On the geometrical optics and the Atiyah-Singer index theorem

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We assume that the curvature in the Atiyah-Singer index theorem is related with the Riemann-Christoffel curvature tensor where the Riemann-Christoffel curvature tensor is decomposed into the unrestricted electric (scalar) potential part and the restricted magnetic (vector) potential part. This decomposition is a consequence of the magnetic symmetry existence for the gauge potential in the geometrical optics.

I. GEOMETRICAL OPTICS

The ray propagation equation of a steady monochromatic wave where the frequency is a constant can be derived from Fermat's principle[1]. It gives[1]

$$\frac{1}{R} = \vec{N} \cdot \frac{\vec{\nabla} n}{n} \tag{1}$$

where R is the radius of curvature, $\vec{N}$ is the unit vector along the principal normal, n is the refractive index, a function of coordinates. Eq.(1) shows that the rays are therefore *bent* in the direction of increasing refractive index[1].

The curvature of space in one dimension, $1/R$, in eq.(1) can be generalized for higher dimension. In case of the dimension is more than two, the curvature can be expressed as the Riemann-Christoffel curvature tensor. It is a generalization of the Gauss' curvature of one and two dimensions[2].

In our previous work on the magnetic symmetry of the geometrical optics, we obtained the result as below[3,4]

$$\begin{aligned} & gN_\sigma\ \partial_\rho \ln \Big| \partial_\nu \Big\{ \frac{c}{\omega} \arccos \\ & \left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g} \hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \Big\} \Big| \\ & = \ R_{\mu\nu\rho\sigma} \end{aligned} \tag{2}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann-Christoffel curvature tensor, $A_\mu^{U(1)}$ is *the unrestricted electric (scalar) potential* of $U(1)$ group and $\hat{m}^{U(1)}$ is *the restricted magnetic (vector) potential* of $U(1)$ group.

Eq.(2) shows that the curvature is decomposed into the unrestricted electric (scalar) potential part and the restricted magnetic (vector) potential part. This decomposition is a consequence of the magnetic symmetry existence for the gauge potential in the geometrical optics[3,4].

II. ELLIPTIC PARTIAL DIFFERENTIAL EQUATION, FREDHOLM INDEX AND TOPOLOGICAL INVARIANT

The equation (1) is a partial differential equation and we assume that eq.(1) is the partial differential equation of *elliptic type*, *the elliptic PDE*. Because the elliptic PDE is an equation which is well suited to describe *equilibrium state*[5,6] and we interpret the equilibrium state here as the steady state.

It is possible to define elliptic equations not just for functions on m variables, but for functions defined on a manifold. Particularly accessible to analysis are the elliptic equations on closed manifolds, that is on manifolds that are finite in extent and that have no boundary[7].

Every elliptic partial differential equation on a closed manifold has a Fredholm index. The Fredholm index of an elliptic PDE (with suitable boundary conditions) is the number of linearly independent solutions of the equation minus the number of linearly independent solutions of its adjoint equation[7].

The Fredholm index is a topological invariant of elliptic PDE. This means that continuous variations in the coefficients of an elliptic PDE leave the Fredholm index unchanged (in contrast, the number of linearly independent solutions of an equation can vary as the coefficients of the equation vary)[7].

III. THE ATIYAH-SINGER INDEX THEOREM

The Atiyah-Singer index theorem is concerned with the existence and uniqueness of *solutions to linear partial differential equations of elliptic type on closed manifolds*[7]. *The Atiyah-Singer index theorem asserts that the Fredholm (analytical) index of an elliptic equation is equal to the topological index of the equation*[7].

Roughly, the Atiyah-Singer index theorem can be written as[7,9]

$$\text{Index} = \int_M I_M\ \cdot\ \text{ch}\ (\sigma) \tag{3}$$

where I_M is a differential form determined by *the curvature of the manifold*, M, on which the equation is defined[8] and ch (σ) is a differential form obtained from *the symbol* of equation. In other words, ch (σ) is Chern character[9]. Here, "Index" is the Fredholm (analytical) index[9].

IV. THE ATIYAH-SINGER INDEX THEOREM IN THE GEOMETRICAL OPTICS

Because the Atiyah-Singer index theorem is related with the elliptic PDE and eq.(1) is also the elliptic PDE, so we have the reason that the Atiyah-Singer index theorem is related with or can be applied to the geometrical optics, eq.(1).

Because I_M is related with the curvature of the manifold, M, on which the equation is defined[7], we assume that[10]

$$I_M \sim \det\left\{\frac{R_{\nu\sigma}}{2}\sinh^{-1}\left(\frac{R_{\nu\sigma}}{2}\right)\right\} \tag{4}$$

where $R_{\nu\sigma}$ is Ricci curvature of M. Explicitly, Ricci curvature tensor is related with the Riemann-Christoffel curvature tensor as

$$R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma} \tag{5}$$

Substituting eq.(2) into (5), we obtain

$$R_{\nu\sigma} = g^{\mu\rho} g N_\sigma \ \partial_\rho \ln \left|\partial_\nu \left\{\frac{c}{\omega}\arccos \left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1} + ct\right\}\right| \tag{6}$$

The Atiyah-Singer index theorem, by substituting eq.(4) into (3), has the form

$$\text{Index} \sim \int_M \det\left\{\frac{R_{\nu\sigma}}{2}\sinh^{-1}\left(\frac{R_{\nu\sigma}}{2}\right)\right\} \cdot \text{ch}\,(\sigma) \tag{7}$$

where $R_{\nu\sigma}$ as given in eq.(6).

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[1] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press Ltd, 1989.

[2] Miftachul Hadi, *Linear and non-linear refractive indices in Riemannian and topological spaces*, https://vixra.org/abs/2105.0163, 2021 and all references therein.

[3] Miftachul Hadi, *Magnetic symmetry of geometrical optics*, https://vixra.org/abs/2104.0188, 2021 and all references therein.

[4] Miftachul Hadi, *Magnetic symmetry, curvature and Gauss-Bonnet-Chern theorem*, https://vixra.org/abs/2106.0019, 2021 and all references therein.

[5] Wikipedia, *Elliptic Partial Differential Equation.*

[6] Elliptic equation is any of a class of partial differential equations describing phenomena that do not change from moment to moment, as when a flow of heat or fluid takes place within a medium with no accumulations (Britannica, *Elliptic equation*, https://www.britannica.com/science/elliptic-equation).

[7] Nigel Higson, John Roe, *The Atiyah-Singer Index Theorem.*

[8] In many cases (for example spheres, toruses, open subsets of Euclidean space) we can ignore the other term in that formula, that we called "I_M", since it is equal to 1. Of course, despite this, mathematicians think it is the most interesting part (Nigel Higson, *Private communication*).

[9] Nigel Higson, *Private communication.*

[10] Dan Berwick-Evans, *The Atiyah-Singer Index Theorem*, https://www.mit.edu/~fengt/282C.pdf