A Lagrangian Proof of the Einsteinian Equivalence between the Mass and the Internal Energy: Additional Analysis

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Abstract

This article contains additional analysis to my article viXra: 2006.0022 “A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the internal energy."

Sommaire

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1. A small reminder of the conclusion of "A straightforward and Lagrangian proof of the Einsteinian equivalence between the mass and the inernal energy (i.e. rest energy) V2"

We have a way to demonstrate the famous Einstein formula $E^*=Mc^2$ directly from an appropriate Lagrangian function selecting the correct variables.

Instead of $L\left(\{\boldsymbol{r_a}\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\}\right)$, we use $L'\left(\{\boldsymbol{r_a^*}\},\left\{\frac{d\boldsymbol{r_a^*}}{dt}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right)=\frac{L^*\left(\{\boldsymbol{r_a^*}\},\left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r_a^*}}{dt}\right\}\right)}{\gamma(\boldsymbol{V}_\mathrm{C})}$.

Instead of $L\left[\{\varphi\},\left\{\frac{\partial\varphi}{\partial r}\right\},\left\{\frac{\partial\varphi}{\partial t}\right\}\right]$, we use $L'\left[\{\varphi^*\},\left\{\frac{\partial\varphi^*}{\partial r^*}\right\},\left\{\frac{\partial\varphi^*}{\partial t}\right\},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C}\right]\equiv\frac{\iiint\Lambda^*\left(\varphi^*,\frac{\partial\varphi^*}{\partial r^*},\gamma(v_\mathrm{C})\frac{\partial\varphi^*}{\partial t}\right)dV^*}{\gamma(v_\mathrm{C})}$.

In the two cases we've calculated directly that $\boldsymbol{P_c}\equiv\frac{\partial L'}{\partial v_\mathrm{C}}=\gamma\frac{E^*}{c^2}\boldsymbol{V}_\mathrm{C}$

In this article, we also showed:

- The strong link with this law and the time dilation formula that highlight the crucial role of the Einstein's requirement of non-universality of time;
- A discussion on the meaning of the new set of variables chosen with an amusing modified velocity addition formula that does not contradict the of Einstein-Poincaré one;
- A discussion of the origin of the energy scale and the link to mass as stated by Landau-Lifchitz;
- Why in Newtonian mechanic Einstein's law is hidden;
- I also add some elements for a Hamiltonian analysis and a discussion about the model of electron that allows the formalism to be applied to a concrete example.

2. Annex

2.1. Elements of Hamiltonian analysis for a material system free

- **Hamiltonian map**

The 4-momentum is $P^i(K^*) = (Mc, \boldsymbol{P}) = \left(\gamma \frac{E^*}{c}, \gamma \frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right)$

Then

$$\left\|P^i(K^*)\right\|^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2 = \left(\gamma\frac{E^*}{c}\right)^2 - \left(\gamma\frac{E^*}{c^2}\boldsymbol{V}_{\mathrm{C}}\right)^2 = \left(\gamma\frac{E^*}{c}\right)^2\left(1-\left(\frac{\boldsymbol{V}_{\mathrm{C}}}{1}\right)^2\right) = \left(\frac{E^*}{c}\right)^2$$

$$\Rightarrow \left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

$$\Leftrightarrow E^2 = {E^*}^2 + c^2\boldsymbol{P}^2$$

$$\Leftrightarrow \boxed{E = \sqrt{{E^*}^2 + c^2\boldsymbol{P}^2}}$$

Having also $E^* = \sum_a E_a^* = \sum_a \sqrt{(m_a c^2)^2 + c^2{\boldsymbol{P_a}^*}^2}$

$$\Rightarrow \boxed{E = \sqrt{\left(\sum_a \sqrt{(m_a c^2)^2 + c^2{\boldsymbol{P_a}^*}^2}\right)^2 + c^2\boldsymbol{P}^2}}$$

Thus the Hamiltonian map $H: (\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) \to H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) \equiv E$ is

$$\boxed{\begin{gathered} H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\})^2 + c^2\boldsymbol{P}^2} \\ with \\ H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}) = \sum_a \sqrt{(m_a c^2)^2 + c^2{\boldsymbol{P_a}^*}^2} \end{gathered}}$$

With $H^*: (\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) \to H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) \equiv E^*$

I give below with evident notation 3 kinds of approximation:

- $H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\})^2 + c^2\boldsymbol{P}^2}$

with

$$H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P_a}^*\}) = \sum_a \sqrt{(m_a c^2)^2 + c^2{\boldsymbol{P_a}^*}^2} = \sum_a (m_a c^2)\sqrt{1 + \frac{c^2{\boldsymbol{P_a}^*}^2}{(m_a c^2)^2}}$$

$$\approx \sum_a m_a c^2\left(1 + \frac{1}{2}\frac{c^2{\boldsymbol{P_a}^*}^2}{(m_a c^2)^2}\right) = M_\Sigma c^2 + \sum_a \frac{{\boldsymbol{P_a}^*}^2}{2m_a}$$

$$=> H_{\{a\}Newtonian}(\{\boldsymbol{r_a^*}\},\{\boldsymbol{P_a}^*\},\boldsymbol{R_C},\boldsymbol{P}) = \sqrt{\left(M_\Sigma c^2 + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)^2 + c^2\boldsymbol{P}^2}$$

$$= \sqrt{(M_\Sigma c^2)^2\left(1 + \frac{1}{M_\Sigma c^2}\sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)^2 + c^2\boldsymbol{P}^2}$$

$$\approx \sqrt{(M_\Sigma c^2)^2\left(1 + \frac{2}{M_\Sigma c^2}\sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\right) + c^2\boldsymbol{P}^2}$$

$$= \sqrt{(M_\Sigma c^2)^2 + c^2\sum_a\left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2} + c^2\boldsymbol{P}^2} = \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\sqrt{1 + \frac{c^2\sum_a\left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2}}{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}}$$

$$\approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\left(1 + \frac{1}{2}\frac{c^2\sum_a\left(\frac{M_\Sigma}{m_a}\right)\boldsymbol{P_a}^{*2}}{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2}\right) = \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}}\sum_a\left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$\boxed{=> H_{\{a\}Newtonian}(\{\boldsymbol{r_a^*}\},\{\boldsymbol{P_a}^*\},\boldsymbol{R_C},\boldsymbol{P}) \approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}}\sum_a\left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right) \quad with\ M_\Sigma \equiv \sum_a m_a}$$

We have also

$$\frac{1}{\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{(M_\Sigma c^2)^2}}} = \frac{1}{\sqrt{1 + \frac{c^2\gamma^2 M^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}} = \frac{1}{\sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2 + \frac{c^2M^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2 + \frac{c^2M^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2 + \frac{c^2\left(M_\Sigma + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_ac^2}\right)^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}}$$

$$= \sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2 + \frac{c^2M_\Sigma^2\left(1 + \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_ac^2M_\Sigma}\right)^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2 + \frac{c^2M_\Sigma^2\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2} + \frac{c^2M_\Sigma^2\sum_a \frac{\boldsymbol{P_a}^{*2}}{m_aM_\Sigma c^2}\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}} = \sqrt{\frac{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}{1 + \frac{M_\Sigma\sum_a \frac{\boldsymbol{P_a}^{*2}}{m_a}\boldsymbol{V_C}^2}{(M_\Sigma c^2)^2}}}$$

$$= \sqrt{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}\left(1 - \frac{1}{2}\frac{\sum_a \frac{\boldsymbol{P_a}^{*2}}{m_a}}{M_\Sigma c^2}\frac{\boldsymbol{V_C}^2}{c^2}\right) = \sqrt{1 - \left(\frac{\boldsymbol{V_C}}{c}\right)^2}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\frac{\boldsymbol{V_C}^2}{c^2}}{M_\Sigma c^2}\right) = \frac{1}{\gamma(\boldsymbol{V_C})}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\left(\frac{\boldsymbol{V_C}}{c}\right)^2}{M_\Sigma c^2}\right)$$

$$=> H_{\{a\}Newtonian}(\{\boldsymbol{r_a^*}\},\{\boldsymbol{P_a}^*\},\boldsymbol{R_C},\boldsymbol{P})$$

$$= \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V_C})}\left(1 - \sum_a \frac{\boldsymbol{P_a}^{*2}}{2m_a}\frac{\left(\frac{\boldsymbol{V_C}}{c}\right)^2}{M_\Sigma c^2}\right)\sum_a\left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$\approx \sqrt{(M_\Sigma c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V_C})}\sum_a\left(\frac{\boldsymbol{P_a}^{*2}}{2m_a}\right)$$

$$=> \boxed{H_{\{a\}Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx \sqrt{(M_{\Sigma}c^2)^2 + c^2\boldsymbol{P}^2} + \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\sum_a\left(\frac{\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}{2m_a}\right)}$$

This result is surprisingly for the second term $\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\sum_a\left(\frac{\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}{2m_a}\right)$ because the dilation of time factor $\gamma(\boldsymbol{V}_{\mathbf{C}})$ divides the internal ("kinematic") energy instead of multiplying it as in the relation E= γ E*.

- $H_{Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx (M_{\Sigma}c^2)\left[1 + \frac{1}{2}\frac{c^2}{(M_{\Sigma}c^2)^2}\left(\sum_a\left(\frac{M_{\Sigma}}{m_a}\right)\boldsymbol{P}_{\boldsymbol{a}}{}^{*2} + \boldsymbol{P}^2\right)\right]$

$$= M_{\Sigma}c^2 + \frac{1}{2}\frac{1}{M_{\Sigma}}\left(\sum_a\left(\frac{M_{\Sigma}}{m_a}\right)\boldsymbol{P}_{\boldsymbol{a}}{}^{*2} + \boldsymbol{P}^2\right) = M_{\Sigma}c^2 + \sum_a\left(\frac{\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}{2m_a}\right) + \frac{\boldsymbol{P}^2}{2M_{\Sigma}}$$

$$=> \boxed{H_{Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx M_{\Sigma}c^2 + \sum_a\frac{\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}{2m_a} + \frac{\boldsymbol{P}^2}{2M_{\Sigma}}}$$

- $H_{C,Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) = \sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^2 + c^2\boldsymbol{P}^2}$

$$= \sqrt{\left[\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}\right]^2 + c^2\boldsymbol{P}^2}$$

$$= \left[\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}\right]\sqrt{1 + \frac{c^2\boldsymbol{P}^2}{\left[\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}\right]^2}}$$

$$= \left[\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}\right]\left(1 + \frac{1}{2}\frac{c^2\boldsymbol{P}^2}{\left[\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}\right]^2}\right)$$

$$=> \boxed{H_{C,Newtonian}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) \approx \sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}} + \frac{1}{2}\frac{c^2\boldsymbol{P}^2}{\sum_a\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}}}$$

- **Hamiltonian equations**

We can verify if the form of the Hamiltonian verifies the Hamilton equation:

$$\frac{\partial H}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P}) = \frac{\partial}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}}\sqrt{H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^2 + c^2\boldsymbol{P}^2} =$$

$$= H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}}\frac{1}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{P})}$$

With

$$\frac{\partial H^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}} = \sum_a\frac{\partial}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}}\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}} = \sum_a c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\frac{\partial\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{\partial\boldsymbol{P}_{\boldsymbol{a}\prime}^{*}}\frac{1}{H_a^*(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}$$

$$= \sum_a c^2\delta_{aa}\frac{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{\sqrt{(m_ac^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}}{}^{*2}}} = \frac{c^2\boldsymbol{P}_{\boldsymbol{a}\prime}{}^{*}}{\sqrt{(m_{\boldsymbol{a}\prime}c^2)^2 + c^2\boldsymbol{P}_{\boldsymbol{a}\prime}{}^{*2}}} = \frac{c^2\boldsymbol{P}_{\boldsymbol{a}\prime}{}^{*}}{H_{a\prime}^*(\boldsymbol{r}_{\boldsymbol{a}\prime}^{*},\boldsymbol{P}_{\boldsymbol{a}\prime}{}^{*})}$$

Then

$$\frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{c^{2}\boldsymbol{P}_{\boldsymbol{a}}^{*}}{H_{a}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},\boldsymbol{P}_{\boldsymbol{a}}{}^{*})}\frac{H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

$$= \frac{c^{2}\boldsymbol{P}_{\boldsymbol{a}}^{*}.H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}).H_{a}^{*}(\boldsymbol{r}_{\boldsymbol{a}}^{*},\boldsymbol{P}_{\boldsymbol{a}}{}^{*})}$$

$$=> \boxed{c^{2}\frac{\boldsymbol{P}_{\boldsymbol{a}}^{*}}{\gamma.E_{a}^{*}} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

But as for the center of mass we can write

$\boldsymbol{P}_{\boldsymbol{a}}^{*} = \frac{E_{a}^{*}}{c^{2}}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^{*}}$ with $E_{a}^{*} = \gamma_{a}^{*}E_{a}^{K_{a}^{*}}$

Where $E_{a}^{K_{a}^{*}}$ is the internal energy of the particle “a” in its own center of mass. This internal energy is equal to its mass only when the particle is free (as for the global center of mass).

$$\left(E_{a}^{K_{a}^{*}}\right)_{a\ is\ free} = m_{a}c^{2}$$

Then we can write in general

$$c^{2}\frac{\frac{E_{a}^{*}}{c^{2}}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^{*}}}{\gamma.E_{a}^{*}} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$$

$$\frac{1}{\gamma}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt^{*}} = \frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$$

$$=> \boxed{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} = \frac{\partial H}{\partial \boldsymbol{P}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

This is again coherent with a first Hamiltonian equation.

$$\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{\partial}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}}\sqrt{H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^{2} + c^{2}\boldsymbol{P}^{2}} =$$

$$= \frac{H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}\frac{\partial H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^{*}}$$

Since $\frac{\partial H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = \frac{\partial\left[\sum_{a}\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt} - L^{*}\right]}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{\partial L^{*}}{\partial \boldsymbol{r}_{\boldsymbol{a'}}^{*}} = -\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^{*}}$

$$\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = -\frac{E^{*}}{\gamma E^{*}}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^{*}} = -\frac{1}{\gamma}\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt^{*}} = -\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt}$$

$$=> \boxed{\frac{d\boldsymbol{P}_{\boldsymbol{a}}{}^{*}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}_{\boldsymbol{a}}^{*}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

This is again consistent with a second Hamiltonian equation.

We then see that for the particular variables chosen in the Lagrangian analysis $\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt},\boldsymbol{r}_{\boldsymbol{a}}^{*}\right)$ we find what we should expect for a Hamiltonian analysis with the variable $(\boldsymbol{P}_{\boldsymbol{a}}{}^{*},\boldsymbol{r}_{\boldsymbol{a}}^{*})$, that is to say the Hamiltonian equation.

For the center of mass we have obviously

$$\frac{\partial H}{\partial \boldsymbol{P}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P}) = \frac{\partial}{\partial \boldsymbol{P}}\sqrt{H^{*}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\})^{2} + c^{2}\boldsymbol{P}^{2}} = \frac{c^{2}\boldsymbol{P}}{H(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

$$= \frac{c^{2}\frac{E}{c^{2}}\boldsymbol{V}_{\mathrm{C}}}{E} = \boldsymbol{V}_{\mathrm{C}}$$

$$=>\boxed{\boldsymbol{V}_{\mathrm{C}}=\frac{\partial H}{\partial \boldsymbol{P}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

The second equation, as already showed (cf. [2]):

$$\boxed{\frac{d\boldsymbol{P}}{dt}=\frac{\partial\sum_{\boldsymbol{a}}\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\frac{d\boldsymbol{r}_{\boldsymbol{a}}^{*}}{dt}+\boldsymbol{P}\boldsymbol{V}_{\mathrm{C}}-H'}{\partial\boldsymbol{R}_{\mathrm{C}}}=-\frac{\partial H'}{\partial\boldsymbol{R}_{\mathrm{C}}}(\{\boldsymbol{r}_{\boldsymbol{a}}^{*}\},\{\boldsymbol{P}_{\boldsymbol{a}}{}^{*}\},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})}$$

We see that if we want to quantize any system in parallel with its center of mass, we should choose the quantum operator associated to the corresponding canonical couples of classical variables:

- $\{(\boldsymbol{r}_{\boldsymbol{a}}^{*},\boldsymbol{P}_{\boldsymbol{a}}{}^{*})\},(\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$ for a system of particles
- $\left\{\left(\varphi^{*},\frac{\partial\varphi^{*}}{\partial\boldsymbol{r}^{*}}\right)\right\},(\boldsymbol{R}_{\mathrm{C}},\boldsymbol{P})$ for a field (scalar for example)
- etc..

2.2. Application: the electromagnetic model of the electron [3], [9], [23]-[25]

Just before and during the construction of the Special Relativity , some theoretical physicists used an electromagnetic model of the electron in order to untangle the ball of wool constituted by different to date of physical theories and experiments about the electrodynamics (and the optic) of moving bodies. The model of electron was used by notably Lorentz and improving by Poincaré (who shares with Einstein the privilege to have realized the last step of the discovery/invention of Special Relativity, both (very different) way of thinking have their own charm) which is interesting to use at least to treat classically the interaction between matter and electromagnetic field without divergence. The latter appears indeed for a material point as showed in [1]. One can always (in a classical universe with matter and electromagnetism living in a static Minkowskian space-time) physically replace a material point by a continuum if one always works for dimensions infinitely larger than the dimension of the continuum. The interest is to have a clear mathematic expression for the mass, even if the model is actually fundamentally wrong (but the ugly last point is here “sufficiently” hidden).

My interest in using this model is to see how a complex system behaves with the particular choice of variables and so thus to see the influence of the dynamics of the center of mass on the internal dynamics, in particular the mass behaviour itself. The model I decided to use is slightly different from the one used by Poincaré since I want to maintain the mass of the continuum without let all the mass to the electromagnetic energy field (as Poincaré & Lorentz & others have done).

I will present the first attempt of the electron model which is unstable and then the one used by Poincaré with his internal “pressure”.

The electron model is:

- a continuum spherical surface in its rest frame K* characterized by a surface density of mass σ;
- the speed of all the material points of the continuum are radial (at an instant t)
- and the mass distribution is spherical K*(at an instant t).

We assume that, the internal spherical behaviour is maintained during motion, although according to [9], this model is in fact unstable.

The Lagrangian is

$$L\left(\{\boldsymbol{r_a}(t)\},\left\{\frac{d\boldsymbol{r_a}}{dt}\right\},t\right)=\sum_a\left[-m_a.c\frac{ds_a}{dt}-\frac{e_a}{c}.A^i(x_{ai})\frac{dx_{ai}}{dt}\right]$$

$$\Rightarrow L'\left(\{\boldsymbol{r}^*_{\boldsymbol{a}}(t)\},\left\{\frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt}\right\},\boldsymbol{R}_{\mathbf{C}},\boldsymbol{V}_{\mathbf{C}},t\right)=-\frac{\sum_a\left[\frac{m_a.c^2}{\gamma_a{}^*}+e_a.\varphi^*(\boldsymbol{r}^*_{\boldsymbol{a}},t^*)\right]}{\gamma(\boldsymbol{V}_{\mathbf{C}})}+\sum_a\frac{e_a}{c}\boldsymbol{A}^*(\boldsymbol{r}^*_{\boldsymbol{a}},t^*).\frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt}$$

$$=-\frac{\sum_a\left[\frac{m_a.c^2}{\gamma_a{}^*}+e_a.\varphi^*(\boldsymbol{r}^*_{\boldsymbol{a}},t^*)\right]}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\text{ via the isotropy hypothesis}$$

$$=-\frac{\iint\left[\frac{\sigma^*_m.c^2}{\gamma^*}+\sigma^*_e.\varphi^*(\boldsymbol{r}^*,t^*)\right]dS^*}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\text{ since we have a continuum}$$

$= -\frac{\left[\frac{\sigma_m^* . c^2}{\gamma^*} + \sigma_e^* . \varphi^*(\boldsymbol{r}^*, t^*)\right] S^*}{\gamma(\boldsymbol{V}_\mathrm{C})}$ via the isotropy of the speed in K*

$= -\frac{\frac{M_\Sigma . c^2}{\gamma^*} + e.\varphi^*(r^*, t^*)}{\gamma(\boldsymbol{V}_\mathrm{C})}$ since the additive mass $M_\Sigma = \sigma_m^* S^*$ and the charge e=$\sigma_e^* S^*$ are relativistic invariants and the distribution of material points is spherical.

We have $e.\varphi^*(r^*, t^*) = E_{em}^*(r^*) = E_{em,eq}^* \frac{r_{eq}^*}{r^*}$ where $E_{em}^*(r^*)$ is the electromagnetic energy and the quantities with index "eq" are the associated quantities for an eventual equilibrium point.

We have also $\gamma^* = \gamma^*\left(\frac{dr^*}{dt^*}\right) = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{dr^*}{dt^*}\right)^2}} = \gamma^*\left(\gamma(\boldsymbol{V}_\mathrm{C})\frac{dr^*}{dt}\right) = \frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_\mathrm{C})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}} = \gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right)$

=> $\boxed{L'\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*(t)\}, \left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}, t\right) = L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}, t\right) = -\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right)} + E_{em,eq}^* \frac{r_{eq}^*}{r^*}\right)}$

with

$$\boldsymbol{P}_{\boldsymbol{c}} = \frac{\partial L'}{\partial \boldsymbol{V}_\mathrm{C}} = \gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a [\gamma_a{}^* . m_a . c^2 + e_a . \varphi^*(\boldsymbol{r}_{\boldsymbol{a}}^*, t^*)]}{c^2}\right)\boldsymbol{V}_\mathrm{C} = \gamma(\boldsymbol{V}_\mathrm{C})\left(\gamma^* . M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}\right)\boldsymbol{V}_\mathrm{C}$$

=> $\boldsymbol{P}_{\boldsymbol{c}} = \gamma(\boldsymbol{V}_\mathrm{C}) M \boldsymbol{V}_\mathrm{C}$

With $M = \gamma^* . M_\Sigma + \frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}$

And $\gamma^* = \gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right) = \frac{1}{1-\frac{\gamma(\boldsymbol{V}_\mathrm{C})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}$

We see that the mass is (modulo c²) the sum of the total internal free energy $\gamma^* . M_\Sigma$ with the electromagnetic energy $\frac{E_{em,eq}^*}{c^2}\frac{r_{eq}^*}{r^*}$ (a potential energy).

Moreover, the value of the mass depends of the "external" dynamics of the center of mass.

The relativistic dynamic is:

$$\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C})\frac{E^*}{c^2}\boldsymbol{V}_\mathrm{C}\right) = \frac{\partial}{\partial \boldsymbol{R}_\mathrm{C}} L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_\mathrm{C}, \boldsymbol{V}_\mathrm{C}, t\right)$$

$$\frac{d}{dt}\left(\gamma_a^* m_a . \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right) = \frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\frac{\partial}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} L^*\left(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \left\{\gamma(\boldsymbol{V}_\mathrm{C})\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}\right\}\right)$$

=> $\frac{d}{dt}\left(\gamma^* M_\Sigma . \frac{dr^*}{dt^*}\right) = \frac{-1}{\gamma(V_c)}\frac{\partial}{\partial r^*}\left(\gamma^*\left(\boldsymbol{V}_\mathrm{C}, \frac{dr^*}{dt}\right) . M_\Sigma c^2 + E_{em,eq}^* \frac{r_{eq}^*}{r^*}\right) = \frac{1}{\gamma(V_c)} E_{em,eq}^* \frac{r_{eq}^*}{r^{*2}}$

=> $\boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_\mathrm{C}) . \gamma^* M_\Sigma . \frac{dr^*}{dt}\right) = \frac{1}{\gamma(V_c)} E_{em,eq}^* \frac{r_{eq}^*}{r^{*2}}}$

One can see that this model is internally radially unstable since there is only a repulsive term.

In order to improve he model we can add to it a truncated cosmological constant [9] which is null everywhere but not into the spherical electron.

The new Lagrangian is ([9] & [3])

$$L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_{\mathrm{C}},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{8\pi k}\iiint \Lambda_P.\theta(R^*-r^*)\sqrt{-g^*}d^3\boldsymbol{R}^*\right)$$

With

$$\theta(R^*-r^*)\equiv 1\, forR^*\leq r^*$$

$$\equiv 0\, forR^*>r^*$$

But the space-time is Minkowskian and the electron is spherical in K*. Then

$$\boxed{L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_{\mathrm{C}},\boldsymbol{V}_{\mathrm{C}},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_{\mathrm{C}},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{8\pi k}\frac{4}{3}\pi\Lambda_P.r^{*3}\right)}$$

A false problem

We can remark that this Lagrangian naively suggests that the interaction terms acts instantaneously which would be inconsistent with Relativity. But actually the interaction terms come from fields that acts just exactly at the points where the material points are located, that is to say on the sphere and not at the center of the sphere.

A digression towards some intriguing Uniform-Energy-Region

The cosmological term in the Lagrangian is not the one used by Einstein since it is not applied to the whole space-time. This is very surprising for me since the general famous theorem (Lovelock) established that the cosmological term à la Einstein (in addition to the Ricci term) is the only one allowed in General Relativity in order to respect the general requirement of this theory: second order equation for dynamics and invariance of physical laws for any transformation of coordinates. A natural question is why the addition of the Poincaré term is authorized in Relativity ? In a more intuitive reasoning (which allows to reveal the solution): saying that a cosmological term applied only to a given fixed region seems to contradict the epistemological views of General Relativity [22] saying in particular that any effect of a phenomenon has to be caused by a direct measurable cause. This direct measurable cause has to be a physical phenomenon, governed by **dynamical** equations, which interact with other fields and matters (that is why reference frame must not be allowed to influence phenomena via inertial forces, the equivalence principle permitting precisely to make the latter dynamic by unifying them with the **dynamical** field of gravity). The solution to my problem is therefore that the boundary of the region, where the Poincaré pressure term is applied, is dynamically coupled with the distribution of the material system localized in the region. This has an interesting consequence: General Relativity allows a priori the existence of an arbitrary number n of deformed closed surfaces surrounding internal regions, of volume $\iiint \theta(\|\boldsymbol{R_n}\|-\|\boldsymbol{r_n}\|)\sqrt{-g}.d^3\boldsymbol{R_n}$, each containing a "Constant-cosmological" term Λ_n with an arbitrary value. This in the condition that all these borders are dynamical coupled with a border variable $\boldsymbol{r_n}$. Explicitly, General Relativity permits an action as

$$S[\{g_{ik}(x,t)\},\{r_n(t)\}] = \frac{-c^3}{16\pi k}\int\iiint(R-2\Lambda_E)\sqrt{-g}d\Omega+\sum_n\frac{c^4}{8\pi k}\int\iiint\Lambda_n.\theta(\|\boldsymbol{R_n}\|-\|\boldsymbol{r_n}\|)\sqrt{-g}.d^3\boldsymbol{R_n}\,dt+S[\{r_n(t),\dots\}]$$

Thus, Lovelock theorem applied, as it should, to a free gravitation field and the other "cosmological" terms are not affected by it since they necessitate the use of other dynamical variables.

Of course, although permit, the other cosmological terms are not very "natural" because we have to add them arbitrary by hand. However they are not more "unnatural" than the complex topologies already often used and a priori allowed. If one accepts such new terms we must therefore complete the action with another part implying the dynamic of a 2D membrane for which every point behaves as a material point, each providing a "ds" term in the action. Hence, this membrane is sensitive to (as it should) the gravitation field (and a priori only to it) and is deformed by it. We can imagine a space time bathed by these Uniform-Energy-Regions. The problem of this kind of Uniform-Energy-Regions is the instabilities of their shape since they behave internally like a dynamical min-de-sitter (or anti-de-sitter) universe and not like a wiser Einstein-static one. Another problem that comes in mind is the possible appearance of gravitational singularities when 2 free point of the same surface (or even several surface) meet at the same point during their "free" movement (but this problem can be maybe cured by a quantum "bandage"). In spite of all these oddities, it is important to keep in mind (surely already known, perhaps by Dirac) all the mathematical possibilities permited by the standard paradigm of physics, which is still today partly constituted by classical General Relativity. After a reading of the Jean Pierre Luminet's book [17], it seems that these speculations look a bit like the concept of gravastars which were conceptually invented in 2001 by Mazur & Mottola: Is "the Uniform-Energy-Regions" the same speculative concept as gravastars ? Is the gravastar the rebirth of the old Lorentz-Poincaré electron in an astrophysical domain? One of the differences would be that the Uniform-Energy-Regions is put by hand as one can put by hand a cosmological constant or the existence of some material points instead of being the result of a dynamical collapse of an existing massive star.

<u>Returning to our initial problem</u>

$$L'\left(r^*,\frac{dr^*}{dt},\boldsymbol{R}_\mathrm{C},\boldsymbol{V}_\mathrm{C},t\right)=-\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\left(\frac{M_\Sigma c^2}{\gamma^*\left(\boldsymbol{V}_\mathrm{C},\frac{dr^*}{dt}\right)}+E^*_{em,eq}\frac{r^*_{eq}}{r^*}-\frac{c^4}{2k}\frac{1}{3}\Lambda_P.r^{*3}\right)$$

Which gives

$$\boldsymbol{P_c}=\frac{\partial L'}{\partial\boldsymbol{V}_\mathrm{C}}=\gamma(\boldsymbol{V}_\mathrm{C})\left(\frac{\sum_a[\gamma_a{}^*.m_a.c^2+e_a.\varphi^*(\boldsymbol{r}^*_{\boldsymbol{a}},t^*)]}{c^2}\right)\boldsymbol{V}_\mathrm{C}$$
$$=\gamma(\boldsymbol{V}_\mathrm{C})\left(\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*}-\frac{c^2}{2k}\frac{1}{3}\Lambda_P.r^{*3}\right)\boldsymbol{V}_\mathrm{C}$$

$$\boxed{\begin{aligned}&=>\boldsymbol{P_c} = \gamma(\boldsymbol{V}_{\mathrm{c}})M\boldsymbol{V}_{\mathrm{c}}\\ &\text{With } M = \gamma^*.M_\Sigma + \frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*} - \frac{c^2}{6k}\Lambda_P.r^{*3}\\ &\text{And } \gamma^* = \gamma^*\left(\boldsymbol{V}_{\mathbf{C}}, \frac{dr^*}{dt}\right) = \frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}}\end{aligned}}$$

We see that the mass is (modulo c²) the sum of the total internal free energy with the electromagnetic energy (which behaves as a potential energy) and with the pressure-Poincaré energy.

Moreover, the value of the mass depends of the “external” dynamics of the center of mass.

The relativistic dynamic for the internal part is now:

$$\frac{d}{dt}\left(\gamma^*_a m_a.\frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt^*}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{\partial}{\partial \boldsymbol{r}^*_{\boldsymbol{a}}}L^*\left(\{\boldsymbol{r}^*_{\boldsymbol{a}}\}, \left\{\gamma(\boldsymbol{V}_{\mathrm{c}})\frac{d\boldsymbol{r}^*_{\boldsymbol{a}}}{dt}\right\}\right)$$

$$=>\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathrm{c}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{-1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{\partial}{\partial r^*}\left(\gamma^*\left(\boldsymbol{V}_{\mathbf{C}}, \frac{dr^*}{dt}\right).M_\Sigma c^2 + E^*_{em,eq}\frac{r^*_{eq}}{r^*} - \frac{c^4}{6k}\Lambda_P.r^{*3}\right)$$

$$= \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}} + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}\Lambda_P.r^{*2}$$

$$=>\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathrm{c}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}} + \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}\Lambda_P.r^{*2}$$

In addition to the repulsive Coulomb term, the Poincaré term add a pressure force

$$S^*.p = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}\Lambda_P.r^{*2}$$

$$=> p = \frac{\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}\Lambda_P.r^{*2}}{4\pi r^{*2}} = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{8\pi k}\Lambda_P$$

In order to stabilize the sphere, we put $\Lambda_P = -|\Lambda_P|$

Hence we have

$$\boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathrm{c}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}} - \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}|\Lambda_P|.r^{*2}}$$

The internal equilibrium is realized when (we put by definition $r^* = r^*_{eq}$):

$$\left(\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}} = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{c}})}\frac{c^4}{2k}|\Lambda_P|.r^{*2}\right)_{r^*=r^*_{eq}}$$

$$<=> E^*_{em,eq} = \frac{c^4}{2k}|\Lambda_P|.{r^*_{eq}}^3$$

$$<=>\boxed{{r^*_{eq}}^3 = 2\frac{k}{|\Lambda_P|c^4}E^*_{em,eq}}$$

Moreover

$$=>\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathbf{C}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right)=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}}-\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\left(\frac{c^4}{2k}|\Lambda_P|\right).r^{*2}=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}E^*_{em,eq}\frac{r^*_{eq}}{r^{*2}}-\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\left(\frac{E^*_{em,eq}}{{r^*_{eq}}^3}\right).r^{*2}$$

$$=>\boxed{\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathbf{C}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right)=\frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{{r^*}^2}{{r^*_{eq}}^3}\right)}$$

$$M=\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*}-\frac{c^2}{6k}\Lambda_P.r^{*3}=\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*}+\frac{1}{3}\left(\frac{c^2}{2k}|\Lambda_P|\right).r^{*3}$$

$$=\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\frac{r^*_{eq}}{r^*}+\frac{1}{3}\left(\frac{E^*_{em,eq}}{c^2{r^*_{eq}}^3}\right).r^{*3}=\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\left(\frac{r^*_{eq}}{r^*}+\frac{1}{3}\frac{{r^*}^3}{{r^*_{eq}}^3}\right)$$

$$=>\boxed{M=\gamma^*.M_\Sigma+\frac{E^*_{em,eq}}{c^2}\left(\frac{r^*_{eq}}{r^*}+\frac{1}{3}\frac{{r^*}^3}{{r^*_{eq}}^3}\right)}$$

The mass is a function $M=M\left(r^*,\frac{dr^*}{dt},\boldsymbol{V}_{\mathbf{C}}\right)$

For the equilibrium point we have the well known result

$$\boxed{M_{eq}=\gamma^*.M_\Sigma+\frac{4}{3}\frac{E^*_{em,eq}}{c^2}=\left(M_\Sigma+\frac{4}{3}\frac{E^*_{em,eq}}{c^2}\right)_{if\ \frac{dr^*}{dt}=0}}$$

Hence, as already many time said, for example in [3] & [9], the a priori astonishing factor mass $\frac{4}{3}$ is due to the necessity of a confining term (Poincaré term) which add an energical contribution.

$$\boxed{M_{eq}=(\gamma^*.M_\Sigma)_{Material\ points}+\left(\frac{E^*_{em,eq}}{c^2}\right)_{Electromagnetic\ field}+\left(\frac{1}{3}\frac{E^*_{em,eq}}{c^2}\right)_{Poinc\quad é\ confinment}}$$

<u>The internal equation of motion of the sphere in second order approximation</u>

We clarify the following factor

$$\gamma^*=\gamma^*\left(\boldsymbol{V}_{\mathbf{C}},\frac{dr^*}{dt}\right)=\frac{1}{\sqrt{1-\frac{\gamma(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\left(\frac{dr^*}{dt}\right)^2}}=\frac{1}{\sqrt{\frac{1-\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2}{1-\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}}}}=\sqrt{\frac{1-\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}}{1-\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2}}$$

$$\approx\left(1-\frac{1}{2}\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\right)\left(1+\frac{1}{2}\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)$$

$$=\left(1+\frac{1}{2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2-\frac{1}{2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}\left(1+\frac{1}{2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\right)$$

$$=1+\frac{1}{2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}+\frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2-\frac{1}{2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}-\frac{1}{4}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}-\frac{1}{4}\frac{{\boldsymbol{V}_{\mathbf{C}}}^2}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2$$

$$=> \gamma^* = 1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 - \frac{1}{4}\frac{(\boldsymbol{V}_{\mathbf{C}})^4}{c^4} - \frac{1}{4}\frac{(\boldsymbol{V}_{\mathbf{C}})^2}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2$$

$$=> \boxed{\gamma^* = 1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 + \theta\left(\frac{v^4}{c^4}\right)}$$

Then

$$\frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathbf{C}}).\gamma^* M_\Sigma.\frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \frac{d}{dt}\left(\gamma(\boldsymbol{V}_{\mathbf{C}})\left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right) M_\Sigma \frac{dr^*}{dt}\right) \approx \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathbf{C}}))\gamma^* M_\Sigma \frac{dr^*}{dt} + \frac{d}{dt}\left(\left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\right)\gamma(\boldsymbol{V}_{\mathbf{C}}) M_\Sigma \frac{dr^*}{dt} + \frac{d}{dt}\left(\frac{dr^*}{dt}\right)\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^* M_\Sigma = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathbf{C}}))\gamma^* M_\Sigma \frac{dr^*}{dt} + \frac{1}{c^2}\frac{dr^*}{dt}\frac{d^2r^*}{dt^2}\gamma(\boldsymbol{V}_{\mathbf{C}}) M_\Sigma \frac{dr^*}{dt} + \frac{d^2r^*}{dt^2}\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^* M_\Sigma = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathbf{C}}))\frac{\gamma^*}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^*} M_\Sigma \frac{dr^*}{dt} + \frac{1}{c^2}\frac{dr^*}{dt}\frac{d^2r^*}{dt^2}\frac{\gamma(\boldsymbol{V}_{\mathbf{C}})}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^*} M_\Sigma \frac{dr^*}{dt} + M_\Sigma \frac{d^2r^*}{dt^2} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^*} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

$$<=> \left(\frac{1}{c^2}\frac{dr^*}{dt}\frac{1}{\gamma^*} M_\Sigma \frac{dr^*}{dt} + M_\Sigma\right)\frac{d^2r^*}{dt^2} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma(\boldsymbol{V}_{\mathbf{C}})\gamma^*} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{dln(\gamma(\boldsymbol{V}_{\mathbf{C}}))}{dt} M_\Sigma \frac{dr^*}{dt}$$

$$<=> \boxed{\left(1 + \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right) M_\Sigma \frac{d^2r^*}{dt^2} \approx \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})^2\gamma^*} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - M_\Sigma \frac{dln(\gamma(\boldsymbol{V}_{\mathbf{C}}))}{dt}\frac{dr^*}{dt}}$$

$$\frac{dln(\gamma(\boldsymbol{V}_{\mathbf{C}}))}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{d\gamma(\boldsymbol{V}_{\mathbf{C}})}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{d\left(1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2\right)^{-1/2}}{dt} = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{-1}{2}\left(1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2\right)^{-3/2}\frac{d\left(1 - \left(\frac{\boldsymbol{V}_{\mathbf{C}}}{c}\right)^2\right)}{dt}$$
$$= \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})}\frac{-1}{2}\gamma(\boldsymbol{V}_{\mathbf{C}})^3\left(-2\left(\frac{1}{c^2}\right)\boldsymbol{V}_{\mathbf{C}}\boldsymbol{a}_{\mathbf{C}}\right) = \frac{1}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})^2\boldsymbol{V}_{\mathbf{C}}\boldsymbol{a}_{\mathbf{C}}$$

$$=> \left(1 + \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right) M_\Sigma \frac{d^2r^*}{dt^2} \approx \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})^2\gamma^*} E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{M_\Sigma}{c^2}\gamma(\boldsymbol{V}_{\mathbf{C}})^2\boldsymbol{V}_{\mathbf{C}}\boldsymbol{a}_{\mathbf{C}}\frac{dr^*}{dt}$$

$$<=> M_\Sigma \frac{d^2r^*}{dt^2} \approx \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})^2\gamma^*}\left(1 - \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right) E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{M_\Sigma}{c^2}\left(1 - \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\gamma(\boldsymbol{V}_{\mathbf{C}})^2\boldsymbol{V}_{\mathbf{C}}\boldsymbol{a}_{\mathbf{C}}\frac{dr^*}{dt}$$

$$M_\Sigma \frac{d^2r^*}{dt^2} = -\alpha.\frac{dr^*}{dt} + \vartheta.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

- $\alpha = \alpha\left(\boldsymbol{V}_{\mathbf{C}}, \frac{dr^*}{dt}\right) = \frac{M_\Sigma}{c^2}\left(1 - \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\gamma(\boldsymbol{V}_{\mathbf{C}})^2\boldsymbol{V}_{\mathbf{C}}\boldsymbol{a}_{\mathbf{C}}$
- $\vartheta = \vartheta\left(\boldsymbol{V}_{\mathbf{C}}, \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathbf{C}})^2\gamma^*}\left(1 - \frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)$

With this effective Newtonian form one can interpret more intuitively the internal relativistic equation where we notice:

- a viscous term $-\alpha.\frac{dr^*}{dt}$ with a coefficient α proportional to $\boldsymbol{V}_{\mathrm{C}}.\boldsymbol{a}_{\mathrm{C}}$,in accordance with the conservation of the energy (exchange between internal energy and the kinetic energy)
- a factor ϑ affecting the repulsive Coulombian force and the pressure force

We see that in general there is a coupling between the external dynamic and the internal dynamic. But this coupling is clearly due to the relativistic regime: outside this regime, the external dynamic does not affect the internal dynamic (since K* is a local Galilean frame, there are no inertial forces).

The case of a non-relativistic internal dynamic ($\frac{1}{c}\frac{dr^*}{dt} \approx 0$) gives:

- $\alpha = \alpha\left(\boldsymbol{V}_{\mathrm{C}}, \frac{dr^*}{dt}\right) \approx \frac{M_\Sigma}{c^2}\gamma(\boldsymbol{V}_{\mathrm{C}})^2\boldsymbol{V}_{\mathrm{C}}\boldsymbol{a}_{\mathrm{C}} = \frac{M_\Sigma}{c^2}\frac{\boldsymbol{V}_{\mathrm{C}}\boldsymbol{a}_{\mathrm{C}}}{1-\frac{\boldsymbol{V}_{\mathrm{C}}^2}{c^2}}$
- $\vartheta = \vartheta\left(\boldsymbol{V}_{\mathrm{C}}, \frac{dr^*}{dt}\right) = \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2} = 1 - \frac{\boldsymbol{V}_{\mathrm{C}}^2}{c^2}$

=> $\alpha\frac{dr^*}{dt} = \frac{M_\Sigma}{c^2}\frac{\boldsymbol{V}_{\mathrm{C}}\boldsymbol{a}_{\mathrm{C}}}{1-\frac{\boldsymbol{V}_{\mathrm{C}}^2}{c^2}}\frac{dr^*}{dt} = K.\frac{1}{c}\frac{dr^*}{dt} \approx 0$

$$\Rightarrow \boxed{\left(M_\Sigma\frac{d^2r^*}{dt^2} \approx \frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)\right)_{None-relativistic\ internal\ dynamic}}$$

To simplify the dynamics, I will assume that the system is close to the equilibrium point. Then we can Taylor the function $f(r^*) \equiv \frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}$ near this point.

$$f(r^*) \approx f(r^*_{eq}) + \frac{df}{dr^*}(r^*_{eq}).(r^* - r^*_{eq}) = \left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right)(r^*_{eq}) + \left(-2\frac{r^*_{eq}}{r^{*3}} - 2\frac{r^*}{{r^*_{eq}}^3}\right)(r^*_{eq}).(r^* - r^*_{eq})$$
$$= (0) - \frac{4}{{r^*_{eq}}^2}.(r^* - r^*_{eq})$$

$$\Rightarrow M_\Sigma\frac{d^2r^*}{dt^2} \approx -\frac{4}{{r^*_{eq}}^2}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}E^*_{em,eq}(r^* - r^*_{eq})$$

If the speed of the center of mass varies sufficiently slowly (adiabatically), we have as desired the case of an effective oscillator around a center of mass velocity ${\boldsymbol{V}_{\mathrm{C}}}^2$:

$$M_\Sigma\frac{d^2r^*}{dt^2} \approx -k_{\boldsymbol{V}_{\mathrm{C}}}.(r^* - r^*_{eq})$$

With $k_{\boldsymbol{V}_{\mathrm{C}}} \equiv \frac{4}{{r^*_{eq}}^2}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^2}E^*_{em,eq}$

Taking into account ${r^*_{eq}}^3 = 2\frac{k}{|\Lambda_P|c^4}E^*_{em,eq}$, the pulsation of the oscillator is then:

$$\omega_{\boldsymbol{V}_{\mathrm{C}}} = \sqrt{\frac{k_{\boldsymbol{V}_{\mathrm{C}}}}{M_\Sigma}} \approx \frac{2}{r^*_{eq}}\sqrt{E^*_{em,eq}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})} = \frac{2\sqrt{E^*_{em,eq}}}{\left(2\frac{k}{|\Lambda_P|c^4}E^*_{em,eq}\right)^{1/3}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})}$$
$$= 2^{2/3}\left(\frac{c^4|\Lambda_P|}{k}\right)^{1/3}{E^*_{em,eq}}^{1/6}\left(1 - \frac{1}{2}\frac{{\boldsymbol{V}_{\mathrm{C}}}^2}{c^2}\right)$$

$$\omega_{V_C} \approx \frac{\omega_0}{\gamma(V_C)} = \omega_0 + \Delta\omega_{V_C}$$

- $\omega_0 \approx 2^{2/3} \left(\frac{c^4|\Lambda_P|}{k}\right)^{1/3} {E^*_{em,eq}}^{1/6}$
- $\Delta\omega_{V_C} \approx -\frac{1}{2}\frac{{V_C}^2}{c^2}\omega_0$

Hence,

- if the internal system has Newtonian dynamics ;
- if the velocity of the center of mass is not negligible relative to the Einstein constant c ([5']);
- and if the speed of the center of mass varies sufficiently slowly with respect to the internal dynamics,

Then the internal oscillator sees its frequency ω_{V_C} decreas to the value:

$$-\Delta\omega_{V_C} \approx \frac{1}{2}\frac{{V_C}^2}{c^2}\omega_0$$

A complex system whose center of mass moves at a sufficiently high speed affects the internal dynamics of the system.

<u>The dynamic effect is actually a kinematic one</u>

This, a priori dynamics effect, is actually rather a kinematic one, Einstein's law of time dilation:

$$dt = \gamma(V_C)dt^* <=> \frac{1}{dt} \approx \frac{1}{\gamma(V_C)}\frac{1}{dt^*}$$

We see in fact that the internal dynamic is frozen by the time dilation.

Indeed, we can re-express the internal dynamics in term of internal time $t^*$by using the fact that:

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{1}{\gamma(V_C)}\frac{d}{dt^*}\right)$$

$$= \frac{1}{\gamma(V_C)}\left[\frac{d\gamma(V_C)^{-1}}{dt^*}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$

$$= \frac{1}{\gamma(V_C)}\left[\frac{dt}{dt^*}\frac{d\gamma(V_C)^{-1}}{dt}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$

$$= \frac{1}{\gamma(V_C)}\left[\gamma(V_C)\frac{d\left(1-\frac{{V_C}^2}{c^2}\right)^{1/2}}{dt}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$

$$= \frac{1}{\gamma(V_C)}\left[\gamma(V_C)\frac{1}{2}\left(-2\frac{V_C}{c^2}\right)a_C\left(1-\frac{{V_C}^2}{c^2}\right)^{-1/2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$

$$= \frac{1}{\gamma(V_C)}\left[-\gamma(V_C)^2\frac{V_C.a_C}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right]$$

$$= -\gamma(V_C)\frac{V_C.a_C}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)$$

$$=> \boxed{\frac{d^2}{dt^2} = -\gamma(V_C)\frac{V_C.a_C}{c^2}\left(\frac{d}{dt^*}\right) + \frac{1}{\gamma(V_C)^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)}$$

Since we know that

$$M_\Sigma\frac{d^2r^*}{dt^2} \approx \frac{1}{\gamma(V_C)^2\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}} - \frac{r^{*2}}{{r^*_{eq}}^3}\right) - \frac{M_\Sigma}{c^2}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\gamma(V_C)^2 V_C a_C\frac{dr^*}{dt}$$

Then we have

$$M_\Sigma\left(-\gamma(\boldsymbol{V}_\mathbf{C})\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\left(\frac{d}{dt^*}\right)+\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{d}{dt^*}\right)\right)r^*$$
$$\approx\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\frac{M_\Sigma}{c^2}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\gamma(\boldsymbol{V}_\mathbf{C}).\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}\frac{dr^*}{dt^*}$$

$$<=> M_\Sigma\left(-\gamma(\boldsymbol{V}_\mathbf{C})\frac{\boldsymbol{V}_\mathbf{C}.\boldsymbol{a}_\mathbf{C}}{c^2}\left(\frac{dr^*}{dt^*}\right)+\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)\right)$$
$$\approx\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\frac{M_\Sigma}{c^2}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)\gamma(\boldsymbol{V}_\mathbf{C}).\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}\frac{dr^*}{dt^*}$$

$$<=> M_\Sigma\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2}\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)$$
$$\approx\frac{1}{\gamma(\boldsymbol{V}_\mathbf{C})^2\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)-\frac{M_\Sigma}{c^2}\left[\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)-1\right]\gamma(\boldsymbol{V}_\mathbf{C})\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}\frac{dr^*}{dt^*}$$

$$<=> M_\Sigma\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)\approx\frac{1}{\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)+\frac{M_\Sigma}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\gamma(\boldsymbol{V}_\mathbf{C})\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}\frac{dr^*}{dt^*}$$

$$<=> M_\Sigma\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)\approx\frac{1}{\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)+\frac{M_\Sigma}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt^*}\right)^2\frac{\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}}{\gamma(\boldsymbol{V}_\mathbf{C})}\frac{dr^*}{dt^*}$$

$$=> M_\Sigma\frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right)=\alpha.\frac{dr^*}{dt^*}+\vartheta.E^*_{em,eq}\left(\frac{r^*_{eq}}{r^{*2}}-\frac{r^{*2}}{{r^*_{eq}}^3}\right)$$

- $\alpha=\alpha\left(\boldsymbol{V}_\mathbf{C},\frac{dr^*}{dt}\right)=\frac{M_\Sigma}{c^2}\frac{1}{c^2}\left(\frac{dr^*}{dt^*}\right)^2\frac{\boldsymbol{V}_\mathbf{C}\boldsymbol{a}_\mathbf{C}}{\gamma(\boldsymbol{V}_\mathbf{C})}$
- $\vartheta=\vartheta\left(\boldsymbol{V}_\mathbf{C},\frac{dr^*}{dt}\right)=\frac{1}{\gamma^*}\left(1-\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right)$

<u>Digression on a (apparent ?) paradox</u>

The viscous term is a little embarrassing for me (Is there an error of sign or worse? I let the reader answer) because it shows a capture of the kinetic energy of the center of mass by the internal oscillator when the center of mass is accelerated. This is the complete opposite, at first sight, of the expected behaviour as seen above from the reference frame K with time t. This seems to contradict the energy conservation law.

But in the case where the whole system is isolated, we know that the center of mass and the total energy has to be constant, so the viscous term disappears and there is no contradiction.

However, let's take this result seriously. One can imagine an external field acting on the whole system in a such way that the field is considered as totally homogenous from the point of view of the internal particles but inhomogeneous from the one of the center of mass. Then the external field accelerates the electron without “touching” the internal part. This results in an acceleration of the center of mass. As we have shown above, this has the effect of injecting energy from the center of mass to the internal energy, and thus increasing the mass. The consequence of this is that any acceleration of our electron increases its mass: “acceleration generates matter”. This remind me of some claims readen in more sophisticated and general physics about creation of matter during “early” Big Bang phase of (more or less) speculative inflation. Our model, although less general (and physically false!), shows how one can understand pedagogically (=simple and explicit) this kind of behaviour in Special Relativity without field theory (if there are not errors of signs...).

To finish this digression, we can interpret the equation in the follows. From the point of view of internal dynamics, an acceleration of the center of mass increases the speed of the oscillator. But from the point of view of the observer K, it is the opposite. As the difference between the 2 point of view comes from the dilation of time, we can say that when we increase the speed of the center of mass, in the point of view of K, the dilation of time increases sufficiently strongly to contrebalance the internal acceleration (seen from K*) and actually provide the result of an decelerated oscillator seen in K. If there is no sign error (or worse), I found this result quite interesting as an apparent paradox in Special Relaticity.

Returning to our Like above, Once again, to simplify the dynamics, I will assume that

- if the internal system has Newtonian dynamics ;
- the system is close to the equilibrium point.
- the speed of the center of mass varies slowly (adiabatically):

$$=> M_\Sigma \frac{d}{dt^*}\left(\frac{dr^*}{dt^*}\right) \approx -k_{\mathbf{0}}.\left(r^* - r^*_{eq}\right)$$

With $k_{\mathbf{0}} \equiv \frac{4}{{r^*_{eq}}^2} E^*_{em,eq}$

Then

$$M_\Sigma \frac{d^2r^*}{dt^2} \approx -\frac{4}{{r^*_{eq}}^2}\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})^2}\frac{E^*_{em,eq}}{M_\Sigma}\left(r^* - r^*_{eq}\right) <=> M_\Sigma \frac{d^2r^*}{{dt^*}^2} \approx -\frac{4}{{r^*_{eq}}^2}\frac{E^*_{em,eq}}{M_\Sigma}\left(r^* - r^*_{eq}\right)$$

In this form, we see that the dynamics is fully expressed in terms of variables in K* without the dilation of time factor.

In this expression, the Hooke force is now characterized by $k_{\mathbf{0}} = \frac{4}{{r^*_{eq}}^2}\frac{E^*_{em,eq}}{M_\Sigma} = \gamma(\boldsymbol{V}_\mathrm{C})^2 k_{\boldsymbol{V}_\mathrm{C}}$

We recover the expression above $\omega_{\mathbf{0}} = \sqrt{\frac{k_{\mathbf{0}}}{M_\Sigma}} = \sqrt{\gamma(\boldsymbol{V}_\mathrm{C})^2\frac{k_{\boldsymbol{V}_\mathrm{C}}}{M_\Sigma}} = \gamma(\boldsymbol{V}_\mathrm{C})\sqrt{\frac{k_{\boldsymbol{V}_\mathrm{C}}}{M_\Sigma}} = \gamma(\boldsymbol{V}_\mathrm{C})\omega_{\boldsymbol{V}_\mathrm{C}}$

Thus, the Newtonian oscillator is seen frozen à la Einstein by our new weird choice of variables that express the dynamics relative to the time t. This indicates naturally that the observer is in the reference frame K.

Anticipating a quantum treatment that might be based on the new choice of variable, one can affirm that the quantum characteristics of the inner part (seen from K) would be:

- The quantum of energy $\Delta E = \hbar\omega_{\boldsymbol{V}_\mathrm{C}} = \hbar\frac{\omega_{\mathbf{0}}}{\gamma(\boldsymbol{V}_\mathrm{C})} \approx \hbar\omega_{\mathbf{0}} + \hbar\Delta\omega_{\boldsymbol{V}_\mathrm{C}}$;
- The zero point energy $E_0 = \frac{1}{2}\hbar\omega_{\boldsymbol{V}_\mathrm{C}} = \frac{1}{2}\hbar\frac{\omega_{\mathbf{0}}}{\gamma(\boldsymbol{V}_\mathrm{C})} \approx \frac{1}{2}\hbar\omega_{\mathbf{0}} + \frac{1}{2}\hbar\Delta\omega_{\boldsymbol{V}_\mathrm{C}}$.

The "external relativistic dynamic" affects, by the relativity of time, the quantum of energy and the lowest energy ("zero point energy") by "renormalizing" them in a kinematic way.

Then if we measure this quantum and this zero point energy among a macroscopic number of such free complex "electrons" we naturally obtain the average values (thanks to the Equirepartition theorem):

- $\langle\Delta E\rangle = \hbar\langle\omega_{\boldsymbol{V}_\mathrm{C}}\rangle = \hbar\omega_{\mathbf{0}}\,\langle\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\rangle \approx \hbar\omega_{\mathbf{0}}\left(1 - \frac{1}{Mc^2}\langle\frac{1}{2}M\boldsymbol{V}_\mathbf{C}{}^2\rangle\right) = \hbar\omega_{\mathbf{0}}\left(1 - \frac{3}{2}\frac{k_BT}{Mc^2}\right)$;
- $\langle E_0\rangle = \frac{1}{2}\hbar\langle\omega_{\boldsymbol{V}_\mathrm{C}}\rangle = \frac{1}{2}\hbar\omega_{\mathbf{0}}\,\langle\frac{1}{\gamma(\boldsymbol{V}_\mathrm{C})}\rangle = \frac{1}{2}\hbar\omega_{\mathbf{0}}\left(1 - \frac{3}{2}\frac{k_BT}{Mc^2}\right)$

And, if all the electrons are each in harmonic-Hooke interaction with a center of force, then the quantum characteristics measured are:

- $\langle\Delta E\rangle = \hbar\langle\omega_{\boldsymbol{V}_\mathrm{C}}\rangle \approx \hbar\omega_{\mathbf{0}}\left(1 - 3\frac{k_BT}{Mc^2}\right)$;
- $\langle E_0\rangle \approx \frac{1}{2}\hbar\omega_{\mathbf{0}}\left(1 - 3\frac{k_BT}{Mc^2}\right)$

Hence, from the point of view of an observer looking at a collection of moving "electron", the dilation of time, expressed by the thermal energy effect, decrease (a little) theses quantum characteristics of that particular electron. Because of the universality of the cause (Einstein dilation of time), this effect must also be true for other kind of material Newtonian oscillators (atoms, molecules...). It will be interesting to see the effect of the dilation of time for the quantum version of "particles" constituted, this time, only by fields (electromagnetism, gravitation fields...) which can be considered (at least for the first when it is free and the second for a weak field regime) as an infinity of oscillators, but this time the last system will be relativistic.
A question: For a wave packet travelling at the constant speed c, can we say that all the zero-point energy would be infinitely dilated and therefore "renormalized" (if internal energy has a meaning for a such system, a priori not but an explicit passage to the limit would be interesting to see) ?

<u>The mass</u>

In this situation the mass is now

$$M = \gamma^*.M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right) \approx \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right).M_\Sigma + \frac{E^*_{em,eq}}{c^2}g(r^*)$$

With the function $g(r^*) \equiv \frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}$

Taking account that for the equilibrium point $\left(\frac{dg}{dr^*}\right)_{r^*_{eq}} = 0$, since the mass is an internal energy near the equilibrium point, we have

$$g(r^*) = g(r^*_{eq}) + \frac{1}{2}\left(\frac{d^2g}{dr^{*2}}\right)_{r^*_{eq}}(r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{1}{2}\left(\frac{d}{dr^*}\left[\frac{d}{dr^*}\left(\frac{r^*_{eq}}{r^*} + \frac{1}{3}\frac{r^{*3}}{{r^*_{eq}}^3}\right)\right]\right)_{r^*_{eq}}(r^* - r^*_{eq})^2$$

$$= \frac{4}{3} + \frac{1}{2}\left(\frac{d}{dr^*}\left(-\frac{r^*_{eq}}{r^{*2}} + \frac{r^{*2}}{{r^*_{eq}}^3}\right)\right)_{r^*_{eq}}(r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{1}{2}\left(2\frac{r^*_{eq}}{r^{*3}} + 2\frac{r^*}{{r^*_{eq}}^3}\right)_{r^*_{eq}}(r^* - r^*_{eq})^2$$

$$= \frac{4}{3} + \frac{1}{2}\left(2\frac{1}{{r^*_{eq}}^2} + 2\frac{1}{{r^*_{eq}}^2}\right)(r^* - r^*_{eq})^2 = \frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2$$

$$=> M \approx \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right).M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left[\frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2\right]$$

$$= \left(1 + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2\right).M_\Sigma + \frac{E^*_{em,eq}}{c^2}\left[\frac{4}{3} + \frac{2}{{r^*_{eq}}^2}(r^* - r^*_{eq})^2\right]$$

$$\boxed{=> M \approx M_{eq} + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2.M_\Sigma + \frac{E^*_{em,eq}}{c^2}2\frac{(r^* - r^*_{eq})^2}{{r^*_{eq}}^2}}$$

With

$$\boxed{M_{eq} = M_\Sigma + \frac{4}{3}\frac{E^*_{em,eq}}{c^2}}$$

The mass is a function $M = M\left(r^*, \frac{dr^*}{dt}, \boldsymbol{V}_{\mathbf{C}}\right)$.

<u>The dynamic of the center of mass</u>

$$\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})M\boldsymbol{V}_{\mathbf{C}}) = M\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) + (\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}})\frac{dM}{dt}$$

and

$$\frac{dM}{dt} = \frac{d}{dt}\left(M_{eq} + \frac{1}{2}\frac{1}{c^2}\left(\frac{dr^*}{dt}\right)^2 .M_\Sigma + \frac{E^*_{em,eq}}{c^2}2\frac{\left(r^* - r^*_{eq}\right)^2}{{r^*_{eq}}^2}\right)$$

$$= \frac{1}{c^2}\frac{dr^*}{dt}\frac{d^2r^*}{dt^2}.M_\Sigma + \frac{E^*_{em,eq}}{c^2}4\frac{\left(r^* - r^*_{eq}\right)}{{r^*_{eq}}^2}\frac{dr^*}{dt}$$

$$\text{But } \frac{d^2r^*}{dt^2} \approx -\left(1 - \frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right)4\frac{E^*_{em,eq}}{M_\Sigma}\frac{(r^* - r^*_{eq})}{{r^*_{eq}}^2}$$

$$=> \frac{dM}{dt} = \frac{dr^*}{dt}\left(-\frac{1}{c^2}\left(1 - \frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right)4E^*_{em,eq}\frac{\left(r^* - r^*_{eq}\right)}{{r^*_{eq}}^2}\right) + \frac{E^*_{em,eq}}{c^2}4\frac{\left(r^* - r^*_{eq}\right)}{{r^*_{eq}}^2}\frac{dr^*}{dt}$$

$$= \frac{1}{c^2}\frac{dr^*}{dt}\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}4E^*_{em,eq}\frac{\left(r^* - r^*_{eq}\right)}{{r^*_{eq}}^2}$$

$$But\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})M\boldsymbol{V}_{\mathbf{C}}) = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}}L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right)$$

$$=> M\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}}) + (\gamma(\boldsymbol{V}_{\mathrm{C}})\boldsymbol{V}_{\mathbf{C}})\frac{dM}{dt} = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}}L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right)$$

Then

$$M\frac{d}{dt}(\gamma(\boldsymbol{V}_{\mathbf{C}})\boldsymbol{V}_{\mathbf{C}}) = \frac{\partial}{\partial \boldsymbol{R}_{\mathbf{C}}}L'\left(r^*, \frac{dr^*}{dt}, \boldsymbol{R}_{\mathrm{C}}, \boldsymbol{V}_{\mathrm{C}}, t\right) - \alpha.\boldsymbol{V}_{\mathbf{C}}$$

with

- $\alpha = \gamma(\boldsymbol{V}_{\mathbf{C}})\frac{dM}{dt} = \frac{dM}{dt^*}$
- $\frac{dM}{dt} = \left(\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right)\left(\frac{4}{{r^*_{eq}}^2}\frac{E^*_{em,eq}}{c^2}\left(r^* - r^*_{eq}\right)\frac{dr^*}{dt}\right)$

The internal dynamics influence the dynamics of the center of mass. This coupling is not due to an eventual relativistic behaviour of the internal dynamics but especially to the relativistic behaviour of the center of mass itself. Indeed, we see that in the Newtonian limit ($\frac{\boldsymbol{V}_{\mathrm{C}}^2}{c^2} = 0$), there is no longer a viscous term where the coupling appears. This coupling is of course due to an exchange between the internal energy M and that of the center of mass. This variation of the internal energy modifies the inertia and then acts on the speed for a given momentum.

We can for example write:

$$\frac{dM}{dt} = \left(\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right)\left(\frac{4}{{r^*_{eq}}^2}\frac{E^*_{em,eq}}{c^2}\left(r^* - r^*_{eq}\right)\frac{dr^*}{dt}\right) = \left(\frac{\boldsymbol{V}_{\mathbf{C}}^2}{c^2}\right)\left(\gamma(\boldsymbol{V}_{\mathrm{C}})^2 k_{\boldsymbol{V}_{\mathrm{C}}}\left(r^* - r^*_{eq}\right)\frac{dr^*}{dt}\right)$$

$$= \left(\frac{\boldsymbol{V}_{\mathbf{C}}{}^{2}}{c^{2}}\right)\gamma(\boldsymbol{V}_{\mathrm{C}})^{2}.F\frac{dr^{*}}{dt} \propto F\frac{dr^{*}}{dt} = P$$

With

- $k_{\boldsymbol{V}_{\mathrm{C}}} \equiv \frac{4}{{r_{eq}^{*}}^{2}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^{2}}E_{em,eq}^{*}$
- The force $F \equiv M_{\Sigma}\frac{d^{2}r^{*}}{dt^{2}} \approx -\frac{4}{{r_{eq}^{*}}^{2}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^{2}}\frac{E_{em,eq}^{*}}{M_{\Sigma}}\left(r^{*} - r_{eq}^{*}\right) = -\frac{4}{{r_{eq}^{*}}^{2}}\frac{1}{\gamma(\boldsymbol{V}_{\mathrm{C}})^{2}}\frac{E_{em,eq}^{*}}{M_{\Sigma}}\left(r^{*} - r_{eq}^{*}\right)$
- The power P= $F\frac{dr^{*}}{dt}$

When the electron absorb energy (P>0), then the viscous force opposes to the movement of the center of mass ($\alpha > 0$).

2.3. Hamiltonian analysis: Hamilton-Jacobi equation(an attempt) for a material system free

As for [1] and [2], we start from the norm equation:

$$\left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

We have to express the different quantities in term of the action. For that, I search the expression of the action as the function of coordinates : that is to say the action resulting from the injection of the equation of motion in its variation. I need the 2 expressions below in term of coordinate:

- By Mixing internal and external degree of freedom $(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$
- And only using internal degree of freedom $(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*)$

- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) \equiv \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}, t]\}_{real\ trajectory} = \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \frac{L^*}{\gamma}.dt = \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} L'.dt$

$=> \delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left(\sum_a \frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt} + \frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}} + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{V}_{\mathrm{C}} + \frac{\partial L'}{\partial t}\delta t\right) dt$$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left(\sum_a \frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{d}{dt}\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* - \delta \boldsymbol{r}_{\boldsymbol{a}}^* \frac{d}{dt}\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\right) + \frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}} \right.$$
$$\left. + \left(\frac{d}{dt}\left(\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right) - \delta \boldsymbol{R}_{\mathrm{C}}\left(\frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\right)\right) + \frac{\partial L'}{\partial t}\delta t\right) dt$$

$$= \int_{t_1}^{t,\{r_a^*\},\boldsymbol{R}_c} \left\{\sum_a \left(\frac{\partial L'}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} - \frac{d}{dt}\frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\right)\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \left(\frac{\partial L'}{\partial \boldsymbol{R}_{\mathrm{C}}} - \frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\right)\delta \boldsymbol{R}_{\mathrm{C}} + \frac{\partial L'}{\partial t}\delta t \right.$$
$$\left. + \frac{d}{dt}\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right)\right\} dt$$

$$=> \delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) = \int_{t_1}^{t,\{\delta r_a^*\},\delta R_{\mathrm{C}}} d\left(\sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}}\delta \boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}}\delta \boldsymbol{R}_{\mathrm{C}}\right)$$

Since

- $\left(\frac{\partial L'}{\partial r_a^*} - \frac{d}{dt}\frac{\partial L'}{\partial \frac{dr_a^*}{dt}} = 0\right)_{for\ a\ real\ trajectory}$
- $\left(\frac{\partial L}{\partial \boldsymbol{R}_{\mathrm{C}}} - \frac{d}{dt}\frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{C}}} = 0\right)_{for\ a\ real\ trajectory}$

Then we have the following result

$$=> dS(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) = \sum_a \left(\frac{\partial S}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} d\boldsymbol{r}_{\boldsymbol{a}}^*\right) + \frac{\partial S}{\partial \boldsymbol{R}_{\mathrm{c}}} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

$$= \sum_a \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt}} d\boldsymbol{r}_{\boldsymbol{a}}^* + \frac{\partial L'}{\partial \boldsymbol{V}_{\mathrm{c}}} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

$$= \sum_a \boldsymbol{P}_{\boldsymbol{a}}^* d\boldsymbol{r}_{\boldsymbol{a}}^* + \boldsymbol{P} d\boldsymbol{R}_{\mathrm{c}} + \frac{\partial S}{\partial t} dt$$

Then

- $\boldsymbol{P}_{\boldsymbol{a}}^* \equiv \frac{\partial L'}{\partial \frac{dr_a^*}{dt}} = \frac{\partial S}{\partial r_a^*}$
- $\boldsymbol{P} \equiv \frac{\partial L'}{\partial V_{\mathrm{c}}} = \frac{\partial S}{\partial R_{\mathrm{c}}}$
- $L' \equiv \frac{dS}{dt}(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t)$
- $H(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \{\boldsymbol{P}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, \boldsymbol{V}_{\boldsymbol{c}}) \equiv \sum_a \boldsymbol{P}_{\boldsymbol{a}}^* \frac{dr_a^*}{dt} + \boldsymbol{P}\boldsymbol{V}_{\boldsymbol{c}} - L' = \frac{\partial S}{\partial t}$
- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, \boldsymbol{R}_{\boldsymbol{c}}, t) \equiv \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, \boldsymbol{R}_{\boldsymbol{c}}(t)]\}_{real\ trajectory}$

- $S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \{S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*), t^*(t)\}]\}_{real\ trajectory} = \int_{t_1^*}^{t^*,\{r_a^*\}} L^* dt^*$

$$\delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \{\delta S[\{\boldsymbol{r}_{\boldsymbol{a}}^*(t^*)\}, t^*(t)]\}_{real\ trajectory} = \int_{t_1^*}^{t^*,\{r_a^*\}} \delta(L^* dt^*)$$

$$= \int_{t_1^*}^{t^*,\{r_a^*\}} \delta_{t^*}(L^* dt^*) + \delta_{\{r_a^*\}}(L^* dt^*) + \delta_{\left\{\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right\}}(L^* dt^*) + \delta_t(L^* dt^*)$$

$$= \int_{t_1^*}^{t^*,\{r_a^*\}} \left\{ L^* \delta_{t^*}(dt^*) + dt^* \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} \delta \boldsymbol{r}_{\boldsymbol{a}}^*\right) dt^* + \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right) dt^* \right\}$$

But dt^* is variable:

$$\delta \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} = \delta\left(\frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\right) = \frac{\delta(d\boldsymbol{r}_{\boldsymbol{a}}^*)}{dt^*} + d\boldsymbol{r}_{\boldsymbol{a}}^* \delta\left(\frac{1}{dt^*}\right) = \frac{d\delta \boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} + d\boldsymbol{r}_{\boldsymbol{a}}^* \left(\frac{-\delta dt^*}{{dt^*}^2}\right) = \frac{d\delta \boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} - d\boldsymbol{r}_{\boldsymbol{a}}^* \left(\frac{d\delta t^*}{{dt^*}^2}\right)$$

$$= \frac{d\delta \boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*} - \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\left(\frac{d\delta t^*}{dt^*}\right)$$

$$\delta S(\{\boldsymbol{r}_{\boldsymbol{a}}^*\}, t^*) = \int_{t_1^*}^{t^*,\{r_a^*\}} \left\{ d(L^* \delta t^*) - \delta t^* dL^* + dt^* \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*} \delta \boldsymbol{r}_{\boldsymbol{a}}^*\right) dt^* \right.$$
$$\left. + \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}} \left(\frac{d\delta \boldsymbol{r}_{\boldsymbol{a}}^*}{ddt^*} - \frac{d\boldsymbol{r}_{\boldsymbol{a}}^*}{dt^*}\left(\frac{d\delta t^*}{dt^*}\right)\right)\right) dt^* \right\}$$

$$= \int_{t_1^*}^{t^*,\{r_a^*\}} \left\{ \frac{d(L^*\delta t^*)}{dt^*} - \delta t^* \frac{dL^*}{dt^*} + \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \delta \boldsymbol{r}_a^* \right) + \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\delta \boldsymbol{r}_a^*}{dt^*} - \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \left(\frac{d\delta t^*}{dt^*} \right) \right\} dt^*$$

$$= \int_{t_1^*}^{t^*,\{r_a^*\}} \left\{ \frac{d(L^*\delta t^*)}{dt^*} - \delta t^* \frac{dL^*}{dt^*} + \delta_{t^*}(L^*) + \left(\sum_a \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \delta \boldsymbol{r}_a^* \right) \right.$$

$$+ \left(\frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \delta \boldsymbol{r}_a^* \right) - \sum_a \delta \boldsymbol{r}_a^* \frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) \right)$$

$$\left. - \left(\frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \delta t^* \right) - \delta t^* \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) \right) \right\} dt^*$$

$$= \int_{t_1^*}^{t^*,\{r_a^*\}} \left\{ \delta t^* \left[-\frac{dL^*}{dt^*} + \frac{\partial L^*}{\partial t^*} + \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) \right] + \sum_a \left(\frac{\partial L^*}{\partial \boldsymbol{r}_a^*} - \frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) \right) \delta \boldsymbol{r}_a^* \right.$$

$$\left. + \frac{d}{dt^*} \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \delta \boldsymbol{r}_a^* - \delta t^* \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* \right) \right) \right\} dt^*$$

- $\left(\frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} \right) - \frac{dL^*}{dt^*} = -\frac{\partial L^*}{\partial t^*} \right)_{for\ a\ real\ trajectory}$
- $\left(\frac{d}{dt^*} \left(\frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \right) = \frac{\partial L^*}{\partial \boldsymbol{r}_a^*} \right)_{for\ a\ real\ trajectory}$

Then we have the following result

$$=> dS(\{\boldsymbol{r}_a^*\}, t^*) = \sum_a \frac{\partial S}{\partial \boldsymbol{r}_a^*} d\boldsymbol{r}_a^* + \frac{\partial S}{\partial t^*} dt^*$$

$$= \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} d\boldsymbol{r}_a^* - \left(\sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* \right) dt^*$$

$$= \sum_a \boldsymbol{P}_a^* d\boldsymbol{r}_a^* - E^* dt^*$$

Then

- $\boldsymbol{P}_a^* \equiv \frac{\partial L'}{\partial \frac{d\boldsymbol{r}_a^*}{dt}} = \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} = \frac{\partial S}{\partial \boldsymbol{r}_a^*}$
- $E^* \equiv \sum_a \frac{\partial L^*}{\partial \frac{d\boldsymbol{r}_a^*}{dt^*}} \frac{d\boldsymbol{r}_a^*}{dt^*} - L^* = \frac{\partial S}{\partial t^*}$
- $L^* \equiv \frac{dS}{dt^*} (\{\boldsymbol{r}_a^*\}, t^*, t)$
- $H^*(\{\boldsymbol{r}_a^*\}, \{\boldsymbol{P}_a^*\}) \equiv \sum_a \boldsymbol{P}_a^* \frac{d\boldsymbol{r}_a^*}{dt} - L^* = \frac{\partial S}{\partial t^*}$
- $S(\{\boldsymbol{r}_a^*\}, t^*) \equiv \{S[\{\boldsymbol{r}_a^*(t^*)\}, t^*]\}_{real\ trajectory}$

$$\left(\frac{E^*}{c}\right)^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{P}^2$$

We have the first equation

$$\boxed{\frac{1}{c^2}\left(\frac{\partial S}{\partial t^*}\right)^2_{t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\}} = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_{\boldsymbol{c}}}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t}}$$

The expression uses the same quantity S but expressed as 2 functions of different variables.

We can also express the equation in term of internal position

$$E^* = \sum_a E_a^*$$

$$\text{With } \left(\frac{m_a c^2}{c}\right)^2 = \left(\frac{E_a^*}{c}\right)^2 - \boldsymbol{P}_{\boldsymbol{a}}^{*\,2}$$

$$=> E_a^* = \sqrt{(m_a c^2)^2 + c^2 \boldsymbol{P}_{\boldsymbol{a}}^{*\,2}}$$

$$=> E^* = \sum_a \sqrt{(m_a c^2)^2 + c^2 \boldsymbol{P}_{\boldsymbol{a}}^{*\,2}}$$

$$=> \boxed{\left(\sum_a \sqrt{\left(\frac{m_a c^2}{c}\right)^2 + \left(\frac{\partial S}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\right)^2_{t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\}}}\right)^2 = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_{\boldsymbol{c}}}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t}}$$

$$<=> \boxed{\sum_a \sqrt{\left(\frac{m_a c^2}{c}\right)^2 + \left(\frac{\partial S}{\partial \boldsymbol{r}_{\boldsymbol{a}}^*}\right)^2_{t^*,\{\boldsymbol{r}_{\boldsymbol{a}}^*\}}} = \sqrt{\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t} - \left(\frac{\partial S}{\partial \boldsymbol{R}_{\boldsymbol{c}}}\right)^2_{\{\boldsymbol{r}_{\boldsymbol{a}}^*\},\boldsymbol{R}_{\boldsymbol{c}},t}}}$$

This equation is pretty complicated. To develop again the analysis, a quantum version (à la Schrödinger) using the action as the phase of a wave function would be interesting to obtain (with the internal degree of freedom and the center of mass as variables). Here we see that is seems not very straightforward (or not possible?).

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