0)}+c^{(\gamma)}} \zeta n_\eta \, d\ell + \gamma \oint_{c^{(\gamma)}+c^{(WL)}} z \zeta n_\eta \, d\ell \\ M_\zeta &= \oint_{c^{(0)}+c^{(\gamma)}} p_0 \eta n_\zeta \, d\ell \\ &\quad + \gamma \int_{c^{(\gamma)}} z \eta n_\zeta \, d\ell + \gamma \int_{c^{(WL)}} z \eta n_\zeta \, d\ell \\ &= p_0 \oint_{c^{(0)}+c^{(\gamma)}} \eta n_\zeta \, d\ell + \gamma \oint_{c^{(\gamma)}+c^{(WL)}} z \eta n_\zeta \, d\ell \end{aligned}\right\} \quad \cdots\cdots(23)$$

Therefore, we can apply Gauss's integral theorem in Eq. (17) to the above contour integrals, as in the case of forces $F_{-\eta}$ and $F_{-\zeta}$ in Section 3.1, and convert them into area integrals. Furthermore, using Eq. (13) for the water depth z, the moments M_η and M_ζ in Eq. (23) can be written, respectively, as follows :

$$\left.\begin{aligned} M_\eta &= p_0 \iint_{A^{(0)}+A^{(\gamma)}} \frac{\partial \zeta}{\partial \eta} \, dA + \gamma \iint_{A^{(\gamma)}} \frac{\partial (z\zeta)}{\partial \eta} \, dA \\ &= \gamma \iint_{A^{(\gamma)}} \frac{\partial (\zeta^2 \cos\theta + \eta \zeta \sin\theta)}{\partial \eta} \, dA \\ &= \gamma \sin\theta \iint_{A^{(\gamma)}} \zeta \, dA \\ M_\zeta &= p_0 \iint_{A^{(0)}+A^{(\gamma)}} \frac{\partial \eta}{\partial \zeta} \, dA + \gamma \iint_{A^{(\gamma)}} \frac{\partial (z\eta)}{\partial \zeta} \, dA \\ &= \gamma \iint_{A^{(\gamma)}} \frac{\partial (\eta^2 \sin\theta + \zeta \eta \cos\theta)}{\partial \zeta} \, dA \\ &= \gamma \cos\theta \iint_{A^{(\gamma)}} \eta \, dA \end{aligned}\right\} \quad \cdots\cdots(24)$$

Here, both moments are proportional to the area moments of the submerged area $A^{(\gamma)}$ of the floating body

about the η - axis or ζ - axis, respectively. This is the result that integrands in the terms for p_0 and the 1st. term for $A^{(\gamma)}$ in the above equations became to zero and vanished.

3.4 Positioning of center of hydrostatic pressure C_P of the floating body with arbitrary form

Since the forces $F_{-\eta}$ and $F_{-\zeta}$ due to the hydrostatic pressure obtained in Section 3.1 act on the pressure center $C_P\,(\eta_P, \zeta_P)$, the clockwise moments M_η and counterclockwise M_ζ obtained in Section 3.3 can be expressed respectively, as follows :

$$\left.\begin{aligned} M_\eta &= F_{-\eta}\,\zeta_P \\ M_\zeta &= F_{-\zeta}\,\eta_P \end{aligned}\right\} \quad \cdots\cdots(25)$$

Here, the total counterclockwise moment M_o around the origin o in Eq. (21) can be calculated as

$$M_o = -F_{-\eta}\,\zeta_P + F_{-\zeta}\,\eta_P \quad \cdots\cdots(26)$$

Then, the moment M_{C_P} around the point C_P at which $F_{-\eta}$ and $F_{-\zeta}$ act is computed as follows, and becomes to zero.

$$M_{C_P} = -F_{-\eta} \times 0 + F_{-\zeta} \times 0 = 0 \quad \cdots\cdots(27)$$

This correctly indicates that C_P is the center of pressure due to hydrostatic pressure.

Therefore, the coordinate (η_P, ζ_P) of this center of pressure C_P can be determined by Eq. (25). Here, the η - coordinate, η_P, can be determined by using the 2nd. part of Eq. (19) for $F_{-\zeta}$ and the 2nd. part of Eq. (24) for M_ζ, as follows :

$$\begin{aligned} \eta_P &= \frac{M_\zeta}{F_{-\zeta}} = \frac{\gamma \cos\theta \iint_{A^{(\gamma)}} \eta\, dA}{\gamma A^{(\gamma)} \cos\theta} \\ &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \eta\, dA \;\left(= \eta_G\right) \end{aligned} \quad \cdots\cdots(28)$$

Further, the ζ - coordinate, ζ_P, can be determined by using the 1st. part of Eq.(19) for $F_{-\eta}$ and the 1st. part of Eq.(24) for M_η, as follows :

$$\begin{aligned} \zeta_P &= \frac{M_\eta}{F_{-\eta}} = \frac{\gamma \sin\theta \iint_{A^{(\gamma)}} \zeta\, dA}{\gamma A^{(\gamma)} \sin\theta} \\ &= \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} \zeta\, dA \;\left(= \zeta_G\right) \end{aligned} \quad \cdots\cdots(29)$$

As a result, both the specific weight γ of water and the heel angle θ have been cancelled out in the denominator and numerator respectively, so that η_P and ζ_P are obtained in the following simple geometrical format. It divides the area-moment about the ζ - axis and the area-moment about the η - axis, respectively, by the area $A^{(\gamma)}$ of the submerged portion. This shows that the center of pressure (η_P, ζ_P) of

the floating body in the inclined state clearly coincides with the centroid (η_G, ζ_G) of the submerged area $A^{(\gamma)}$, that is, the well-known center of buoyancy.

Considering the above, ζ_P of vertical component can be obtained by offsetting the zero factor $\sin\theta$ at the heel angle $\theta \to 0$ with the denominator and numerator, as shown in Eq. (29). Here, if we start and calculate as the upright state $\theta = 0$, both the denominator $F_{-\eta}$ and the numerator M_{η} are in equilibrium and become zero, so the fraction becomes indeterminate forms and ζ_P cannot be determined. This is the reason why we were able to determine the position of the center of pressure in the ζ direction as $\zeta_P = \zeta_G$ by inclining the floating body laterally.

On the other hand, in the calculation of η_P in Eq. (28), even if the heel angle is $\theta = 0$ from the beginning, the denominator $F_{-\zeta}$ takes a finite value as the cosine component of the buoyancy. Therefore, the horizontal component η_P can be determined as $\eta_P = \eta_G$, if we start and calculate as the upright state.

These situations described above are exactly the same as in Eq. (11) of Section 2.5 for a rectangular cross section

As a final step, let's find the center of pressure in the upright state by setting the heel angle to $\theta \to 0$, in order to make this result clearer. Then, since the $\eta\zeta$-coordinates tilted and fixed on the floating body coincide with the yz-coordinates fixed in space, the Eq. (28) and Eq. (29) become as

$$\left.\begin{aligned} (y_P,\ z_P) &= \left(\frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} y\, dA\ ,\ \frac{1}{A^{(\gamma)}} \iint_{A^{(\gamma)}} z\, dA \right) = (y_G,\ z_G) \\ \therefore\ C_P &= B \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (30)$$

Therefore, this proves that the center of pressure C_P due to hydrostatic pressure coincides with the well-known "Center of Buoyancy, B".

In addition, the reason why the consequence of z_P shown in Eq. (30) could be derived more easily than the author's previous paper[27] is that Gauss's integral theorem was applied to an inclined $o-\eta\zeta$ coordinate system fixed to a floating body.

4. Conclusions

In this paper, we elucidated an unsolved problem in physics and naval architecture by proving that "the center of hydrostatic pressure is equal to the well-known ship's center of buoyancy".

To solve this problem, we dared to assume the left-right asymmetric pressure field by inclining the ship. In that state, the force and moment due to hydrostatic pressure were calculated correctly with respect to the tilted coordinate system fixed to the floating body. By doing so, we succeeded in determining the center of hydrostatic pressure. Finally, by setting the heel angle to zero, the result of the upright state was obtained and the proof was clarified.

As for the shape of the floating body, the simplest rectangular cross-section was proved first, and then the arbitrary cross-sectional shape was proved by applying Gauss's integral theorem.

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Appendix

A-1. Centroid of the trapezoidal area, which is the underwater sectional shape

In this appendix, the centroid of trapezoidal area, which is the cross-sectional shape under the water surface when inclined laterally, is geometrically obtained from the area moment.

As shown in Fig. A-1, let's analyze in an inclined $o-\eta\zeta$ coordinate system with the origin o at the center of the bottom of the floating body and fixed to the body. This is the same coordinate system as Fig. 1 in Chapter 2. Here, the draft of upright state is f, the half-width is b, and the heel angle is θ.

Then, we consider that the trapezoidal region under the water is divided into a rectangle (centroid g_1) and a triangle (centroid g_2) by a single dotted line.

If the area of the rectangular part is A_1 and the area of the triangular part is A_2, each of them and their sum can be obtained as

$$\left.\begin{aligned} A_1 &= 2b\,(f - b\tan\theta) \\ A_2 &= \frac{1}{2}\cdot 2b\cdot 2b\tan\theta = 2b^2\tan\theta \\ A_1 + A_2 &= 2bf \end{aligned}\right\} \cdots\cdots \text{(A-1)}$$

First, we calculate the area moment M'_η about the η-axis. Here, dashes are added to distinguish them

from the moments caused by forces shown in Chapters 2 and 3. Then, M'_η can be calculated as

$$\begin{aligned} M'_\eta &= A_1 \times \frac{f - b\tan\theta}{2} \\ &\quad + A_2 \times \left\{ (f - b\tan\theta) + \frac{2b\tan\theta}{3} \right\} \\ &= b f^2 + \frac{1}{3} b^3 \tan^2\theta \end{aligned} \qquad \text{(A-2)}$$

Next, the area moment M'_ζ about the ζ -axis can be calculated as

$$\begin{aligned} M'_\zeta &= A_1 \times 0 + A_2 \times \left(b - \frac{2b}{3} \right) = A_2 \times \frac{b}{3} \\ &= \frac{2}{3} b^3 \tan\theta \end{aligned} \qquad \text{(A-3)}$$

If the coordinate of the centroid position G of the trapezoid is (η_G, ζ_G), the above area moments M'_η and M'_ζ can be written as the product of the total area and the lever, respectively, as follows :

$$\left. \begin{aligned} M'_\eta &= (A_1 + A_2)\,\zeta_G \\ M'_\zeta &= (A_1 + A_2)\,\eta_G \end{aligned} \right\} \qquad \text{(A-4)}$$

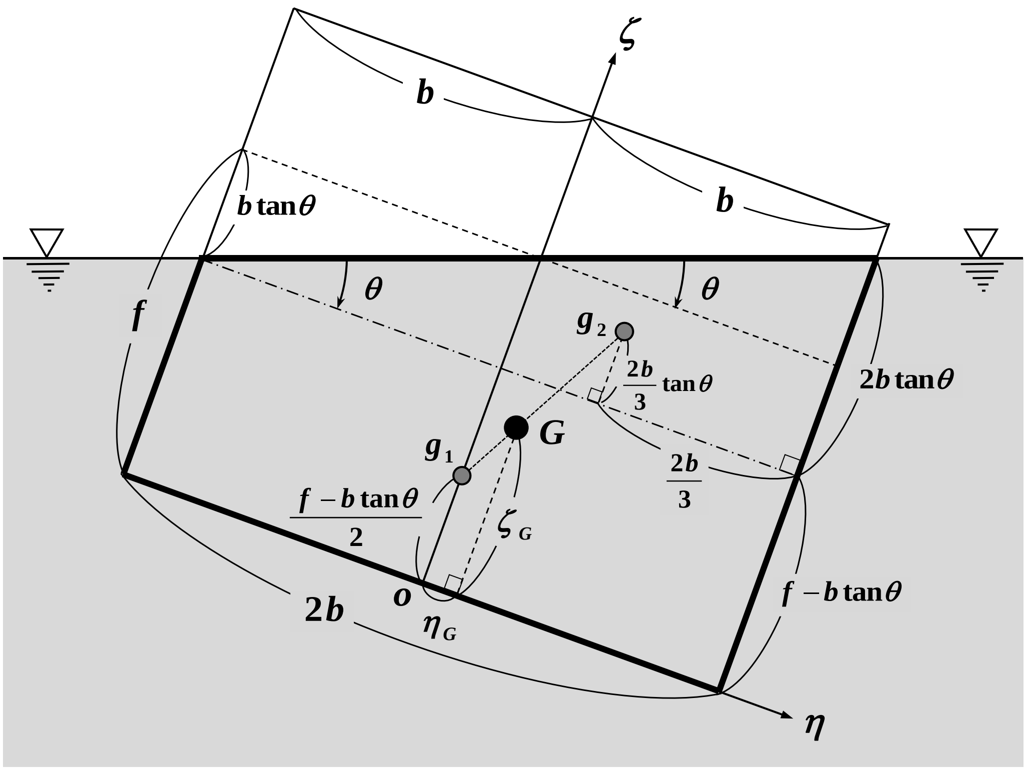

Fig. A-1 Centroid G of area of the underwater trapezoid.

Therefore, the coordinates η_G and ζ_G of the centroid G of area can be calculated and determined as follows :

$$\left.\begin{aligned} \eta_G &= \frac{M'_\zeta}{A_1+A_2} = \frac{b^2}{3f}\tan\theta \\ \zeta_G &= \frac{M'_\eta}{A_1+A_2} = \frac{f}{2}+\frac{b^2}{6f}\tan^2\theta \end{aligned}\right\} \cdots\cdots\cdots \text{(A-5)}$$

Here, g_1, g_2 and G in Fig. A-1 are drawn the correct positions in this state, and the three points are on the same straight line.

A-2. Introduction of lecture videos uploaded to YouTube

The content of Chapter 2, which proves that "Center of Buoyancy = Center of Pressure" by inclining a floating body of rectangular cross-section laterally, is lectured to second-year students of the naval architecture course(41) at the "Stability of the Ship" of the university where the author works.

With the recent trend of remote lectures, the situation is shot in two parts, the 1st. half(42) and the 2nd. half(43), and on-demand teaching materials are created and uploaded as YouTube videos. The explanation is in Japanese, but if you are interested, please have a look.