Rigorous proof of Riemann hypothesis that rigidly comply with Principle of Maximum Density for Integer Number Solutions

J. Ting, 21 July 2021 Preprint submitted to viXra

Dedicated to my daughter Jelena prematurely born 13 weeks early on May 14, 2012 and all front-line health workers globally fighting against the deadly 2020 Coronavirus Pandemic.

Abstract
The 1859 Riemann hypothesis conjectured all nontrivial zeros in Riemann zeta function are uniquely located on sigma = 1/2 critical line. Derived from Dirichlet eta function [proxy for Riemann zeta function] are, in chronological order, simplified Dirichlet eta function and Dirichlet Sigma-Power Law. Computed Zeroes from the former uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (–sigma) that is twice present in this function to be integer –1. Computed Pseudo-zeroes from the later uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (1 – sigma) that is twice present in this law to be integer 1. All nontrivial zeros are, respectively, obtained directly and indirectly as one specific type of Zeroes and Pseudo-zeroes only when sigma = 1/2. Thus, it is proved that Riemann hypothesis is true whereby this function and law must rigidly comply with Principle of Maximum Density for Integer Number Solutions.

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Figure 1: INPUT for $\sigma = \frac{1}{2}, \frac{2}{5},$ and $\frac{3}{5}$. $\zeta(s)$ has countable infinite set of Completely Predictable trivial zeros located at $\sigma = \text{all negative even numbers}$ and conjectured countable infinite set of Incompletely Predictable nontrivial zeros located at $\sigma = \frac{1}{2}$ given by various $t$ values.

1. **Introduction**

Riemann hypothesis is an intractable open problem in Number theory that was proposed in 1859 by famous German mathematician Bernhard Riemann (September 17, 1826 - July 20, 1866). This hypothesis conjectured all nontrivial zeros in Riemann zeta function are uniquely located on $\sigma = \frac{1}{2}$ critical line. By applying Euler formula to Dirichlet eta function [proxy for Riemann zeta function], we obtain simplified Dirichlet eta function whereby its computed Zeros uniquely occur at $\sigma = \frac{1}{2}$ resulting in total summation of fractional exponent ($-\sigma$) that is twice present in this function to be integer $-1$. Dirichlet Sigma-Power Law is the solution from performing integration on simplified Dirichlet eta function whereby its computed Pseudo-zeros uniquely occur at $\sigma = \frac{1}{2}$ resulting in total summation of fractional exponent ($1 - \sigma$) that is twice present in this law to be integer $1$. All nontrivial zeros are, respectively, obtained directly and indirectly as one specific type of Zeroes and Pseudo-zeroes only when $\sigma = \frac{1}{2}$. Then nonexistent virtual nontrivial zeros and virtual Pseudo-nontrivial zeros cannot be obtained directly and indirectly as a type of virtual Zeros and virtual Pseudo-zeros when $\sigma \neq \frac{1}{2}$. Successfully solving Theorems 1, 2 and 3, and the associated Lemmas 1 and 2 [with crucial groundings in our so-called Mathematics for Incompletely Predictable problems]; it is proved that Riemann hypothesis is true whereby this function and law rigidly comply with Principle of Maximum Density for Integer Number Solutions. In addition, they serendipitously obey trigonometric identities and manifest Principle of Equidistant for Multiplicative Inverse.

In Appendix A, we provide an assortment of information on various useful topics that need not form an essential part of our proof for Riemann hypothesis e.g. brief discussions involving certain types of infinite series.

2. **Sketch of the Proof for Riemann hypothesis including the Modified Equations for simplified Dirichlet eta function and Dirichlet Sigma-Power Law that are expressed using trigonometric identities**

**Abbreviations.**
- CFS: countable finite set
- CIS: countable infinite set
- UIS: uncountable infinite set
- CP: Completely Predictable – see section 3 on CP entities
Figure 2: OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Schematically depicted polar graph of $\zeta\left(\frac{1}{2} + it\right)$ plotted along critical line for real values of $t$ running from 0 to 34, horizontal axis: $Re\{\zeta\left(\frac{1}{2} + it\right)\}$, and vertical axis: $Im\{\zeta\left(\frac{1}{2} + it\right)\}$. Total presence of Origin intercept points.

Figure 3: OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re\{\zeta\left(\frac{2}{5} + it\right)\}$, and vertical axis: $Im\{\zeta\left(\frac{2}{5} + it\right)\}$. Total absence of Origin intercept points.

IP: Incompletely Predictable – see section 3 on IP entities
DA: Dimensional analysis – see section 4 on exact and inexact DA homogeneity
NTZ: nontrivial zeros (Gram $x=0,y=0$ points) = Origin intercept points when $\sigma = \frac{1}{2}$

$\zeta(s)$: $f(n)$ Riemann zeta function
$\eta(s)$: $f(n)$ Dirichlet eta function
$\text{sim-}\eta(s)$: $f(n)$ simplified Dirichlet eta function

DSPL: $F(n)$ Dirichlet Sigma-Power Law = $\int sim - \eta(s)dn$

Symbolically named after German mathematician Gustav Lejeune Dirichlet (February 13, 1805 - May 5, 1859), the word “Law” in DSPL represent a convenient terminology to describe this function – viz, there is resemblance to Zipf’s law via power law functions in $\sigma$ from $s = \sigma + it$ being exponent of a power function as similar format to $n^\sigma$, logarithm scale use, and $\zeta(s)$ harmonic series connection. Respectively, we use Zeros (as three types of Gram points) and Pseudo-zeros (as three types of Pseudo-Gram points) at $\sigma = \frac{1}{2}$ to collectively refer to corresponding $f(n)$’s and $F(n)$’s x-axis intercept points, y-axis intercept points and Origin intercept points. Respectively, we use virtual Zeros (as two types of virtual Gram points) and
Figure 4: OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points with horizontal axis: $Re\{\zeta(\frac{3}{5} + it)\}$, and vertical axis: $Im\{\zeta(\frac{3}{5} + it)\}$. Varying Loops are shifted to right of Origin. Total absence of Origin intercept points.

virtual Pseudo-zeroes (as two types of virtual Pseudo-Gram points) at $\sigma \neq \frac{1}{2}$ to collectively refer to corresponding $f(n)$‘s and $F(n)$‘s x-axis intercept points and y-axis intercept points [with absent Origin intercept points].

Geometrical and mathematical definitions for Gram points and virtual Gram points. Figure 1 depicts complex variable $s (= \sigma \pm it)$ as INPUT with x-axis denoting real part $Re\{s\}$ associated with $\sigma$, and y-axis denoting imaginary part $Im\{s\}$ associated with $t$. The critical line: $\sigma = \frac{1}{2}$; non-critical lines: $\sigma \neq \frac{1}{2}$ viz, $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$; and critical strip: $0 < \sigma < 1$. Both unique $\sigma = \frac{1}{2}$ value and non-unique $\sigma \neq \frac{1}{2}$ values $\in$ Set all $\sigma$ values whereby Set all $\sigma$ values $= \sigma \mid \sigma$ is a real number, and $0 < \sigma < 1$. With including its complex conjugate, $s = \sigma \pm it$ is present in all our well-defined $f(n)$ and $F(n)$ whereby these are continuous [complex] functions that are always defined for any arbitrarily chosen intervals $[a,b]$. With $f(n) = 0$ and $F(n) = 0$ giving rise to relevant derived equations that are dependently-related [via Varying Loops], they generate corresponding types of IP entities. These IP entities inherently belong to the correctly assigned independent mutually exclusive CIS constituted by $t$ values as transcendental numbers except for first Gram[y=0] point (and first virtual Gram[y=0] point) given by $t = 0$. Origin intercept points, x-axis intercept points and y-axis intercept points are geometrical definitions for IP entities of Gram[x=0,y=0] points, Gram[y=0] points and Gram[x=0] points at $\sigma = \frac{1}{2}$. These geometrical definitions are equivalent to mathematical definitions as given by the equations below in this section.

Origin intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[x=0,y=0] points or NTZ are computed directly from equations $\eta(s) = 0$ and sim-$\eta(s) = 0$; and indirectly from equation DSPL = 0. x-axis intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[y=0] points or (traditional) ‘usual’ Gram points are computed directly from equation Gram[y=0] points-sim-$\eta(s) = 0$; and indirectly from equation Gram[y=0] points-DSPL = 0. y-axis intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[x=0] points are computed directly from equation Gram[x=0] points-sim-$\eta(s) = 0$; and indirectly from equation Gram[x=0] points-DSPL = 0.

Relevant functions and equations are unique mathematical objects usefully classified as three types of infinite series: Harmonic series, Alternating harmonic series or Alternating series with trigonometric terms. We perform crucial de novo analysis on these functions and equations by noting their manifested intrinsic properties. Without loss of validity in our correct and complete set of mathematical arguments, we adopt the convention of providing focused analysis predominantly on appropriately chosen Alternating series with trigonometric terms throughout our presentation. The complex $f(n) \zeta(s)$ is a Harmonic series that does not converge in critical
strip. The complex \( f(n) \eta(s) \) is an Alternating harmonic series that converge in critical strip. Through analytic continuation, \( \eta(s) \) must act as proxy function for \( \zeta(s) \) in this strip. [Caveat: the limit of an analytic continuation is not the analytic continuation of the limit.] Derived as Euler formula application to \( \eta(s) \) is the complex \( f(n) \) sim-\( \eta(s) \), and derived as \( \int \text{sim} - \eta(s) \, dn \) is the complex \( F(n) \) DSPL. Both \( \text{sim-} \eta(s) \) and DSPL are Alternating series with trigonometric terms that converge in critical strip.

The \( f(n) \eta(s) \) will converge infinitely often to a zero value as \( \eta(s) = 0 \) equation giving rise to all NTZ or \( \text{Gram}[x=0,y=0] \) points. This event will only happen when \( \eta(s) \) is substituted with one unique \( \sigma \) value which is conjectured to be \( \sigma = \frac{1}{2} \) by Riemann hypothesis. Being an Alternating harmonic series [without trigonometric terms that graphically cater for all possible types of x-axis and y-axis intercept points], we inherently cannot derive valid functions to obtain corresponding equations \( \text{Gram}[y=0] \) points-\( \eta(s) = 0 \) and \( \text{Gram}[x=0] \) points-\( \eta(s) = 0 \) that will enable mathematical computations of \( \text{Gram}[y=0] \) points as x-axis intercept points and \( \text{Gram}[x=0] \) points as y-axis intercept points. Then, computed Zeroes are mathematically defined as \( \eta(s) = 0 \) and sim-\( \eta(s) = 0 \) when variable \( \sigma = \frac{1}{2} \); computed virtual Zeroes are mathematically defined as \( \eta(s) \neq 0 \) and sim-\( \eta(s) = 0 \) when variable \( \sigma \neq \frac{1}{2} \); computed Pseudo-Zeroes are mathematically defined as DSPL = 0 when variable \( \sigma = \frac{1}{2} \); and computed virtual Pseudo-zeroes are mathematically defined as DSPL = 0 when variable \( \sigma \neq \frac{1}{2} \).

For \( 0 \leq \delta \leq 1 \), let \( f(n) = \sin(n) \pm \delta \) and \( f(n) = \cos(n) \pm \delta \) represent two [simple] trigonometric functions which are periodic transcendental-type functions. Both \( \sin(n) \pm \delta = 0 \) and \( \cos(n) \pm \delta = 0 \) as equations will generate infinitely many CP x-axis intercept points (Zeroes) for any given values of \( \delta \). This will additionally include the solitary Origin intercept point (Zero) obtained from \( \sin(n) \pm \delta = 0 \) when \( \delta = 0 \). For both \( \sin(n) \pm \delta \) and \( \cos(n) \pm \delta \), only when \( \delta = 0 \) will their progressive / cumulative Areas Above the horizontal axis be overall identical to Areas Below the horizontal axis. Otherwise, these mentioned Areas will not be overall identical to each other when \( \delta \neq 0 \). We now provide analogical reasoning for existence of infinitely many substituted \( \sigma \) values (including \( \sigma = \frac{1}{2} \)) that will all contribute to two conditions sim-\( \eta(s) = 0 \) and DSPL = 0 being satisfied while simultaneously giving rise to (i) IP Zeroes and IP Pseudo-zeroes [when \( \sigma = \frac{1}{2} \)], and (ii) IP virtual Zeroes and IP virtual Pseudo-zeroes [when \( \sigma \neq \frac{1}{2} \)]. With (complex) sine and/or cosine terms present in \( f(n) \) sim-\( \eta(s) \) and \( F(n) \) DSPL also being periodic transcendental-type functions, we intuitively deduce \( \sigma = \frac{1}{2} \) and \( \sigma \neq \frac{1}{2} \) must respectively act as the analogical equivalence of \( \delta = 0 \) and \( \delta \neq 0 \). This deduction allows intuitive and valid explanations for our two conditions to be satisfied by the infinitely many substituted \( \sigma \) values. Consequently, we must rigorously prove additional property of sim-\( \eta(s) \) and DSPL that they will characteristically, inevitably and uniquely comply with Principle of Maximum Density for Integer Number Solutions only when \( \sigma = \frac{1}{2} \) with this Principle signifying complete presence of NTZ in sim-\( \eta(s) \) or Pseudo-NTZ in DSPL as one unique type of Gram points or Pseudo-Gram points [which are otherwise totally absent when \( \sigma \neq \frac{1}{2} \)].

Figures 2, 3 and 4 are \( \zeta(\sigma + it) \) Polar Graphs [see Remark 14 on intimate relationship between Cartesian Coordinates and Polar Coordinates] with x-axis denoting real part Re\{\( \zeta(s) \)\} and y-axis denoting imaginary part Im\{\( \zeta(s) \)\} generated by \( \zeta(s) \)'s output as real values of \( t \) running from 0 to 34. There are infinite types-of-spirals (Varying Loops) possibilities associated with each \( \sigma \) value arising from all infinite \( \sigma \) values in \( 0 < \sigma < 1 \) critical strip whereby the unique and solitary \( \sigma = \frac{1}{2} \) value that denote critical line is located in this strip. We observe that Figure 3 [with \( \sigma = \frac{2}{3} \)] and Figure 4 [with \( \sigma = \frac{3}{4} \)] show associated shifts of Varying Loops that manifest Principle of Equidistant for Multiplicative Inverse - see Lemma 2 from section 7. From observing Figure 2, we can geometrically define NTZ (or \( \text{Gram}[x=0,y=0] \) points) as Origin intercept points occurring when \( \sigma = \frac{1}{2} \). Then, two remaining types of Gram points as part of continuous Varying Loops are consequently defined as x-axis intercept points and y-axis intercept points occurring when \( \sigma = \frac{1}{2} \).
From Remark 6 below, corresponding three types of \( F(n)'s \) Pseudo-zeroes or Pseudo-Gram points and two types of \( F(n)'s \) virtual Pseudo-zeroes or virtual Pseudo-Gram points can be precisely converted to three types of \( f(n)'s \) Zeroes or Gram points and two types of \( f(n)'s \) virtual Zeroes or virtual Gram points. Then, Statement (I) – (IV) below are valid whereby \( \sigma = \frac{1}{2} \)'s derived entities from Statement (III) can be precisely converted to those from Statement (I), and \( \sigma \neq \frac{1}{2} \)'s derived virtual entities from Statement (IV) can be precisely converted to those from Statement (II):

Statement (I) The \( f(n)'s \) Zeroes at \( \sigma = \frac{1}{2} \) [directly] equates to three types of Gram points.

Statement (II) The \( f(n)'s \) virtual Zeroes at \( \sigma \neq \frac{1}{2} \) [directly] equates to two types of virtual Gram points.

Statement (III) The \( F(n)'s \) Pseudo-zeroes at \( \sigma = \frac{1}{2} \) [indirectly] equates to three types of Gram points.

Statement (IV) The \( F(n)'s \) virtual Pseudo-zeroes at \( \sigma \neq \frac{1}{2} \) [indirectly] equates to two types of virtual Gram points.

Remark 1. Of particular relevance to Riemann hypothesis, we mathematically deduce from above materials that \( f(n)'s \) NTZ or Gram\([x=0,y=0]\) points as one type of Gram points will conjecturally only exist at unique \( \sigma = \frac{1}{2} \) critical line [but not at non-unique \( \sigma \neq \frac{1}{2} \) non-critical lines]. This can be equivalently stated as: \( F(n)'s \) Pseudo-NTZ or Pseudo-Gram\([x=0,y=0]\) points as one type of Pseudo-Gram points will conjecturally only exist at unique \( \sigma = \frac{1}{2} \) critical line [but not at non-unique \( \sigma \neq \frac{1}{2} \) non-critical lines].

Useful analogy for Remark 2: A line consists of infinitely many points. Graphically, the Origin is a zero-dimensional [single] point; x-axis or horizontal axis and y-axis or vertical axis are one-dimensional lines [containing infinitely many points].

Remark 2. From Figures 3 and 4, we note Origin intercept points as Gram\([x=0,y=0]\) points or NTZ cannot exist when \( \sigma \neq \frac{1}{2} \). From Figure 2, we note Origin intercept points as Gram\([x=0,y=0]\) points or NTZ only exist when \( \sigma = \frac{1}{2} \). Of particular relevance to Riemann hypothesis, we deduce sim-\( \eta(s) \) as periodic transcendental-type function only contain one solitary \( \sigma \)-valued type of Origin intercept points (when \( \sigma = \frac{1}{2} \) for Gram\([x=0,y=0]\) points or NTZ as conjectured by Riemann hypothesis) but infinitely many different \( \sigma \)-valued types of x-axis intercept points and y-axis intercept points (constituted by solitary \( \sigma = \frac{1}{2} \) value for Gram\([y=0]\) points and Gram\([x=0]\) points as well as infinitely many \( \sigma \neq \frac{1}{2} \) values for virtual Gram\([y=0]\) points and virtual Gram\([x=0]\) points). We can conjure up an equivalent statement for DSPL as periodic transcendental-type function whereby we replace NTZ and (virtual) Gram points with their counterparts Pseudo-NTZ and (virtual) Pseudo-Gram points.

We can now propose Theorem 1 (with \( \sigma = \frac{1}{2} \) connoting exact DA homogeneity) and its corollary Theorem 2 (with \( \sigma \neq \frac{1}{2} \) connoting inexact DA homogeneity) to fully represent Remark 1 and Remark 2. Their successful proofs will firstly, denote rigorous proof for Riemann hypothesis that involves conjecture on location of NTZ as one type of Gram points [viz, Origin intercept points] and secondly, provide precise explanations for remaining two types of Gram points [viz, x-axis intercept points and y-axis intercept points]. In addition, we incorporate Theorem 3 on rigid compliance by sim-\( \eta(s) \) and DSPL with Principle of Maximum Density for Integer Number Solutions whereby its successful proof will only eventuate when \( \sigma = \frac{1}{2} \).

Theorem 1. Rigidly complying with exact DA homogeneity, \( f(n) \) sim-\( \eta(s) \) and \( F(n) \) DSPL as relevant equations can incorporate three types of Gram points and Pseudo-Gram points onto solitary \( \sigma = \frac{1}{2} \) critical line thus fully supporting Riemann hypothesis to be true.
Proof. Using \( f(n) \) sim-\( \eta \) and \( F(n) \) DSPL, Riemann hypothesis propose all NTZ are located on \( \sigma = \frac{1}{2} \) critical line in these functions. The three types of Gram points and Pseudo-Gram points are each infinite in magnitude consisting of mutually exclusive entities. Amounting to direct Proof by Positive, we show CIS of Gram\( [x=0,y=0] \) points or NTZ constitutes one type of Gram points only when \( \sigma = \frac{1}{2} \) thus fully supporting Riemann hypothesis to be true. The preceding sentence is equally valid when we replace Gram\( [x=0,y=0] \) points, NTZ and Gram points with corresponding Pseudo-Gram\( [x=0,y=0] \) points, Pseudo-NTZ and Pseudo-Gram points. Respectively, the conveniently defined term of exact DA homogeneity denote \( [\text{exact}] \) integer \(-1\) and \( 1 \) derived from \( \sum \) (all fractional exponents) = \( 2(-\sigma) \) and \( 2(1-\sigma) \). These act as surrogate markers in sim-\( \eta \) and DSPL on \([\text{solitary}]\ \sigma = \frac{1}{2} \) situation. Generated by relevant functions and laws when \( \sigma = \frac{1}{2} \), the three types of Gram points are mathematically defined as equations sim-\( \eta \) = 0, Gram\( [y=0] \) points-sim-\( \eta \) = 0 and Gram\( [x=0] \) points-sim-\( \eta \) = 0; and the three types of Pseudo-Gram points are mathematically defined as equations DSPL = 0, Gram\( [y=0] \) points-DSPL = 0 and Gram\( [x=0] \) points-DSPL = 0. They all correspond to relevant geometrically defined Origin intercept points, x-axis intercept points and y-axis intercept points. Thus, three types of IP Gram points [IP Zeroes] and IP Pseudo-Gram points [IP Pseudo-Zeroes] are mathematically and geometrically defined to be located on \( \sigma = \frac{1}{2} \) critical line. Based solely on these definitive definitions, we can uniquely incorporate three types of IP Gram points [IP Zeroes] and IP Pseudo-Gram points [IP Pseudo-zeroes] onto \( \sigma = \frac{1}{2} \) critical line. The proof is now complete for Theorem 1\( \square \).

Theorem 2. Rigidly complying with inexact DA homogeneity, \( f(n) \) sim-\( \eta \) and \( F(n) \) DSPL as relevant equations can incorporate two types of virtual Gram points and virtual Pseudo-Gram points onto infinitely many \( \sigma \neq \frac{1}{2} \) non-critical lines thus also fully supporting Riemann hypothesis to be true.

Proof. Using \( f(n) \) sim-\( \eta \) and \( F(n) \) DSPL, Riemann hypothesis equivalently propose all NTZ are not located on \( \sigma \neq \frac{1}{2} \) non-critical lines in these functions. The two types of virtual Gram points and virtual Pseudo-Gram points are each infinite in magnitude consisting of mutually exclusive entities. Amounting to indirect Proof by Contrapositive, we show [non-existent] virtual Gram\( [x=0,y=0] \) points or virtual NTZ will not constitute one type of [non-existent] virtual Gram points when \( \sigma \neq \frac{1}{2} \) thus also fully supporting Riemann hypothesis to be true. The preceding sentence is equally valid when we replace virtual Gram\( [x=0,y=0] \) points, virtual NTZ and virtual Gram points with corresponding virtual Pseudo-Gram\( [x=0,y=0] \) points, virtual Pseudo-NTZ and virtual Pseudo-Gram points. Respectively, conveniently defined term of inexact DA homogeneity denote [inexact] fractional (non-integer) number \( \neq -1 \) and \( \neq 1 \) derived from \( \sum \) (all fractional exponents) = \( 2(-\sigma) \) and \( 2(1-\sigma) \). These act as surrogate markers in sim-\( \eta \) and DSPL on \([\text{solitary}]\ \sigma \neq \frac{1}{2} \) situations. Generated by relevant functions and laws when \( \sigma \neq \frac{1}{2} \), the two types of virtual Gram points are mathematically defined as equations virtual Gram\( [y=0] \) points-sim-\( \eta \) = 0 and virtual Gram\( [x=0] \) points-sim-\( \eta \) = 0; and the two types of virtual Pseudo-Gram points are mathematically defined as equations virtual Gram\( [y=0] \) points-DSPL = 0 and virtual Gram\( [x=0] \) points-DSPL = 0. They all correspond to relevant geometrically defined x-axis intercept points and y-axis intercept points. Thus, two types of IP virtual Gram points [IP Zeroes] and IP virtual Pseudo-Gram points [IP Pseudo-Zeroes] are mathematically and geometrically defined to be located on \( \sigma \neq \frac{1}{2} \) non-critical lines. Based solely on these definitive definitions, we can uniquely incorporate two types of IP virtual Gram points [IP virtual Zeroes] and IP virtual Pseudo-Gram points [IP virtual Pseudo-zeroes] onto \( \sigma \neq \frac{1}{2} \) non-critical lines. The proof is now complete for Theorem 2\( \square \).

Theorem 3. Conforming to the solitary \( \sigma = \frac{1}{2} \) critical line [and not the infinitely many \( \sigma \neq \frac{1}{2} \) non-critical lines e.g. \( \sigma = \frac{1}{3} \) or \( \frac{2}{3} \)] whereby \( \sigma \) forms part of relevant fractional exponents from base quantities \( (2n) \) and \( (2n-1) \) in sim-\( \eta \) [as Riemann sum \( \Delta n \rightarrow 1 \) with variable \( n \)]
involving all integers \( \geq 1 \) or DSPL [as definite integral \( \Delta n \rightarrow 0 \) with variable \( n \) involving all real numbers \( \geq 1 \)]; square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when applied to combined base quantities \((2n)\) and \((2n-1)\) in sim-\(\eta(s)\) or DSPL will generate the maximum number of integer solutions (constituted by all integers \( \geq 1 \)) that uniquely comply with Principle of Maximum Density for Integer Number Solutions while also manifesting Principle of Equidistant for Multiplicative Inverse.

**Proof.** \( \text{sim-}\eta(s)dn = \text{DSPL} \). Whereas the two subsets of rational roots as integers and irrational roots as irrational numbers can be generated by combined base quantities \((2n)\) and \((2n-1)\) from sim-\(\eta(s)\) [as Riemann sum \( \Delta n \rightarrow 1 \) with variable \( n \) involving all integers \( \geq 1 \)], so must these two exact same subsets be generated by combined base quantities \((2n)\) and \((2n-1)\) from DSPL [as definite integral \( \Delta n \rightarrow 0 \) with variable \( n \) involving all real numbers \( \geq 1 \)]. Thus in sim-\(\eta(s)\) or DSPL, its computed CIS rational roots (subset) as integers [rational numbers] + computed CIS irrational roots (subset) as irrational numbers = computed CIS total roots. These two mutually exclusive subsets belong to UIS real numbers. Using subset rational roots as integers at \( \sigma = \frac{1}{2} \) critical line, and by comparing and contrasting this subset with [different] subset rational roots as integers at \( \sigma = \frac{1}{3} \) or \( \frac{2}{3} \) non-critical lines corollary situation; we will show that square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when applied to combined base quantities \((2n)\) and \((2n-1)\) from sim-\(\eta(s)\) or DSPL giving rise to maximum number of integer solutions (constituted by all integers \( \geq 1 \)) must uniquely comply with Principle of Maximum Density for Integer Number Solutions (see Lemma 1 in section 6) while also manifesting Principle of Equidistant for Multiplicative Inverse (see Lemma 2 in section 7). We apply concepts from elegant Gauss Circle Problem and Primitive Circle Problem in section 5 onto materials on aptly-named Gauss Areas of Varying Loops to justifiably obtain correct and complete set of mathematical arguments that fully support Theorem 3. *The proof is now complete for Theorem 3.*

By conveniently employing only sim-\(\eta(s)\) for analysis here [with analysis using DSPL being equally valid], Theorem 1 and Theorem 2 above can also be insightfully combined as follows. Let Set \( G \) = all Gram points = Gram\([x=0,y=0]\) points + Gram\([y=0]\) points + Gram\([x=0]\) points and Set \( vG \) = all virtual Gram points = virtual Gram\([y=0]\) points + virtual Gram\([x=0]\) points with virtual Gram\([x=0,y=0]\) points = null set \( \emptyset \). We can apply **inclusion-exclusion principle** \( |G \cup vG| = |G| + |vG| - |G \cap vG| = |G| + |vG| \) because \( |G \cap vG| = 0 \). Since exclusive presence of Gram points and absence of virtual Gram points on critical line denotes exclusive absence of Gram points and exclusive presence of virtual Gram points on non-critical lines; then Gram points and virtual Gram points as mutually exclusive entities must mathematically and geometrically be incorporated, respectively, onto unique (solitary) critical line and non-unique (infinitely many) non-critical lines of sim-\(\eta(s)\).

**Derived f(n) = 0 and F(n) = 0 equations** – see \( \sigma = \frac{1}{2} \) (via Proposition 4.3 and Proposition 5.3) and \( \frac{2}{3} \) (via Corollary 4.4 and Corollary 5.4) representative examples given in [1], p. 27-28, 29-30 and section 4 below – comply with exact DA homogeneity at \( \sigma = \frac{1}{2} \) critical line and inexact DA homogeneity at \( \sigma \neq \frac{1}{2} \) non-critical lines. NTZ are synonymous with Gram\([x=0,y=0]\) points which is one type of Gram points. Whenever applicable, all modified equations below are expressed using trigonometric identities. Together with Gram\([y=0]\) points and Gram\([x=0]\) points as remaining two types of Gram points, these three types of Gram points are **intrinsically located** in their complex equations (akin to Complex Containers) as IP entities whereby their actual location [but not actual positions] are **intrinsically incorporated** in these complex equations – see section 3 below. Eqs. (2.1), (2.3), (2.5), (2.6), (2.7) and (2.8) that comply with exact DA homogeneity at \( \sigma = \frac{1}{2} \) all have fractional exponents \( \frac{1}{2} \); Eqs. (2.2) and (2.4) that comply with inexact DA homogeneity at \( \sigma = \frac{2}{3} \) have fractional exponents \( \frac{2}{5} \) in the former and \( \frac{3}{5} \) in the later that are mixed with fractional exponents \( \frac{1}{2} \).
\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n - 1) + \frac{1}{4} \pi) = 0 \quad (2.1)
\]

With exact DA homogeneity, Eq. (2.1) is \( f(n) \) sim-\( \eta \) that will incorporate all NTZ [as Zeroes]. There is total absence of (non-existent) virtual NTZ [as virtual Zeroes].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{3}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{3}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n - 1) + \frac{1}{4} \pi) = 0 \quad (2.2)
\]

With inexact DA homogeneity, Eq. (2.2) is \( f(n) \) sim-\( \eta \) that will incorporate all (non-existent) virtual NTZ [as virtual Zeroes]. There is total absence of NTZ [as Zeroes].

\[
\frac{1}{2^{\frac{1}{2}}} \left( t^2 + \frac{9}{25} \right)^\frac{1}{2} \cdot \left[ (2n)^{\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - (2n - 1)^{\frac{1}{2}} \cos(t \ln(2n - 1) - \frac{1}{4} \pi) + C \right]_1^{\infty} = 0
\]

(2.3)

\[
\frac{1}{2^{\frac{1}{2}}} \cdot \left[ (2n)^{\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - (2n - 1)^{\frac{1}{2}} \cos(t \ln(2n - 1) - \frac{1}{4} \pi) + C \right]_1^{\infty} = 0
\]

(2.4)

With exact DA homogeneity, Eq. (2.3) is \( F(n) \) DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion]. There is total absence of (non-existent) virtual NTZ [as virtual Zeroes].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{4}} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{1}{4}} \sin(t \ln(2n - 1)) = 0 \quad (2.5)
\]

Eq. (2.5) can also be equivalently written as
\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{4}} \cos(t \ln(2n) - \frac{1}{2} \pi) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{1}{4}} \cos(t \ln(2n - 1) - \frac{1}{2} \pi) = 0.
\]

With exact DA homogeneity, Eq. (2.5) is \( f(n) \) Gram\([y=0]\) points-sim-\( \eta \) at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\([y=0]\) points [as Zeroes]. There is total absence of virtual Gram\([y=0]\) points [as virtual Zeroes].

\[
-\frac{1}{2^{\frac{1}{2}}} \cdot \left[ (2n)^{\frac{1}{4}} \cos(t \ln(2n) - \frac{1}{4} \pi) - \cos(t \ln(2n - 1) - \frac{1}{4} \pi) + C \right]_1^{\infty} = 0
\]

(2.6)

Eq. (2.6) can also be equivalently written as
\[
\frac{1}{2^{\frac{1}{2}}} \cdot \left[ (2n)^{\frac{1}{4}} \cos(t \ln(2n) + \frac{3}{4} \pi) - \cos(t \ln(2n - 1) + \frac{3}{4} \pi) + C \right]_1^{\infty} = 0.
\]

With exact DA homogeneity, Eq. (2.6) is \( F(n) \) Gram\([y=0]\) points-DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\([y=0]\) points [as virtual Pseudo-zeroes to virtual Zeroes conversion]. There is total absence of virtual Gram\([y=0]\) points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{1}{2}} \cos(t \ln(2n - 1)) = 0 \quad (2.7)
\]
With exact DA homogeneity, Eq. (2.7) is $f(n)$ Gram[x=0] points-sim-$\eta(s)$ at $\sigma = \frac{1}{2}$ that will incorporate all Gram[x=0] points [as Zeroes]. There is total absence of virtual Gram[x=0] points [as virtual Zeroes]

$$\frac{1}{2(t^2 + \frac{1}{4})^{\frac{3}{2}}} \left[ (2n)^{\frac{1}{2}}(\cos(t \ln(2n) - \frac{3}{4} \pi) - \cos(t \ln(2n - 1) - \frac{3}{4} \pi)) + C \right]_{1}^{\infty} = 0 \quad (2.8)$$

With exact DA homogeneity, Eq. (2.8) is $F(n)$ Gram[x=0] points-DSPL at $\sigma = \frac{1}{2}$ that will incorporate all Gram[x=0] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[x=0] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

We outline sim-$\eta(s)$ as Eq. (2.2) and DSPL as Eq. (2.4) that comply with inexact DA homogeneity at $\sigma = \frac{2}{5}$ non-critical line (depicted by Figure 3) whereby $\sigma = \frac{2}{5}$ [instead of $\sigma = \frac{1}{2}$] is substituted into these two equations. Using [selective] trigonometric identity for linear combination of sine and cosine function whenever applicable to relevant $f(n) = 0$ and $F(n) = 0$ equations, we outline exact DA homogeneity at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points as Eq. (2.5) and Gram[x=0] points as Eq. (2.7) whereby we will only manifest solitary [unmixed] $\neq \frac{1}{2}$ fractional exponents. We provide [self-explanatory] corresponding $f(n) = 0$ equations below for Gram[y=0] points and Gram[x=0] points corollary situation when $\sigma = \frac{2}{5}$.

$$\sum_{n=1}^{\infty} (2n)^{-\frac{3}{2}} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{3}{2}} \sin(t \ln(2n - 1)) = 0$$

$$\sum_{n=1}^{\infty} (2n)^{-\frac{3}{2}} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\frac{3}{2}} \cos(t \ln(2n - 1)) = 0$$

We arbitrarily chose single cosine wave with format $R \cos(n \pm \alpha)$ to use above where $R$ is scaled amplitude and $\alpha$ is phase shift. For equations regarding NTZ, Gram[y=0] points and Gram[x=0] points; all their approximate Areas of Varying Loops $\propto$ precise Areas of Varying Loops with $R$ validly treated as a proportionality factor. We analyze $f(n) = 0$ and $F(n) = 0$ equations at $\sigma = \frac{1}{2}$ critical line for NTZ situation where $R = 2^{\frac{1}{2}}(2n)^{-\frac{1}{2}}$ or $2^{\frac{1}{2}}(2n - 1)^{-\frac{1}{2}}$ in $f(n)$’s Eq. (2.1) and $R = \frac{1}{2^{\frac{1}{2}}(t^2 + \frac{1}{4})^{\frac{3}{2}}} (2n)^{\frac{1}{2}}$ or $\frac{1}{2^{\frac{1}{2}}(t^2 + \frac{1}{4})^{\frac{3}{2}}} (2n - 1)^{\frac{1}{2}}$ in $F(n)$’s Eq. (2.3).

**Remark 3.** Whereas for NTZ $F(n)$ Eq. (2.3) that exactly represent precise Areas of Varying Loops and $f(n)$ Eq. (2.1) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude $R$ from Eq. (2.3) which is dependent on parameter $t$ and Eq. (2.1) which is independent of parameter $t$ represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

We analyze $f(n) = 0$ equations [relevant to approximate Areas of Varying Loops] at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points as Eq. (2.5) and Gram[x=0] points as Eq. (2.7) whereby we validly designate $R = (2n)^{-\frac{1}{2}}$ or $(2n - 1)^{-\frac{1}{2}}$ as the assigned scaled amplitude and [unwritten] $\alpha = 0$ as the assigned phase shift.

Relevant to precise Areas of Varying Loops at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points $F(n)$ Eq. (2.6) with $R = -\frac{1}{2(t^2 + \frac{1}{4})^{\frac{3}{2}}} (2n)^{\frac{1}{2}}$ or $-\frac{1}{2(t^2 + \frac{1}{4})^{\frac{3}{2}}} (2n - 1)^{\frac{1}{2}}$ and Gram[x=0] points $F(n)$
Eq. (2.8) with \( R = \frac{1}{2 (t^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2 (t^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \), we observe the former \( R \) to be the negative of the later \( R \). However, this observation is context-sensitive because when Eq. (2.6) is written in its equivalent format above, the former \( R \) is identical to the later \( R \). Both \( R \) are now just given by \( \frac{1}{2 (t^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2 (t^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \).

**Remark 4.** Whereas for Gram\( [y=0] \) points \( F(n) \) Eq. (2.6) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (2.5) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) in Eq. (2.6) which is dependent on parameter \( t \) and Eq. (2.5) which is independent of parameter \( t \) represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

**Remark 5.** Whereas for Gram\( [x=0] \) points \( F(n) \) Eq. (2.8) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (2.7) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) in Eq. (2.8) which is dependent on parameter \( t \) and Eq. (2.7) which is independent of parameter \( t \) represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Finally, we analyze \( f(n) = 0 \) and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for NTZ situation where phase shift \( \alpha = \frac{1}{4} \pi \) in NTZ \( f(n) \) Eq. (2.1) and \( -\frac{1}{4} \pi \) in NTZ \( F(n) \) Eq. (2.3); and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for Gram\( [y=0] \) points and Gram\( [x=0] \) points situations where phase shift \( \alpha = -\frac{1}{4} \pi \) (or \( \frac{3}{4} \pi \)) when written in its equivalent format above in Gram\( [y=0] \) points \( F(n) \) Eq. (2.6) and \( -\frac{3}{4} \pi \) in Gram\( [x=0] \) points \( F(n) \) Eq. (2.8). Always being \( \frac{1}{2} \pi \) out-of-phase with each other, trigonometric functions sine and cosine are cofunctions with sin \( n = \cos (\frac{\pi}{2} - n) \), cos \( n = \sin (\frac{\pi}{2} - n) \) or \( \sin (n + \frac{\pi}{2}) \), \( \frac{d}{dn} \sin n = \cos n \), \( \frac{d}{dn} \cos n = -\sin n \), \( \int \sin n \cdot dn = -\cos n + C \) [= \( \sin n - \frac{\pi}{2} \) + C] and \( \int \cos n \cdot dn = \sin n + C \) [= \( \cos n - \frac{\pi}{2} \) + C]. Last two integrals explain relation between \( f(n) \)'s Zeroes and \( F(n) \)'s Pseudo-zeroes when they involve simple sine and/or cosine terms viz, \( f(n) \)'s CP Zeroes = \( F(n) \)'s CP Pseudo-zeroes – \( \frac{1}{2} \pi \) with CP Zeroes and CP Pseudo-zeroes being \( \frac{1}{2} \pi \) out-of-phase with each other.

**Remark 6.** Involving trigonometric functions as complex sine and/or cosine terms: \( f(n) \)'s IP NTZ or [non-existent] \( f(n) \)'s IP virtual NTZ (in \( t \) values) = \( F(n) \)'s IP Pseudo-NTZ or [non-existent] \( F(n) \)'s IP virtual Pseudo-NTZ (in \( t \) values) – \( \frac{1}{2} \pi \); \( f(n) \)'s IP virtual Gram\( [y=0] \) points or \( f(n) \)'s IP virtual Gram\( [y=0] \) points (in \( t \) values) = \( F(n) \)'s IP Pseudo-Gram\( [y=0] \) points or \( F(n) \)'s IP virtual Pseudo-Gram\( [y=0] \) points (in \( t \) values) – \( \frac{3}{4} \pi \); and \( f(n) \)'s IP Gram\( [x=0] \) points or \( f(n) \)'s IP Gram\( [x=0] \) points (in \( t \) values) = \( F(n) \)'s IP Pseudo-Gram\( [x=0] \) points or \( F(n) \)'s IP virtual Pseudo-Gram\( [x=0] \) points (in \( t \) values) – \( \frac{3}{4} \pi \).

\[ \int f(n) dn = F(n) + C \] where \( F'(n) = f(n) \). \( f(n) \) and \( F(n) \) are literally [connected] bijective (both injective and surjective or a one-to-one correspondence) functions. Underlying \( f(n) \) as equation and \( F(n) \) as law (equation) that generate their CIS of IP Zeroes, IP virtual
Zeroes, IP Pseudo-zeroes and IP virtual Pseudo-zeroes are precisely related as $\frac{1}{2}\pi$ (for NTZ case) or $\frac{3}{4}\pi$ (for Gram[y=0] points and Gram[x=0] points cases) out-of-phase with each other. Peculiar to IP NTZ as Origin intercept points, we crucially note only they will uniquely behave in accordance with complex sine and/or cosine terms present in their equations that generate corresponding IP Zeroes and IP Pseudo-zeroes which are $\frac{1}{2}\pi$ out-of-phase with each other.

The x-axis and y-axis are orthogonal to each other with angle between them = $\frac{1}{2}\pi$ radian. Involving trigonometric functions as complex sine and/or cosine terms: $f(n)$’s IP Gram[y=0] points or $f(n)$’s IP virtual Gram[y=0] points (in t values) = $f(n)$’s IP Gram[x=0] points or $f(n)$’s IP virtual Gram[x=0] points (in t values) + $\frac{1}{2}\pi$; and $F(n)$’s IP Pseudo-Gram[y=0] points or $F(n)$’s IP virtual Pseudo-Gram[y=0] points (in t values) = $F(n)$’s IP Pseudo-Gram[x=0] points or $F(n)$’s IP virtual Pseudo-Gram[x=0] points (in t values) + $\frac{1}{2}\pi$.

**Remark 7.** These observations imply underlying $f(n)$ as equation and $F(n)$ as law (equation) that generate corresponding paired IP two types of Gram points [as Zeroes], paired IP two types of virtual Gram points [as virtual Zeroes], paired IP two types of Pseudo-Gram points [as Pseudo-zeroes], and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] being always $\frac{1}{2}\pi$ out-of-phase with each other.

3. The Completely Predictable and Incompletely Predictable entities

The word “number” [singular noun] or “numbers” [plural noun] used in reference to prime and composite numbers, NTZ and two other types of Gram points can interchangeably be replaced with the word “entity” [singular noun] or “entities” [plural noun]. We outline an innovative method to classify certain appropriately chosen equations or algorithms in two ways by using relevant locational properties of its output. This output consist of generated entities either from function-based equations or from algorithms. Our classification system as part of conveniently [albeit loosely] named “Mathematics for Completely Predictable problems” and “Mathematics for Incompletely Predictable problems” is formalized by providing definitions for CP entities obtained from CP equations or algorithms, and IP entities obtained from IP equations or algorithms.

CP simple equation or algorithm generates CP numbers. A generated CP number is **locationally defined** as a number whose position is independently determined by simple calculations without needing to know related positions of all preceding numbers in neighborhood. IP complex equation or algorithm generates IP numbers. A generated IP number is **locationally defined** as a number whose position is dependently determined by complex calculations with needing to know related positions of all preceding numbers in neighborhood. Container is a useful analogical term that metaphorically group CP entities (e.g. even and odd numbers) and IP entities (e.g. nontrivial zeros, prime and composite numbers) to be exclusively located in, respectively, Simple Container and Complex Container. Gram points and virtual Gram points are t-valued transcendental numbers.

Simple properties are inferred from a sentence such as “This simple equation or algorithm by itself will intrinsically incorporate actual location [and actual positions] of all CP numbers”. Examples: simple equations $E = (2 \times i)$ and $O = (2 \times i) - 1$ for $i = \text{all real numbers} \geq 0$ or $i = \text{all integers} \geq 0$ will respectively and intrinsically incorporate or generate CIS of all CP even number $E_i = 0, 2, 4, 6,...$ and CIS of all CP odd numbers $O_i = 1, 3, 5, 7,...$ whereby even
number \((n)\) is defined as “Any integer that can be divided exactly by 2 with last digit always being 0, 2, 4, 6 or 8” and odd number \((n)\) is defined as “Any integer that cannot be divided exactly by 2 with last digit always being 1, 3, 5, 7 or 9". Congruence \(n \equiv 0 \text{ (mod 2)}\) holds for even \(n\) and congruence \(n \equiv 1 \text{ (mod 2)}\) holds for odd \(n\). We note the zeroth even number is given by \(E_0 = 0\).

Complex properties, or meta-properties, are inferred from a sentence such as “This complex equation or algorithm by itself will intrinsically incorporate actual location [but not actual positions] of all IP numbers”. Examples: complex algorithms \(P_{i+1} = P_i + p\text{Gap}_i\) and \(C_{i+1} = C_i + c\text{Gap}_i\) for \(i = 1, 2, 3, \ldots\) \(\infty\) with \(P_1 = 2\) and \(C_1 = 4\) will respectively and intrinsically incorporate CIS of all IP prime number 2, 3, 5, 7, \ldots and CIS of all IP composite numbers 4, 6, 8, 9, \ldots whereby prime numbers are defined as “All Natural numbers apart from 1 that are evenly divisible by itself and by 1” and composite numbers are defined as “All Natural numbers apart from 1 that are evenly divisible by numbers other than itself and 1”. E.g. via computed Pseudo-zeroes that can be converted to Zeros at \(\sigma = \frac{1}{2}\) critical line, complex equation DSPL will intrinsically incorporate the CIS of all IP NTZ [given as \(t\) values rounded off to six decimal places]: 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178, \ldots and complex equation \(\text{Gram}[y=0]\) points-DSPL will intrinsically incorporate the CIS of all IP \(\text{Gram}[y=0]\) points [given as \(t\) values rounded off to six decimal places]: 0, 3.436218, 9.66908, 17.845999, 23.170282, 27.670182, \ldots Choice of index \(n\) for \(\text{Gram}[y=0]\) points is crudely chosen in the past to be -3, -2, -1, 0, 1, 2, 3, \ldots whereby the first \(\text{Gram}[y=0]\) point is historically denoted by \(n = 1\) with \(t\) value 17.845999 (on critical line) being larger than first NTZ’s \(t\) value of 14.134725 (on critical line). \(\text{Gram}[y=0]\) points or ‘usual’ Gram points are named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 - April 29, 1916) – see brief exposition on \(\text{Gram}[y=0]\) points obeying Gram’s Law in [1], p. 44-45.

**The Even-Odd Pairing.** For \(i = 1, 2, \ldots, \infty\); let mutually exclusive \(i^{th}\) Even and \(i^{th}\) Odd numbers = \(E_i\) and \(O_i\), and \(i^{th}\) even and \(i^{th}\) odd number gaps = \(e\text{Gap}_i\) and \(o\text{Gap}_i\). The positions of \(E_i\) and \(O_i\) are CP and their independence from each other is shown below.

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We employ simple equations \(E = (2 \times i)\) and \(O = (2 \times i) - 1\). We can precisely, easily and independently calculate \(E_5 = (2 \times 5) = 10\) and \(O_5 = (2 \times 5) - 1 = 9\).

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**The Prime-Composite Pairing.** For \(i = 1, 2, \ldots, \infty\); let mutually exclusive \(i^{th}\) Prime and \(i^{th}\) Composite numbers = \(P_i\) and \(C_i\), and \(i^{th}\) prime and \(i^{th}\) composite number gaps = \(p\text{Gap}_i\) and \(c\text{Gap}_i\). The positions of \(P_i\) and \(C_i\) are IP and their dependence on each other is shown below.

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We employ complex algorithms \(P_{i+1} = P_i + p\text{Gap}_i\) and \(C_{i+1} = C_i + c\text{Gap}_i\). We can precisely, tediously and dependently compute \(P_6 = 13\): 2 is 1\(^{st}\) prime number, 3 is 2\(^{nd}\) prime number, 4 is 1\(^{st}\) composite number, 5 is 3\(^{rd}\) prime number, 6 is 2\(^{nd}\) composite number, 7 is 4\(^{th}\) prime number, 8 is 3\(^{rd}\) composite number, 9 is 4\(^{th}\) composite number, 10 is 5\(^{th}\) composite number, 11 is 5\(^{th}\) prime number, 12 is 6\(^{th}\) composite number, and our desired 13 is 6\(^{th}\) prime number.
The $\sigma = \frac{1}{2}$ NTZ from Eq. (2.1) – $\sigma \neq \frac{1}{2}$ (non-existent) virtual NTZ from Eq. (2.2)

**Pairing.** We create this metaphoric pairing: For $i = 1, 2, 3, \ldots$, let mutually exclusive $i^{th}$ NTZ and $i^{th}$ virtual NTZ = NTZ$_i$ and vNTZ$_i$, and $i^{th}$ NTZ and $i^{th}$ virtual NTZ gaps = NTZ-Gap$_i$ and vNTZ-Gap$_i$. The positions of NTZ$_i$ and vNTZ$_i$ are IP and their dependence is seen as Eq. (2.1) and Eq. (2.2) being identical except for their different associated $\sigma$ values.

<table>
<thead>
<tr>
<th>$C_i$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>.....</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$Gap$_i$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
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4. The exact and inexact Dimensional analysis homogeneity for Equations

For ‘base quantities’ length, mass and time; their fundamental SI ‘units of measurement’ meter (m) is defined as distance travelled by light in vacuum for time interval $1/299,792,458 \text{ s}$ with speed of light $c = 299,792,458 \text{ ms}^{-1}$, kilogram (kg) is defined by taking fixed numerical value Planck constant $h$ to be $6.626 070 15 \times 10^{-34} \text{ Joules-second (Js)}$ [whereby Js is equal to $\text{kgm}^2\text{s}^{-1}$] and second (s) is defined in terms of $\Delta vCs = \Delta(133\text{Cs})_{hfs} = 9,192,631,770 \text{ s}^{-1}$. Derived SI units such as J and $\text{ms}^{-1}$ respectively represent ‘base quantities’ energy and velocity. ‘Dimension’ is commonly used to indicate ‘units of measurement’ in well-defined equations. DA is a traditional analytic tool with DA homogeneity and DA non-homogeneity (respectively) denoting valid and invalid equation occurring when ‘units of measurements’ for ‘base quantities’ are “balanced” and “unbalanced” across both sides of equation. E.g. equation $2 \text{ m} + 3 \text{ m} = 5 \text{ m}$ is valid but equation $2 \text{ m} + 3 \text{ kg} = 5 \text{ m}$ is invalid (respectively) manifesting DA homogeneity and non-homogeneity.

We conveniently adopt concepts from DA which are mathematically correct and valid. Let $(2n)$ and $(2n-1)$ be ‘base quantities’ in DSPL formatted in simplest forms as equations. E.g. DA on exponent $\frac{1}{2}$ in $(2n)^{\frac{1}{2}}$ when depicted in simplest form is desirable for our purpose but DA on exponent $\frac{1}{2}$ in equivalent $(2n^2)^{\frac{1}{4}}$ not depicted in simplest form is undesirable for our purpose. Fractional exponents as ‘units of measurement’ given by $(1 - \sigma)$ for equations when $\sigma = \frac{1}{2}$ coincide with exact DA homogeneity; and $(1 - \sigma)$ for equations when $\sigma \neq \frac{1}{2}$ coincide with exact DA homogeneity. Respectively, exact DA homogeneity at $\sigma = \frac{1}{2}$ denotes $\sum$ (all fractional exponents) as $2(1 - \sigma)$ equates to [exact] integer 1; and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ denotes $\sum$ (all fractional exponents) as $2(1 - \sigma)$ equates to [inexact] fractional number $\neq 1$ [Range: $0 < 2(1 - \sigma) < 1$ and $1 < 2(1 - \sigma) < 2$]. Computations based on exact and inexact DA homogeneity in DSPL explicitly give rise to $\sigma = \frac{1}{2}$ critical line Gram points (given indirectly as Pseudo-zeroes t-values which can be converted to Zeros t-values) and $\sigma \neq \frac{1}{2}$ non-critical lines virtual Gram points (given indirectly as virtual Pseudo-zeroes t-values which can be converted to virtual Zeros t-values). For calculations involving $2(1 - \sigma)$ or $2(-\sigma)$, it is inconsequential whether $\sigma$ values in these fractional exponents are depicted in simplest form or not in simplest form. Performing exact and inexact DA homogeneity on sim-$\eta(s)$ is valid. With same ‘base quantities’, fractional exponents as ‘units of measurement’ are now given by $(-\sigma)$. Respectively, exact DA homogeneity at $\sigma = \frac{1}{2}$ denotes $\sum$ (all fractional exponents) as $2(-\sigma)$ equates to [exact] integer $-1$; and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ denotes $\sum$ (all fractional exponents) as $2(-\sigma)$ equates to [inexact] fractional number $\neq -1$ [Range: $-2 < 2(-\sigma) < -1$ and $-1 < 2(-\sigma) < 0$]. Computations using sim-$\eta(s)$ [when interpreted as Riemann sum] explicitly give rise to $\sigma = \frac{1}{2}$ critical line Gram points (given as Zeros t-values) while representing exact DA homogeneity and $\sigma \neq \frac{1}{2}$ non-critical lines virtual Gram points (given as virtual Zeros t-values) while representing inexact DA homogeneity.
5. Gauss Circle Problem and Primitive Circle Problem

Equation of a circle centered at Origin with radius $r$ and precise Area $= \pi r^2$ is given in Cartesian coordinates as $x^2 + y^2 = r^2$. The number of integer lattice points $N(r)$ on and inside a circle [viz, pairs of integers $(m,n)$ such that $m^2 + n^2 \leq r^2$] can be exactly determined by following two equations whereby $N(r)$ is considered the most accurate surrogate marker of approximate Area for a given circle. Named after German mathematician Carl Friedrich Gauss (April 30, 1777 - February 23, 1855), Gauss Circle Problem is the problem of determining how many integer lattice points as approximate Area for a given circle. For $i$ and $r = 0, 1, 2, 3, \ldots$ in terms of a sum involving the floor function, $N(r)$ can be expressed as equation

$$N(r) = 1 + 4 \sum_{i=0}^{\infty} \left( \left\lfloor \frac{r^2}{4i+1} \right\rfloor - \left\lfloor \frac{r^2}{4i+3} \right\rfloor \right)$$

whereby it is a consequence of Jacobi’s two-square theorem which follows almost immediately from the Jacobi triple product. A much simpler sum appears if sum of squares function $r_2(n)$ that is defined as number of ways of writing number $n$ as sum of two squares is used. Then, we have alternative equation $N(r) = \sum_{n=0}^{r^2} r_2(n)$.

The first few $N(r)$ values for $r = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ are $1, 5, 13, 29, 49, 81, 113, 149, \ldots$ whereby these are IP entities complying with relationship: [simple] equation for precise Area of circle $= \pi r^2$ is proportional to above two most accurate and equivalent [complex] equations for approximate Area of circle $= N(r)$.

$N(r)$ is closely connected with Leibniz series since $\frac{1}{4} \left| \frac{N(r)}{r^2} - \frac{1}{r^2} \right| = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} \cdots \pm \frac{1}{r} \pm \frac{E(r)}{r} = \frac{1}{4} [\pi + 2\Phi(-1,1,\frac{1}{2} + r)] \pm \frac{E(r)}{r}$, where $E(r)$ is an error term, $\Phi(z,s,a)$ is a Lerch transcendent and $\psi_0(x)$ is a digamma function, so taking the limit $r \to \infty$ gives $\pi = \frac{1}{4} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$. Gauss showed $N(r) = \pi r^2 + E(r)$, where $|E(r)| \leq 2\sqrt{2\pi r}$. The identity

$$N(x) - \frac{r_2(x^2)}{2} = \pi x^2 + x \sum_{n=1}^{\infty} \frac{r_2(n)}{n} J_1(2\pi x \sqrt{n})$$

is implicitly related to number of integer lattice points, $N(r)$, where $J_1$ denotes Bessel function of first kind with order 1. It was discovered by English mathematician Godfrey H. Hardy (February 7, 1877 - December 1, 1947)[2].

Primitive Circle Problem as least accurate surrogate marker of approximate Area for a given circle involves calculating the number of coprime integer solutions $(m,n)$ to the inequality $m^2 + n^2 \leq r^2$. If the number of such solutions is denoted $V(r)$ then the values of $V(r)$ for $r$ taking small integer values are $0, 4, 8, 16, 32, 48, 72, 88, 120, 152, 192, \ldots$. Using the same ideas as usual Gauss Circle Problem and the fact that probability two integers are coprime is $\frac{6}{\pi^2}$, it is relatively straightforward to show $V(r) = \frac{6}{\pi^2} r^2 + O(r^{1+\epsilon})$. We solve problematic part of Primitive Circle Problem by reducing the exponent in the error term. This exponent is presently best known to be $221/304 + \varepsilon$ since we can now validly assume Riemann hypothesis to be true in this paper.

**Remark 8** Let $A$ denote Area of a given circle with radius $r$. The computed precise $A$ using $A = \pi r^2$ method, computed approximate $A$ using [most accurate] approximate $N(r)$ method of Gauss Circle Problem and computed approximate $A$ using [least accurate] approximate $A(r)$ method of Primitive Circle Problem will explicitly confirm $A \propto r^2$ for all three methods.
6. Gauss Areas of Varying Loops and Principle of Maximum Density for Integer Number Solutions

We translate concepts from Gauss Circle Problem and Primitive Circle Problem outlined in section 5 onto Gauss Areas of Varying Loops to fully support all materials below.

**Lemma 1.** We can validly and fully demonstrate that only when \( \sigma = \frac{1}{2} \) [and not when \( \sigma \neq \frac{1}{2} \)] in \( \Delta n \rightarrow 1 \) or \( n \) classically involving all integers \( \geq 1 \) in DSPL as \( \Delta n \rightarrow 0 \); their base quantities \( (2n) \) and \( (2n-1) \), respectively, generate CIS even numbers commencing from 2 and CIS odd numbers commencing from 1. These base quantities are subjected to algebraic function square roots at \( \sigma = \frac{1}{2} \) critical line [viz, when \( \sigma = \frac{1}{2} \)] and cube roots at \( \sigma = \frac{2}{3} \) non-critical line or twice cube roots at \( \sigma = \frac{3}{4} \) non-critical line [viz, when \( \sigma \neq \frac{1}{2} \)] thus giving rise to corresponding subset of rational roots and subset of irrational roots. Relevant to Remark 9, we now concentrate on combined \( (2n) \)’s and \( (2n-1) \)’s obtained integer lattice points \( \geq 1 \) to derive solitary subset of rational roots for \( n = 1 \) to 100 range in \( \text{sim-}\eta(s) \) or DSPL when:

(I) \( \sigma = \frac{1}{2} \) involving a **neither even nor odd function** with no symmetry viz, \( f(-n) \neq f(n) \) and \( f(-n) \neq -f(n) \) by applying \( f(n) \) as fractional exponent \( \frac{1}{2} \) or square root on \( n \) = ten perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 giving rise to (maximum) ten rational roots as consecutive integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

(II) \( \sigma = \frac{1}{2} \) involving a **odd function** with Origin symmetry viz, \( f(-n) = -f(n) \) by applying \( f(n) \) as fractional exponent \( \frac{1}{2} \) or cube root on \( n \) = four perfect cubes 1, 8, 27, 64 giving rise to (non-maximum) four rational roots as consecutive integer solutions 1, 2, 3, 4.

(III) \( \sigma = \frac{2}{3} \) involving an **even function** with y-axis symmetry viz, \( f(-n) = f(n) \) by applying \( f(n) \) as fractional exponent \( \frac{2}{3} \) or squared cube root on \( n \) = four perfect cubes 1, 8, 27, 64 giving rise to (non-maximum) four rational roots as non-consecutive integer solutions 1, 4, 9, 16.

**Remark 9.** Only at \( \sigma = \frac{1}{2} \) critical line which involves applying \( f(n) \) as fractional exponent \( \frac{1}{2} \) or square root on \( n \) = all perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100... will we obtain maximum number of rational roots as consecutive integer solutions 1, 2, 3, 4, 5, 6, 7, 8, 9, 10... (viz, all integers \( \geq 1 \)). This observation uniquely comply with **Principle of Maximum Density for Integer Number Solutions** at \( \sigma = \frac{1}{2} \) critical line.

The proof is now complete for Lemma 1.\( \square \).

Notation: **Term-(2n)** denote \( (2n) \)-complex term with algebraic functions \( X \) \( (2n) \)-complex term with transcendental functions; and **Term-(2n-1)** denote \( (2n-1) \)-complex term with algebraic functions \( X \) \( (2n-1) \)-complex term with transcendental functions. \( \text{sim-}\eta(s) \) or DSPL is complex function or law with single variable \( n \) and parameters \( \sigma \), t. Their derived equations [Eqs. (2.1) to (2.8)] have **(2n)-** or **(2n-1)-complex term with algebraic functions** consisting of powers, fractional powers, root extraction and scaled amplitude \( R \) that are dependent on parameter \( \sigma \), and **(2n)-** or **(2n-1)-complex term with transcendental functions** consisting of sine, cosine, single cosine wave, single sine wave, natural logarithm that are independent of parameter \( \sigma \).

**Remark 10.** Corresponding to Areas of Varying Loops = 0 in \( f(n) \text{ sim-}\eta(s) \) or \( F(n) \) DSPL, **Term-(2n)** must precisely cancel **Term-(2n-1)** in order to obtain \( \sigma = \frac{1}{2} \) \( f(n) \)'s Zeroes and \( F(n) \)'s Pseudo-zeroes or to obtain \( \sigma \neq \frac{1}{2} \) \( f(n) \)'s virtual Zeroes and \( F(n) \)'s virtual Pseudo-zeroes.
In sim-$\eta(s)$ or DSPL, the computed CIS rational roots (subset) as integers [rational numbers] + CIS irrational roots (subset) as irrational numbers = CIS total roots.

**Remark 11.** Complex function $F(n) = DSPL$ [representative of precise Area under the Curve] generates the most accurate precise Areas of Varying Loops [when all rational and irrational roots from combined base quantities $(2n)$ and $(2n-1)$ are utilized] and the least accurate precise Areas of Varying Loops [when only rational roots from combined base quantities $(2n)$ and $(2n-1)$ are utilized]; and complex function $f(n) = sim-\eta(s)$ when interpreted as Riemann sum [representative of approximate Area under the Curve] generates the most accurate approximate Areas of Varying Loops [when all rational and irrational roots from combined base quantities $(2n)$ and $(2n-1)$ are utilized] and the least accurate approximate Areas of Varying Loops [when only rational roots from combined base quantities $(2n)$ and $(2n-1)$ are utilized].

Our [metaphoric] varying radius $r$ in sim-$\eta(s)$ or DSPL is defined as $r = \text{Term-(2n)} - \text{Term-(2n-1)}$ whereby perpetually recurring $r = 0$ will correspond to Areas of Varying Loops = 0 in order to obtain $\sigma = \frac{1}{2}$, $f(n)$’s Zeroes and $F(n)$’s Pseudo-zeroes or to obtain $\sigma \neq \frac{1}{2}$, $f(n)$’s virtual Zeroes and $F(n)$’s virtual Pseudo-zeroes. In effect, Areas of Varying Loops is conceptionally synonymous with varying radius $r$ whereby varying radius $r$ could also be visualized as [metaphoric] varying distance $d$ between Term-(2n) and Term-(2n-1).

**Remark 12.** Whether involving the most accurate method using total roots or the least accurate method using rational roots to determine DSPL’s precise or sim-$\eta(s)$’s approximate Areas of Varying Loops, we can explicitly conclude all these (infinitely many) obtained Areas of Varying Loops $\propto$ varying radius $r$ with these Varying Loops being synthesized in a perfectly dynamic, cyclical and IP manner.

7. **Simple observation on Shift of Varying Loops in $\zeta(\sigma+it)$ Polar Graph and Principle of Equidistant for Multiplicative Inverse including the General Equations for simplified Dirichlet eta function and Dirichlet Sigma-Power Law**

We reiterate that Gram[$x=0,y=0$] points, Gram[$y=0$] points and Gram[$x=0$] points are three types of IP Gram points [Zeroes] occurring at $\sigma = \frac{1}{2}$ critical line (Figure 2) based on, respectively, Origin intercept points, x-axis intercept points and y-axis intercept points. They can be dependently computed from relevant types of sim-$\eta(s) = 0$ equations whereby sim-$\eta(s)$ is obtained by applying Euler formula to $\eta(s)$. Gram[$x=0,y=0$] points are synonymous with NTZ and Gram[$y=0$] points are synonymous with ‘usual’ Gram points. Virtual Gram[$y=0$] points and virtual Gram[$x=0$] points are two types of IP virtual Gram points [virtual Zeroes] occurring at $\sigma \neq \frac{1}{2}$ non-critical lines based on, respectively, x-axis intercept points and y-axis intercept points – see Figure 3 for $\sigma = \frac{2}{5}$ and Figure 4 for $\sigma = \frac{3}{5}$. They are also dependently computed from these same equations.

**Lemma 2.** Both $f(n)$ sim-$\eta(s)$ and $F(n)$ DSPL will manifest Principle of Equidistant for Multiplicative Inverse.

**Proof.** Let $\delta = \frac{1}{10}$. This will generate in Figure 3 and Figure 4 the $\delta$ induced shift of [infinitely many] Varying Loops in reference to Origin; viz, the simple relationship of [more negative] left-shift given by $\zeta(\frac{1}{2} - \delta + it)$ [Figure 3] < [neutral] nil-shift given by $\zeta(\frac{1}{2} + it)$ [Figure 2] < [more positive] right-shifted given by $\zeta(\frac{1}{2} + \delta + it)$ [Figure 4] will always be consistently true.

Given $\delta = \frac{1}{10}$, the $\sigma = \frac{3}{2} - \delta$ non-critical line (represented by Figure 3) and $\sigma = \frac{1}{2} + \delta$ non-critical line (represented by Figure 4) are equidistant from $\sigma = \frac{1}{2}$ critical line (represented by
The additive inverse operation of \( \sin(\delta) + \sin(-\delta) = 0 \) indicating symmetry with respect to Origin \[ \text{or } \cos(\delta) \cdot \cos(-\delta) = 0 \] indicating symmetry with respect to y-axis is not applicable to our complex single sine wave [or single cosine wave] since \((2n)\)- or \((2n-1)\)-complex term with transcendental functions consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are independent of parameter \( \sigma \). However, \((2n)\)- or \((2n-1)\)-complex term with algebraic functions consisting of powers, fractional powers, root extraction [and scaled amplitude R as alluded to by previous Remarks 3, 4 and 5 on its (in)dependency on parameter \( t \)] are dependent on parameter \( \sigma \). Let \( x = (2n) \) or \( (2n - 1) \) and \( \frac{1}{(2n)} \) or \( \frac{1}{(2n-1)} \). With multiplicative inverse operation of \( x^\delta \cdot x^{-\delta} = 1 \) or \( \frac{1}{x^\delta} \cdot \frac{1}{x^{-\delta}} = 1 \) that is applicable, this imply intrinsic presence of Multiplicative Inverse in \( \sin-\eta(s) \) or DSPL for all \( \sigma \) values with this function or law rigidly obeying relevant trigonometric identity.

**Remark 13.** This phenomenon is Principle of Equidistant for Multiplicative Inverse. We note by letting \( \delta = 0 \), we will generate Figure 2 representing \( \sigma = \frac{1}{2} \) critical line.

The proof is now complete for Lemma 2□.

For complex functions and complex equations in this paper, \( s = \sigma \pm it \) whereby we commonly invoke \( s = \sigma + it \) for discussion. For all \( f(n) \) and \( F(n) \) general equations depicted below without trigonometric identity application, we note presence of mixed sine and cosine terms in these general equations except for \( f(n)’s \) Gram\[y=0\] points-sim-\( \eta(s) \) and \( f(n)’s \) Gram\[x=0\] points-sim-\( \eta(s) \).

**I. NTZ or Gram \[x=0,y=0\] points** as geometrical Origin intercept points are mathematically defined by \( \sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0 \). General equation for \( f(n)’s \) sim-\( \eta(s) \) as Zeroes is given by \( \sum_{n=1}^{\infty} -(2n)^{-\sigma}(\sin(t \ln(2n)) - \cos(t \ln(2n))) = 0 \) \[ (7.1) \]

General equation for \( F(n)’s \) DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by \( \frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot [(2n)^{1-\sigma}(t + \sigma - 1) \sin(t \ln(2n)) + (t - \sigma + 1) \cos(t \ln(2n))] \[ \cdot \sin(t \ln(2n) - (2n - 1)^{1-\sigma}(t + \sigma - 1) \cos(t \ln(2n) - (2n - 1)^{1-\sigma}(t - \sigma + 1) \cos(t \ln(2n) + C)]_{1}^{\infty} = 0 \) \[ (7.2) \]

**II. Gram\[y=0\] points** as geometrical x-axis intercept points are mathematically defined by \( \sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + 0 \), or simply \( Im\{\eta(s)\} = 0 \). General equation for \( f(n)’s \) Gram\[y=0\] points-sim-\( \eta(s) \) as Zeroes is given by \( \sum_{n=1}^{\infty} -\sum_{n=1}^{\infty} (2n)^{-\sigma} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \sin(t \ln(2n - 1)) \[ (7.3) \]

General equation for \( F(n)’s \) Gram\[y=0\] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by \( -\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot [(2n)^{1-\sigma}(\sigma - 1) \sin(t \ln(2n)) + t \cos(t \ln(2n))] \[ - (2n - 1)^{1-\sigma}(\sigma - 1) \sin(t \ln(2n - 1)) + t \cos(t \ln(2n - 1))) + C]_{1}^{\infty} = 0 \) \[ (7.4) \]

**III. Gram\[x=0\] points** as geometrical y-axis intercept points are mathematically defined by \( \sum ReIm\{\eta(s)\} = 0 + Im\{\eta(s)\} \), or simply \( Re\{\eta(s)\} = 0 \). General equation for
f(n)’s Gram[x=0] points-sim-η(s) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{-\sigma} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \cos(t \ln(2n - 1)) = 0 \quad (7.5)
\]

General equation for F(n)’s Gram[x=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

\[
\frac{1}{2(\pi^2 + (\sigma - 1)^2)} \cdot \left[ (2n)^{-\sigma} (t \sin(t \ln(2n))) - (\sigma - 1) \cos(t \ln(2n)) \right] - \\
(2n - 1)^{-\sigma} (t \sin(t \ln(2n - 1))) - (\sigma - 1) \cos(t \ln(2n - 1)) + C \right|_{1}^{\infty} = 0 \quad (7.6)
\]

**Remark 14.** The Cartesian Coordinates (x,y) is intimately related to Polar Coordinates (r,θ) with \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \). In anti-clockwise direction, it has four quadrants defined by the + or - of (x,y); viz, Quadrant I as (+,+), Quadrant II as (-,+), Quadrant III as (-,-), and Quadrant IV as (+,-).

NTZ are Origin intercept points or Gram [x=0,y=0] points. With ‘gap’ being synonymous with ‘interval’, NTZ gap is given by initial NTZ t-value minus next NTZ t-value. Running a Full cycle from 0π to 2π, size of each IP Varying Loop in Figure 2 is proportional to magnitude of its corresponding IP NTZ varying gap. We note the 2π here as observed in Figure 2 [on Gram points at \( \sigma = \frac{1}{2} \)], Figure 3 [on virtual Gram points at \( \sigma = \frac{3}{5} \)] and Figure 4 [on virtual Gram points at \( \sigma = \frac{3}{5} \)] refers to IP Varying Loops transversed by parameter t with NTZ (Gram [x=0,y=0] points) corresponding to t values as Origin intercept on Origin’s solitary (0,0) part (point); Gram [y=0] points and virtual Gram [y=0] points corresponding to t values as x-axis intercept on x-axis’ (+ve) 0π part and (-ve) 1π part; and Gram [x=0] points and virtual Gram [x=0] points corresponding to t values as y-axis intercept on y-axis’ (+ve) π2 part and (-ve) 3π2 part. Virtual NTZ entities do not exist; viz, Origin intercept points do not occur in Figure 3 and Figure 4.

With η(s) being proxy function for ζ(s), NTZ are defined by η(s) = 0 or sim-η(s) = 0. This mathematically-defined NTZ (or Gram[x=0,y=0] points) are precisely equivalent to the geometrically-defined Origin intercept points. Then, NTZ given by relevant computed IP t values are validly deduced to be infinite in magnitude since the sim-η(s) = 0 equation contains [complex] sine and/or cosine functions which are well-defined continuous functions having infinitely many computed Origin intercept points located on infinitely many Varying Loops generated by \( 0 < t < +\infty \) or [its complex conjugate] \( -\infty < t < 0 \) domain with unlimited range.

Riemann hypothesis is the original 1859-dated conjecture that all NTZ are located on \( \sigma = \frac{1}{2} \) critical line of ζ(s). Mathematically proving all NTZ location on critical line as denoted by solitary \( \sigma = \frac{1}{2} \) value equates to geometrically proving all Origin intercept points occurrence at solitary \( \sigma = \frac{1}{2} \) value. Both result in rigorous proof for Riemann hypothesis. Locations of first 10,000,000,000,000 NTZ on critical line have previously been computed to be correct. Hardy[2], and with Littlewood[3], showed infinitely many NTZ on \( \sigma = \frac{1}{2} \) critical line by considering moments of certain functions related to ζ(s).

**Remark 15.** This discovery by Hardy and Littlewood showing infinitely many NTZ on \( \sigma = \frac{1}{2} \) critical line cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of NTZ located away from critical line [when \( \sigma \neq \frac{1}{2} \)]. Furthermore, it is literally a mathematical impossibility (“mathematical impasse”) to be able to computationally check [in a successful manner] locations of all the infinitely many NTZ to be on the critical line.
The monumental task of solving Riemann hypothesis is completed by deriving \( F(n) \) DSPL from \( f(n) \) sim-\( \eta(s) \) with its computed Pseudo-zeroes and virtual Pseudo-zeroes which can all be converted to corresponding Zeroes and virtual Zeroes since \( F(n) \)'s IP Pseudo-zeroes and IP virtual Pseudo-zeroes \((t \text{ values}) = f(n)'s \) IP Zeroes and IP virtual Zeroes \((t \text{ values}) + \frac{\pi}{2} \) [for NTZ situation] whereby both \( f(n) \) and \( F(n) \) have parameters \( \sigma \) and \( t \). Correctly deducing exact DA homogeneity in DSPL symbolizes rigorous proof for Riemann hypothesis which is depicted as Pseudo-zeroes to Zeroes conversion that obeys relevant trigonometric identities.

Three types of \([\text{traditionally}]\) finite-interval Riemann Sums: Left / Right / Midpoint Riemann Sum uses left endpoints / right endpoints / midpoint of the subintervals. With \( n = 1, 2, 3, \ldots, \infty \) and therefore \( \Delta n = 1, \) we note \( f(n) \) can analogically be interpreted as approximate Area under the Curve \( (\text{AUC}) \) \([\text{right infinite-interval}]\) Riemann sum \( \sum_{n=1}^{\infty} f(n) \Delta n = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{2} f(n) \) + \( \sum_{n=3}^{4} f(n) + \cdots + \sum_{n=\infty-1}^{\infty} f(n) \). Corresponding solution to exact AUC improper integral \( \int_{n=1}^{n=\infty} f(n) \text{dn} \) can be validly expanded as \( \int_{n=1}^{n=2} f(n) \text{dn} + \int_{n=2}^{n=3} f(n) \text{dn} + \int_{n=3}^{n=4} f(n) \text{dn} + \cdots + \int_{n=\infty-1}^{n=\infty} f(n) \text{dn} = [F(n) + C]_{1}^{2} + [F(n) + C]_{2}^{3} + [F(n) + C]_{3}^{4} + \cdots + [F(n) + C]_{\infty-1}^{\infty} \) which, for all sufficiently large \( n \) as \( n \rightarrow \infty \), will manifest divergence by oscillation \( (\text{viz. for all sufficiently large } n \text{ as } n \rightarrow \infty, \text{this cumulative total will not diverge in a particular direction to a solitarily well-defined limit value since the [complex] sine and/or cosine terms present in sim-\( \eta(s) \) and DSPL are periodic transcendental-type functions). \) Evaluation of definite integrals Eq. (2.3) or Eq. (7.2), Eq. (2.6) or Eq. (7.4) and Eq. (2.8) or Eq. (7.6) using limit as \( n \rightarrow +\infty \) for \( 0 < t < +\infty \) enable countless computations resulting in \( t \) values for \( \text{(respectively) CIS NTZ, CIS Gram}[y=0] \) points and CIS Gram\([x=0]\) points \([\text{all as Pseudo-zeroes to Zeroes conversion}]. \) Larger \( n \) values used for computations will correspond to increasing accuracy of these entities.

**Remark 16.** Whereas exact AUC from \( F(n) \) given by DSPL = \( \int_{n=1}^{n=\infty} \text{sim} - \eta(s) \text{dn} \) and approximate AUC from \( f(n) \) given by sim-\( \eta(s) = \sum_{n=1}^{\infty} \text{sim-}\eta(s) \) \([\text{when interpreted as Riemann sum} \text{ are proportional; the Zeroes when indirectly derived from DSPL [as Pseudo-zeroes converted to Zeroes] and the Zeroes when directly derived from sim-}\eta(s) \text{ must agree with each other at } \sigma = \frac{1}{2} \text{ critical line.} \)

8. **Riemann zeta function, Dirichlet eta function, simplified Dirichlet eta function and Dirichlet Sigma-Power Law**

Named after German mathematician Adolf Hurwitz (March 26, 1859 - November 18, 1919), Hurwitz zeta function is one of the many zeta functions. It is formally defined for complex arguments \( s \) with \( \text{Re}(s) > 1 \) and \( q \) with \( \text{Re}(q) > 0 \) by \( \zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n + q)^s}. \) This series is absolutely convergent for given values of \( s \) and \( q \), and can be extended to a meromorphic function defined for all \( s \neq 1 \). With this scheme, our Riemann zeta function \( \zeta(s) \) can be equivalently given as \( \zeta(s, 1) \).

An L-function consists of a Dirichlet series with a functional equation and an Euler product. Examples of L-functions come from modular forms, elliptic curves, number fields, and Dirichlet characters, as well as more generally from automorphic forms, algebraic varieties, and Artin representations. They form an integrated component of ‘L-functions and Modular Forms Database’ (LMFDB) with far-reaching implications. In perspective, \( \zeta(s) \), being the simplest
example of an L-function, is a function of complex variable \( s = \sigma \pm it \) that analytically continues sum of infinite series \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \). The common convention is to write \( s \) as \( \sigma \pm it \) with \( t = \sqrt{-1} \), and with \( \sigma \) and \( t \) real. Valid for \( \sigma > 0 \), we write \( \zeta(s) \) as \( \text{Re}\{\zeta(s)\} + i\text{Im}\{\zeta(s)\} \) and note that \( \zeta(\sigma + it) \) when \( 0 < t < +\infty \) is the complex conjugate of \( \zeta(\sigma - it) \) when \(-\infty < t < 0 \).

Also known as alternating zeta function, \( \eta(s) \) must act as proxy for \( \zeta(s) \) in critical strip (viz. \( 0 < \sigma < 1 \)) containing critical line (viz. \( \sigma = \frac{1}{2} \)) because \( \zeta(s) \) only converges when \( \sigma > 1 \). This implies \( \zeta(s) \) is undefined to left of this region in critical strip which then requires \( \eta(s) \) representation instead. They are related to each other as \( \zeta(s) = \gamma \cdot \eta(s) \) with proportionality factor \( \gamma = \frac{1}{(1 - 2^{1-s})} \) and \( \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \).

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} = \frac{1}{(1 - 2^{-s}) \cdot (1 - 3^{-s}) \cdot (1 - 5^{-s}) \cdot (1 - 7^{-s}) \cdot (1 - 11^{-s}) \cdots (1 - p^{-s}) \cdots}
\]

Eq. (8.1) is defined for only \( 1 < \sigma < \infty \) region where \( \zeta(s) \) is absolutely convergent with no zeros located here. In Eq. (8.1), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents \( \zeta(s) \Rightarrow \) all prime and, by default, composite numbers are (intrinsically) encoded in \( \zeta(s) \). Brief diversion: On April 17, 2013, Zhang[5] announced a ground-breaking proof stating there are infinitely many pairs of prime numbers that differ by 70 million or less. This result implies the existence of an infinitely repeatable prime 2-tuple, thus establishing a theorem akin to the twin prime conjecture.

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \cdot \Gamma(1 - s) \cdot \zeta(1 - s)
\]  

(8.2)

With \( \sigma = \frac{1}{2} \) as symmetry line of reflection, Eq. (8.2) is Riemann’s functional equation valid for \(-\infty < \sigma < \infty \). It can be used to find all trivial zeros on horizontal line at \( st = 0 \) occurring when \( \sigma = -2, -4, -6, -8, -10, \ldots, \infty \) whereby \( \zeta(s) = 0 \) because factor \( \sin(\frac{\pi s}{2}) \) vanishes. \( \Gamma \) is gamma function, an extension of factorial function [a product function denoted by \( ! \) notation whereby \( n! = n(n-1)(n-2)\ldots(n-(n-1)) \)] with its argument shifted down by 1, to real and complex numbers. That is, if \( n \) is a positive integer, \( \Gamma(n) = (n-1)! \).

\[
\zeta(s) = \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{(1 - 2^{1-s})} \left( \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \right)
\]  

(8.3)

Eq. (8.3) is defined for all \( \sigma > 0 \) values except for simple pole at \( \sigma = 1 \). As alluded to above, \( \zeta(s) \) without \( \frac{1}{(1 - 2^{1-s})} \) viz. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \) is \( \eta(s) \). It is a holomorphic function of \( s \) defined by analytic continuation and is mathematically defined at \( \sigma = 1 \) whereby analogous trivial
zeros with presence for \( \eta(s) \) [but not for \( \zeta(s) \)] on vertical straight line \( \sigma = 1 \) are found at 
\[
s = 1 \pm \frac{2\pi k}{\ln(2)} \quad \text{where } k = 1, 2, 3, 4, \ldots, \infty.
\]

Euler formula can be stated as 
\[ e^{i\pi} = \cos \pi + i \cdot \sin \pi = -1 + 0 \quad \text{or stated as } e^{i\pi} + 1 = 0. \]
The \( n^s \) of \( \zeta(s) \) is expanded to \( n^s = n^{(\sigma+i\epsilon)} = n^\sigma e^{i\epsilon \ln(n)} \).

Apply Euler formula to \( n^s \) result in \( n^s = n^\sigma (\cos(t \ln(n)) + i \cdot \sin(t \ln(n))). \)
This is written in trigonometric form [designated by short-hand notation \( n^s(\text{Euler}) \)] whereby \( n^\sigma \) is modulus and \( t \ln(n) \) is polar angle (argument).

We apply \( n^s(\text{Euler}) \) to Eq. (8.3) to obtain \( f(n) \) general sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) NTZ versus (non-existent) \( \sigma \neq \frac{1}{2} \) virtual NTZ[1], section 4, p. 24 - 28. This is given as Eq. (7.1) and with relevant trigonometric identity application [at \( \sigma = \frac{1}{2} \)] as Eq. (2.1). Integrate \( f(n) \) general sim-\( \eta(s) \) to obtain \( F(n) \) general DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeroes versus (non-existent) \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeroes. Pseudo-zeroes and (non-existent) virtual Pseudo-zeroes can be converted to Zeroses (NTZ) and (non-existent) virtual Zeros (virtual NTZ).
This is given as Eq. (7.2) and with relevant trigonometric identity application [at \( \sigma = \frac{1}{2} \)] as Eq. (2.3).

We provide \( f(n) \) general Gram[\( y=0 \)] points-sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) Gram[\( y=0 \)] points versus \( \sigma \neq \frac{1}{2} \) virtual Gram[\( y=0 \)] points[1], section 5, p. 28 - 30. This is given as Eq. (7.3) but we are unable to apply relevant trigonometric identity. Integrate \( f(n) \) general Gram[\( y=0 \)] points-sim-\( \eta(s) \) to obtain \( F(n) \) general Gram[\( y=0 \)] points-DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeroes versus \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeros (Gram[\( y=0 \)] points) and virtual Zeros (virtual Gram[\( y=0 \)] points). This is given as Eq. (7.4) and with relevant trigonometric identity application [at \( \sigma = \frac{1}{2} \)] as Eq. (2.6).

We provide \( f(n) \) general Gram[\( x=0 \)] points-sim-\( \eta(s) \) for determining \( \sigma = \frac{1}{2} \) Gram[\( x=0 \)] points versus \( \sigma \neq \frac{1}{2} \) virtual Gram[\( x=0 \)] points[1], section 5, p. 28 - 30. This is given as Eq. (7.5) but we are unable to apply relevant trigonometric identity. Integrate \( f(n) \) general Gram[\( x=0 \)] points-sim-\( \eta(s) \) to obtain \( F(n) \) general Gram[\( x=0 \)] points-DSPL for determining \( \sigma = \frac{1}{2} \) Pseudo-zeroes versus \( \sigma \neq \frac{1}{2} \) virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeros (Gram[\( x=0 \)] points) and virtual Zeros (virtual Gram[\( x=0 \)] points). This is given as Eq. (7.6) and with relevant trigonometric identity application [at \( \sigma = \frac{1}{2} \)] as Eq. (2.8).

9. Conclusions including brief discussion on p-adic Riemann zeta function \( \zeta_p(s) \)

The \( \sigma = \frac{1}{2} \) NTZ from Eq. (2.1) – \( \sigma \neq \frac{1}{2} \) (non-existent) virtual NTZ from Eq. (2.2)
Pairing outlined in section 3 crucially serve to exemplify nontrivial zeros and (non-existent) virtual nontrivial zeros as mutually exclusive and Incompletely Predictable entities. In this paper, we intrinsically treat and analyze in a de novo fashion relevant simple and complex single-variable function \( f(n) \) or \( F(n) \) and their simple and complex single-variable equation \( f(n) = 0 \) or \( F(n) = 0 \) as unique Completely Predictable or Incompletely Predictable mathematical objects. Dirichlet Sigma-Power Law symbolizes the end-product proof on Riemann hypothesis. The critical line of Riemann zeta function is denoted by \( \sigma = \frac{1}{2} \) whereby all nontrivial zeros are proposed to be located in 1859 Riemann hypothesis.

Using innovative research method of p-adic analysis; a p-adic zeta function, or more generally a p-adic L-function, is a function analogous to the Riemann zeta function, or more general L-functions, but whose domain and target are p-adic (where p is a prime number). In p-adic Riemann zeta function \( \zeta_p(s) \), values at negative odd integers are those of Riemann zeta function \( \zeta(s) \) at negative odd integers (up to an explicit correction factor). The p-adic L-functions are typically referred to as analytic p-adic L-functions. The other major source of
p-adic L-functions is from the arithmetic of cyclotomic fields, or more generally, certain Galois modules over towers of cyclotomic fields or even more general towers.

With groundings in Mathematics for Incompletely Predictable problems and invoking the discussion in preceding paragraph on p-adic zeta function, we advocate that we have now provided an elementary and rigorous proof on Riemann hypothesis while explaining existence of mutually exclusive three types of Gram points and two types of virtual Gram points. These achievements are completed with appropriate analysis on complex (meta-) properties present in Dirichlet Sigma-Power Law, Gram[y=0] points-Dirichlet Sigma-Power Law and Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant Pseudo-Gram points; and in virtual Gram[y=0] points-Dirichlet Sigma-Power Law and virtual Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant virtual Pseudo-Gram points. Exact Dimensional analysis homogeneity [occurring only once at \( \sigma = \frac{1}{2} \) critical line] in these Laws is endowed with ability to convert their computed Pseudo-zeroes to Zeroes resulting in nontrivial zeros (Origin intercept points or Gram[x=0,y=0] points) as one type of Gram points plus two remaining types of Gram points. Inexact Dimensional analysis homogeneity [occurring infinitely often at \( \sigma \neq \frac{1}{2} \) non-critical lines] in these Laws is endowed with ability to convert their computed virtual Pseudo-zeroes to virtual Zeroes resulting in two types of virtual Gram points.

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References

Appendix A. Miscellaneous materials

Cardinality: With increasing size, arbitrary Set \( X \) can be CFS, CIS or UIS. Cardinality of Set \( X \), \( |X| \), measures number of elements in Set \( X \). E.g. Set negative Gram[y=0] point has CFS of negative Gram[y=0] point with |negative Gram[y=0] point| = 1, Set even Prime number has CFS of even Prime number with |even Prime number| = 1, Set Natural numbers has CIS of Natural numbers with |Natural numbers| = \( \aleph_0 \), and Set Real numbers has UIS of Real numbers with |Real numbers| = \( c \) (cardinality of the continuum). Let \( C \) = UIS complex numbers, \( R \) = UIS real numbers, \( Q \) = CIS rational numbers that include fractional numbers and rational roots, \( R-Q \) = UIS total irrational numbers, \( A \) = CIS algebraic numbers, \( R-A \) = UIS transcendental irrational numbers, \( Z \) = CIS integers which are literally fractional numbers with denominator 1, \( W \) = CIS whole numbers, \( N \) = CIS natural numbers, \( E \) = CIS even numbers, \( O \) = CIS odd numbers, \( P \) = CIS prime numbers, and \( C \) = CIS composite numbers. CIS \( N \) = Set \( E \) [whereby we did not include the zeroth even number \( E_0 = 0 \)] + Set \( O \); CIS \( N \) = CIS \( P \) + CIS \( C \) + CFS Number 1; and CIS \( N \subset CIS W \subset \)
CIS $\mathbb{Z} \subset$ CIS $\mathbb{Q} \subset$ UIS $\mathbb{R} \subset$ CIS $\mathbb{C}$. CIS $\mathbb{A}$ as $\mathbb{C}$ (including $\mathbb{R}$) = CIS $\mathbb{Q}$ that include fractional numbers and rational roots + CIS irrational roots whereby both rational and irrational roots are derived from non-zero polynomials.

The following refined definitions are useful: UIS total irrational numbers = CIS irrational roots (numbers) + UIS transcendental irrational numbers whereby transcendental irrational numbers $\gg$ [algebraic] irrational numbers. Whereas CIS rational roots (numbers), CIS irrational roots (numbers) and UIS transcendental numbers are treated separately as mutually exclusive numbers; so must the existing algebraic functions that generate CIS rational roots (numbers) and CIS irrational roots (numbers), and the existing transcendental functions that generate UIS transcendental numbers be treated separately as mutually exclusive functions.

An algebraic function [such as rational functions, square root, cube root function, etc] satisfies a polynomial equation. A transcendental function [such as exponential function, natural logarithm, trigonometric functions, hyperbolic functions, gamma, elliptic, zeta functions, etc] is an analytic function that does not satisfy a polynomial equation. Thus a transcendental function “transcends” algebra since it cannot be expressed in terms of a finite sequence of algebraic operations consisting of addition, subtraction, multiplication, division, powers, and fractional powers or root extraction. All integers, rational numbers, rational or irrational roots of real and complex numbers are algebraic numbers e.g. a root of polynomial $x^2 - x - 1 = 0$. Golden ratio $\varphi = \frac{1 + \sqrt{5}}{2} = 1.618033\ldots$, square root of 2 viz, $\sqrt{2}$ or $\sqrt{2} = 2^{\frac{1}{2}} = 1.414213\ldots$, or cube root of 2 viz, $\sqrt[3]{2} = 2^{\frac{1}{3}} \approx 1.259921$. Real and complex numbers that are not algebraic numbers e.g. $\pi$ and $e$ are transcendental numbers. However, we note sine and cosine as transcendental functions generally give rise to mutually exclusive sets of transcendental numbers except at discrete points such as $\sin \frac{\pi}{6} = \sin 30^\circ = \cos \frac{2\pi}{6} = \cos 60^\circ = \frac{\sqrt{3}}{2} = \frac{1}{2}$ [viz, transcendental functions generating an algebraic number as rational root (number) at certain discrete points].

Following [side-note] treatise of interest involve infinite series. A property of irrational number $\sqrt{2}$ is $\frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$ since $(\sqrt{2} + 1) (\sqrt{2} - 1) = 2 - 1 = 1$. This is related to the property of silver ratios. $\sqrt{2}$ can also be expressed in terms of copies of imaginary unit $i$ using only square root and arithmetic operations, if the square root symbol is interpreted suitably for complex numbers $i$ and $-i$: $\sqrt{i + i\sqrt{i}}$ and $\sqrt{-i} - i\sqrt{-i}$. Multiplicative inverse (reciprocal) of $(2)^{\frac{1}{2}}$ or $\sqrt{2}$ is $(2)^{-\frac{1}{2}}$ or $\sqrt{\frac{1}{2}}$ which is a unique [irrational number] constant since $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2} \sqrt{2} = \cos \frac{\pi}{4} = \sin \frac{\pi}{4}$. Transcendental numbers such as $\frac{\pi}{4}$ (given by Leibniz series $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \approx 0.785398167$); and $\frac{\pi^2}{6}$ (given by $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \approx 1.6449340668482$), respectively, encode complete set of alternating odd and, by default, alternating even numbers; and natural numbers. Also known as alternating zeta function, Dirichlet eta function $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s}$ when expanded, will intrinsically encode complete set of alternating natural numbers e.g. $\eta(1) = \ln(2)$ (given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{1}{2n} [\zeta(n) - 1] + \frac{1}{2} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \approx 0.69314718056$). Equivalent Euler product formula for $\zeta(s)$ with product over prime numbers [instead of summation over natural numbers] will intrinsically encode complete set of prime and, by default, composite numbers. As an extra point, complete set of alternating prime and, by default, alternating composite numbers is encoded in
converging alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{p_k} \approx -0.2696063519$ (transcendental number) when fully expanded whereby $p_k$ is $k\text{th}$ prime number.

**Zeroes and Pseudo-zeroes:** There are three types of stationary points in a given [simple] periodic $f(n)$ involving sine and/or cosine functions that act as x-axis intercept points via three types of $f(n)$’s Zeroes with corresponding three types of $F(n)$’s Pseudo-zeroes: maximum points e.g. with $f(n)$ or $F(n) = \sin n - 1$; minimum points e.g. with $f(n)$ or $F(n) = \sin n + 1$; and points of inflection e.g. with $f(n)$ or $F(n) = \sin n$ [which also has Origin intercept point as a Zero or Pseudo-zero]. A fourth type of $f(n)$’s Zeroes and $F(n)$’s Pseudo-zeroes consist of non-stationary points occurring e.g. with $f(n)$ or $F(n) = \sin n + 0.5$. One can analogically assimilate these concepts to aesthetically explain the more “exotic” characteristics manifested by complex points occurring e.g. with $f(n)$ or $F(n) = \sin n + 0.5$. A fourth type of $f(n)$’s Zeroes and $F(n)$’s Pseudo-zeroes consist of non-stationary points occurring e.g. with $f(n)$ or $F(n) = \sin n + 0.5$. One can analogically assimilate these concepts to aesthetically explain the more “exotic” characteristics manifested by complex points occurring e.g. with $f(n)$ or $F(n) = \sin n + 0.5$.

For single-term trigonometric function $f(n) = \sin(n)$, it is an odd function with Origin symmetry since -$f(n)$ = $f(-n)$ for all $n$. The $f(n) = \sin(n)$ has an infinite number of CP x-axis intercept points (Zeroes) and a solitary unique Origin intercept point (Zero) since it belong to a class of odd functions that is defined at $n = 0$ and must pass through the Origin. Otherwise, the other class of odd functions such as $f(n) = \sin(\frac{1}{n})$ with infinite number of CP x-axis intercept points (Zeroes) but without Origin intercept point [since $\sin(\frac{1}{n})$ is undefined at $n = 0$] can remain symmetrical about the Origin without actually passing through it. For
single term trigonometric function \( f(n) = \cos(n) \) with symmetry about the \( y \)-axis, it is an even function since \( f(n) = f(-n) \) for all \( n \). It has an infinite number of CP \( x \)-axis intercept points (Zeroes). Being undefined at \( n = 0 \), it will never have Origin intercept point.

For dual terms trigonometric functions \( f(n) = \cos(n) - \sin(n) \) and \( f(n) = \cos(n) + \sin(n) \), they are neither even nor odd without any symmetry. They both have an infinite number of CP \( x \)-axis intercept points (Zeroes). Being undefined at \( n = 0 \), they will never have Origin intercept point.

Special properties are given below for Addition and Multiplication: The sum or difference of two even functions is even. The sum or difference of two odd functions is odd. The product of an even function and an odd function is an odd function. The product of two even functions is an even function. The product of two odd functions is an odd function. The sum or difference of an even and odd function is neither even nor odd unless one function is zero; viz, there is (exactly) one function that is both even and odd, and it is the zero function \( f(n) = 0 \). The product of two even functions is an even function. The product of two odd functions is an odd function. The sum or difference of an even and odd function is neither even nor odd unless one function is zero; viz, there is (exactly) one function that is both even and odd, and it is the zero function \( f(n) = 0 \).

**Trigonometric identity for linear combination of sine and cosine function:** Here, we use the notation \( f(x) \) instead of \( f(n) \). The trigonometric identity for linear combination of sine and cosine \( \cos(x) + b\sin(x) \) can be freely, arbitrarily and interchangeably written as either [simple] single cosine wave \( R\cos(x - \alpha) \) or [simple] single sine wave \( R\sin(x + \alpha) \) whereby \( R \) is the scaled amplitude and \( \alpha \) is the phase shift. \( R = \sqrt{a^2 + b^2} = (a^2 + b^2)^{\frac{1}{2}} \). Since \( \sin(\alpha) = \frac{b}{\sqrt{a^2 + b^2}} \) and \( \cos(\alpha) = \frac{a}{\sqrt{a^2 + b^2}} \), then \( \alpha = \tan^{-1} \frac{b}{a} \). Below, we assign \( \sqrt{2} \) to equivalently denote \( 2^{\frac{1}{2}} \).

With \( a = 1, b = -1, R = \sqrt{2}; \cos(x) - \sin(x) = \sqrt{2} \sin \left( x + \frac{1}{4} \right) = \sqrt{2} \sin \left( x + \frac{3}{4} \pi \right) \).

With \( a = -1, b = 1, R = \sqrt{2}; -\cos(x) + \sin(x) = \sqrt{2} \sin \left( \frac{1}{4} \right) = \sqrt{2} \cos \left( \frac{3}{4} \pi \right) \).

With \( a = 1, b = 1, R = \sqrt{2}; \cos(x) + \sin(x) = \sqrt{2} \sin \left( \frac{1}{4} \right) = \sqrt{2} \sin \left( \frac{1}{4} \pi \right) \).

With \( a = -1, b = -1, R = \sqrt{2}; -\cos(x) - \sin(x) = \sqrt{2} \cos \left( \frac{1}{4} \right) = \sqrt{2} \sin \left( \frac{3}{4} \pi \right) \).

\[
\int f(x)dx = F(x) + C \text{ with } F'(x) = f(x). \text{ With } |a| = 1 \text{ and } |b| = 1, \text{ consider single-term [simple] trigonometric functions: } f(x) = acos(x) \text{ which belongs to an even function and } f(x) = bsin(x) \text{ which belongs to an odd function. Whereas all linear combination of [simple] cos(x) and [simple] sin(x) as sum or difference such as } f(x) = \cos(x) + \sin(x) \text{ and } f(x) = \cos(x) - \sin(x) \text{ belong to neither even nor odd functions, then their corresponding } F(x) \text{ being linear combination of [simple] cos(x) and [simple] sin(x) as sum or difference must also belong to neither even nor odd functions. With both } f(x) \text{ and corresponding } F(x) \text{ considered as [simple] functions and relevant trigonometric identities being applied, they can intrinsically and arbitrarily be expressed as either [simple] single cosine wave or [simple] single sine wave containing a phase shift } \frac{1}{4} \pi \text{ or } \frac{3}{4} \pi \text{ and a scaled amplitude } \sqrt{2} = 2^{\frac{1}{2}} \text{ which is base 2 endowed with exponent } \frac{1}{2}. \text{ Respectively, } F(x) \text{ and } f(x) \text{ have an infinite number of x-axis intercept points called Pseudo-zeroes and Zeroes but nil Origin intercept points.}

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