

# Values of the Barnes function

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## Abstract

In this paper, we give conjectural values for the Barnes function  $G$  at unusual points as  $G(1/5)$ ,  $G(2/5)$  or  $G(3/10)$  for example.

Expressions for  $G(k/2)$ ,  $G(k/3)$ ,  $G(k/4)$  or  $G(k/6)$  with  $k$  natural integer are well-known but what about  $G(k/5)$  or  $G(k/10)$ ?

We give eight conjectural integral formulas and we see several applications.

There are 2 cases:

- Either there is an integral with the bounds ( 0 and  $-1/5$  ) and two trigamma functions.

- Or there are just only two trigamma functions at  $1/5$  and at  $2/5$ .

(1)

## 1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G(1+z) = (2\pi)^{\frac{z}{2}} e^{-\frac{z(1+z)}{2} - \frac{\gamma z^2}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k e^{-z + \frac{z^2}{2k}} \quad (2)$$

where gamma is the Euler-Mascheroni constant.

The following properties of  $G$  are well-known.

## 2 Properties

$$G(1) = 1 \quad (3)$$

$$G(1+z) = G(z)\Gamma(z) \quad (4)$$

$$\log(G(1+z)) = \frac{z \log(2\pi)}{2} - \frac{z(1+z)}{2} + z \log(\Gamma(1+z)) - \int_0^z \log(\Gamma(t+1)) dt \quad (5)$$

$$\int_0^z \log(\Gamma(t+1)) dt = \frac{z \log(2\pi)}{2} - \frac{z(1+z)}{2} + z \log(\Gamma(1+z)) - \log(G(z)) - \log(\Gamma(z)) \quad (6)$$

## 3 List of conjectural formulas (found experimentally)

Let  $A$  be the Glaisher–Kinkelin constant. Then

$$\begin{aligned} \log\left(G\left(\frac{1}{5}\right)\right) &= \frac{2}{25} - \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{\log(\pi)}{10} - \frac{\log(2)}{10} \\ &\quad + \frac{4\pi^2 + 4\pi^2\sqrt{5} - 2\sqrt{5}\Psi(1, 2/5) + \Psi(1, \frac{1}{5})(-\sqrt{5}-5)}{50\pi\sqrt{10+2\sqrt{5}}} - \frac{4\log(\Gamma(\frac{1}{5}))}{5} \end{aligned}$$

$$\begin{aligned} \log\left(G\left(\frac{2}{5}\right)\right) &= \frac{3}{25} + \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{12\log(A)}{5} + \frac{\log(\pi)}{10} + \frac{\log(2)}{10} + \frac{\log(5)}{120} \\ &\quad + \frac{(5-\sqrt{5})\Psi(1, \frac{1}{5}) + (3\sqrt{5}+5)\Psi(1, \frac{2}{5}) - 2\pi^2\sqrt{5} - 6\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} - \frac{3\log(\Gamma(\frac{2}{5}))}{5} \end{aligned}$$

$$\begin{aligned} \log\left(G\left(\frac{3}{5}\right)\right) &= \frac{3}{25} + \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{12\log(A)}{5} - \frac{3\log(\pi)}{10} - \frac{\log(2)}{2} + \frac{13\log(5)}{120} \\ &\quad + \frac{\log(\sqrt{5}+1)}{5} + \frac{(3\sqrt{5}+5)\Psi(1, \frac{1}{5}) + (\sqrt{5}-5)\Psi(1, \frac{2}{5}) - 6\pi^2\sqrt{5} - 2\pi^2}{100\sqrt{10+2\sqrt{5}}\pi} + \frac{2\log(\Gamma(\frac{2}{5}))}{5} \end{aligned}$$

$$\begin{aligned} \log\left(G\left(\frac{4}{5}\right)\right) &= \frac{2}{25} - \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{3\log(\pi)}{10} - \frac{\log(2)}{5} + \frac{\log(5)}{20} \\ &\quad - \frac{\log(\sqrt{5}+1)}{10} + \frac{\log(\Gamma(\frac{1}{5}))}{5} \end{aligned}$$

$$\begin{aligned}
\log \left( G \left( \frac{1}{10} \right) \right) &= \frac{9}{200} - \frac{3}{2} \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt + \frac{9 \log(A)}{10} + \frac{3 \log(\pi)}{10} \\
&+ \frac{289 \log(2)}{600} - \frac{7 \log(5)}{30} - \frac{9 \log(\sqrt{5}+1)}{20} \\
&+ \frac{(-4\sqrt{5}-10) \Psi(1, \frac{1}{5}) + (-3\sqrt{5}+5) \Psi(1, \frac{2}{5}) + 10 \pi^2 \sqrt{5} + 6 \pi^2}{100 \sqrt{10+2\sqrt{5}} \pi} \\
&- \frac{9 \log(\Gamma(\frac{1}{5}) \Gamma(\frac{2}{5}))}{10}
\end{aligned}$$

$$\begin{aligned}
\log \left( G \left( \frac{3}{10} \right) \right) &= \frac{21}{200} + \frac{3}{2} \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{27 \log(A)}{10} - \frac{\log(\pi)}{5} \\
&- \frac{509 \log(2)}{600} + \frac{\log(5)}{240} + \frac{7 \log(\sqrt{5}+1)}{10} \\
&+ \frac{(3\sqrt{5}+5) \Psi(1, \frac{1}{5}) + (\sqrt{5}-5) \Psi(1, \frac{2}{5}) - 6 \pi^2 \sqrt{5} - 2 \pi^2}{200 \sqrt{10+2\sqrt{5}} \pi} \\
&+ \frac{7 \log \left( \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})} \right)}{10}
\end{aligned}$$

$$\begin{aligned}
\log \left( G \left( \frac{7}{10} \right) \right) &= \frac{21}{200} + \frac{3}{2} \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt - \frac{27 \log(A)}{10} - \frac{29 \log(2)}{600} + \frac{\log(5)}{240} \\
&+ \frac{(3\sqrt{5}+25) \Psi(1, \frac{1}{5}) + (11\sqrt{5}+5) \Psi(1, \frac{2}{5}) - 18 \pi^2 \sqrt{5} - 22 \pi^2}{200 \sqrt{10+2\sqrt{5}} \pi} \\
&- \frac{3 \log \left( \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})} \right)}{10}
\end{aligned}$$

$$\begin{aligned}
\log \left( G \left( \frac{9}{10} \right) \right) &= \frac{9}{200} - \frac{3}{2} \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt + \frac{9 \log(A)}{10} - \frac{3 \log(\pi)}{10} \\
&- \frac{131 \log(2)}{600} + \frac{\log(5)}{60} - \frac{\log(\sqrt{5}+1)}{20} \\
&+ \frac{(\sqrt{5}-5) \Psi(1, \frac{1}{5}) + (-3\sqrt{5}-5) \Psi(1, \frac{2}{5}) + 2 \pi^2 \sqrt{5} + 6 \pi^2}{100 \sqrt{10+2\sqrt{5}} \pi} \\
&+ \frac{\log(\Gamma(\frac{1}{5}) \Gamma(\frac{2}{5}))}{10}
\end{aligned}$$

## 4 Examples of applications

First case

Consider the sum

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1) z^{2k+2} (2^{2k+1} - 1)}{k+1} \quad (8)$$

and let  $z = 3/10$ . We obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1) (2^{2k+1} - 1)}{k+1} \left(\frac{3}{10}\right)^{2k+2} = \\ -\frac{9}{100} - \frac{9\gamma}{100} - \frac{29 \log(2)}{300} - \frac{\log(\Gamma(\frac{4}{5}))}{2} - \frac{\log(\Gamma(\frac{1}{5}))}{2} \\ - \log\left(G\left(\frac{1}{5}\right)\right) - \log\left(G\left(\frac{4}{5}\right)\right) + 3\zeta(1, -1). \end{aligned}$$

With the definition (5), we have:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1) (2^{2k+1} - 1)}{k+1} \left(\frac{3}{10}\right)^{2k+2} = \int_0^{\frac{1}{5}} \log(\Gamma(t+1)) dt + \int_0^{\frac{4}{5}} \log(\Gamma(t+1)) dt \\ + 3\zeta(1, -1) + \frac{3}{4} - \frac{9\gamma}{100} - \frac{4 \log(\pi)}{5} \\ - \frac{176 \log(2)}{75} + \frac{43 \log(5)}{40} - \frac{3 \log(\sqrt{5}+1)}{20} + \frac{3 \log(\Gamma(\frac{1}{5}))}{5} \end{aligned}$$

With the rule (6) and using formulas, finally

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1) (2^{2k+1} - 1)}{k+1} \left(\frac{3}{10}\right)^{2k+2} = 2 \int_0^{-1/5} \log(\Gamma(t+1)) dt - 3 \log(A) - \frac{9\gamma}{100} \\ - \frac{\log(\pi)}{10} - \frac{7 \log(2)}{150} + \frac{3 \log(5)}{40} - \frac{3 \log(\sqrt{5}+1)}{20} \\ + \frac{(5 + \sqrt{5}) \Psi(1, \frac{1}{5}) + 2\sqrt{5} \Psi(1, 2/5) - 4\pi^2 - 4\pi^2 \sqrt{5}}{50 \sqrt{10} + 2\sqrt{5}\pi} \\ + \frac{3 \log(\Gamma(\frac{1}{5}))}{5} \end{aligned}$$

### Other example

Consider the sum

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1) z^{2k+2}}{k+1} \quad (9)$$

and let  $z = 9/10$ . We obtain

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} \left(\frac{9}{10}\right)^{2k+2} = -\frac{81}{100} - \frac{81\gamma}{100} - \log\left(G\left(\frac{1}{10}\right) G\left(\frac{19}{10}\right)\right)$$

With the definition (5), we have:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} \left(\frac{9}{10}\right)^{2k+2} &= \int_0^{\frac{9}{10}} \log(\Gamma(t+1)) dt + \int_0^{-\frac{9}{10}} \log(\Gamma(t+1)) dt - \frac{81\gamma}{100} \\ &\quad - \frac{9 \log(\pi)}{5} - \frac{9 \log(2)}{25} - \frac{9 \log(3)}{5} \\ &\quad + \frac{27 \log(5)}{20} + \frac{9 \log\left(\Gamma\left(\frac{1}{5}\right) \Gamma\left(\frac{2}{5}\right)\right)}{5} \end{aligned}$$

With the rule (6) and using formulas, finally

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} \left(\frac{9}{10}\right)^{2k+2} &= 3 \int_0^{-1/5} \log(\Gamma(t+1)) dt - \frac{9 \log(A)}{5} - \frac{9}{10} - \frac{81\gamma}{100} \\ &\quad - \frac{3 \log(\pi)}{2} - \frac{289 \log(2)}{300} + \frac{7 \log(5)}{15} \\ &\quad + \frac{(3\sqrt{5}+15)\Psi\left(1, \frac{1}{5}\right) + 6\Psi\left(1, 2/5\right)\sqrt{5} - 12\pi^2\sqrt{5} - 12\pi^2}{100\sqrt{10} + 2\sqrt{5}\pi} \\ &\quad + \frac{9 \log\left(\Gamma\left(\frac{1}{5}\right) \Gamma\left(\frac{2}{5}\right)\right)}{5} \end{aligned}$$

**Second case** (there are only two trigamma functions at the end)

Consider and calculate the closed form of

$$2 \int_0^{-\frac{9}{10}} \log(\Gamma(t+1)) dt + 3 \int_0^{-3/5} \log(\Gamma(t+1)) dt$$

We obtain

$$\begin{aligned} &\frac{27 \log(A)}{5} - \frac{9 \log(\pi)}{5} - \frac{541 \log(2)}{300} - \frac{\log(5)}{120} \\ &+ \frac{(11\sqrt{5}+5)\Psi\left(1, \frac{1}{5}\right) + (-3\sqrt{5}-25)\Psi\left(1, \frac{2}{5}\right) - 14\pi^2\sqrt{5} + 6\pi^2}{100\sqrt{10} + 2\sqrt{5}\pi} \end{aligned}$$

Other example with three integrals

Consider and calculate the closed form of

$$6 \int_0^{\frac{7}{10}} \log(\Gamma(t+1)) dt - 4 \int_0^{\frac{13}{10}} \log(\Gamma(t+1)) dt + 3 \int_0^{9/5} \log(\Gamma(t+1)) dt$$

We obtain

$$\begin{aligned} & \frac{27 \log(A)}{5} - \frac{28}{5} + \frac{11 \log(\pi)}{5} + \frac{2771 \log(2)}{300} \\ & + \frac{48 \log(3)}{5} - \frac{673 \log(5)}{120} + \frac{21 \log(7)}{5} - \frac{26 \log(13)}{5} \\ & + \frac{(-3\sqrt{5} - 65) \Psi\left(1, \frac{1}{5}\right) + (-31\sqrt{5} - 25) \Psi\left(1, \frac{2}{5}\right) + 42\pi^2\sqrt{5} + 62\pi^2}{100\sqrt{10} + 2\sqrt{5}\pi} \end{aligned}$$

**Remark and conclusion:** this is a first approach with these particular values of the Barnes function and any improvement for these eight expressions is welcome.

## 5 References

(1): <https://en.wikipedia.org/wiki/Trigamma-function>

(2),(3),(4),(5) and (6): <https://dlmf.nist.gov/5.17>

(7): <https://mathworld.wolfram.com/Glaisher-KinkelinConstant.html>

(8) and (9): Junesang Choi and H.M Srivastava, Zeta and q-zeta Functions and Associated Series and Integrals.