No Formal System Containing Sets Arithmetic and Relations between the Rational Numbers is Consistent

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Abstract: We present two arguments that are contradictory. These arguments can be developed in any formal system containing sets, arithmetic and relations between the rational numbers.

Introduction. For all rational numbers $a$ in the open interval $(0, 1)$ let the collection of all $R_a = \{ y \text{ a rational number } | 0 \leq y < a \}$

Each set in the collection of $R_a$ contains a largest element.
Select a single $R_a$ taken from the collection of all $R_a$.

Create $S_a$ a group of all proper subsets of our selected $R_a$ taken from the collection of all $R_a$. Since the sets of $S_a$ are nested within each other, each set of $S_a$ contains all the elements of the sets beneath it in the nested set hierarchy. Yet no set in $S_a$ contains all the elements of our selected $R_a$. Since the union of all the sets in $S_a$ cannot contain an element that is larger than all the elements of the individual sets in the $S_a$ nested set hierarchy, our selected $R_a$ contains an element that is not in the union of the sets in $S_a$.

For any two elements of the selected $R_a$ the smaller element will be in a set contained in $S_a$. Thus, each $R_a$ must contain a single largest element that is not contained in any set of $S_a$.

No Set in the collection of $R_a$ has a largest element.

Suppose there is a largest element $a'$ in $R_a$.
$a' < (a + a')/2 < a$. Let $b = (a + a')/2$. Then $b$ is in $R_a$ and $a' < b$. 

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