On the Generalized Uncertainty Principle

Ming-Cheng Chen[1,2], Chao-Yang Lu[1,2], Jian-Wei Pan[1,2]

[1]Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

[2]CAS Centre for Excellence and Synergetic Innovation Centre in Quantum Information and Quantum Physics, University of Science and Technology of China, Shanghai 201315, China

Abstract

Generalized Uncertainty Principle (GUP), which manifests a minimal Planck length in quantum spacetime, is central in various quantum gravity theories and has been widely used to describe the Planck-scale phenomenon. Here, we propose a thought experiment based on GUP – as a quantum version of Galileo's falling bodies experiment – to show that the experimental results cannot be consistently described in quantum mechanics. This paradox arises from the interaction of two quantum systems in an interferometer, a photon and a mirror, with different effective Planck constants. Our thought experiment rules out the widely used GUP, and establishes a Quantum Coupling Principle that two physical systems of different effective Planck constants cannot be consistently coupled in quantum mechanics. Our results point new directions to quantum gravity.

One-sentence summary: The widely used Generalized Uncertainty Principle is proven inconsistent with a proposed thought experiment with Planck interferometer.

Quantum mechanics predicts that there is a smallest unit in the phase space of quantum systems (*1*), manifested by the Heisenberg uncertainty principle $[x,p]=i\hbar$. Is there also a smallest unit in the real physical space? Various approaches to quantum gravity have predicted a minimal observable length (*2-7*). These lead to a correction of the original Heisenberg uncertainty principle into Generalized Uncertainty Principle (GUP) (*8-12*). Based on the GUP, numerous interesting quantum gravity effects have been predicted, and recently, several proposals of experimental tests have also been suggested (*13-18*). However, due to the very stringent experimental requirements, no non-trivial result has been observed.

In this work, we implement a thought experiment to reveals a paradox between the GUP and quantum mechanics. The paradox is analog to Galileo's falling bodies experiment in classical gravity (*19*). Galileo's classical experiment (Fig. 1A) coupled two falling balls with different masses and revealed a paradox in their landing time, thus logically ruling out the possibility that heavy objects fall faster than the lighter ones in gravity. Here, in our experiment, we couple two quantum objects with different effective Planck constants in a quantum optical interferometer (Fig. 1B). The two quantum objects are a photon and a mirror, which have different effective Planck constants derived from the GUP. In this thought experiment, we reveal an inconsistency of predicted interference visibilities caused by the GUP.

The GUP can be expressed as a deformed commutator. Considering a quantum object of rest mass m in one-dimension space, its two canonical observables, coordinate x and momentum p, obey the following generalized version of commutator (*8*),

$$[x,p]=i\hbar\sqrt{1+2\beta\frac{m^2+(p/c)^2}{M_P^2}},$$

where $M_P=\sqrt{\hbar c/G}\approx 22\mu g$ is the Planck mass, $\hbar$ is the Planck constant, c is the light speed, G is the gravitational constant, and $\beta\approx 1$ is a free constant. Importantly, the GUP gives rise to a modified, state-dependent effective Planck constant in the form

$$\tilde{\hbar} = \hbar\sqrt{1 + 2\beta\frac{m^2 + (p/c)^2}{M_P^2}} .$$

In our thought experiment, we restrict the system's momentum in the realm $p << M_P c$, so we can expand the commutator in the first order of $p / M_P c$. For a photon of zero rest mass, we have the effective Planck constant

$$\tilde{\hbar}_{\text{photon}} \approx \hbar\left(1 + \beta\left\langle p_{\text{photon}}^2\right\rangle / \left(M_P^2 c^2\right)\right).$$

For the sake of convenience, we use a mirror of a rest mass $m = \sqrt{(\gamma^2 - 1)/(2\beta)} M_P$, where $\gamma > 1$ is a constant related to the mass, we have

$$\tilde{\hbar}_{\text{mirror}} \approx \hbar\left(\gamma + \beta\left\langle p_{\text{mirror}}^2\right\rangle / \left(\gamma^2 M_P^2 c^2\right)\right).$$

In the realm $p << M_P c$, the effective Planck constants of the photon and the mirror can be conveniently written as $\hbar$ and $\gamma\hbar$, respectively. The different effective Planck constants are due to the system's different rest masses.

The thought experiment is carried out on a quantum optics interferometer in Michelson configuration (*20*). A Planck photon of momentum $\tilde{\hbar}_{\text{photon}} k$ is sent to a balanced beam splitter and split into two path modes, $|a\rangle$ (vertical) and $|b\rangle$ (horizontal). The paths are coupled to the mirror through radiation-pressure-induced interacting Hamiltonian $H_{\text{int}} = g(t) p_{\text{photon}} \otimes x_{\text{mirror}}$ with a coupling strength $\int g(t) dt = 2$ (ref. *21*). The mirror is a quantum harmonic oscillator of eigenfrequency ω and is prepared in the quantum ground state.

The single photon after the first beam splitter is in a superposition state $(|a\rangle + |b\rangle)/\sqrt{2}$. Before reflection, the composite system of the photon and the mirror is in a product state $\Psi_1^{(x)} = (1/\sqrt{2})(|a\rangle + |b\rangle) \otimes \int \psi(x)|x\rangle dx$, where the mirror's wave function $\psi(x)$ is represented in the position representation. After the reflection, the photon is entangled with the mirror in a wave function $\Psi_2^{(x)} = \int \psi(x)(e^{-2ikx}|a\rangle|x\rangle + |b\rangle|x\rangle)/\sqrt{2}dx$. The photon's phase factor e^{-2ikx} comes from the mirror-position-dependent phase shift. Quantum mechanics also promises to equivalently describe the same physical process in the momentum representation of the mirror. The state of the composite system $\Psi_1^{(p)} = (1/\sqrt{2})(|a\rangle + |b\rangle) \otimes \int \phi(p)|p\rangle dp$, where $\phi(p)$ is the mirror's wave function in the momentum representation, evolves to $\Psi_2^{(p)} = \int \phi(p)(|a\rangle|p - 2\hbar k\rangle + |b\rangle|p\rangle)/\sqrt{2}dp$ after the reflection. The change of the mirror's momentum $-2\hbar k$ comes from the kick of the photon.

Then, the photon is recombined on the same beam splitter. Scanning a phase shift over the path, we can observe an interference fringe at the output. The quantum entanglement between the photon and the mirror will reduce the interference visibility from the ideal 100%. It is straightforward to derive that in the position and momentum representations, the visibility is $V^{(x)} = e^{-2k^2\Delta x^2}$ and $V^{(p)} = e^{-(\hbar k)^2/2\Delta p^2}$, where $\Delta x = \sqrt{\gamma\hbar/2m\omega}$ and $\Delta p = \sqrt{\gamma\hbar m\omega/2}$ is the mirror's position and momentum uncertainty, respectively (*22*).

One would expect $V^{(x)} = V^{(p)}$ as the quantum mechanics is consistent. However, in this thought experiment, remarkable, we observe that

$$V^{(x)}/V^{(p)} = e^{\frac{\hbar k^2}{m\omega}\left(\frac{1}{\gamma} - \gamma\right)}.$$

Only when $\gamma = 1$, these two visibilities at positon and momentum representations equal to each other. For any GUP correction of $\gamma \neq 1$, our thought experiment reveals a direct contradiction. Figure 2 illustrates examples of $V^{(x)}$ and $V^{(p)}$ at the regime where the mirror's momentum uncertainty is comparable to the photon's momentum kick, which show evident differences (*22*).

Intuitively, this conflict can be also seen from the mirror's position and momentum uncertainty, $\Delta x = \sqrt{\gamma\hbar / 2m\omega}$ and $\Delta p = \sqrt{\gamma\hbar m\omega / 2}$, respectively, as a function of the GUP correction γ. When the γ increases, it would give both a larger Δx and $\Delta \mathrm{p}$. However, while a larger Δx would result in a lower interference visibility, a larger $\Delta \mathrm{p}$ would imply a higher interference visibility. The spirit of our paradox is similar to Galileo's falling bodies experiment in classical gravity: different views on the two-body interaction process can generate different results (*19*).

Therefore, our thought experiment reveals an inconsistency between standard quantum mechanics and the GUP. We can summarize our observation with a Quantum Coupling Principle (QCP) that two physical systems with different effective Planck constants cannot be consistently coupled in quantum mechanics. An immediate inspiration from the QCP is that classical physics which assumes $\tilde{\hbar} = 0$ cannot consistently couple to quantum systems ($\tilde{\hbar} \neq 0$). This reinforces the view that we need to quantize everything, including the classical gravity (*23*).

References

1. P. A. M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, 1981).
2. D. Amati, M. Ciafaloni, G. Veneziano, Can spacetime be probed below the string size? Physics Letters B. **216**, 41–47 (1989).
3. L. J. Garay, Quantum gravity and minimum length. Int. J. Mod. Phys. A. **10**, 145–165 (1995).
4. K. Konishi, G. Paffuti, P. Provero, Minimum physical length and the generalized uncertainty principle in string theory. Physics Letters B. **234**, 276–284 (1990).
5. M. Maggiore, The algebraic structure of the generalized uncertainty principle. Physics Letters B. **319**, 83–86 (1993).
6. F. Scardigli, Generalized Uncertainty Principle in Quantum Gravity from Micro-

Black Hole Gedanken Experiment. Physics Letters B. **452**, 39–44 (1999).

7. X. Calmet, M. Graesser, S. D. H. Hsu, Minimum Length from Quantum Mechanics and Classical General Relativity. Phys. Rev. Lett. **93**, 211101 (2004).
8. M. Maggiore, A Generalized Uncertainty Principle in Quantum Gravity. Physics Letters B. **304**, 65–69 (1993).
9. S. Hossenfelder, Minimal Length Scale Scenarios for Quantum Gravity. Living Rev. Relativ. **16**, 2 (2013).
10. A. Kempf, G. Mangano, R. B. Mann, Hilbert Space Representation of the Minimal Length Uncertainty Relation. Phys. Rev. D. **52**, 1108–1118 (1995).
11. A. F. Ali, S. Das, E. C. Vagenas, Discreteness of Space from the Generalized Uncertainty Principle. Physics Letters B. **678**, 497–499 (2009).
12. A. N. Tawfik, A. M. Diab, Generalized Uncertainty Principle: Approaches and Applications. Int. J. Mod. Phys. D. **23**, 1430025 (2014).
13. S. Das, E. C. Vagenas, Universality of Quantum Gravity Corrections. Phys. Rev. Lett. **101**, 221301 (2008).
14. A. F. Ali, S. Das, E. C. Vagenas, Proposal for testing quantum gravity in the lab. Phys. Rev. D. **84**, 044013 (2011).
15. I. Pikovski, M. R. Vanner, M. Aspelmeyer, M. S. Kim, Č. Brukner, Probing Planck-scale physics with quantum optics. Nat Phys. **8**, 393–397 (2012).
16. G. Amelino-Camelia, Challenge to Macroscopic Probes of Quantum Spacetime Based on Noncommutative Geometry. Phys. Rev. Lett. **111**, 101301 (2013).
17. F. Marin, F. Marino, M. Bonaldi, M. Cerdonio, L. Conti, P. Falferi, R. Mezzena, A. Ortolan, G. A. Prodi, L. Taffarello, G. Vedovato, A. Vinante, J.-P. Zendri, Gravitational bar detectors set limits to Planck-scale physics on macroscopic variables. Nature Phys. **9**, 71–73 (2013).
18. M. Bawaj, C. Biancofiore, M. Bonaldi, F. Bonfigli, A. Borrielli, G. Di Giuseppe, L. Marconi, F. Marino, R. Natali, A. Pontin, G. A. Prodi, E. Serra, D. Vitali, F. Marin, Probing deformed commutators with macroscopic harmonic oscillators. Nat Commun. **6**, 7503 (2015).
19. S. Drake, Galileo at Work: His Scientific Biography (Courier Corporation, 2003).

20. A. A. Michelson, E. W. Morley, On the relative motion of the Earth and the luminiferous ether. American Journal of Science. s3-34, 333–345 (1887).

21. M. Aspelmeyer, T. J. Kippenberg, F. Marquardt, Cavity Optomechanics. Rev. Mod. Phys. **86**, 1391–1452 (2014).

22. In the Supplemental Materials, the quantum state of the other mirror is explicitly considered and the central conclusion remains the same.

23. C. Marletto, V. Vedral, Why we need to quantise everything, including gravity. npj Quantum Inf. 3, 1–5 (2017).

Figures:

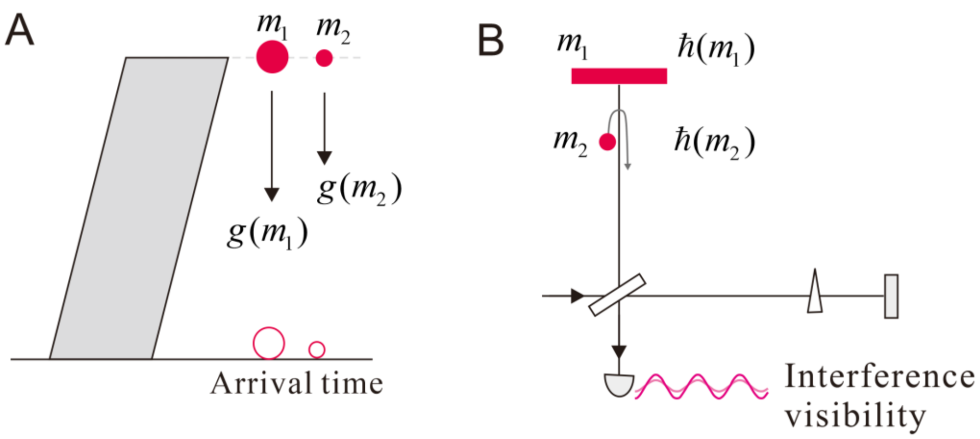

Fig. 1. A quantum version of Galileo's falling bodies thought experiment. (A) The classical Galileo's falling bodies experiment. If falling objects of different masses have different gravitational accelerations, a logic paradox will arise in their landing time. (B) The Planck quantum optics interferometer. If quantum objects of different masses have different quantum-gravity-corrected effective Planck constants, a paradox will raise in the interference visibility.

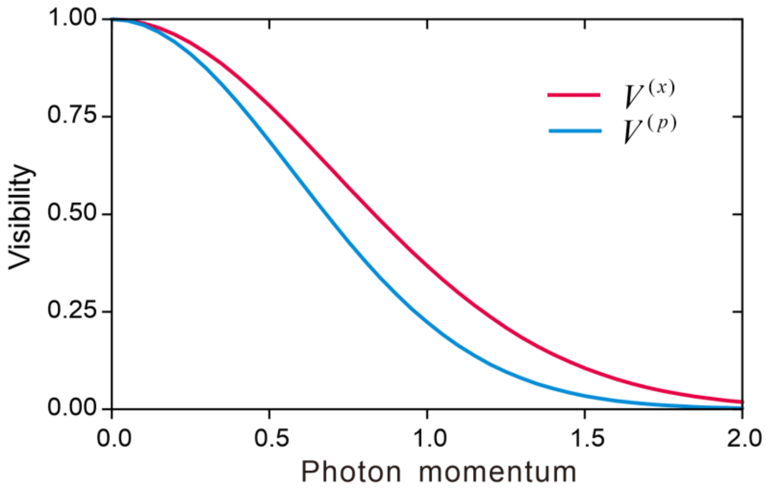

Fig 2. Examples of the inconsistency of the interference visibility due to GUP. For illustrative purpose, we choose a GUP correction of $\gamma = 1.5$ for the effective Planck constant, and a regime where the momentum uncertainty of the mirror is comparable to the single photon momentum kick. The photon momentum shown in horizontal axis is normalized to the momentum uncertainty of the mirror. The plotted interference visibilities $V^{(x)}$ and $V^{(p)}$ are estimated using the position and the momentum representations, respectively, which show a pronounced gap.

Supplementary Materials

Interference Visibility

In the interferometer, when the photon is input to the balanced beam splitter, it is in a path superposition state $\Psi_1 = (|a\rangle + |b\rangle)/\sqrt{2}$, where $|a\rangle$ and $|b\rangle$ are the vertical and horizontal paths (Fig. 1B), respectively. After the interaction between the photon and the Planck mirror, the photon is entangled with the mirror in an entangled state $\Psi_2 = (|a\rangle|\tilde{\varphi}\rangle + |b\rangle|\varphi\rangle)/\sqrt{2}$, where $|\varphi\rangle$ and $|\tilde{\varphi}\rangle$ are the quantum states of the mirror before and after the interaction, respectively. The photon then passes a tunable phase shifter $e^{i\theta}$ and combines on the balanced beam splitter. The output quantum state is

$\Psi_{\text{output}} = (|a\rangle(|\tilde{\varphi}\rangle + e^{i\theta}|\varphi\rangle) + |b\rangle(|\tilde{\varphi}\rangle - e^{i\theta}|\varphi\rangle))/2$. Therefore, the photon counting probability at the output port $|a\rangle$ is $P_a(\theta) = 1/2 + (1/2)|\langle\tilde{\varphi}|\varphi\rangle|\cos(\theta + \arg\langle\tilde{\varphi}|\varphi\rangle)$ and the interference visibility is $V = |\langle\tilde{\varphi}|\varphi\rangle|$.

In the position representation, the mirror's quantum state $|\varphi\rangle$ and $|\tilde{\varphi}\rangle$ are $|\varphi\rangle = \int \psi(x)|x\rangle dx$ and $|\tilde{\varphi}\rangle = \int \psi(x)e^{-2ikx}|x\rangle dx$, respectively. As the mirror's ground state is $\psi(x) = \left(\frac{1}{2\pi\Delta x^2}\right)^{1/4} e^{-\left(\frac{x}{2\Delta x}\right)^2}$, the interference visibility is $V^{(x)} = |\langle\tilde{\varphi}|\varphi\rangle| = e^{-2k^2\Delta x^2}$.

In the momentum representation, as $|\varphi\rangle = \int \phi(p)|p\rangle dp$, $|\tilde{\varphi}\rangle = \int \phi(p)|p - 2\hbar k\rangle dp$, and $\phi(p) = \left(\frac{1}{2\pi\Delta p^2}\right)^{1/4} e^{-\left(\frac{p}{2\Delta p}\right)^2}$, the interference visibility is $V^{(p)} = |\langle\tilde{\varphi}|\varphi\rangle| = e^{-\frac{(\hbar k)^2}{2\Delta p^2}}$.

Two Planck mirrors

In the main text, we only consider one mirror at one optical path for simplicity. Here, we present a full calculation including the other mirror. The central conclusion remains the same.

After the interaction between the photon and the two Planck mirrors, the photon is entangled with the mirrors in an entangled state $\Psi_2 = (|a\rangle|\tilde{\varphi}\rangle_a|\varphi\rangle_b + |b\rangle|\varphi\rangle_a|\tilde{\varphi}\rangle_b)/\sqrt{2}$, where $|\varphi\rangle_a$ and $|\tilde{\varphi}\rangle_a$ ($|\varphi\rangle_b$ and $|\tilde{\varphi}\rangle_b$) are the quantum states of the mirror on the path $|a\rangle$ ($|b\rangle$) before and after the interaction, respectively. Thus, the interference visibility becomes $V = |\langle\tilde{\varphi}|_a\langle\varphi|_b \cdot |\varphi\rangle_a|\tilde{\varphi}\rangle_b| = V_aV_b$, where $V_a = |\langle\tilde{\varphi}|_a \cdot |\varphi\rangle_a|$ and $V_b = |\langle\varphi|_b \cdot |\tilde{\varphi}\rangle_b|$.

We estimate the interference visibilities in two selected composite representations: the first mirror in the position or the momentum representations, and the second mirror always in the position representations. Therefore, the interference visibilities are $V^{(x)} = V_a^{(x)}V_b^{(x)}$ and $V^{(p)} = V_a^{(p)}V_b^{(x)}$, respectively. In this case, we will have the value

$V^{(x)}/V^{(p)} = V_a^{(x)}/V_a^{(p)}$, which is the same to the result in the main text.

Effective Rest Mass

In our thought experiment, the effective Planck constant is dependent on the rest mass, which is the key to the quantum paradox. There are alternative deformed commutators without explicit rest mass (*10*), for example:

$$[x_i, p_i] = i\hbar\left(1 + \frac{\beta}{(M_P c)^2} p_i^2 + \frac{\beta'}{(M_P c)^2}\sum_{i=1,2,3} p_i^2\right),$$

where i is the index of the spatial dimensions, β and β' are correction constants. In this case, we can use the momentum component perpendicular to the first dimension as the effective rest mass. By setting $\langle p_2^2 + p_3^2 \rangle = \frac{\gamma - 1}{\beta'}(M_P c)^2$, the commutator for the first spatial dimension becomes:

$$[x_1, p_1] \approx i\hbar\left(\gamma + \frac{\beta + \beta'}{(M_P c)^2} p_1^2\right),$$

where γ describes the effective rest mass. Therefore, the effective Planck constant is $\gamma\hbar$ when $p_1 << M_P c$.