Harmonisation of Classical Wave Equation

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Abstract

In this short letter we present a technique using which one attributes frequency and wavevector to (almost) arbitrary scalar fields. Our proposed definition is then applied to the classical wave equation to yield a novel nonlinear PDE.

Keywords— harmonic analysis, nonlinear partial differential equation

Harmonic waves posses a special significance in the foundations of modern physics as the notions of frequency and wavevector which are characteristics of harmonic waves appear *explicitly* in Einstein-Planck relation $E = \hbar\omega$ and de Broglie hypothesis $\mathbf{p} = \hbar\mathbf{k}$. Being the cornerstones of Fourier analysis, importance of harmonic waves is not limited to physics and the study of such waves is a lively field of mathematical research to the extent that *harmonic analysis* is a major branch of analysis.

The monumental importance of harmonic waves in foundations of quantum mechanics suggests that they must be thought of as being *more* than mere Fourier transform variables. Indeed a strong reading of $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$ suggests that all waves are harmonic and if one is not it must be *enforced* to become a harmonic wave. This is the maxim that we follow in this letter: we propose a definition using which one can attribute frequency and wavevector to all sufficiently smooth non-zero scalar fields. To motivate our definition we start from the simplest case of a forward-in-time[1] harmonic wave $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$

$$\phi(\mathbf{x}, t) = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)};$$

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[1] Basically the reason for this assumption is the notion of *arrow of time*.

now observe that

$$\frac{\partial \phi}{\partial t} = -i\omega\phi$$

and

$$\nabla\phi = i\mathbf{k}\phi.$$

The *prima facie* approach that has been thoroughly pursued both in quantum physics and mathematics[1] is to take these equations as defining *eigenvalue* problems for operators $\hat{\omega}$ and $\hat{\mathbf{k}}$. This eigenvalue perspective however need not *necessarily* be the case: notice that one can well have

$$\frac{\partial \phi}{\partial t}\frac{1}{\phi} = -i\omega$$

and

$$\frac{\nabla\phi}{\phi} = i\mathbf{k}$$

for a non-zero ϕ. This observation suggests the following

Definition 1 (*Harmonisation* of a non-zero scalar field)**.** *Let* $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}\backslash\{0\}$ *and* $\phi \in \mathcal{C}^2$*. Then*

$$\boxed{\omega := \frac{i}{\phi}\frac{\partial \phi}{\partial t}, \quad \boldsymbol{k} := -i\frac{\nabla\phi}{\phi}} \tag{1}$$

This definition can now be substituted in

Definition 2 (The Classical Wave Equation)**.**

$$c^2\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2}, \tag{2}$$

where

$$c := \frac{\omega}{\|\boldsymbol{k}\|} \tag{3}$$

for a harmonic wave[2]*,*

to arrive at the following novel nonlinear PDE:

Corollary.

$$\boxed{\left|\frac{\partial \phi}{\partial t}\right|^2 \nabla^2\phi = \frac{\partial^2\phi}{\partial t^2}|\nabla\phi|^2} \tag{4}$$

Proof. To substitute (1) in (2) we need to utilise (3) which requires us to specify the norm of $\mathbf{k}$, which is naturally seen to be the usual norm of $\mathbb{C}^3$: Since according to (1), $\mathbf{k}$ is a complex vector we have

$$\|\mathbf{k}\|^2 = \langle \mathbf{k}, \mathbf{k}\rangle = \mathbf{k}\cdot\bar{\mathbf{k}} = (-i\frac{\nabla\phi}{\phi})\cdot(i\frac{\nabla\bar{\phi}}{\bar{\phi}}) = \frac{|\nabla\phi|^2}{|\phi|^2}$$

where dot denotes euclidean inner product and $\bar{\mathbf{k}}$ is the complex conjugate of $\mathbf{k}$; as usual $|\cdot|$ denotes the modulus of a complex number. By a similar rationale

$$\omega^2 = \langle \omega, \omega\rangle = \omega\bar{\omega} = (\frac{i}{\phi}\frac{\partial\phi}{\partial t})(-\frac{i}{\bar{\phi}}\frac{\partial\bar{\phi}}{\partial t}) = \frac{1}{|\phi|^2}\left|\frac{\partial\phi}{\partial t}\right|^2$$

[2] The norm depends on the space in which ϕ lives. ϕ can well be a *real* harmonic wave, like $\phi(\mathbf{x}, t) = \sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$. Therefore we do not here in this definition specify a particular norm.

therefore

$$c^2 = \frac{\left|\frac{\partial \phi}{\partial t}\right|^2}{|\nabla \phi|^2}. \tag{5}$$

Substituting (5) in (2) now yields (4). □

Conflict of interest declaration

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Brian C. Hall. *Quantum Theory for Mathematicians.* Springer-Verlag New York, 2013.