

Refractive index and mass of kink in curved space

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The proposals that mass-energy is topological charge and topological charge is winding number are given. The refractive index-mass relation using the decomposed form of refractive index is showed where we replace mass with topological charge and winding number.

I. KINK, TOPOLOGICAL CHARGE AND WINDING NUMBER

A kink is a topological soliton in one-dimensional space, $\phi(x)$ ¹. Its energy density, at any given location, does not vanish with time in the long time limit. By definition, the kink is a map¹

$$\phi : Z_2 \rightarrow Z_2 \quad (1)$$

where Z_2 denotes the group of integer with size or modulo 2. In general, Z_p is group of integer with size or modulo p . The elements of Z_p are $0, \pm 1, \dots, \pm(p-1)$. So, the subscript 2 in Z_2 of eq.(1) indicates the modulo of the group of integer where the elements or members are 0 and ± 1 ²⁻⁴.

In the sine-Gordon model⁵, the topological charge is given by⁶

$$Q = \frac{\phi(x = +\infty) - \phi(x = -\infty)}{2\pi} \quad (2)$$

Refer to Skyrme⁷, the field configuration with boundary conditions for the sine-Gordon model⁸ is given by

$$\phi(x = +\infty) - \phi(x = -\infty) = 2N\pi \quad (3)$$

where N is the winding number.

If we accommodate the idea of Skyrme in eq.(3), by substituting eq.(3) into (2), we obtain

$$Q = \frac{2N\pi}{2\pi} = N \quad (4)$$

It means that the topological charge, Q , is equal to the winding number, N ⁹.

Topological charge is related to total energy¹⁰. The total energy or mass¹¹ of the kink (anti-kink)¹² in the sine-Gordon model is⁶

$$M = 8|Q| \quad (5)$$

In harmony with the Skyrme's idea⁷, if we treat the topological charge, Q , is equal to the winding number, N , then eq.(5) becomes

$$M = 8|N| \quad (6)$$

Here, we should remember that the mass of kink (anti-kink), eq.(5) or (6), is the mass which is formulated in the curved space.

II. THE REFRACTIVE INDEX AND MASS OF KINK

There exist the relationship between the refractive index and the mass in curved space, as below¹³

$$n(r) = \left\{ 1 - \frac{2G}{c^2 r} M \right\}^{-1} \quad (7)$$

or

$$M = X (1 - n^{-1}) \quad (8)$$

where $X = c^2 r / 2G$.

By substituting eqs.(5), (6), into (7) we obtain the relationship between the refractive index and the mass of the kink (anti-kink) in the sine-Gordon model

$$n(r) = \left\{ 1 - \frac{2G}{c^2 r} (8|Q|) \right\}^{-1} = \left\{ 1 - \frac{2G}{c^2 r} (8|N|) \right\}^{-1} \quad (9)$$

Eq.(9) show that the refractive index can be formulated related to the topological charge and the winding number.

How about the linear refractive index formulation related to the topological charge and the winding number in the sine-Gordon model? In case of the linear optics for sine-Gordon model, we treat the refractive index as the second rank tensor, and because G , c , r , m , λ are scalars, so the topological charge and the winding number are the second rank tensors. Then, eqs.(9) become¹³

$$n_{\mu\nu} = \left\{ 1 - \frac{2G}{c^2 r} (8|Q^{\mu\nu}|) \right\}^{-1} = \left\{ 1 - \frac{2G}{c^2 r} (8|N^{\mu\nu}|) \right\}^{-1} \quad (10)$$

It means that the mass of kink (anti-kink) in the sine-Gordon model for the linear optics can be related with the topological charge and winding number as

$$M^{\mu\nu} = 8|Q^{\mu\nu}| = 8|N^{\mu\nu}| \quad (11)$$

We see from eqs.(8), (9), (10), (11), then the refractive index-mass of kink (anti-kink) relation can be written explicitly as

$$M^{\mu\nu} = X (1 - n_{\mu\nu}^{-1}) \quad (12)$$

$$|Q^{\mu\nu}| = \frac{X}{8} (1 - n_{\mu\nu}^{-1}) \quad (13)$$

$$|N^{\mu\nu}| = \frac{X}{8} (1 - n_{\mu\nu}^{-1}) \quad (14)$$

where¹⁴

$$\left| \partial_\nu \left\{ \frac{c}{\omega} \arccos \left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g} \hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \right\} \right| = n_{\mu\nu} \quad (15)$$

$n_{\mu\nu}$ is the second rank tensor of refractive index, $A_\mu^{U(1)}$ is the unrestricted electric (scalar) potential of $U(1)$ group and $\hat{m}^{U(1)}$ is the restricted magnetic (vector) potential of $U(1)$ group.

We see from eqs.(12)-(15), there exist relation between the unrestricted electric (scalar) potential of $U(1)$ group, $A_\mu^{U(1)}$, the restricted magnetic (vector) potential of $U(1)$ group, $\hat{m}^{U(1)}$, with the mass, $M^{\mu\nu}$, the topological charge, $Q^{\mu\nu}$, and the winding number, $N^{\mu\nu}$, of kink (anti-kink).

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¹Tanmay Vachaspati, *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons*, Cambridge University Press, 2015.

²Geoff Smith, *Integers modulo N*
<https://people.bath.ac.uk/masgcs/book1/amplifications/g.pdf>

³Group of integers, https://groupprops.subwiki.org/wiki/Group_of_integers.

⁴Sze-Tsen Hu, *Homotopy Theory*, Academic Press, 1965.

⁵If we compare the sine-Gordon and ϕ^4 models, the formulation of the topological charge for both models looked different. Does it mean that the formulation of the topological charge in the ϕ^4 model is more general than the sine-Gordon (e.g. if we take $m = \pi$)? Here, m is a arbitrary normalisation parameter for the topological charge. For the ϕ^4 model, it is convenient to fix it to 1 so that the topological charge is in units of 1 (0, +1, -1). For the sine-Gordon model, we can replace 2π by m so that the topological charge will be in units of π (if our fields go from 0 to π) (Wojtek Zakrzewski, *Private communication*.)

⁶Lidia Stocker, *Kinks*, ETH Zurich, 2018.

⁷T.H.R. Skyrme, *The Origins of Skyrmions*, International Journal of Modern Physics A, Vol. 3, No.12, 2745-2751, 1988.

⁸Actually, Skyrme uses $\alpha(x)$ as a notation for describing a single angle-type field variable instead of $\phi(x)$.

⁹In order to be classically stable, soliton should have energy with special lower bound. The bound usually involves the topological index: $\varepsilon_{\phi \in Q_n} \geq C|N|$, where C is a constant and N is topological index (A.P. Balachandran, G. Marmo, B.S. Skagerstam, A. Stern, *Classical Topology and Quantum States*, World Scientific, 1991, p.103). N is topological index which is similar with the winding number (A.P. Balachandran, *Private communication*).

¹⁰Tanmay Vachaspati, *Private communication*.

¹¹Here we use natural unit $c = 1$, so $E = Mc^2$ gives $E = M$.

¹²In this case, the kink (anti-kink) is a solution of the sine-Gordon equation.

¹³Miftachul Hadi, *A refractive index of a kink in curved space*
<https://osf.io/preprints/inarxiv/x2qtw>, 2019.

¹⁴Miftachul Hadi, *Magnetic symmetry of geometrical optics*, <https://vixra.org/abs/2104.0188>, 2021 and all references therein.