Modules and Universal Constructions

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ABSTRACT. The concept of module subsumes that of category and provides a more general and abstract framework to explore the theory of structures, in particular that of universal properties. In this book, we present the basic theory of modules and redevelop the notion of universal constructions in category theory using the framework of modules.

About modules and profunctors

The terms “module” and “profunctor” are synonymous in the sense that both are defined as a functor of the form $F: X^{op} \times A \to \text{Set}$. The notion given by this definition is seen by many as a generalization of the notion of functor. The term “profunctor” is quite appropriate in this regard. However, the notion has another, probably more important, aspect: it is also seen as an extension of categories. We use the term “module” when this aspect of the notion is emphasized; roughly, modules are to categories what, in the sense of abstract algebra, modules are to rings (or vector spaces are to fields). Modules constitute the objects of the category $\text{MOD}$, while profunctors constitute the morphisms of the category $\text{Prof}$. This book studies modules, not profunctors; to put it more precisely, this book studies modules as an extension of categories.

Modules as an extension of categories

A module from a category $X$ to $A$ consists of arrows whose domain resides in $X$ and codomain in $A$. To say that the notion of module is an extension of the notion of category is to say that module arrows are an extension of category arrows; the foregoing analogy between modules and categories versus vector spaces and fields may be restated as follows: module arrows are to category arrows what vectors are to scalars. Module arrows and category arrows share some notions: for example, epicity and monicity. On the other hand, module arrows lack the notion of isomorphism—a crucial notion associated with category arrows. However, to compensate for this lack, they possess a very essential notion in category theory, namely, universality. It is mainly for this property of module arrows that this book is written.

Topics

Chapter 1 introduces modules and cells among them. A module $M: X \to A$ from a category $X$ to $A$ is a functor of the form $M: X^{op} \times A \to \text{Set}$, assigning a set of “arrows” to each pair of objects $x \in X$ and $a \in A$. A cell from a module $M$ to $N$ sends each arrow of $M$ to an arrow of $N$. The hom-functor of a category $C$ defines the hom-module $\text{hom}_C: C \to C$, and the arrow function of a functor $F: C \to D$ defines the hom-cell $(F): (C) \to (D)$ from the hom-module of $C$ to that of $D$. The concepts of categories and cells subsume those of categories and functors in this way and set up a more general and conceptual framework to explore the structure of mathematics. Presheaves and copresheaves are called right and left modules respectively in this book and studied as special instances of modules.

Chapter 2 discusses the action of a module on its domain and codomain—the operation that yields the Yoneda embedding functor in the case of a hom-module. The chapter introduces an important class of modules called representable, which each functor produces by composition with the hom-module of its codomain.

1Some use the term “heteromorphism” for arrows of a module in contrast with “homomorphism” used for arrows of a category. In this book, we just call them module arrows and category arrows.
Chapter 3 presents two variants of modules, namely collages and commas\(^2\), which are special sorts of cospans and spans between two categories. We will establish an isomorphism between the category of modules and the category of collages, and, later in Chapter 11, construct an adjoint equivalence between the category of commas and the category of collages. Two forgetful functors from \(\text{MOD}\) (the category of modules and cells) to \(\text{CAT}\) (the category of categories and functors) are defined through constructions of collages and commas, and it is shown that they form left and right adjoints of the embedding \(\text{CAT} \hookrightarrow \text{MOD}\) given by the hom-module assignment \(C \mapsto (\mathcal{C})\) (and thus that \(\text{CAT}\) is a reflective and coreflective subcategory of \(\text{MOD}\)).

Chapter 4 introduces the notion of frames of a module. A cylindrical frame of an endomodule abstracts the notion of a natural transformation between two functors, and a conical frame of a right (resp. left) module abstracts the notion of a cone between an object and a functor. Ordinary and extraordinary cylinders—natural transformations spanning a module\(^3\), so to speak—are defined as instances of cylindrical frames. Likewise, cones are defined along a module as instances of conical frames. The section also introduces the notion of orbits of a module as a generalization of that of a group action. Orbits may be seen as the dual of frames.

Chapter 5 discusses the actions of the domain and codomain of a module on its arrows and frames. It is shown, as a generalization of the Yoneda embedding, that these actions embed a module \(X \rightarrow A\) in the category of right modules (i.e. presheaves) over \(X\) and in that of left modules (i.e. copresheaves) over \(A\). The Yoneda lemma is presented in a general form to state that the morphisms from the representable module of a functor \(F : X \rightarrow A\) to an arbitrary module \(M : X \rightarrow A\) correspond one-to-one with the cylinders defined between \(F\) and \(M\). Using the lemma, we establish a variety of bijective correspondences between frames and cells.

The second part of the book, starting with Chapter 6, explores universal constructions in the framework of modules using the theory and machinery built in the first part. Universality is formulated in terms of universal arrows of a module; the embedding of a module in the category of presheaves makes the definition of a universal arrow simple enough: an arrow in a module is universal if its image under the embedding is an isomorphism. Chapter 6 presents a general theory of universals, formulated as universal arrows, in the abstract context of modules and cells. We then, in the remaining chapters, look at specific universal constructions (limits, ends, extensions, adjoint functors, etc.) as examples of universals, defining them as universal arrows of a module—whenever we define a specific universal construction, we define a specific module for it—so that the general theory developed in Chapter 6 can be applied. Here are some highlights.

- Limits and ends are defined along a module as universal cones and as universal extraordinary cylinders, respectively. Preservation of limits and ends are treated as an instance of the general notion of a cell preserving universal arrows.

- Extensions are defined as universal cells, and lifts are defined as universal (ordinary) cylinders. Kan extensions and Kan lifts are presented as special instances of the general notions of extensions and lifts, respectively.

- The notion of pointwise lift is introduced. We show that the notion subsumes that of pointwise extension and vice versa. Parameterized limits, ends, and adjunctions are all treated as instances of pointwise lifts.

- The notion of a symmetric cell is introduced to define adjunctions between two modules as well as between two categories. In fact, adjunctions are proved to constitute universal arrows of the

\(^2\)The term “two-sided discrete fibration” is used in the literature to refer to what this book calls a comma. The term “comma” is adopted because every two-sided discrete fibration is given by a comma category and its projection functors (see [LR18] Theorem 2.3.3).

\(^3\)The notion is called “het natural transformation” in [Ell07].
module of symmetric cells, allowing the treatment of them in the general framework of modules and universal arrows.

- Adjoins are defined not only for a functor but for a cell as well. We show that a cell preserves universal arrows if it has adjoints, and deduce RAPL (right adjoints preserve limits) as a corollary of this general mechanism.

- The notion of an equivalence of categories is extended to that of an equivalence of modules. It is shown that an equivalence cell preserves and reflects universal arrows.

- Epicity and monicity are defined not only for arrows in a category but for arrows in a module. A proof of the special adjoint functor theorem is given using epi-mono-factorizations for modules.

- The concept of density is also generalized for modules (in fact, density is most naturally defined with modules). The fact “every presheaf is a colimit of representables” is proved using the concept of density to show how the concept works in the framework of modules.

**Advice on reading**

The exposition is carried out within the framework of set-enriched 1-dimensional categories. Chapter 0 presents the notations and some elementary facts of category theory used in the sequel.

Size issues are not treated rigorously. The specification of a universe is almost always implicit; unless otherwise stated, a universe $\mathcal{U}$ is chosen so that given categories and modules become locally $\mathcal{U}$-small. We just say “small” instead of “$\mathcal{U}$-small” for $\mathcal{U}$ chosen implicitly. The choice of a universe is fixed in each context unless otherwise stated.

This book is made up of sequences of “Definition” - “Proposition” - “Theorem” - “Corollary”, with each optionally preceded by “Note” and followed by “Remark”. Each “Proposition” is tightly associated with the preceding “Definition” and states some straightforward consequences of it. Many “Theorem”’s in fact state facts too obvious to deserve such a title (“there are no theorems in category theory”).

Two statements dual to each other are indicated by the symbols $\triangleright$ and $\triangleleft$; for example, we write

- Right adjoints preserve limits.
- Left adjoints preserve colimits.

The proof is usually given only for the assertion indicated by $\triangleright$. We also combine two dual statements into one using the abbreviation “op.”; the foregoing example is alternatively written as

Right [op. left] adjoint preserves limits [op. colimits].

Parentheses and brackets serve mainly as punctuation to enhance readability; parentheses are used to delimit objects and arrows, whereas square (resp. angle) brackets are used to delimit categories and functors (resp. modules and cells).
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0 Preliminaries

0.0.1 Language of categories

1. A category is defined in terms of hom-sets. Pairwise disjointness of hom-sets is not required. Given a category $\mathbf{C}$, an element $f$ of a hom-set $\text{hom}_{\mathbf{C}}(a,b)$ or a triple $(a,f,b)$ such that $f \in \text{hom}_{\mathbf{C}}(a,b)$ is called an arrow of $\mathbf{C}$ (or a $\mathbf{C}$-arrow) and written $f : a \to b$, $a \xrightarrow{f} b$, or just $f$ if its domain and codomain are understood or unimportant. The composite of arrows $f : a \to b$ and $g : b \to c$ is written as

$$f \circ g = g \circ f.$$ 

The identity arrow $x \to x$ is denoted by $1_x$ or $x \xrightarrow{1} x$, or by $x \xrightarrow{x} x$ using the name of the object. An invertible arrow is called iso (or an isomorphism).

2. The set of objects of a category $\mathbf{C}$ is denoted by $\|\mathbf{C}\|$ and the object function of a functor $F$ is denoted by $\|F\|$.

3. Italic small letters $a, b, c, \ldots$ vary over objects and arrows. When we write $c \in \mathbf{C}$ for a category $\mathbf{C}$, $c$ stands for an object or arrow of $\mathbf{C}$.

4. The value of a functor $F : \mathbf{C} \to \mathbf{B}$ at $c \in \mathbf{C}$ is written variously as

$$c : F = cF = (c)F = F(c) = F_c = F \circ c,$$

and the component of a natural transformation $\tau : S \to T : \mathbf{C} \to \mathbf{B}$ at an object $c \in \|\mathbf{C}\|$ is written as

$$c : \tau = \tau_c = \tau \circ c.$$

5. The opposite of a category $\mathbf{C}$ is denoted by $\mathbf{C}^-$; for a $\mathbf{C}$-arrow $f : a \to b$, the corresponding $\mathbf{C}^-$-arrow (the opposite of $f$) is written as $f^- : b \to a$. The opposite of a functor $F : \mathbf{C} \to \mathbf{B}$ is denoted by $F^- : \mathbf{C}^- \to \mathbf{B}^-$, or often by $F : \mathbf{C}^- \to \mathbf{B}^-$ using the same name as its original counterpart.

6. Given categories $\mathbf{A}$ and $\mathbf{B}$, the product, coproduct, and functor categories are denoted by $\mathbf{A} \times \mathbf{B}$, $\mathbf{A} + \mathbf{B}$, and $[\mathbf{A}, \mathbf{B}]$ respectively. The terminal category $\mathbf{1}$ consists of a single object 0 and a single arrow $0 : 0 \to 0$, and the interval category $\mathbf{2}$ consists of two objects 0 and 1 with only one non-identity arrow $0 \to 1$. The terminal category and its sole object are also (in fact, almost always in this book) denoted by $*$.

7. Italic capital letters $F$, $G$, $\ldots$ vary over both functors and natural transformations. When we write $F \in [\mathbf{A}, \mathbf{B}]$, $F : \mathbf{A} \to \mathbf{B}$, or $A \xrightarrow{F} B$ for categories $\mathbf{A}$ and $\mathbf{B}$, $F$ stands for an object or an arrow of the functor category $[\mathbf{A}, \mathbf{B}]$, i.e. a functor or a natural transformation.

8. For a universe $\mathcal{U}$, a category $\mathbf{C}$ is called locally $\mathcal{U}$-small if each of its hom-sets is $\mathcal{U}$-small, and called $\mathcal{U}$-small if it is locally $\mathcal{U}$-small and the set of objects is $\mathcal{U}$-small. By “small” we mean “$\mathcal{U}$-small” for $\mathcal{U}$ chosen implicitly. Note that the functor category $[\mathbf{A}, \mathbf{B}]$ is in general large (i.e. not locally small) even if given categories $\mathbf{A}$ and $\mathbf{B}$ are locally small; if we are to require $[\mathbf{A}, \mathbf{B}]$ to be locally small, we require $\mathbf{A}$ to be small.

9. $\textbf{Set}_\mathcal{U}$, $\textbf{Cat}_\mathcal{U}$, and $\textbf{CAT}_\mathcal{U}$ denote the categories of $\mathcal{U}$-small sets, $\mathcal{U}$-small categories, and locally $\mathcal{U}$-small categories respectively; by $\textbf{Set}$ we mean $\textbf{Set}_\emptyset$ for some universe $\mathcal{U}$, and similarly for $\textbf{Cat}$ and $\textbf{CAT}$. The categories $\textbf{Set}$ and $\textbf{Cat}$ are locally small, while $\textbf{CAT}$ is large.
0.0.2 Precomposition and postcomposition functors

1. Let $A$, $B$, and $C$ be categories. The composite of $F : A \to B$ and $G : B \to C$ is written as $F \circ G = G \circ F$,
and its value $a \cdot [F \circ G] = (a \cdot F) \circ G$ at $a \in A$ is often written as $a \cdot F \circ G = G \circ F \cdot a$.

The composition $(F,G) \mapsto F \circ G$ defines a bifunctor $[A,B] \times [B,C] \to [A,C]$, and the right and left exponential transpositions yield functors

$\left[ A, -, \right] : [B,C] \to \left[ \left[ A, B \right], \left[ A, C \right] \right]$ ; $G \mapsto (F \mapsto F \circ G)$

and

$[\left[ -, C \right] : [A,B] \to \left[ \left[ B, C \right], \left[ A, C \right] \right]$ ; $F \mapsto (G \mapsto F \circ G)$.

The functor $[A,-]$ sends each functor $G : B \to C$ to the postcomposition functor $[A,G] : [A,B] \to [A,C]$ ; $F \mapsto F \circ G$ and sends each natural transformation $\tau : S \to T : B \to C$ to the postcomposition natural transformation

$\left[ A, \tau \right] : [A,S] \to [A,T] : [A,B] \to [A,C]$ defined by $\left[ A, \tau \right]_F = F \circ \tau : F \circ S \to F \circ T : A \to C$ for $F$ a functor $A \to B$.

The functor $[\left[ -, C \right]$ sends each functor $F : A \to B$ to the precomposition functor $[F,C] : [B,C] \to [A,C]$ ; $G \mapsto F \circ G$ and sends each natural transformation $\sigma : S \to T : A \to B$ to the precomposition natural transformation

$\left[ \sigma, C \right] : [S,C] \to [T,C] : [B,C] \to [A,C]$ defined by $\left[ \sigma, C \right]_G = \sigma \circ G : S \circ G \to T \circ G : A \to C$ for $G$ a functor $B \to C$.

2. The bifunctorial operation $[\left[ -, - \right]$ on $\mathbf{CAT}^\circ \times \mathbf{CAT}$ is defined in the following way:

a) for a pair of categories $X$ and $A$, $[X,A]$ is the category of functors $X \to A$;

b) for a functor $P : X \to Y$ and a category $B$, $[P,B] : [Y,B] \to [X,B]$ is the precomposition functor;

c) for a functor $Q : B \to A$ and a category $X$, $[X,Q] : [X,B] \to [X,A]$ is the postcomposition functor.

The commutativity of

$[Y,B] \xrightarrow{[P,B]} [X,B]$

$\downarrow_{[Y,Q]}$

$[Y,A] \xrightarrow{[P,A]} [X,A]$

expresses the associativity of functor composition.

0.0.3 Inclusion and restriction functors

The identity functor $1_E : E \to E$ is also denoted by $E$ using the name of the category. More generally, if $D$ is a subcategory $E$, the inclusion functor $D \hookrightarrow E$ is also denoted by $D$ using the name of its
domain; the restriction of \( F : E \to C \) to \( D \) is written as
\[
D \circ F = F \circ D
\]; precomposition with the inclusion \( D : D \to E \) yields the functor \([D, C] : [E, C] \to [D, C]\), “restriction to \( D' \), for any category \( C \).

### 0.0.4 Terminal and product categories

1. By the obvious isomorphism \([*, E] \cong E\), a functor (resp. natural transformation) \(* \to E\) is identified with an object (resp. arrow) of \( E\). Given categories \( E \) and \( D\) and given an object \( e \in [E] \) [op. \( d \in [D] \)], the product functor \( e \times D \) [op. \( E \times d \)] as in
\[
\begin{array}{ccc}
* & \times D & \to D \\
\downarrow e \times D & \downarrow D & \downarrow d \\
E \times D & \to D & E
\end{array}
\]
gives the section
\[
e \times D : D \to E \times D \quad \text{[op.]} \quad E \times d : E \to E \times D
\]
of \( E \times D \) at \( e \in [E] \) [op. \( d \in [D] \)] under the identification \(* \times D \cong D\) [op. \( E \times * \cong E\)].

2. Given a category \( E\), the unique functor \( E \to *\) is denoted by \(!_E\) or just by \(!\). Given categories \( E \) and \( D\), the product functor \( !_E \times D \) [op. \( E \times !_D \)] as in
\[
\begin{array}{ccc}
E & \times D & \to D \\
\downarrow !_E \times D & \downarrow D & \downarrow _D \\
E \\
E \\
E \times D & \to E & E \times * \to *
\end{array}
\]
gives the projection
\[
!_E \times D : E \times D \to D \quad \text{[op.]} \quad E \times !_D : E \times D \to E
\]
of \( E \times D \) on \( D\) [op. \( E\)] under the identification \(* \times D \cong D\) [op. \( E \times * \cong E\)].

### 0.0.5 Exponential and twist transposition of bifunctors

The right [op. left] exponential transposition of bifunctors and its inverse are denoted by \( \triangleright \) and \( \triangleleft \) [op. \( \triangleright \) and \( \triangleleft \)] and twist transposition is denoted by \( \uparrow \) as shown in the commutative diagrams:

\[
\begin{array}{ccc}
[E \times D, C] & \xrightarrow{\triangleright} & [D, [E, C]] \\
\uparrow \uparrow \uparrow \triangleright & & \uparrow \uparrow \uparrow \triangleright \\
[D \times E, C] & \xrightarrow{\triangleleft} & [D, [E, C]]
\end{array}
\]

\[
\begin{array}{ccc}
[E \times D, C] & \xrightarrow{\uparrow \uparrow \uparrow \triangleleft} & [D, [E, C]] \\
\uparrow \uparrow \uparrow \uparrow \triangleleft & & \uparrow \uparrow \uparrow \uparrow \triangleleft \\
[D \times E, C] & \xrightarrow{\triangleright \uparrow \uparrow \triangleright} & [D, [E, C]]
\end{array}
\]

, all natural in \( E, D, \) and \( C\). We note that

1. the right [op. left] exponential transpose of a bifunctor \( K : E \times D \to C\) is the functor
\[
[K \triangleright] : D \to [E, C] \quad \text{[op.]} \quad [K \triangleleft] : E \to [D, C]
\]
which sends each object \( d \in [D] \) [op. \( e \in [E] \)] to the functor
\[
[K \triangleright d] : E \to C ; e \mapsto (e, d) \quad \text{[op.]} \quad [e \triangleleft K] : D \to C ; d \mapsto K(e, d)
\]
and—given by \( \triangleright \) [op. \( \triangleleft \)] slice of \( K : E \times D \to C\) at \( d \in [D] \) [op. \( e \in [E] \)]—and sends each \( D \)-arrow
\[
f : d \to d' \quad [\text{op.]} \quad E \text{-arrow } f : e \to e' \]

\[
K \triangleright f : [K \triangleright d] \to [K \triangleright d'] : E \to C \quad \text{[op.]} \quad [f \triangleleft K] : [e \triangleleft K] \to [e' \triangleleft K] : D \to C
\]
whose component at $e \in \|E\|$ is defined by

$$[K \cdot f]_e = K(e, f) \quad \text{op.} \quad [f \cdot K]_d = K(f, d) .$$

2. The right [op. left] exponential transpose of a natural transformation $\tau : S \to T : E \times D \to C$ is the natural transformation

$$[\tau \cdot] : [S \cdot] \to [T \cdot] : D \to [E, C] \quad \text{op.} \quad [\cdot \tau] : [\cdot S] \to [\cdot T] : E \to [D, C]$$

whose component at $d \in \|D\|$ is the natural transformation

$$[\tau \cdot d] : [S \cdot d] \to [T \cdot d] : E \to C \quad \text{op.} \quad [\cdot e \tau] : [\cdot e S] \to [\cdot e T] : D \to C$$

— the right [op. left] slice of $\tau : S \to T : E \times D \to C$ at $d \in \|D\|$ defined by

$$[\tau \cdot d]_e = \tau(e, d) \quad \text{op.} \quad [\cdot e \tau]_d = \tau(e, d) .$$

3. The precomposition functor $[\cdot, C] : [E, D, C] \to [D \times E, C]$ with the twist isomorphism $D \times E \to E \times D$ is called the twist transposition (or twisting) and denoted $[E, D, C] \overset{\tau}{\to} [D \times E, C]$; the twist transpose $K^\tau : D \times E \to C$ of a bifunctor $K : E \times D \to C$ is given by the composition

$$D \times E \overset{\tau}{\to} E \times D \overset{K}{\to} C .$$

4. The twist transposition $[D, [E, C]] \overset{\tau}{\to} [E, [D, C]]$ is the exponentially transposed twisting $[E \times D, C] \overset{\tau}{\to} [D \times E, C]$: the twist transpose $K^\tau : E \to [D, C]$ of a functor $K : D \to [E, C]$ sends each object $e \in \|E\|$ to the functor

$$[e : K^\tau] : D \to C ; d \mapsto e : [d : K]$$

—the slice of $K : D \to [E, C]$ at $e \in \|E\|$.

0.0.6 Evaluation of functors

1. Given categories $E$ and $C$, the evaluation

$$(e, F) \mapsto e : F : E \times [E, C] \to C$$

is identified with the composition

$$(e, F) \mapsto e : F : [*] \times [E, C] \to [*] .$$

Given an object $e \in \|E\|$, the precomposition functor

$$[e, C] : [E, C] \to C$$

, “evaluation at $e$”, takes each $F \in [E, C]$ and yields the composite

$$e \mapsto E \overset{F}{\to} C$$

, i.e. the value of $F$ at $e$.

2. The naturality square

$$[E \times D, C] \overset{\sim}{\longrightarrow} [E, [D, C]]$$

$$[\ast \times D, C] \overset{\sim}{\longrightarrow} [\ast, [D, C]]$$

of left exponential transposition shrinks to the commutative triangle

$$[E \times D, C] \overset{\sim}{\longrightarrow} [E, [D, C]]$$

$$[\ast \times D, C] \overset{\sim}{\longrightarrow} [\ast, [D, C]]$$

by the identifications $[\ast \times D, C] \cong [D, C] \cong [\ast, [D, C]]$; the precomposition

$$D \overset{e \times D}{\longrightarrow} E \times D \overset{K}{\longrightarrow} C$$
of $K$ with the section $e \times D$ thus yields the same functor as the one given by the evaluation
\[ * \xrightarrow{e} E \xrightarrow{K^*} [D, C] \]
, i.e. the left slice of $K$ at $e$.

3. Similarly, the naturality square
\[ [D, [E, C]] \xrightarrow{\tau} [E, [D, C]] \]
\[ [D, [*, C]] \xrightarrow{\tau} [*, [D, C]] \]
of twist transposition shrinks to the commutative triangle
\[ [D, [E, C]] \xrightarrow{\tau} [E, [D, C]] \]
\[ [D, C] \]
by the identifications $[D, [*, C]] \cong [D, C] \cong [*, [D, C]]$; the postcomposition
\[ D \xrightarrow{K} [E, C] \xrightarrow{[e, C]} C \]
of $K$ with the evaluation $[e, C]$ thus yields the same functor as the one given by the evaluation
\[ * \xrightarrow{e} E \xrightarrow{K^*} [D, C] \]
, i.e. the slice of $K$ at $e$.

0.0.7 Diagonal functors
Given categories $D$, $E$, and $C$, the $D$-ary diagonal functor of the functor category $[E, C]$ is the precomposition functor
\[ [E \times !_D, C] : [E, C] \to [E \times D, C] \]
sending each functor $F : E \to C$ to the composite bifunctor
\[ E \times D \xrightarrow{E \times !} E \xrightarrow{F} C. \]
The diagonal functor $[E \times !_D, C]$ thus duplicates each $F : E \to C$ across $D$ by introducing to it a dummy variable varying over $D$. As a special case, the $D$-ary diagonal functor
\[ ![D, C] : C \to [D, C] \]
of a category $C$ sends each object $c \in [C]$ to the constant functor $\Delta_D c : D \to C$ given by the composition
\[ D \xrightarrow{!} * \xrightarrow{e} C. \]
The commutativity of
\[ [E \times !_D, C] \]
\[ [E \times D, C] \]
\[ [E \times * , C] \]
\[ [E \times !_D, C] \]
follows from the commutativity of the naturality square
\[ [E \times !_D, C] \]
\[ [E \times D, C] \]
\[ [E \times * , C] \]
\[ [E \times !_D, C] \]
by the identification
\[ [E \times * , C] \cong [E, C] \cong [*, [E, C]] \]
\[ [E \times * , C] \cong [E, C] \cong [*, [E, C]]. \]
0.0.8 Slice categories

The slice [op. coslice] category of a category \( C \) over [op. under] an object \( c \in \| C \| \) and its forgetful functor is denoted by

\[
\Sigma_c : C/c \to C \quad \text{op.} \quad \Sigma^c : c\backslash C \to C
\]

; any \( C \)-arrow \( f : c \to d \) induces the postcomposition [op. precomposition] functor

\[
C/f : C/c \to C/d \quad \text{op.} \quad f\backslash : d\backslash C \to c\backslash C
\]

over [op. under] \( f \) given by the composition

\[
h \mapsto h \circ f \quad \text{op.} \quad h \mapsto f \circ h
\]

such that the triangle

\[
\begin{array}{ccc}
C/c & \xrightarrow{C/f} & C/d \\
\Sigma_{c} & \searrow & \downarrow \Sigma_{d} \\
\downarrow \Sigma_{c} & & \downarrow \Sigma_{d}
\end{array}
\]

\[
\begin{array}{ccc}
d\backslash C & \xrightarrow{f\backslash} & c\backslash C \\
\Sigma_{d} & \searrow & \downarrow \Sigma_{c} \\
\downarrow \Sigma_{d} & & \downarrow \Sigma_{c}
\end{array}
\]

commutes, and any functor \( F : C \to D \) induces the slice [op. coslice] functor\(^1\)

\[
F/c : C/x \to D/(c\backslash F) \quad \text{op.} \quad c\backslash F : c\backslash C \to (c\backslash F)\backslash D
\]

over [op. under] \( c \in \| C \| \) by the evaluation

\[
h \mapsto h\cdot F
\]

such that the diagram

\[
\begin{array}{ccc}
C/c & \xrightarrow{F/c} & D/(c\backslash F) \\
\Sigma_{c} \downarrow & & \downarrow \Sigma_{c\backslash F} \\
C & \xrightarrow{F} & D
\end{array}
\]

\[
\begin{array}{ccc}
c\backslash C & \xrightarrow{c\backslash F} & (c\backslash F)\backslash D \\
\Sigma_{c} \downarrow & & \downarrow \Sigma_{c\backslash F} \\
C & \xrightarrow{F} & D
\end{array}
\]

commutes.

0.0.9 Pullback of functors

A pullback of a span \( A \xrightarrow{F} C \xleftarrow{G} B \) of functors is seen as a product in the slice category \( \text{CAT}/C \) with the canonical pullback diagram written as

\[
\begin{array}{ccc}
F \times G & \xrightarrow{F \times G} & B \\
\downarrow & & \downarrow \text{G} \\
A & \xrightarrow{F} & C
\end{array}
\]

(cf. 0.0.4(2)); \( F \times G \) is the subcategory of the product category \( A \times B \) consisting of all \((f, g) \in A \times B\) such that \( f \cdot F = g \cdot G \).

0.0.10 Connected categories

Two objects \( c \) and \( c' \) in a category \( C \) are said to be connected, written \( c \approx c' \), if there is a finite sequence of \( C \)-arrows

\[
c \to c_1 \leftarrow c_2 \ldots \leftarrow c_n \leftarrow c'
\]

(the direction of each arrow is arbitrary) connecting \( c \) to \( c' \). The relation \( c \approx c' \) is an equivalence relation and each equivalence class under \( \approx \) is called a connected component of \( C \). A category \( C \) is called connected if it is non-empty and any two objects of \( C \) are connected; that is, if \( C \) consists of exactly one connected component.

\(^1\)There does not seem to be any standard on the terminology. Different authors use the term “slice functor” to mean different things.
0.0.11 Isomorphism-dense subcategories

A subcategory $D \subseteq E$ is called isomorphism-dense if every object $e \in \|E\|$ is isomorphic to some object $d \in \|D\|$. A functor is called essentially surjective if its image is an isomorphism-dense subcategory of the codomain. Note that a subcategory is isomorphism-dense if and only if the inclusion functor is essentially surjective. We use the following lemma in the sequel.

**Lemma.** Let $\tau : S \to T : E \to C$ be a natural transformation and $D$ be an isomorphism-dense subcategory of $E$. Then $\tau$ is a natural isomorphism if and only if its restriction to $D$ is; that is, if and only if the component $\tau_d$ is an isomorphism for each $d \in \|D\|$.

**Proof.** Since a natural transformation is iso iff each component is an isomorphism, the forward implication is obvious. Assume now that $\tau$ is iso on $D$. We need to show that the component $\tau_e$ is an isomorphism for each $e \in \|E\|$. Since $D$ is isomorphism-dense in $E$, there is an object $d \in \|D\|$ and an iso $E$-arrow $h : d \to e$, giving a naturality square

$$
\begin{array}{ccc}
d \cdot S & \xrightarrow{\tau_d} & T \cdot d \\
h \cdot S \downarrow & & \downarrow T \cdot h \\
e \cdot S & \xrightarrow{\tau_e} & T \cdot e
\end{array}
$$

with $h \cdot S$ and $T \cdot h$ iso (because any functor preserves isomorphisms). Now, since $\tau_d$ an isomorphism by assumption, so is $\tau_e = (h \cdot S)^{-1} \circ \tau_d \circ (T \cdot h)$ as required. \qed

(We will see in Theorem 8.9.21 that the lemma still holds with the condition “isomorphism-dense” weakened to “retract-dense”.)
1 Modules and Cells

The first three sections in this chapter introduce the basic language of modules, which is summarized below. As we will see, the concept of module subsumes that of category.

A left (one-sided) module $M : \ast \to A$ and a right (one-sided) module $M : X \to \ast$ are defined by a functor $M : A \to \text{Set}$ and a contravariant functor $M : X^\to \to \text{Set}$, and a two-sided module $M : X \to A$ is defined by a bifunctor $M : X^\times A \to \text{Set}$ (a one-sided module is identified with a special instance of a two-sided module $X \to A$ where $X$ or $A$ is the terminal category). The category $[\cdot : A]$ of left modules over $A$, the category $[X : \cdot]$ of right modules over $X$, and the category $[X : A]$ of two-sided modules $X \to A$ are defined by the functor categories $[A, \text{Set}]$, $[X^\to, \text{Set}]$, and $[X^\times A, \text{Set}]$ with module morphisms defined by natural transformations between set-valued functors. A left module $M : \ast \to A$ assigns the set $\langle M \rangle a$ of arrows $\ast \to a$ to each object $a \in A$ and defines the composition of an $M$-arrow $m : \ast \to a$ with an $A$-arrow $f : a \to b$ as shown in the commutative triangle

$$
\begin{array}{c}
\ast \xrightarrow{m} a \\
\downarrow \circ \downarrow f \\
\ast \xrightarrow{m \circ f} b
\end{array}
$$

Similarly, a two-sided module $M : X \to A$ assigns the set $x(M)a$ of arrows $x \to a$ to each pair of objects $x \in X$ and $a \in A$ and defines the composition of an $M$-arrow $m : x \to a$ with an $X$-arrow $g : y \to x$ and an $A$-arrow $f : a \to b$ as shown in the commutative square

$$
\begin{array}{c}
x \xrightarrow{m} a \\
\downarrow g \\
y \xrightarrow{g \circ m \circ f} b
\end{array}
$$

A module $M : X \to A$ also yields the composite module $G(M)F : E \to D$ with functors $G$ and $F$ as in $E \xrightarrow{G} X \xrightarrow{M} A \xrightarrow{F} D$.

A cell $\ast \xrightarrow{\psi} A$ between left modules $M$ and $N$ is defined by a functor $Q$ and a left module $\downarrow \psi \downarrow \downarrow Q$ and a cell $X \xrightarrow{\psi} A$ between two-sided modules $M$ and $N$ is defined by a pair of functors $P$ and $Q$ and a module morphism $\psi : M \to P(N)Q : X \to A$. Modules and cells form the category $\text{MOD}$, and $\text{CAT}$ is embedded in $\text{MOD}$; the embedding assigns the hom-module $\langle C \rangle : C \to C$ to each category $C$ and assigns the hom-cell $C \xrightarrow{(\text{C})} C$ to each functor $\downarrow \text{H} \downarrow \downarrow \text{H}$. Given a pair of right modules $M, N : X \to \ast$, a morphism $\psi$ from $M$ to $N$, written $\psi : M \to N : X \to \ast$, the embedding assigns the hom-cell morphism $(\tau) : (G) \to (F) : (C) \to (B)$ to each natural transformation $\tau : G \to F : C \to B$.

1.1 Modules

1.1.1 Definition.

- A right module $M$ over a category $X$, written $M : X \to \ast$, is a functor $M : X^\to \to \text{Set}$. Given a pair of right modules $M, N : X \to \ast$, a morphism $\psi$ from $M$ to $N$, written $\psi : M \to N : X \to \ast$,
is a natural transformation $\psi : M \to N : X^\to \to \text{Set}$.

- A left module $M$ over a category $A$, written $M : \ast \to A$, is a functor $M : A \to \text{Set}$. Given a pair of left modules $M, N : \ast \to A$, a morphism $\psi$ from $M$ to $N$, written $\psi : M \to N : \ast \to A$, is a natural transformation $\psi : M \to N : A \to \text{Set}$.

1.1.2 Remark.

(1) If a category $A$ consists of a single object, i.e. if $A$ is a monoid, then a module over $A$ is the same thing as a monoid action on a set (as a special case, a module over a group is the same thing as a group action on a set). If we work in $\text{Ab}$ (the category of abelian groups) instead of $\text{Set}$, we have a module over an $\text{Ab}$-category $A$ defined by an $\text{Ab}$-functor $A \to \text{Ab}$;\(^1\) if $A$ consists of a single object, i.e. if $A$ is a ring, we then have a ring action on an abelian group—an algebraic structure known as a module.

(2) For a right module $M : X^\to \to \ast$, the image of an object/arrow $x \in X$ under the functor $M : X^\to \to \text{Set}$ is written $(x)(M)$ or just $x(M)$, and for a left module $M : \ast \to A$, the image of an object/arrow $a \in A$ under the functor $M : A \to \text{Set}$ is written $(M)(a)$ or just $(M)a$.

(3) For a right module morphism $\psi : M \to N : X^\to \to \ast$, the component of the natural transformation $\psi : M \to N : X^\to \to \text{Set}$ at $x \in [X]$ is written $x(\psi)$, and for a left module morphism $\psi : M \to N : \ast \to A$, the component of the natural transformation $\psi : M \to N : A \to \text{Set}$ at $a \in \|A\|$ is written $(\psi)a$.

(4) For a universe $U$, a right module $M : X^\to \to \ast$ is called $U$-small (resp. locally $U$-small) if $X$ is $U$-small (resp. locally $U$-small) and $M$ is a functor $X^\to \to \text{Set}_U$ (i.e. the sets $x(M)$ are $U$-small for all objects $x \in \|X\|$), and similarly a left module $M : \ast \to A$ is called $U$-small (resp. locally $U$-small) if $A$ is $U$-small (resp. locally $U$-small) and $M$ is a functor $A \to \text{Set}_U$. We just say “small” instead of “$U$-small” for $U$ chosen implicitly.

(5) Right and left modules are referred to as one-sided modules to distinguish them from two-sided modules to be introduced below in Definition 1.1.11.

1.1.3 Definition. The notation $[X^\to, \text{Set}]$ [op. $[A, \text{Set}]$] (see Preliminary 0.0.2(2)) is abbreviated to $[X:]$ [op. $[:A]$]:

- for a category $X$, $[X:]$ (the abbreviation of $[X^\to, \text{Set}]$) denotes the category of right modules over $X$, and for a functor $G : E \to X$, $[G:] : [X:] \to [E:]$ (the abbreviation of $[G^\to, \text{Set}] : [X^\to, \text{Set}] \to [E^\to, \text{Set}]$) denotes the precomposition functor given by the assignment $M \mapsto G \circ M$.
- for a category $A$, $[:A]$ (the abbreviation of $[A, \text{Set}]$) denotes the category of left modules over $A$, and for a functor $F : E \to A$, $[:F] : [:A] \to [:E]$ (the abbreviation of $[F, \text{Set}] : [A, \text{Set}] \to [E, \text{Set}]$) denotes the precomposition functor given by the assignment $M \mapsto F \circ M$.

1.1.4 Remark.

(1) If $X$ [op. $A$] is a small category, then the category $[X:]$ [op. $[:A]$] is locally small.

(2) The assignment $G \mapsto [G:]$ [op. $F \mapsto [:F]$] is contravariant functorial.

(3) By definition,

$$[X:] = [:X^\to] \quad \text{and} \quad [:A] = [A^\to]$$

: a right module over $X$ is the same thing as a left module over the opposite category $X^\to$, and a left module over $A$ is the same thing as a right module over the opposite category $A^\to$.

1.1.5 Definition.

- A right module morphism $\psi : M \to N : X \to \ast$ is called iso (or an isomorphism) if it is invertible in the category $[X:]$.
- A left module morphism $\psi : M \to N : \ast \to A$ is called iso (or an isomorphism) if it is invertible in the category $[:A]$.

\(^1\)Any further study on this enrichment is beyond the scope of this book.
1.1.6 Proposition.
- A right module morphism \( \psi : M \to N : X \to * \) is iso if and only if each component \( \psi(M) : x(M) \to x(N) \) is a bijection.
- A left module morphism \( \psi : M \to N : * \to A \) is iso if and only if each component \( (\psi a : (M) a \to (N) a) \) is a bijection.

Proof. Since a right module morphism \( \psi : M \to N : X \to * \) is the same thing as a natural transformation \( \psi : M \to N : X^\to \to \text{Set} \), this is an instance of the general fact that a natural transformation is iso iff each component is an isomorphism. \( \square \)

1.1.7 Definition.
- If \( M : X \to * \) is a right module, for any object \( x \in \|X\| \), an element \( m \) of the set \( x(M) \), or a pair \( (x, m) \) such that \( m \in x(M) \), is called an arrow of \( M \) (or an \( M \)-arrow) at \( x \), and written \( m : x \to * \) or \( x \xrightarrow{m} * \) (or just \( m \) if its domain is understood or unimportant).
- If \( M : * \to A \) is a left module, for any object \( a \in \|A\| \), an element \( m \) of the set \( (M) a \), or a pair \( (m, a) \) such that \( m \in (M) a \), is called an arrow of \( M \) (or an \( M \)-arrow) at \( a \), and written \( m : * \to a \) or \( * \xrightarrow{m} a \) (or just \( m \) if its codomain is understood or unimportant).

1.1.8 Remark. For a right module morphism \( \psi : M \to N : X \to * \), the image of an \( M \)-arrow \( m : x \to * \) under the function \( x(\psi) : x(M) \to x(N) \) is written variously as

\[
m : x(\psi) = m : x(\psi) = \psi(m) = \psi(\cdot)m = x(\psi) \cdot m
\]

; similarly, for a left module morphism \( \psi : M \to N : * \to A \), the image of an \( M \)-arrow \( m : * \to a \) under the function \( (\psi) a : (M) a \to (N) a \) is written as

\[
m : (\psi) a = m : (\psi) a = \psi(m) = \psi(\cdot)m = (\psi) a \cdot m.
\]

1.1.9 Definition.
- For a right module \( M : X \to * \), the composite

\[
g \circ m = m \circ g
\]

of an \( X \)-arrow \( g : y \to x \) and an \( M \)-arrow \( m : x \to * \), as shown in

\[
\begin{array}{ccc}
    x & \xrightarrow{m} & * \\
    \downarrow{g} & & \downarrow{g \circ m} \\
    y
\end{array}
\]

, is the \( M \)-arrow \( y \to * \) defined by

\[
g \circ m = g \cdot m
\]

— the image of \( m \) under the function \( g(M) : x(M) \to y(M) \).
- For a left module \( M : * \to A \), the composite

\[
m \circ f = f \circ m
\]

of an \( M \)-arrow \( m : * \to a \) and an \( A \)-arrow \( f : a \to b \), as shown in

\[
\begin{array}{ccc}
    * & \xrightarrow{m} & a \\
    \downarrow{m \circ f} & & \downarrow{f} \\
    b
\end{array}
\]

, is the \( M \)-arrow \( * \to b \) defined by

\[
m \circ f = m \cdot (M)f
\]

— the image of \( m \) under the function \( (M)f : (M) a \to (M) b \).

1.1.10 Remark. The naturality of a right module morphism \( \psi : M \to N : X \to * \) is expressed by the identity

\[
\psi(g \circ m) = g \circ \psi(m)
\]
there are obvious isomorphisms \( \psi : \mathcal{M} \to \mathcal{N} : \ast \to A \) is expressed by the identity
\[
\psi(m \circ f) = \psi(m) \circ f
\]
for every composable pair of an \( \mathcal{M} \)-arrow \( m \) and an \( A \)-arrow \( f \).

1.1.11 Definition. A (two-sided) module \( \mathcal{M} \) from a category \( X \) to a category \( A \), written \( \mathcal{M} : X \to A \), is a bifunctor \( \mathcal{M} : X^\ast \times A \to \text{Set} \). Given a pair of modules \( \mathcal{M}, \mathcal{N} : X \to A \), a morphism \( \psi \) from \( \mathcal{M} \) to \( \mathcal{N} \), written \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \), is a natural transformation \( \psi : \mathcal{M} \to \mathcal{N} : X^\ast \times A \to \text{Set} \).

1.1.12 Remark.
(1) If \( \mathcal{M} : X \to A \) is a module, then the image of an object/arrow \( (x, a) \in X^\ast \times A \) under the functor \( \mathcal{M} : X^\ast \times A \to \text{Set} \) is written \( \langle x \rangle \langle \mathcal{M} \rangle (a) \) or just \( x \langle \mathcal{M} \rangle a \).
(2) If \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \) is a module morphism, then the component of the natural transformation \( \psi : \mathcal{M} \to \mathcal{N} : X^\ast \times A \to \text{Set} \) at \( (x, a) \in \|X^\ast \times A\| \) is written \( x \langle \psi \rangle a \).
(3) For a universe \( U \), a module \( \mathcal{M} : X \to A \) is called \( U \)-small (resp. locally \( U \)-small) if \( X \) and \( A \) are \( U \)-small (resp. locally \( U \)-small) and \( \mathcal{M} \) is a functor \( X^\ast \times A \to \text{Set}_U \) (i.e. the sets \( x \langle \mathcal{M} \rangle a \) are \( U \)-small for all pairs of objects \( x \in \|X\| \) and \( a \in \|A\| \)). We just say “small” instead of “\( U \)-small” for \( U \) chosen implicitly.
(4) For a two-sided module \( \mathcal{M} : X \to A \), the categories \( X \) and \( A \) are called respectively the domain and the codomain of \( \mathcal{M} \). A two-sided module is called an endomodule when its domain and codomain coincide; an arrow \( m : e \to e \) of an endomodule \( \mathcal{M} : E \to E \) is called an endoarrow when its domain and codomain coincide.

1.1.13 Definition. The notation \([X^\ast \times A, \text{Set}]\) (see Preliminary 0.0.2(2)) is abbreviated to \([X : A]\):
(1) for a pair of categories \( X \) and \( A \), \([X : A]\) (an abbreviation of \([X^\ast \times A, \text{Set}]\)) denotes the category of modules \( X \to A \);
(2) for a pair of functors \( G : E \to X \) and \( F : D \to A \), \([G : F] : [X : A] \to [E : D]\) (an abbreviation of \([G^\ast \times F, \text{Set}] : [X^\ast \times A, \text{Set}] \to [E^\ast \times D, \text{Set}]\)) denotes the precomposition functor given by the assignment \( \mathcal{M} \mapsto [G^\ast \times F] \circ \mathcal{M} \).

1.1.14 Remark.
(1) If \( X \) and \( A \) are small categories, then the category \([X : A]\) is locally small.
(2) The assignment \((G, F) \mapsto [G : F]\) is contravariant bifunctorial.
(3) By definition,
\[
[X \times A^\ast :] = [X : A] = [\cdot : X^\ast \times A]
\]
: a two-sided module \( \mathcal{M} : X \to A \) is the same thing as a right module \( \mathcal{M} : X \times A^\ast \to \ast \) [op. left module \( \mathcal{M} : \ast \to X^\ast \times A \)].
(4) The canonical isomorphisms \( X^\ast \cong X^\ast \times \ast \) and \( A \cong \ast \times A \) yield canonical isomorphisms
\[
[X :] \cong [X : \ast] \quad \text{and} \quad [: A] \cong [\ast : A]
\]
; by these isomorphisms, a right module over \( X \) is identified with a two-sided module from \( X \) to the terminal category, and a left module over \( A \) is identified with a two-sided module from the terminal category to \( A \).
(5) There are obvious isomorphisms
\[
\text{Set} \cong [\ast :] \cong [: \ast] \cong [\ast : \ast]
\]
, by which a set is identified with a module \( \ast \to \ast \) over the terminal category.

\(^2\)Despite this terminology, modules are never treated as morphisms in this book. We adopted the notation \( \mathcal{M} : X \to A \) to suggest the direction of arrows of \( \mathcal{M} \) (see Definition 1.1.17): the domain of each \( \mathcal{M} \)-arrow resides in the domain of \( \mathcal{M} \), and the codomain of each \( \mathcal{M} \)-arrow resides in the codomain of \( \mathcal{M} \). Note, however, that many authors reverse the direction \( X \to A \) and call a bifunctor \( X^\ast \times A \to \text{Set} \) a (bi)module, profunctor, or distributor from \( A \) to \( X \).
1.1.15 Definition. A module morphism \( \psi : M \to N : X \to A \) is called iso (or an isomorphism) if it is invertible in the category \( [X : A] \).

1.1.16 Proposition. A module morphism \( \psi : M \to N : X \to A \) is iso if and only if each component \( x(\psi) a : x(M) a \to x(N) a \) is a bijection.

Proof. See the proof of Proposition 1.1.6. \( \square \)

1.1.17 Definition. If \( M : X \to A \) is a module, for any pair of objects \( x \in \|X\| \) and \( a \in \|A\| \), an element \( m \) of the set \( x(M) a \), or a triple \( (x, m, a) \) such that \( m \in x(M) a \), is called an arrow of \( M \) (or an \( M \)-arrow), and written \( m : x \to a \) or \( x \overset{m}{\to} a \) (or just \( m \) if its domain and codomain are understood or unimportant).

1.1.18 Remark. For a module morphism \( \psi : M \to N : X \to A \), the image of an \( M \)-arrow \( m : x \to a \) under the function \( x(\psi) a : x(M) a \to x(N) a \) is written variously as

\[
m' : x(\psi) a = m' \psi = \psi(m) = \psi : m = x(\psi) a : m.
\]

1.1.19 Definition. For a module \( M : X \to A \),

- the composite

\[
g \circ m = m \circ g
\]

of an \( X \)-arrow \( g : y \to x \) and an \( M \)-arrow \( m : x \to a \), as shown in

\[
\begin{array}{c}
\xymatrix{ x \ar@{-->}[r]^m \ar@{-->}[d]^{g} & a \\
y \ar@{-->}[r]_{g \circ m} & }
\end{array}
\]

is the \( M \)-arrow \( y \to a \) defined by

\[
g \circ m = g(M) a \cdot m
\]

—the image of \( m \) under the function \( g(M) a : x(M) a \to y(M) a \).

- the composite

\[
m \circ f = f \circ m
\]

of an \( M \)-arrow \( m : x \to a \) and an \( A \)-arrow \( f : a \to b \), as shown in

\[
\begin{array}{c}
\xymatrix{ x \ar@{-->}[r]^m \ar@{-->}[d]^{m \circ f} & a \\
y \ar@{-->}[r]_{g \circ m \circ f} & b }
\end{array}
\]

is the \( M \)-arrow \( x \to b \) defined by

\[
m \circ f = m \cdot x(M) f
\]

—the image of \( m \) under the function \( x(M) f : x(M) a \to x(M) b \).

1.1.20 Remark. (1) A module \( M : X \to A \) thus induces a composition law among the arrows of \( X, A, \) and \( M \).

Conversely, a module \( M : X \to A \) may be defined by giving a set of \( M \)-arrows and a composition law of them with \( X \)-arrows and \( A \)-arrows satisfying the associativity and identity axioms, i.e. by giving a collage (see Section 3.1) from \( X \) to \( A \).

(2) The composite

\[
(g \circ m) \circ f = g \circ m \circ f = g \circ (m \circ f)
\]

of an \( X \)-arrow \( g : y \to x \), an \( M \)-arrow \( m : x \to a \), and an \( A \)-arrow \( f : a \to b \), as shown in

\[
\begin{array}{c}
\xymatrix{ x \ar@{-->}[r]^m \ar@{-->}[d]^{g} & a \\
y \ar@{-->}[r]_{g \circ m \circ f} & b }
\end{array}
\]

is the \( M \)-arrow \( y \to b \) defined by

\[
g \circ m \circ f = m \cdot g(M) f
\]
When a two-sided module

For a pair of objects

For categories

In a diagram, a module

For specific categories (such as assignment for every composable triple of an X-arrow g, an M-arrow m, and an A-arrow f.

When a two-sided module \( M : X \to A \) is regarded as a right module \( M : X \times A^\text{op.} \to * \) [op. left module \( M : * \to X \times A \)], an \( M \)-arrow \( m : x \to a \) is written as \( m : (a, x) \to * \) [op. \( m : * \to (x, a) \)], and the compositions

are written as

\[
\begin{align*}
(x, a) & \sim_{m} (x, a) \\
(y, a) & \sim_{m} (y, a)
\end{align*}
\]

\[
\begin{align*}
(x_a) & \sim_{m} (x, a) \\
(y, a) & \sim_{m} (y, a)
\end{align*}
\]

\[
\begin{align*}
(x, a) & \sim_{m} (x, a) \\
(y, a) & \sim_{m} (y, a)
\end{align*}
\]

\[
\begin{align*}
(x, a) & \sim_{m} (x, a) \\
(y, a) & \sim_{m} (y, a)
\end{align*}
\]

\[
\begin{align*}
(x, a) & \sim_{m} (x, a) \\
(y, a) & \sim_{m} (y, a)
\end{align*}
\]

1.1.21 Definition. The hom-module of a category \( C \) is the endomodule \( \langle C \rangle : C \to C \) given by the assignment \( (x, a) \to \text{hom}_{C}(x, a) \) for \( x, a \in C \).

1.1.22 Remark.

(1) For a pair of objects \( a, b \in |C| \), the hom-set \( \text{hom}_{C}(a, b) \) of a category \( C \) is the same thing as the hom-set \( a(C)b \) of the endomodule \( \langle C \rangle \).

(2) Hereafter a hom-set is written as \( a(C)b \) rather than \( \text{hom}_{C}(a, b) \); likewise, for an object \( c \in |C| \) and a \( C \)-arrow \( f : a \to b \), the functions

\[
\begin{align*}
\text{hom}_{C}(c, f) : \text{hom}_{C}(c, a) & \to \text{hom}_{C}(c, b) \\
\text{and} \quad \text{hom}_{C}(f, c) : \text{hom}_{C}(b, c) & \to \text{hom}_{C}(a, c)
\end{align*}
\]

are written as

\[
\begin{align*}
c(C)f : c(C)a & \to c(C)b \\
\text{and} \quad f(C)c : b(C)c & \to a(C)c
\end{align*}
\]

respectively.

(3) For categories \( A \) and \( B \), the hom-module of the functor category \([A, B]\) is denoted by \( \langle A, B \rangle \); given a pair of functors \( F, G : A \to B \), \( \langle F, A, B \rangle \) denotes the set of natural transformations from \( F \) to \( G \).

(4) For categories \( X \) and \( A \), the hom-module of the category \([X : A]\) is denoted by \( \langle X : A \rangle \); given a pair of modules \( \mathcal{M}, \mathcal{N} : X \to A \), \( \mathcal{M}(X : A) \mathcal{N} \) denotes the set of module morphism from \( \mathcal{M} \) to \( \mathcal{N} \). Likewise, the hom-module of the category \([X : ]\) [op. \( \mathcal{M} : A \rightarrow ]\) is denoted by \( \langle X : [ \mathcal{N} \rightarrow ] \rangle \).

(5) For specific categories (such as \( \text{Set} \) or \( \text{CAT} \)), a hom-set \( a(C)b \) is often written as \( C[a, b] \) or, when \( C \) is understood, just \( [a, b] \). For example, the set of functions from a small set \( S \) to a small set \( T \) is written as \([S, T] \) rather than \( S(\text{Set})T \).

1.1.23 Notation.

(1) In a diagram, a module \( \mathcal{M} : X \to A \) and a module morphism \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \) are depicted as below:

\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} A \\
& \xrightarrow{\mathcal{N}} A
\end{align*}
\]

(2) Italic capital letters \( M, N, \ldots \) vary over both modules and module morphisms. When we write \( M \in [X : A] \), \( M : X \to A \), or \( X \xrightarrow{M} A \), \( M \) stands for an object or an arrow of the category \([X : A] \), i.e. a module or a module morphism \( X \to A \) (cf. Preliminary0.0.1(7)).
1.1.24 Definition.
- Given a right module (or module morphism) $M$ and a functor (or natural transformation) $G$ as in

$$E \xrightarrow{G} X \xrightarrow{M} *$$

their composite, written $[G](M)$ or just $G(M)$, is the right module (or module morphism) $E \to *$ defined by the composition

$$E \xrightarrow{G} X \xrightarrow{M} * \xrightarrow{* \to} \text{Set}.$$

- Given a left module (or module morphism) $M$ and a functor (or natural transformation) $F$ as in

$$* \xrightarrow{M} A \xleftarrow{F} E$$

their composite, written $(M)[F]$ or just $(M)F$, is the left module (or module morphism) $* \to E$ defined by the composition

$$E \xrightarrow{F} A \xrightarrow{M} \text{Set}.$$

1.1.25 Remark.
- The composition $(G,M) \mapsto G(M) : [E,X] \times [X:] \to [E:]$ is functorial in each variable, contravariant in $G$ and covariant in $M$. If $G : E \to X$ is a functor, then

$$G(M) = G \circ M = M : [G:]$$

for any $M \in [X:]$.

- The composition $(M,F) \mapsto (M)F : [:A] \times [E,A] \to [:E]$ is functorial in each variable, covariant in both $M$ and $F$. If $F : E \to A$ is a functor, then

$$(M)F = F \circ M = M : [:F]$$

for any $M \in [:A]$.

1.1.26 Definition. Given a module (or module morphism) $M$, a functor (or natural transformation) $G$, and a second functor (or natural transformation) $F$, all as in

$$E \xrightarrow{G} X \xrightarrow{M} A \xleftarrow{F} D$$

their composite, written $[G](M)[F]$ or just $G(M)F$, is the module (or module morphism) $E \to D$ defined by the composition

$$E \xrightarrow{G \times F} X \times A \xrightarrow{M} D \xrightarrow{\text{Set}}.$$

1.1.27 Remark. The composition $(G,M,F) \mapsto G(M)F : [E,X] \times [X:A] \times [D,A] \to [E:D]$ is functorial in each variable, contravariant in $G$ and covariant in $M$ and $F$. If $G : E \to X$ and $F : D \to A$ are functors, then

$$G(M)F = [G \times F] \circ M = M : [G:F]$$

for any $M \in [X:A]$.

1.1.28 Definition. As a special case of Definition 1.1.26 where $F \ [\text{op.} \ G]$ is the identity, given

$$E \xrightarrow{G} X \xrightarrow{M} A$$

their composite $G(M) : E \to A$ is defined by the composition

$$E \xrightarrow{G \times A} X \times A \xrightarrow{M} \text{Set}.$$
\begin{itemize}
  \item given
  \[
  \begin{array}{cccc}
    X & \xrightarrow{M} & A & \xleftarrow{F} E \\
  \end{array}
  \]
  , their composite \( (M) F : X \rightarrow E \) is defined by the composition
  \[
  X \times E \xrightarrow{X \times F} X \times A \xrightarrow{M} \text{Set}.
  \]
\end{itemize}

1.1.29 Remark. Under the identification in Remark 1.1.14(4), Definition 1.1.24 is regarded as a special case of Definition 1.1.28 where \( A \) \([\text{op.} \ X]\) is the terminal category.

1.1.30 Proposition.
(1) Given

\[
\begin{array}{ccc}
  \text{E}' & \xrightarrow{G'} & \text{E} \\
  \downarrow & & \downarrow \\
  \text{X} & \xrightarrow{M} & \text{A} \\
  \downarrow & & \downarrow \\
  \text{D} & \xleftarrow{F'} & \text{D}'
\end{array}
\]

, the associative law

\[ G'(G(M) F) F' = [G' \circ G](M) [F \circ F'] \]

holds.

(2) Given

\[
\begin{array}{ccc}
  \text{E} & \xrightarrow{G} & \text{X} \\
  \downarrow & & \downarrow \\
  \text{A} & \xleftarrow{F} & \text{D}
\end{array}
\]

, the associative law

\[ G(\{M\} F) = G(M) F = \{G(M)\} F \]

holds.

Proof.
(1) Indeed,

\[
G'(G(M) F) F' = [G' \times F'] \circ [[G \times F] \circ M]
\]

\[
= [[G' \times F'] \circ [G \times F]] \circ M
\]

\[
= [[G' \circ G] \times [F' \circ F]] \circ M
\]

\[= [G' \circ G](M) [F \circ F'].\]

(2) By what we have just seen,

\[ G(\{M\} F) = G(\{1_X\} (M) F) = [G \circ 1_X](M) F = G(M) F \]

and

\[ \{G(M)\} F = \{G(M) [1_A]\} F = G(M) [1_A \circ F] = G(M) F. \]

1.1.31 Example.
(1) Given a module \( \mathcal{M} \) and functors \( G \) and \( F \) as in

\[
\begin{array}{ccc}
  \text{E} & \xrightarrow{G} & \text{X} \\
  \downarrow & & \downarrow \\
  \mathcal{M} & \xleftarrow{F} & \text{D}
\end{array}
\]

, the composition yields the module \( G(\mathcal{M}) F : \text{E} \rightarrow \text{D} \) defined by

\[ e \langle G(\mathcal{M}) F \rangle d = (e \circ G)(\mathcal{M}) (F \circ d) \]

for \( e \in \text{E} \) and \( d \in \text{D} \). A \( G(\mathcal{M}) F \)-arrow \( m : e \rightarrow d \) is given by an \( \mathcal{M} \)-arrow \( m : e \circ G \rightarrow F \circ d \); for an \( \text{E} \)-arrow \( h : e' \rightarrow e \), a \( G(\mathcal{M}) F \)-arrow \( m : e \rightarrow d \), and a \( \text{D} \)-arrow \( k : d \rightarrow d' \), their composite \( h \circ m \circ k : e' \rightarrow d' \) is given by the \( \mathcal{M} \)-arrow \( (h \circ G) \circ m \circ (F \circ k) : (e' \circ G) \rightarrow F \circ d' \) as indicated in

\[
\begin{array}{ccc}
  e & \xrightarrow{m} & F \circ d \\
  h \downarrow & & \downarrow \circ k \\
  e' & \xrightarrow{e'} & F \circ d'
\end{array}
\]
(2) Given a module morphism $\psi$ and functors $G$ and $F$ as in
\[
\begin{array}{c}
\begin{array}{c}
 E \\
 \downarrow G
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 X \\
 \downarrow M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 A \\
 \downarrow F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 B \\
 \downarrow N
\end{array}
\end{array}
\]
, the composition yields the module morphism
\[
G(\psi) F : G(\mathcal{M}) F \to G(\mathcal{N}) F : E \to D
\]
defined by
\[
e(G(\psi) F) d = (e \cdot G)(\psi)(F \cdot d)
\]
for $e \in \|E\|$ and $d \in \|D\|$.

(3) Given a module $\mathcal{M}$ and natural transformations $\tau : G' \to G$ and $\sigma : F \to F'$ as in
\[
\begin{array}{c}
\begin{array}{c}
 E \\
 \downarrow G'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 X \\
 \downarrow M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 A \\
 \downarrow F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 D \\
 \downarrow F'
\end{array}
\end{array}
\]
, the composition yields the module morphism
\[
\tau(\mathcal{M}) \sigma : G(\mathcal{M}) F \to G'(\mathcal{M}) F' : E \to D
\]
which maps each $G(\mathcal{M})$-arrow $m : e \to d$ to the $G'(\mathcal{M})$-$F'$-arrow $m \cdot \tau(\mathcal{M}) \sigma : e \to d$ given by
\[
m \cdot \tau(\mathcal{M}) \sigma = \tau_e \circ m \circ \sigma_d
\]
as indicated in the commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
 E \\
 \downarrow G'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 X \\
 \downarrow M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 A \\
 \downarrow F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 D \\
 \downarrow F'
\end{array}
\end{array}
\]

(4) More generally, given a module morphism $\psi$ and natural transformations $\tau : G' \to G$ and $\sigma : F \to F'$ as in
\[
\begin{array}{c}
\begin{array}{c}
 E \\
 \downarrow G'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 X \\
 \downarrow M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 A \\
 \downarrow F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 D \\
 \downarrow F'
\end{array}
\end{array}
\]
, the composition yields the module morphism
\[
\tau(\psi) \sigma : G(\mathcal{M}) F \to G'(\mathcal{N}) F' : E \to D
\]
which maps each $G(\mathcal{M})$-arrow $m : e \to d$ to the $G'(\mathcal{N})$-$F'$-arrow $m \cdot \tau(\psi) \sigma : e \to d$ given by
\[
m \cdot \tau(\psi) \sigma = (\tau_e \circ m \circ \sigma_d) \cdot \psi = \tau_e \circ (m \cdot \psi) \circ \sigma_d
\]
as indicated in the commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
 E \\
 \downarrow G'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 X \\
 \downarrow M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 A \\
 \downarrow F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 D \\
 \downarrow F'
\end{array}
\end{array}
\]

; note that the module morphism $\tau(\psi) \sigma : G(\mathcal{M}) F \to G'(\mathcal{N}) F'$ is given by the diagonal of the commutative square
\[
\begin{array}{c}
\begin{array}{c}
 G(\mathcal{M}) F \\
 \downarrow \tau(\mathcal{M}) \sigma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 G(\mathcal{N}) F \\
 \downarrow \tau(\mathcal{N}) \sigma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 G'(\mathcal{M}) F' \\
 \downarrow \tau(\mathcal{M}) \sigma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 G'(\mathcal{N}) F' \\
 \downarrow \tau(\mathcal{N}) \sigma
\end{array}
\end{array}
\]
consisting of the module morphisms we looked at in (2) and (3) above (the square commutes by the naturality of $\psi$).

(5) The restriction of a module $\mathcal{M} : X \to A$ to subcategories $Y \subseteq X$ and $B \subseteq A$ is given by the
composition

\[ Y \xrightarrow{\psi} X \xrightarrow{M} A \xleftarrow{\psi'} B \]

of \( \mathcal{M} \) with the two inclusion functors. Similarly, the restriction of a module morphism \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \) to \( Y \) and \( B \) is given by the composition

\[ Y \xrightarrow{\psi} X \xrightarrow{M} A \xleftarrow{\psi'} B . \]

(6) The evaluation of a right module \( \mathcal{M} : X \to \ast \) at \( x \in X \) is identified with the composition

\[ \ast \xrightarrow{x} X \xrightarrow{M} \ast \]

, the evaluation of a left module \( \mathcal{M} : \ast \to A \) at \( a \in A \) is identified with the composition

\[ X \xrightarrow{M} A \xrightarrow{a} \ast \]

, and the evaluation of a two-sided module \( \mathcal{M} : X \to A \) at \( (x, a) \in X \times A \) is identified with the composition

\[ \ast \xrightarrow{x} X \xrightarrow{M} A \xrightarrow{a} \ast . \]

(7) Given a module \( \mathcal{M} : X \times Y \to A \times B \) and an object \((x, a) \in [X \times A]\), the composition

\[ Y \xrightarrow{\times Y} X \times Y \xrightarrow{M} A \times B \xrightarrow{a \times B} B \]

of \( \mathcal{M} \) with the sections \( x \times Y \) and \( a \times B \) yields the slice of \( \mathcal{M} \) at \((x, a)\), i.e. the module

\[ [x \times Y] \langle \mathcal{M} \rangle [a \times B] : Y \to B \]

such that

\[ y \langle [x \times Y] \langle \mathcal{M} \rangle [a \times B] \rangle b = (x, y) \langle \mathcal{M} \rangle (a, b) \]

for \( y \in Y \) and \( b \in B \).

(8) Given a right module \( \mathcal{M} : X \to \ast \) [op. left module \( \mathcal{M} : \ast \to A \)] and a category \( E \), the composition

\[ X \xrightarrow{M} \ast \xleftarrow{\parallel ALT_1} E \quad \text{op.} \quad E \xleftarrow{\parallel ALT_1} \ast \xrightarrow{M} A \]

yields the two-sided module

\[ \langle \mathcal{M} \rangle [1_E] : X \to E \quad \text{op.} \quad [1_E] \langle \mathcal{M} \rangle : E \to A \]

by duplicating \( \mathcal{M} \) across \( E \). The \( E \)-ary diagonal functor of \([X:] \) [op. \([ : A]\)] is the precomposition functor

\[ [X : 1_E] : [X :] \to [X : E] \quad \text{op.} \quad [1_E : A] : [: A] \to [E : A] \]

sending each \( \mathcal{M} : X \to \ast \) [op. \( \mathcal{M} : \ast \to A \)] to the composite module \( \langle \mathcal{M} \rangle [1_E] : X \to E \) [op. \( [1_E] \langle \mathcal{M} \rangle : E \to A \)].

(9) Given a right module \( \mathcal{M} : X \to \ast \) [op. left module \( \mathcal{M} : \ast \to A \)] and a category \( E \), the composition

\[ X \times \overline{E} \xrightarrow{X \times [1_E]} X \xrightarrow{M} \ast \quad \text{op.} \quad X \times \overline{E} \xrightarrow{X \times [1_E]} \overline{X} \xrightarrow{M} \ast \]

yields the right [op. left] module

\[ [X \times [1_E] \langle \mathcal{M} \rangle] : X \times \overline{E} \to \ast \quad \text{op.} \quad \langle \mathcal{M} \rangle [1_E \times : A] : \ast \to \overline{E} \times A \]

by introducing to \( \mathcal{M} \) a dummy variable varying over \( \overline{E} \). Note that this right [op. left] module is the same thing as the two-sided module \( \langle \mathcal{M} \rangle [1_E] : X \to E \) [op. \( [1_E] \langle \mathcal{M} \rangle : E \to A \)] in (8).

(10) If \( X \) [op. \( A \)] in (8) is replaced by the terminal category, the composition

\[ \ast \xrightarrow{S} \ast \xleftarrow{\parallel ALT_1} E \quad \text{op.} \quad E \xleftarrow{\parallel ALT_1} \ast \xrightarrow{S} \ast \]
1.1. Modules

yields the constant left [op. right] module
\[ \Delta E S : * \to E \quad \text{op.} \quad \Delta E S : E \to * \]
such that
\[ S = (\Delta E S) e \quad \text{op.} \quad e (\Delta E S) = S \]
for every object \( e \in \| E \| \). The \( E \)-ary diagonal functor
\[ [: l E] : \text{Set} \to [: E] \quad \text{op.} \quad \lfloor E \rfloor : \text{Set} \to \lfloor E \rfloor \]
sends each small set \( S \) to the constant module \( \Delta E S : * \to E \quad \text{op.} \quad \Delta E S : E \to * \).

(11) As a special case of (10) where \( S \) is given by the singleton set \( 1 = \{ 0 \} \), the composition
\[ \ast \ast \Delta E : \ast \to E \quad \text{op.} \quad \Delta E \ast : E \to * \]
(where \( \ast \) denotes the hom-module of the terminal category) yields the constant left [op. right] module
\[ \ast \Delta E : \ast \to E \quad \text{op.} \quad \Delta E \ast : E \to * \]
such that
\[ 1 = (\ast \Delta E) e \quad \text{op.} \quad e (\Delta E \ast) = 1 \]
for every object \( e \in \| E \| \); that is,
\[ \Delta E \ast = \Delta E 1 \quad \text{op.} \quad \Delta E \ast = \Delta E \ast. \]

(12) As a special case of (1) above, given a pair of functors
\[ X \xrightarrow{G} C \xrightarrow{(C)} C \xrightarrow{F} A \]
, the composition yields the module \( G(C) F : X \to A \) defined by
\[ x (G(C) F) a = (x^i G) (C) (F^i a) \]
for \( x \in X \) and \( a \in A \). A \( G(C) F \)-arrow \( h : x \to a \) is given by a \( C \)-arrow \( h : x^i G \to F^i a \); for an \( X \)-arrow \( g : x' \to x \), a \( G(C) F \)-arrow \( h : x \to a \), and an \( A \)-arrow \( f : a \to a' \), their composite \( g \circ h \circ f : x' \to a' \) is given by the \( C \)-arrow \((g^i G) \circ h \circ (F^i f) : x'^i G \to F^i a' \) as indicated in
\[
\begin{array}{ccc}
  x & \xrightarrow{G} & x^i G \\
  g & \xrightarrow{G} & g^i G \\
  x' & \xrightarrow{G} & x'^i G \\
\end{array}
\]

\[
\begin{array}{ccc}
  \downarrow f & & \downarrow f \\
  a & \xrightarrow{F^i f} & a' \\
\end{array}
\]

1.1.32 Proposition. In Example 1.1.31(5), suppose that \( Y \) and \( B \) are isomorphism-dense (see Preliminary 0.0.11) subcategories of \( X \) and \( A \). Then \( \psi : M \to N : X \to A \) is an isomorphism if and only if its restriction to \( Y \) and \( B \) are isomorphisms; that is, if and only if the component \( y (\psi) b : y (M) b \to y (N) b \) is a bijection for each pair of objects \( y \in \| Y \| \) and \( b \in \| B \| \) (cf. Proposition 1.1.16).

Proof. Since a module morphism \( \psi : M \to N : X \to A \) is the same thing as a natural transformation \( \psi : M \to N : X \times A \to \text{Set} \), this is an instance of the lemma in Preliminary 0.0.11. \( \Box \)

1.1.33 Proposition. If \( \psi \) in Example 1.1.31(2) is a module isomorphism, so is the composite \( G(\psi) F \). The converse holds if \( G \) and \( F \) are essentially surjective.

Proof. Note that \( G(\psi) F \) is given by the image of \( \psi \) under the precomposition functor \([G : F]\). Since any functor preserves isomorphisms, if \( \psi \) is an isomorphism, so is \( G(\psi) F \). Since the image of an essentially surjective functor is an isomorphism-dense subcategory of its codomain, the second assertion follows from Proposition 1.1.32. \( \Box \)

1.1.34 Remark. Proposition 1.1.32 is a special case of Proposition 1.1.33 where \( G \) and \( F \) are inclusion functors.
1.1.35 Proposition. If, in Example 1.1.31(12), h, f, and k are isomorphisms, so is the C-arrow (h:G) ⋆ f ⋆ (F−k).

Proof. Immediate because any functor preserves isomorphisms. □

1.1.36 Definition. Given categories X and A, the twisting transposition

\[ \text{[X × A, Set]} \xrightarrow{\sim} \text{[A × X, Set]} \]

(see Preliminary 0.0.5) is denoted by

\[ \text{[X: A]} \xrightarrow{\sim} \text{[A: X]} \]

using the abbreviation in Definition 1.1.13; the functor sends a module \( M : X \to A \) to its opposite module \( M^\sim : A^\sim \to X^\sim \) given by the composition

\[ A \times X^\sim \xrightarrow{\sim} X^\sim \times A \xrightarrow{M} \text{Set} \]

with the twist isomorphism, and sends a module morphisms \( \psi : M \to N : X \to A \) to its opposite module morphism \( \psi^\sim : M^\sim \to N^\sim : A^\sim \to X^\sim \) given by the composition

\[ A \times X^\sim \xrightarrow{\sim} X^\sim \times A \xrightarrow{M^\sim} \text{Set}. \]

1.1.37 Remark.
1. For any module \( M \) and any module morphism \( \psi \),
\[ (M^\sim)^\sim = M \quad \text{and} \quad (\psi^\sim)^\sim = \psi. \]
2. For any category \( C \),
\[ \langle C^\sim \rangle = \langle C \rangle^\sim \]
; that is, the hom-module of the opposite of \( C \) is the opposite of the hom-module of \( C \).
3. For any composite module \( G\langle M \rangle F \),
\[ \langle G\langle M \rangle F \rangle^\sim = F\langle M^\sim \rangle G \]
; that is, the opposite of the composite
\[ E \xrightarrow{G} X \xrightarrow{M} A \xleftarrow{F} D \]
is given by the composite
\[ D^\sim \xrightarrow{F} A^\sim \xrightarrow{M^\sim} X^\sim \xleftarrow{G} E^\sim. \]
4. The right [op. left] module
\[ G\langle M \rangle a : E \to * \quad \text{op.} \quad x\langle M \rangle F : * \to E \]
given by the composition
\[ E \xrightarrow{G} X \xrightarrow{M} A \xleftarrow{a} * \quad \text{op.} \quad * \xrightarrow{x} X \xrightarrow{M} A \xleftarrow{F} E \]
is the same thing as the left [op. right] module
\[ a\langle M^\sim \rangle G : * \to E^\sim \quad \text{op.} \quad F\langle M^\sim \rangle x : E^\sim \to * \]
given by the composition
\[ * \xrightarrow{a} A^\sim \xrightarrow{M^\sim} X^\sim \xleftarrow{G} E^\sim \quad \text{op.} \quad E^\sim \xrightarrow{F} A^\sim \xrightarrow{M^\sim} X^\sim \xleftarrow{x} *. \]
5. The opposite of the module morphisms \( \psi : M \to N : X \to A \) is often denoted by \( \psi : M^\sim \to N^\sim : A^\sim \to X^\sim \) using the same name as its original counterpart (cf. Preliminary 0.0.1(5)).
1.2 Cells

As we saw at the beginning of the chapter, the concept of a cell is an extension of that of a functor. Notions associated with a functor, such as fully faithfulness and equivalence, extend to a cell.

Cells between two modules $\mathcal{J} : E \to D$ and $\mathcal{M} : X \to A$ form the module $(\mathcal{J}, \mathcal{M}) : [E, X] \to [D, A]$, and a cell

\[
\begin{array}{c}
X \xrightarrow{\mathcal{M}} A \\
P \downarrow \psi \downarrow Q \\
Y \xrightarrow{\mathcal{N}} B
\end{array}
\]

yields the postcomposition cell

\[
\begin{array}{c}
[E, X] \xrightarrow{[\mathcal{J}, \mathcal{M}]} [D, A] \\
[E, P] \downarrow (\mathcal{J}, \psi) \downarrow [D, Q] \\
[E, Y] \xrightarrow{[\mathcal{J}, \mathcal{N}]} [D, B]
\end{array}
\]

that sends each cell $E \xrightarrow{\mathcal{J}} D$ to its composite $E \xrightarrow{\mathcal{J}} D$ with $\psi$. In Section 12.4, an extension is defined by a universal arrow of the module $(\mathcal{J}, \mathcal{M})$.

What is introduced here is an ordinary cell. We will introduce a different type of module cell in Section 7.1.

1.2.1 Definition. Given a pair of modules $\mathcal{M} : X \to A$ and $\mathcal{N} : Y \to B$, and given a pair of functors $P : X \to Y$ and $Q : A \to B$, a module cell (or just a cell) $\psi : P \to Q : \mathcal{M} \to \mathcal{N}$, written diagrammatically as $X \xrightarrow{\mathcal{M}} A$ or $A \xrightarrow{\mathcal{N}} B$, is defined by a module morphism $\psi : \mathcal{M} \to P \langle \mathcal{N} \rangle Q : X \to A$.

1.2.2 Remark.

(1) The functors $P$ and $Q$ are called the left and right components of a cell $\psi : P \to Q : \mathcal{M} \to \mathcal{N}$.

A cell is sometimes denoted just by $\psi : \mathcal{M} \to \mathcal{N}$, and when this is the case, its left and right components are denoted by $\psi_0 : \mathcal{M}_0 \to \mathcal{N}_0$ and $\psi_1 : \mathcal{M}_1 \to \mathcal{N}_1$, respectively.

(2) A cell $\psi : P \to Q : \mathcal{M} \to \mathcal{N}$ sends each $\mathcal{M}$-arrow $m : x \to a$ to the $\mathcal{N}$-arrow $m : x : P \to Q : a$—the image of $m$ under the function

\[
x(\mathcal{M}) a \xrightarrow{[\psi]_a} x(\mathcal{N}) Q a = (x : P) (\mathcal{N}) (Q : a) .
\]

(3) Cells and module morphisms are regarded as special instances of each other. A cell $\psi : P \to Q : \mathcal{M} \to \mathcal{N}$ is thought of as a module morphism from $\mathcal{M}$ to the composite module $P \langle \mathcal{N} \rangle Q$.

Conversely, a module morphism $\psi : \mathcal{M} \to \mathcal{N} : X \to A$ is expressed by a cell $X \xrightarrow{\mathcal{M}} A$.

(4) The identity module morphism $\mathcal{M} \to \mathcal{M} : X \to A$ yields the identity cell $X \xrightarrow{\mathcal{M}} A$.

(5) Any composition

\[
\begin{array}{c}
X \xrightarrow{P} Y \xrightarrow{\mathcal{N}} B \xrightarrow{Q} A
\end{array}
\]

trivially yields a cell

\[
\begin{array}{c}
X \xrightarrow{P(\mathcal{N})Q} A \\
P \downarrow \psi \downarrow Q \\
Y \xrightarrow{\mathcal{N}} B
\end{array}
\]
1.2. Cells

\[ X \xrightarrow{M} A \]
\[ P \xleftarrow{\mathbf{1}} \xrightarrow{Q} \]
\[ Y \xrightarrow{N} B \]

expresses an identity \( M = P(N)Q \).

1.2.3 Proposition. A cell is compatible with composition of arrows. Specifically, a cell

\[ X \xrightarrow{M} A \]
\[ P \xleftarrow{\psi} \xrightarrow{Q} \]
\[ Y \xrightarrow{N} B \]

sends a commutative diagram

\[ \begin{array}{ccc}
    x \xrightarrow{m} & a \\
    g \uparrow & \downarrow f \\
    y \xrightarrow{g \circ m \circ f} b
\end{array} \]

in \( \mathcal{M} \) to a commutative diagram

\[ \begin{array}{ccc}
    x \xrightarrow{m \circ \psi} & Q \circ a \\
    g \circ P \uparrow & \downarrow Q \circ f \\
    y \xrightarrow{(g \circ m \circ f) \circ \psi} Q \circ b
\end{array} \]

in \( \mathcal{N} \); that is, the identity

\[(g \circ m \circ f) \circ \psi = (g \circ P) \circ (m \circ \psi) \circ (Q \circ f)\]

holds.

Proof. The commutativity of the second diagram follows from the naturality of \( \psi \) (see Remark 1.1.20(3)) and noting Example 1.1.31(1). \( \square \)

1.2.4 Definition.

\( \triangleright \) Let \( \mathcal{M} : X \rightrightarrows * \) and \( \mathcal{N} : Y \rightrightarrows * \) be right modules. Given a functor \( P : X \rightrightarrows Y \), a right module cell \( \psi : P \rightrightarrows * : \mathcal{M} \rightrightarrows \mathcal{N} \), written diagrammatically as \( X \xrightarrow{M} * \), is defined by a right module morphism \( \psi : \mathcal{M} \rightrightarrows \mathcal{P} / (\mathcal{N}) : X \rightrightarrows * \).

\( \triangleright \) Let \( \mathcal{M} : * \rightrightarrows A \) and \( \mathcal{N} : * \rightrightarrows B \) be left modules. Given a functor \( Q : A \rightrightarrows B \), a left module cell \( \psi : * \rightrightarrows Q : \mathcal{M} \rightrightarrows \mathcal{N} \), written diagrammatically as \( * \xrightarrow{M} * \), is defined by a left module morphism \( \psi : \mathcal{M} \rightrightarrows \mathcal{Q} / (\mathcal{N}) : * \rightrightarrows A \).

1.2.5 Remark.

(1) Under the identification in Remark 1.1.14(4), a right \( [\text{op. left}] \) module cell \( \mathcal{M} \rightrightarrows \mathcal{N} \) in Definition 1.2.4 is regarded as a special instance of a two-sided module cell \( \mathcal{M} \rightrightarrows \mathcal{N} \) in Definition 1.2.1 where the codomains \( [\text{op. domains}] \) of \( \mathcal{M} \) and \( \mathcal{N} \) are the terminal category.

(2) Conversely, by Remark 1.1.14(3), a two-sided module cell \( X \xrightarrow{M} A \) in Definition 1.2.1 is the same thing as a right module cell \( X \times A \xrightarrow{M} * \) \( [\text{op. left module cell} \quad * \xrightarrow{M} X \times A \] \).
The composition in the module
\[ \xymatrix{ X \ar[r]^-{\mathcal{M}} & * } \]
for a pair of functors
\[ \xymatrix{ \mathcal{M} \ar[r] & \mathcal{M} : X \to * } \]
and the identity right module morphism \( \mathcal{M} \to \mathcal{M} : * \to A \) yields the identity left module cell
\[ \xymatrix{ * \ar[r]^-{\mathcal{M}} & A } \]

1.2.6 Definition. Given a cell \( \theta : G \to F \) and natural transformations \( \tau : G' \to G \) and \( \sigma : F \to F' \) as in
\[ \xymatrix{ E \ar[r]^-{\tau} & D \\ G' \ar[r]^-{\tau \circ \theta \circ \sigma} & F' \\ X \ar[r]^-{\mathcal{M}} & A } \]
their composite is the cell
\[ \xymatrix{ E \ar[r]^-{\tau} & D \\ G' \ar[r]^-{\tau \circ \theta \circ \sigma} & F' \\ X \ar[r]^-{\mathcal{M}} & A } \]
defined by the module morphism \( \tau \circ \theta \circ \sigma : J \to \mathcal{M}(A) \) given by the composition
\[ J \xrightarrow{\theta} \mathcal{M}(A) \mathcal{F} \xrightarrow{\tau(M) \sigma} \mathcal{M}(A) \mathcal{F}'. \]

1.2.7 Remark. By Example 1.1.31(3), the cell \( \tau \circ \theta \circ \sigma \) sends each \( J \)-arrow \( j : e \to d \) to the \( M \)-arrow
\[ j : (\tau \circ \theta \circ \sigma) = \tau_e \circ (j : \theta) \circ \sigma_d : e : G' \to F' : d \]
as indicated in the commutative diagram
\[ \xymatrix{ e : G \ar[r]^-{j \circ \theta} & F' : d \\ e : G' \ar[r]^-{j \circ (\tau \circ \theta \circ \sigma)} & F' : d \ar[l]_-{\sigma_d} \ar[u]^-{\tau_e} } \]

Note. The composition in Definition 1.2.6 yields the module of cells \( J \to M \); the functoriality in Remark 1.2.7 allows the following definition.

1.2.8 Definition. Given a pair of modules \( J : E \to D \) and \( M : X \to A \), the module
\[ \langle J, M \rangle : [E, X] \to [D, A] \]
of cells \( J \to M \) is defined by
\[ (G) \langle J, M \rangle (F) = (J) (E : D) (M A) \]
for \( G \in [E, X] \) and \( F \in [D, A] \), where \( (E : D) \) is the hom-module of the category \([E, D]\) (see Definition 1.1.13).

1.2.9 Remark.
(1) For a pair of functors \( G : E \to X \) and \( F : D \to A \), the set
\[ (G) \langle J, M \rangle (F) = (J) (E : D) (M A) \]
consists of all module morphisms \( J \to \mathcal{M}(A) \mathcal{F} : E \to D \), i.e. all cells \( G \sim F : J \to M \).

(2) The composition in the module \( \langle J, M \rangle \) is that defined in Definition 1.2.6; indeed, by definition, the composition
\[ \xymatrix{ G \ar[r]^-{\theta} & F \\ G' \ar[r]^-{\tau \circ \theta \circ \sigma} & F' \ar[u]_-{\sigma} \ar[l]_-{\tau_e} } \]
in \(\{J, M\}\) is given by
\[
\tau \cdot \theta \cdot \sigma = \theta \cdot \tau \cdot \sigma.
\]
(3) If \(J\) is small and \(M\) is locally small, then the module \(\{J, M\}\) is locally small.

1.2.10 Proposition. Given a module \(J\) and a composite module \(P\langle N\rangle Q\) as in
\[
\begin{array}{c}
E \xleftarrow{J} D \\
X \xleftarrow{P\langle N\rangle Q} A \\
\downarrow P \quad \downarrow 1 \quad \downarrow Q \\
\downarrow \quad \downarrow \quad \downarrow \\
Y \xleftarrow{N} B
\end{array}
\]
, the identity
\[
\begin{array}{c}
[E, X] \xleftarrow{\langle J, P\langle N\rangle Q \rangle} [D, A] \\
\langle E, P \rangle \quad 1 \\
\downarrow \quad \downarrow \\
[D, Q] \\
\downarrow \\
[E, Y] \xleftarrow{\langle J, N \rangle} [D, B]
\end{array}
\]
(i.e.
\[
\langle J, P\langle N\rangle Q \rangle = \langle E, P \rangle \langle J, N \rangle [D, Q]
\]
) holds.

Proof. For any \(G \in [E, X]\) and \(F \in [D, A]\),
\[
G \langle J, P\langle N\rangle Q \rangle F = \langle J \rangle (E : D) (G \langle P\langle N\rangle Q \rangle F)
\]
\[
= \langle J \rangle (E : D) ([G \cdot P] \langle N \rangle [Q \cdot F])
\]
\[
= (G \cdot P) \langle J, N \rangle (Q \cdot F)
\]
\[
= (G \cdot [E, P]) \langle J, N \rangle ([D, Q] \cdot F)
\]
\[
= (G) ([E, P] \langle J, N \rangle [D, Q]) (F)
\]

\(\square\)

1.2.11 Remark. The cell \(\langle J, P\langle N\rangle Q \rangle \xrightarrow{1} \langle J, N \rangle\) above sends each cell
\[
\begin{array}{c}
E \xleftarrow{J} D \\
G \downarrow \theta \downarrow F \\
X \xleftarrow{P\langle N\rangle Q} A
\end{array}
\]
to the cell
\[
\begin{array}{c}
E \xleftarrow{J} D \\
G \cdot P \downarrow \theta \downarrow Q \cdot F \\
Y \xleftarrow{N} B
\end{array}
\]
defined by the same module morphism.

1.2.12 Definition. If \(\theta : J \rightarrow M\) is a cell and \(\psi : M \rightarrow N\) is a module morphism as in
\[
\begin{array}{c}
E \xleftarrow{J} D \\
G \downarrow \theta \downarrow F \\
X \xleftarrow{M \cdot \psi} A
\end{array}
\]
, then their composite \(\theta \cdot \psi = \psi \cdot \theta\) is the cell
\[
\begin{array}{c}
E \xleftarrow{J} D \\
G \downarrow \theta \cdot \psi \downarrow F \\
X \xleftarrow{N} A
\end{array}
\]
defined by the module morphism \(\theta \circ \psi : J \to G(\mathcal{N})F\) given by the composition
\[
J \xrightarrow{\theta} G(\mathcal{M})F \xrightarrow{G(\psi)F} G(\mathcal{N})F.
\]

1.2.13 Remark. See Example 1.1.31(2) for the module morphism \(G(\psi)F\). The cell \(\theta \circ \psi\) sends each \(J\)-arrow \(j : e \to d\) to the \(\mathcal{N}\)-arrow
\[
j : (\theta \circ \psi) = j : \theta \circ \psi : e : G \to F : d
\]
— the image of \(j\) under the composite function
\[
e(\mathcal{J})d \xrightarrow{e(\theta)d} e(G(\mathcal{M})F)d = (e : G)(\mathcal{M})(F : d) \xrightarrow{(e : G)(\psi)(F : d)} (e : G)(\mathcal{N})(F : d).
\]

Note. For any module \(\mathcal{J}\) and any module morphism \(\psi : \mathcal{M} \to \mathcal{N}\), the composition in Definition 1.2.12 yields the postcomposition module morphism \(\langle \mathcal{J}, \psi \rangle : \langle \mathcal{J}, \mathcal{M} \rangle \to \langle \mathcal{J}, \mathcal{N} \rangle\) from the module of cells \(\mathcal{J} \to \mathcal{M}\) to the module of cells \(\mathcal{J} \to \mathcal{N}\). Here is the formal definition:

1.2.14 Definition. Given a module \(\mathcal{J} : E \to D\) and a module morphism \(\psi : \mathcal{M} \to \mathcal{N} : X \to A\), the postcomposition module morphism
\[
\langle \mathcal{J}, \psi \rangle : \langle \mathcal{J}, \mathcal{M} \rangle \to \langle \mathcal{J}, \mathcal{N} \rangle : [E, X] \to [D, A]
\]
, “postcomposition with \(\psi\)”, is defined by
\[
(G)\langle \mathcal{J}, \psi \rangle(F) = \langle \mathcal{J} \rangle(E : D)(G(\psi)F)
\]
for each pair of functors \(G : E \to X\) and \(F : D \to A\).

1.2.15 Remark.
(1) The composite module morphism \(G(\psi)F : G(\mathcal{M})F \to G(\mathcal{N})F : E \to D\) induces the function
\[
\langle \mathcal{J} \rangle(E : D)(G(\psi)F) : \langle \mathcal{J} \rangle(E : D)(G(\mathcal{M})F) \to \langle \mathcal{J} \rangle(E : D)(G(\mathcal{N})F)
\]
or
\[
(G)\langle \mathcal{J}, \psi \rangle(F) : (G)\langle \mathcal{J}, \mathcal{M} \rangle(F) \to (G)\langle \mathcal{J}, \mathcal{N} \rangle(F)
\]
in the notation introduced in Definition 1.2.8. Since \(G(\psi)F\) is natural in \(G\) and \(F\) by Remark 1.1.27, so is \((G)\langle \mathcal{J}, \psi \rangle(F) = \langle \mathcal{J} \rangle(E : D)(G(\psi)F)\). Hence \(\langle \mathcal{J}, \psi \rangle\) so defined does form a module morphism \(\langle \mathcal{J}, \mathcal{M} \rangle \to \langle \mathcal{J}, \mathcal{N} \rangle\).

(2) The module morphism \(\langle \mathcal{J}, \psi \rangle\) maps each cell \(\theta : G \to F : \mathcal{J} \to \mathcal{M}\) to the cell \(\theta \circ \psi : G \to F : \mathcal{J} \to \mathcal{N}\) defined in Definition 1.2.12.

(3) The assignment \(\psi \mapsto \langle \mathcal{J}, \psi \rangle\) is functorial; indeed, the functor
\[
\langle \mathcal{J}, - \rangle : [X : A] \to [[E, X] : [D, A]]
\]
is defined by
\[
G(\langle \mathcal{J}, \mathcal{M} \rangle)F = \langle \mathcal{J} \rangle(E : D)(G(\mathcal{M})F)
\]
for \(G \in [E, X]\), \(F \in [D, A]\), and \(M \in [X : A]\).

Note. By Remark 1.2.2(3), the following definition is regarded as a special case of Definition 1.2.12, and vice versa:

1.2.16 Definition. Given a pair of cells as in
\[
\begin{array}{c}
E \xrightarrow{\mathcal{J}} D \\
\downarrow^\theta \downarrow_\mathcal{M} \\
X \xrightarrow{\mathcal{M}} A \\
\downarrow^\psi \downarrow_\mathcal{N} \\
Y \xrightarrow{\mathcal{N}} B
\end{array}
\]
, their composite \(\theta \circ \psi = \psi \circ \theta\) is the cell
\[
\begin{array}{c}
E \xrightarrow{\mathcal{J}} D \\
\downarrow^{G \circ \mathcal{P}} \downarrow^\mathcal{Q} \downarrow^\delta \downarrow \mathcal{F} \\
Y \xrightarrow{\mathcal{N}} B
\end{array}
\]
defined by the module morphism \( \theta \circ \psi : J \to [G \circ P] \langle N \rangle [Q \circ F] \) given by the composition

\[
J @> \theta \circ \psi >> G \langle M \rangle F \xrightarrow{G(\psi)F} G \langle P \langle N \rangle Q \rangle F = [G \circ P] \langle N \rangle [Q \circ F].
\]

1.2.17 Remark.
(1) The cell \( \theta \circ \psi \) sends each \( J \)-arrow \( j : e \rightarrow d \) to the \( N \)-arrow

\[
j : \langle \theta \circ \psi \rangle = j : \theta : \psi : e : G : P \rightarrow Q : F : d
\]

—the image of \( j \) under the composite function

\[
e(J) d
\]

\[
\downarrow e(\theta)d
\]

\[
e(G(\langle M \rangle) F) d = (e : G) (\langle M \rangle) (F : d)
\]

\[
\downarrow (e : G)(\psi)(F : d)
\]

\[
(e : G) \langle P \langle N \rangle Q \rangle (F : d) = (e : G) \langle P \rangle \langle N \rangle (Q : F : d).
\]

(2) Similarly, given a pair of right \([\text{op. left}]\) module cells as in

\[
\begin{array}{ccc}
E & \xrightarrow{-} & \ast \\
G \downarrow \theta & & \downarrow F \\
X & \xrightarrow{-} & M \xrightarrow{=} \ast \\
P \downarrow \psi & & \downarrow Q \\
Y & \xrightarrow{-} & N \xrightarrow{=} \ast
\end{array}
\]

the composite

\[
\begin{array}{ccc}
E & \xrightarrow{-} & \ast \\
G \circ P \downarrow \psi \theta & & \downarrow Q \circ F \\
Y & \xrightarrow{-} & N \xrightarrow{=} \ast
\end{array}
\]

is defined by the right \([\text{op. left}]\) module morphism

\[
\theta \circ \psi : J \rightarrow [G \circ P] \langle N \rangle
\]

\[
\text{op.}
\]

\[
\theta \circ \psi : J \rightarrow \langle N \rangle [Q \circ F]
\]

given by the composition

\[
J @> \theta \circ \psi >> G \langle M \rangle \xrightarrow{G(\psi)} G \langle P \langle N \rangle \rangle = [G \circ P] \langle N \rangle \text{ op.}
\]

\[
J @> \theta \circ \psi >> \langle N \rangle Q \xrightarrow{(\psi)F} \langle N \rangle Q = \langle N \rangle [Q \circ F].
\]

1.2.18 Proposition. Modules and cells among them form a category with the composition given in Definition 1.2.16 and the identities given in Remark 1.2.2(4).

Proof. The only non-trivial part is the verification of the associativity of the composition. Given cells

\[
\begin{array}{ccc}
X & \xrightarrow{=} & A \\
P \downarrow \psi & & \downarrow Q \\
X' & \xrightarrow{=} & A' \\
P' \downarrow \psi' & & \downarrow Q' \\
X'' & \xrightarrow{=} & A'' \\
P'' \downarrow \psi'' & & \downarrow Q''
\end{array}
\]

the two cell compositions \((\psi \circ \psi') \circ \psi''\) and \(\psi \circ (\psi' \circ \psi'')\) are defined by the module morphisms

\[
\langle \psi \circ \psi \rangle \circ \psi' = [P \circ P'] \langle \psi' \rangle \circ \psi''\]

\[
\text{and \( \psi \circ \psi \circ \psi' \circ \psi'' \)}\text{. But by the functoriality (see Remark 1.1.27) and associativity (see Proposition 1.1.30) of the composition, (we have}
\]

\[
\langle \psi \circ \psi \rangle \circ \psi' = [P \circ P'] \langle \psi' \rangle \circ \psi''\]

\[
= \psi \circ \psi \circ \psi' \circ \psi''.
\]

\(\square\)
1.2.19 Remark.
(1) Given a universe \( \mathcal{U} \), \( \text{MOD}_{\mathcal{U}} \) denotes the category consisting of all locally \( \mathcal{U} \)-small modules and all cells among them. By \( \text{MOD} \) we mean \( \text{MOD}_{\mathcal{U}} \) for \( \mathcal{U} \) chosen implicitly.

(2) Given a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \), there is a canonical embedding \([\mathbf{X} : \mathbf{A}] \hookrightarrow \text{MOD}\), identical on objects, given by the obvious arrow function (see Remark 1.2.2(3)). The embedding is not, in general, full.

(3) Similarly, right \([\text{op. left}]\) modules and cells among them form a category, denoted by \( \text{MODR \ [\text{op. MODL}] } \), with the composition given in Remark 1.2.17(2) and the identities given in Remark 1.2.5(3). By Remark 1.2.5(1), \( \text{MODR} \) and \( \text{MODL} \) are fully embedded in \( \text{MOD} \).

1.2.20 Definition. A cell is called iso or an isomorphism of modules if it is invertible in the category \( \text{MOD} \). Similarly, a right \([\text{op. left}]\) module cell is called iso or an isomorphism of right \([\text{op. left}]\) modules if it is invertible in the category \( \text{MODR \ [\text{op. MODL}] } \).

1.2.21 Proposition. A cell \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is iso if and only if the functors \( \mathcal{P} \) and \( \mathcal{Q} \) are iso and the module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \mathcal{Q} \) is iso. Specifically, if \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) has the inverse \( \xymatrix{ \mathbf{Y} & \mathbf{B} \ar[l]_{\psi}^{\mathcal{N}} } \), \( \xymatrix{ \mathbf{Y} & \mathbf{B} \ar[l]_{\psi}^{\mathcal{N}} } \) is iso. Similarly, \( \mathcal{P}(\mathcal{Q}) \mathcal{Q} \) gives the inverses of \( \mathcal{P}, \mathcal{Q} \), and \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \mathcal{Q} \), and, conversely, if \( \mathcal{P}, \mathcal{Q} \), and \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \mathcal{Q} \) are iso, then the inverse of \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is given by \( \xymatrix{ \mathbf{Y} & \mathbf{B} \ar[l]_{\psi}^{\mathcal{N}} } \) is iso, and a left module cell \( \xymatrix{ * & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is iso if and only if the functor \( \mathcal{Q} \) is iso and the left module morphism \( \psi : \mathcal{M} \rightarrow (\mathcal{N}) \mathcal{Q} \) is iso.

Proof. Recalling Definition 1.2.16, this is easily verified. \( \square \)

1.2.22 Remark. Similarly,

- a right module cell \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is iso if and only if the functor \( \mathcal{P} \) is iso and the right module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \) is iso.

- a left module cell \( \xymatrix{ * & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is iso if and only if the functor \( \mathcal{Q} \) is iso and the left module morphism \( \psi : \mathcal{M} \rightarrow (\mathcal{N}) \mathcal{Q} \) is iso.

1.2.23 Definition. A cell \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is called fully faithful if the module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \mathcal{Q} \) is iso. Similarly, a right module cell \( \xymatrix{ \mathbf{X} & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is called fully faithful if the right module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \) is iso, and a left module cell \( \xymatrix{ * & \mathbf{A} \ar[l]_{\psi}^{\mathcal{M}} } \) is called fully faithful if the left module morphism \( \psi : \mathcal{M} \rightarrow (\mathcal{N}) \) is iso.

1.2.24 Remark.
(1) By Proposition 1.2.21, an iso cell is fully faithful.

(2) We will see in Proposition 1.2.31 that the notion is a generalization of fully faithfulness of functors.
Note. For any module $\mathcal{J}$ and any cell $\psi : \mathcal{M} \to \mathcal{N}$, the composition in Definition 1.2.16 yields the postcomposition cell $\langle \mathcal{J}, \psi \rangle : \langle \mathcal{J}, \mathcal{M} \rangle \to \langle \mathcal{J}, \mathcal{N} \rangle$ from the module of cells $\mathcal{J} \to \mathcal{M}$ to the module of cells $\mathcal{J} \to \mathcal{N}$; the postcomposition module morphism in Definition 1.2.14 and the identity in Proposition 1.2.10 allow the following definition.

1.2.25 Definition. Given a module $\mathcal{J}$ and a cell $\psi$ as in

\[
\begin{align*}
E & \xrightarrow{\mathcal{J}} D \\
X & \xrightarrow{\mathcal{M}} A \\
P \downarrow \psi & \xrightarrow{Q} \\
Y & \xrightarrow{\mathcal{N}} B
\end{align*}
\]

, the postcomposition cell

\[
[\mathcal{E}, \mathcal{X}] \xrightarrow{\langle \mathcal{J}, \mathcal{M} \rangle} [D, A] \\
[\mathcal{E}, \mathcal{P}] \xrightarrow{\langle \mathcal{J}, \psi \rangle} [D, Q] \\
[\mathcal{E}, \mathcal{Y}] \xrightarrow{\langle \mathcal{J}, \mathcal{N} \rangle} [D, B]
\]

is defined by the postcomposition module morphism

\[
\langle \mathcal{J}, \mathcal{M} \rangle \xrightarrow{\langle \mathcal{J}, \psi \rangle} \langle \mathcal{J}, \mathcal{N} \rangle = \langle \mathcal{J}, \mathcal{P}(\mathcal{N}) \rangle
\]

(postcomposition with $\psi : \mathcal{M} \to \mathcal{P}(\mathcal{N}) \mathcal{Q}$).

1.2.26 Remark. The cell $\langle \mathcal{J}, \psi \rangle$ sends each cell $\theta : G \to \mathcal{F} : \mathcal{J} \to \mathcal{M}$ to the cell $\theta \circ \psi : G \circ \mathcal{P} \to \mathcal{Q} \circ \mathcal{F} : \mathcal{J} \to \mathcal{N}$ defined in Definition 1.2.16.

1.2.27 Proposition. The assignment $\psi \mapsto \langle \mathcal{J}, \psi \rangle$ of the postcomposition cell is functorial.

Proof. Clearly, the assignment $\psi \mapsto \langle \mathcal{J}, \psi \rangle$ preserves the identities. To verify that it preserves the composition, let $\psi$ and $\varphi$ be a composable pair of cells and consider the cells $\langle \mathcal{J}, \psi \rangle$, $\langle \mathcal{J}, \varphi \rangle$, and $\langle \mathcal{J}, \psi \circ \varphi \rangle$ depicted in

\[
\begin{align*}
X & \xrightarrow{\mathcal{M}} A \\
P \downarrow \psi & \xrightarrow{Q} \\
Y & \xrightarrow{\mathcal{N}} B \\
P \downarrow \varphi & \xrightarrow{Q'} \\
Z & \xrightarrow{\mathcal{N}} C
\end{align*}
\]

\[
\begin{align*}
[\mathcal{E}, \mathcal{X}] & \xrightarrow{\langle \mathcal{J}, \mathcal{M} \rangle} [D, A] \\
[\mathcal{E}, \mathcal{P}] & \xrightarrow{\langle \mathcal{J}, \psi \rangle} [D, Q] \\
[\mathcal{E}, \mathcal{Y}] & \xrightarrow{\langle \mathcal{J}, \mathcal{N} \rangle} [D, B] \\
[\mathcal{E}, \mathcal{P}'] & \xrightarrow{\langle \mathcal{J}, \varphi \rangle} [D, Q'] \\
[\mathcal{E}, \mathcal{Z}] & \xrightarrow{\langle \mathcal{J}, \mathcal{L} \rangle} [D, C]
\end{align*}
\]

; we need to verify that the composition of the cells $\langle \mathcal{J}, \psi \rangle$ and $\langle \mathcal{J}, \varphi \rangle$ yields the cell $\langle \mathcal{J}, \psi \circ \varphi \rangle$. First note that $[\mathcal{E}, \mathcal{P}'] = [\mathcal{E}, \mathcal{P}] \circ [\mathcal{E}, \varphi]$ and $[D, Q' \circ Q] = [D, Q'] \circ [D, Q]$ by the functoriality of the operations $[\mathcal{E}, -]$ and $[D, -]$. The cell $\langle \mathcal{J}, \psi \circ \varphi \rangle$ is defined by the module morphism $\langle \mathcal{J}, \psi \circ \varphi \rangle : [\mathcal{E}, \mathcal{P}] \langle \mathcal{J}, \varphi \rangle [D, Q]$ and the cell $\langle \mathcal{J}, \psi \circ \varphi \rangle$ is defined by the module morphism $\langle \mathcal{J}, \psi \circ \mathcal{P}(\varphi) \rangle$. But by the functoriality of the operation $\langle \mathcal{J}, - \rangle$ (see Remark 1.2.15(3)) and Proposition 1.2.10,

\[
\langle \mathcal{J}, \psi \circ \mathcal{P}(\varphi) \rangle = \langle \mathcal{J}, \psi \rangle \circ \langle \mathcal{J}, \mathcal{P}(\varphi) \rangle = \langle \mathcal{J}, \psi \rangle \circ [\mathcal{E}, \mathcal{P}] \langle \mathcal{J}, \varphi \rangle [D, Q] \cdot
\]

\[\square\]

1.2.28 Remark. Given a small module $\mathcal{J} : \mathcal{E} \to \mathcal{D}$, the functor

\[
\langle \mathcal{J}, - \rangle : \text{MOD} \to \text{MOD}
\]
is defined by the object function \( \mathcal{M} \mapsto (\mathcal{J}, \mathcal{M}) \) and the arrow function \( \psi \mapsto (\mathcal{J}, \psi) \), extending the functor \( \langle \mathcal{J}, - \rangle \) in Remark 1.2.15(3) as shown in

\[
\begin{align*}
[X : A] \xrightarrow{(\mathcal{J}, -)} &\quad [[E, X] : [D, A]] \\
&\quad \downarrow \\
\mathsf{MOD} \xrightarrow{(\mathcal{J}, -)} &\quad \mathsf{MOD}
\end{align*}
\]

, where \( \rightarrow \) denotes the canonical embedding in Remark 1.2.19(2).

Note. We defined in Definition 1.1.21 the hom-module \( \mathsf{C} \) of a category \( C \). Now we define the hom-cell \( \langle (\mathbb{C}) \rangle \mapsto \langle (\mathbb{B}) \rangle \) of a functor \( H : C \to B \).

1.2.29 Definition. The hom-cell of a functor \( H : C \to B \) is the cell

\[
\begin{array}{c}
\mathcal{C} \\
\mathcal{H} \\
\mathcal{B}
\end{array} \xrightarrow{\langle (\mathbb{C}) \rangle} \langle (\mathbb{B}) \rangle
\]

given by the arrow function of \( H \); that is, for each pair of objects \( x, a \in |C| \), the function

\[
a(H)b : a(\mathcal{C})b \to (a \cdot H)(\mathcal{B})(H \cdot b)
\]

is given by

\[
H_{a,b} : \mathsf{hom}_C(a, b) \to \mathsf{hom}_B(H \cdot a, H \cdot b) ; f \mapsto H \cdot f.
\]

1.2.30 Remark.
(1) The naturality of the module morphism \( \langle (\mathbb{C}) \rangle \mapsto H(\mathcal{B})H : C \to C \) follows from the functoriality of \( H \).
(2) Hereafter, given a functor \( H \), each component of the arrow function of \( H \) is written as \( a(H)b \).

1.2.31 Proposition. A functor \( H : C \to B \) is iso (resp. fully faithful) if and only if the hom-cell \( \langle (\mathbb{C}) \rangle \mapsto H(\mathcal{B})H : C \to C \) is iso (resp. fully faithful) (see Definition 1.2.23).

Proof. Immediate from the definitions. \( \square \)

1.2.32 Theorem. The assignment of the hom-module defined in Definition 1.1.21 and the assignment of the hom-cell defined in Definition 1.2.29 embed \( \mathsf{CAT} \) in \( \mathsf{MOD} \). Specifically, the assignment \( C \mapsto \langle (\mathbb{C}) \rangle \) forms a faithful functor \( \langle - \rangle : \mathsf{CAT} \to \mathsf{MOD} \), injective on objects.

Proof. The verification of the functoriality is straightforward. The faithfulness and the injectivity on objects are evident. \( \square \)

1.2.33 Definition. Given a cell \( \psi \) and commutative squares of functors as in

\[
\begin{array}{c}
E \xrightarrow{G} X \xrightarrow{\mathcal{M}} A \xrightarrow{F} D \\
\downarrow K \quad \downarrow P \quad \downarrow \psi \quad \downarrow Q \quad \downarrow L \\
E' \xrightarrow{G'} X' \xrightarrow{\mathcal{M}'} A' \xrightarrow{F'} D'
\end{array}
\]

, their pasting composite is the cell

\[
\begin{array}{c}
E \xrightarrow{G(\mathcal{M})F} D \\
\downarrow K \quad \downarrow G(\psi)F \quad \downarrow L \\
E' \xrightarrow{G'(\mathcal{M}')F} D'
\end{array}
\]

defined by the module morphism

\[
G(\psi)F : G(\mathcal{M})(\mathcal{F}) \to G(\mathcal{P}(\mathcal{M}')\mathcal{Q})(\mathcal{F}) = K(\mathcal{G'(\mathcal{M}')F'})\mathcal{L}
\]

— the composite of

\[
\begin{array}{c}
E \xrightarrow{G} X \xrightarrow{\mathcal{M}} A \xrightarrow{F} D \\
\downarrow P(\mathcal{M}')\mathcal{Q}
\end{array}
\]
1.2.34 Remark. A pasting composition

\[
\begin{array}{c}
E \xrightarrow{G} X \xrightarrow{M} A \xrightarrow{F} D \\
\downarrow \quad \downarrow \quad \downarrow \\
E' \xrightarrow{G'} X' \xrightarrow{M'} A' \xrightarrow{F'} D'
\end{array}
\]

with both ends being identities, yields a cell

\[
\begin{array}{c}
E \xrightarrow{G(M)F} D \\
\downarrow \quad \downarrow \\
E' \xrightarrow{G'(M')F} D'
\end{array}
\]

, i.e. a module morphism \( G(\psi)F : G(M)F \rightarrow G'(M')F' : E \rightarrow D \).

1.2.35 Proposition. In Definition 1.2.33, if \( \psi \) is fully faithful, so is the cell \( G(\psi)F \).

Proof. Immediate from Proposition 1.1.33. \( \square \)

1.2.36 Definition. Given a cell \( \psi \) and natural transformations \( \tau \) and \( \sigma \) as in

\[
\begin{array}{c}
E \xrightarrow{G} X \xrightarrow{M} A \xrightarrow{F} D \\
\downarrow \quad \downarrow \quad \downarrow \\
E' \xrightarrow{G'} X' \xrightarrow{M'} A' \xrightarrow{F'} D'
\end{array}
\]

, their pasting composite is the cell

\[
\begin{array}{c}
E \xrightarrow{G(M)F} D \\
\downarrow \quad \downarrow \\
E' \xrightarrow{G'(M')F} D'
\end{array}
\]

defined by the module morphism \( \tau \circ \psi \circ \sigma : G(M)F \rightarrow K(G'(M')F')L \) given by the composition

\[
G(M)F \xrightarrow{G(\psi)F} G(P(M')Q)F = [G \circ P] \langle M' \rangle [Q \circ F] \xrightarrow{\tau(M')} [K \circ G'] \langle M' \rangle [F' \circ L] = K(G'(M')F')L
\]

, where \( G(\psi)F \) is as in Definition 1.2.33.

1.2.37 Remark.
(1) The cell \( \tau \circ \psi \circ \sigma \) sends each \( G(M)F \)-arrow \( m : e \rightarrow d \) to the \( G'(M')F' \)-arrow \( m' : (\tau \circ \psi \circ \sigma) : e : K \rightarrow L : d \) given by

\[
m' : (\tau \circ \psi \circ \sigma) = \tau e \circ (m : \psi) \circ \sigma d
\]

as indicated in the commutative diagram

\[
\begin{array}{c}
e : G : P \xrightarrow{m : \psi} Q : F : d \\
\downarrow \quad \downarrow \\
\tau e \quad \sigma d
\end{array}
\]

(cf. Example 1.1.31(4)).

(2) Definition 1.2.33 is a special case of Definition 1.2.36 where \( \tau \) and \( \sigma \) are identities.

(3) Definition 1.2.6 is a special case of Definition 1.2.36 where \( G, G', F, \) and \( F' \) are identities.

(4) Example 1.1.31(3) is a special case of Definition 1.2.36 where \( K, L, \) and \( \psi \) are identities.

1.2.38 Theorem. Given cells and natural transformations as in

\[
\begin{array}{c}
E \xrightarrow{G} X \xrightarrow{M} A \xrightarrow{F} D \\
\downarrow \quad \downarrow \quad \downarrow \\
E' \xrightarrow{G'} X' \xrightarrow{M'} A' \xrightarrow{F'} D'
\end{array}
\]

\[
\begin{array}{c}
E'' \xrightarrow{G''} X'' \xrightarrow{M''} A'' \xrightarrow{F''} D''
\end{array}
\]
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, the interchange law

\[ \tau \circ \tau' \circ (\psi \circ \psi') \circ [\sigma \circ \sigma'] = (\tau \circ \psi \circ \sigma) \circ (\tau' \circ \psi' \circ \sigma') \]

holds for the vertical and horizontal compositions.

**Proof.** Since the composite cell

\[
\begin{array}{c}
X \xrightarrow{M} A \\
P \xmapsto{P'} \psi \circ \psi' \xrightarrow{Q} Q' \circ Q
\end{array}
\]

\[X'' \xrightarrow{M''} A''\]

is defined by the module morphism \(\psi \circ P(\psi')Q\), the composite cell

\[
\begin{array}{c}
E \xrightarrow{G(M)^F} D \\
K \xmapsto{K'} (\tau \circ \tau') \circ (\psi \circ \psi') \circ [\sigma \circ \sigma'] \xrightarrow{L' \circ L}
\end{array}
\]

\[E'' \xrightarrow{G(M'')^F} D''\]

is defined by the module morphism

\[G(\psi \circ P(\psi')Q) \circ F \circ (\tau \circ \tau')(\mathcal{M}''), [\sigma \circ \sigma'] = G(\psi) \circ F \circ G(P(\psi')Q) \circ F \circ (\tau' \circ \mathcal{M}'') \circ (\psi' \circ \sigma') \circ L, \]

and since the composite cells

\[
\begin{array}{c}
E \xrightarrow{G(M)^F} D \\
K \xmapsto{K'} (\tau \circ \tau') \circ (\psi \circ \psi') \circ [\sigma \circ \sigma'] \xrightarrow{L' \circ L}
\end{array}
\]

\[E'' \xrightarrow{G(M'')^F} D''\]

are defined by the module morphisms \(G(\psi) \circ F \circ (\tau \circ \mathcal{M}') \circ \sigma \circ K(\tau' \circ \mathcal{M}'') \circ \sigma' \circ L\), and since the composite cells

\[
\begin{array}{c}
E \xrightarrow{G(M)^F} D \\
K \xmapsto{K'} (\tau \circ \tau') \circ (\psi \circ \psi') \circ [\sigma \circ \sigma'] \xrightarrow{L' \circ L}
\end{array}
\]

\[E'' \xrightarrow{G(M'')^F} D''\]

are defined by the module morphism

\[G(\psi) \circ F \circ (\tau' \circ \mathcal{M}') \circ \sigma \circ K(\tau' \circ \mathcal{M}'') \circ \sigma' \circ L\]

; but \(G(P(\psi')Q) \circ F \circ (\tau \circ \mathcal{M}) \circ (\psi' \circ \sigma') \circ L\) because

\[G(P(\psi')Q) \circ F \circ (\tau' \circ \mathcal{M}') \circ (\psi' \circ \sigma') \circ L = G(\psi) \circ F \circ (\tau' \circ \mathcal{M}') \circ (\psi' \circ \sigma') \circ L \]

(see Example 1.1.31(4)).

\[\square\]

**1.2.39 Corollary.** Cells and commutative squares of functors as in

\[
\begin{array}{c}
E \xrightarrow{G(M)^F} X \xrightarrow{M} A \xrightarrow{F} D \\
K \xmapsto{K'} \psi \xrightarrow{Q} Q \xrightarrow{L}
\end{array}
\]

\[E' \xrightarrow{G(M')^F} X' \xrightarrow{M'} A' \xrightarrow{F'} D' \\
K' \xmapsto{K'} \psi' \xrightarrow{Q'} Q' \xrightarrow{L'}
\]

\[E'' \xrightarrow{G(M'')^F} X'' \xrightarrow{M''} A'' \xrightarrow{F''} D'' \]

yield the same cell

\[
\begin{array}{c}
E \xrightarrow{G(M)^F} D \\
K \xmapsto{K'} \psi \xrightarrow{Q} Q \xrightarrow{L}
\end{array}
\]

\[E'' \xrightarrow{G(M'')^F} D'' \]

irrespective of the order of the horizontal and vertical compositions.

**Proof.** This is a special case of Theorem 1.2.38 where \(\sigma, \sigma', \tau, \) and \(\tau'\) are identities.

\[\square\]
1.3 Cell morphisms

If the notion of a cell is seen as an extension of that of a functor, the notion of a cell morphism is seen as an extension of that of a natural transformation.

*Note.* In the following, the left and right components of a cell $\psi : \mathcal{M} \to \mathcal{N}$ are denoted by $\psi_0 : \mathcal{M}_0 \to \mathcal{N}_0$ and $\psi_1 : \mathcal{M}_1 \to \mathcal{N}_1$ respectively (see Remark 1.2.2(1)). (This is a general practice when discussing cell morphisms.)

1.3.1 Definition. Given a parallel pair of cells $\psi, \varphi : \mathcal{M} \to \mathcal{N}$, a morphism from $\psi$ to $\varphi$, written $\tau : \psi \to \varphi : \mathcal{M} \to \mathcal{N}$, is defined by a pair of natural transformations

$$
\tau_0 : \psi_0 \to \varphi_0 \quad \text{and} \quad \tau_1 : \psi_1 \to \varphi_1
$$

such that the square

$$
\begin{array}{ccc}
\mathbf{x} : \psi_0 & \xrightarrow{m_\psi} & \psi_1 \cdot \mathbf{a} \\
\downarrow \tau_0 & & \downarrow \tau_1 : \mathbf{a} \\
\mathbf{x} : \varphi_0 & \xrightarrow{m_\varphi} & \varphi_1 \cdot \mathbf{a}
\end{array}
$$

commutes for every $\mathcal{M}$-arrow $\mathbf{m} : \mathbf{x} \sim \mathbf{a}$.

1.3.2 Remark.  
(1) The natural transformations $\tau_0$ and $\tau_1$ are called the left and right components of a cell morphism $\tau$.

(2) The identity morphism of a cell $\psi : \mathcal{M} \to \mathcal{N}$ is given by the pair of identity natural transformations; that is,

$$
1_\psi := (1_{\psi_0}, 1_{\psi_1}).
$$

(3) The composition of two cell morphisms $\tau : \psi \to \varphi : \mathcal{M} \to \mathcal{N}$ and $\sigma : \varphi \to \Omega : \mathcal{M} \to \mathcal{N}$ is given componentwise; that is

$$
\tau \circ \sigma := (\tau_0 \circ \sigma_0, \tau_1 \circ \sigma_1).
$$

(4) Given a pair of modules $\mathcal{M}$ and $\mathcal{N}$, all cells $\mathcal{M} \to \mathcal{N}$ and morphisms among them define the cell category $[\mathcal{M} : \mathcal{N}]$ with the identities and the composition defined above.

(5) If $\mathcal{M}$ is small and $\mathcal{N}$ is locally small, then the category $[\mathcal{M} : \mathcal{N}]$ is locally small.

1.3.3 Definition. A cell morphism $\tau : \psi \to \varphi : \mathcal{M} \to \mathcal{N}$ is called iso (or an isomorphism) if it is invertible in the category $[\mathcal{M} : \mathcal{N}]$.

1.3.4 Proposition. A cell morphism $\tau : \psi \to \varphi : \mathcal{M} \to \mathcal{N}$ is iso if and only if its components $\tau_0$ and $\tau_1$ are natural isomorphisms.

*Proof.* Immediate from the definitions. \(\square\)

*Note.* We defined in Definition 1.2.29 the hom-cell $\langle H \rangle : \langle C \rangle \to \langle B \rangle$ of a functor $H : C \to B$. Now we define the hom-cell morphism $\langle \tau \rangle : \langle G \rangle \to \langle F \rangle : \langle C \rangle \to \langle B \rangle$ of a natural transformation $\tau : G \to F : C \to B$.

1.3.5 Definition. The hom-cell morphism of a natural transformation $\tau : G \to F : C \to B$ is the cell morphism $\langle \tau \rangle : \langle G \rangle \to \langle F \rangle : \langle C \rangle \to \langle B \rangle$ given by the pair $(\tau, \tau)$.

1.3.6 Theorem. Given a pair of categories $C$ and $B$, the assignment of the hom-cell defined in Definition 1.2.29 and the assignment the hom-cell morphism defined in Definition 1.3.5 embed the functor category $[C, B]$ in the cell category $[[C] : [B]]$. Specifically, the assignment $H \mapsto \langle H \rangle$ forms a faithful functor $\langle - \rangle : [C, B] \to [[C] : [B]]$, injective on objects. Moreover, the functor reflects isomorphisms.
Proof. The verification of the functoriality is straightforward. The faithfulness and the injectivity on objects are evident. The last assertion is immediate from Definition 1.3.5 and Proposition 1.3.4. □

Note. Recall from Definition 1.2.25 that, given a module \( J \) and a cell \( \psi : M \to N \), \( (J, \psi) \) denotes the postcomposition cell \( (J, M) \to (J, N) \). The following definition turns the assignment \( \psi \mapsto (J, \psi) \) into a functor.

1.3.7 Definition. Given a module \( J \) and a cell morphisms \( \tau : \psi \to \varphi : M \to N \), the postcomposition cell morphism

\[
(J, \tau) : (J, \psi) \to (J, \varphi) : (J, M) \to (J, N)
\]

“postcomposition with \( \tau \)”, is defined by the pair of postcomposition natural transformations

\[
\{[J_0, \tau_0] : [J_0, \psi_0] \to [J_0, \varphi_0] \quad [J_1, \tau_1] : [J_1, \psi_1] \to [J_1, \varphi_1]
\]

(see Preliminary 0.0.2(1)).

1.3.8 Remark.

(1) Given a cell \( \theta : J \to M \), the commutativity of

\[
\theta_0 \delta \psi_0 \overrightarrow{\theta \delta \psi} \psi_1 \delta \theta_1
\]

(i.e.

\[
\theta_0 \delta \psi_0 \theta_0 \delta \tau_0 \overrightarrow{\theta \delta \psi} \psi_1 \theta_1 \delta \theta_1
\]

) follows from the commutativity of

\[
e : \theta_0 : \psi_0 \overrightarrow{e : \theta_0 \delta \psi} \psi_1 : \theta_1 \delta : d
\]

for each \( J \)-arrow \( m : e \to d \).

(2) The assignment \( \tau \mapsto (J, \tau) \) defines a functor

\[
(J, -) : [M : N] \to [(J, M) : (J, N)]
\]

; indeed, by the definition of the cell morphism \( (J, \tau) \), the functoriality of \( (J, -) \) is reduced to that of \( [J_i, -] : [M_i, N_i] \to [[J_i, M_i], [J_i, N_i]] \) for \( i = 0, 1 \).

1.4 Conic cells

This section introduces a conic cell—a cell from a one-sided module to a two-sided module. A conic cell \( E \overset{\mathcal{J}}{\to} \star \) from a right module \( J \) to a two-sided module \( M \) is defined by a functor \( G : E \to X \)

\[
\begin{array}{ccc}
G & \overset{\theta}{\Rightarrow} & A \\
\downarrow \theta & & \downarrow \alpha \\
X & \overset{\mathcal{M}}{\Rightarrow} & A
\end{array}
\]

an object \( a \in A \), and a right module morphism \( \theta : J \to G(M) a : E \to \star \).

Conic cells between a right module \( J : E \to \star \) and a two-sided module \( M : X \to A \) form the module \( (J, M) : [E, X] \to A \), and a cell

\[
\begin{array}{ccc}
X & \overset{\mathcal{M}}{\Rightarrow} & A \\
\downarrow \psi & & \downarrow \psi \\
Y & \overset{\mathcal{N}}{\Rightarrow} & B
\end{array}
\]
1.4. Conic cells

yields the postcomposition cell

\[ \begin{array}{ccc}
\mathbb{E}, \mathbb{X} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{A} \\
\downarrow \mathbb{P} \quad \downarrow \mathbb{Q} \\
\mathbb{E}, \mathbb{Y} \xrightarrow{\mathbb{J}, \mathbb{N}} \mathbb{B}
\end{array} \]

that sends each conic cell \( \mathbb{E} \xrightarrow{\mathbb{J}} \mathbb{X} \) to its composite \( \mathbb{E} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{A} \) with \( \mathbb{M} \). In Section 12.3, a weighted limit is defined by a universal arrow of the module \( \langle \mathbb{J}, \mathbb{M} \rangle \).

1.4.1 Definition.

- Let \( \mathbb{J} : \mathbb{E} \to * \) be a right module and \( \mathbb{M} : \mathbb{X} \to \mathbb{A} \) be a two-sided module. Given a functor \( \mathbb{G} : \mathbb{E} \to \mathbb{X} \) and an object \( \mathbb{a} \in \| \mathbb{A} \| \), a right conic cell \( \mathbb{G} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{a} \), written diagrammatically as \( \mathbb{G} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{a} \), is defined by a right module morphism \( \mathbb{G} : \mathbb{J} \to \mathbb{G}(\mathbb{M}) \mathbb{a} : \mathbb{E} \to * \).

- Let \( \mathbb{J} : * \to \mathbb{E} \) be a left module and \( \mathbb{M} : \mathbb{X} \to \mathbb{A} \) be a two-sided module. Given an object \( \mathbb{x} \in \| \mathbb{X} \| \) and a functor \( \mathbb{F} : \mathbb{E} \to \mathbb{A} \), a left conic cell \( \mathbb{x} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{F} \), written diagrammatically as \( \mathbb{x} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{F} \), is defined by a left module morphism \( \mathbb{x} : \mathbb{J} \to \mathbb{x}(\mathbb{M}) \mathbb{F} : * \to \mathbb{E} \).

1.4.2 Remark.

1. Under the identification in Remark 1.1.14(4), a right [op. left] conic cell \( \mathbb{J} \to \mathbb{M} \) is regarded as a special instance of a two-sided module cell \( \mathbb{J} \to \mathbb{M} \) where the codomain [op. domain] of \( \mathbb{J} \) is the terminal category.

2. A right [op. left] module cell \( \mathbb{M} \to \mathbb{N} \) in Definition 1.2.4 is regarded as a special instance of a right [op. left] conic cell in Definition 1.4.1 where the codomain [op. domain] of \( \mathbb{N} \) is the terminal category. Conversely, a right [op. left] conic cell in Definition 1.4.1 yields a right module cell \( \mathbb{E} \xrightarrow{\mathbb{J}} \mathbb{X} \) [op. left module cell * \xrightarrow{\mathbb{J}} \mathbb{E} *].

3. A right conic cell \( \mathbb{G} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{a} \) is called fully faithful if the right module morphism \( \mathbb{G} : \mathbb{J} \to \mathbb{G}(\mathbb{M}) \mathbb{a} : \mathbb{E} \to * \) is iso, and a left conic cell \( \mathbb{x} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{F} \) is called fully faithful if the left module morphism \( \mathbb{x} \xrightarrow{\mathbb{J}, \mathbb{M}} \mathbb{x}(\mathbb{M}) \mathbb{F} : * \to \mathbb{E} \) is iso (cf. Definition 1.2.23).

Note. The following definition is identified with a special case of Definition 1.2.8 where the codomain [op. domain] of \( \mathbb{J} \) is the terminal category (cf. Remark 1.4.2(1)).

1.4.3 Definition.

- Given a right module \( \mathbb{J} : \mathbb{E} \to * \) and a module \( \mathbb{M} : \mathbb{X} \to \mathbb{A} \), the module

\( \langle \mathbb{J}, \mathbb{M} \rangle : [\mathbb{E}, \mathbb{X}] \to \mathbb{A} \)

of right conic cells \( \mathbb{J} \to \mathbb{M} \) is defined by

\( (\mathbb{G}) (\langle \mathbb{J}, \mathbb{M} \rangle (\mathbb{a})) = (\mathbb{G}) (\langle \mathbb{E} : \mathbb{J} \rangle (\mathbb{G}(\mathbb{M}) \mathbb{a})) \)

for \( \mathbb{G} \in [\mathbb{E}, \mathbb{X}] \) and \( \mathbb{a} \in \mathbb{A} \), where \( (\mathbb{E} : \mathbb{J}) \) is the hom-module of the category \([\mathbb{E} : \mathbb{J}]\) of right modules \( \mathbb{E} \to * \).

- Given a left module \( \mathbb{J} : * \to \mathbb{E} \) and a module \( \mathbb{M} : \mathbb{X} \to \mathbb{A} \), the module

\( \langle \mathbb{J}, \mathbb{M} \rangle : \mathbb{X} \to [\mathbb{E}, \mathbb{A}] \)

of left conic cells \( \mathbb{J} \to \mathbb{M} \) is defined by

\( (\mathbb{x}) (\langle \mathbb{J}, \mathbb{M} \rangle (\mathbb{F})) = (\mathbb{x}) (\langle \mathbb{J} \rangle (\mathbb{F} (\mathbb{x}(\mathbb{M}) \mathbb{F}))) \)
for \( F \in [E,A] \) and \( x \in X \), where \( \langle E \rangle \) is the hom-module of the category \([E] \) of left modules \( * \to E \).

1.4.4 Remark. For a functor \( G: E \to X \) and an object \( a \in \|A\| \), the set

\[
(G)(\mathcal{J},M)(a) = (\mathcal{J})(\langle E \rangle)(G(M)a)
\]

consists of all right module morphisms \( \mathcal{J} \to G(M)a : E \to * \), i.e. all right conic cells \( G \sim a : \mathcal{J} \sim M \), and for a functor \( F: E \to A \) and an object \( x \in \|X\| \), the set

\[
(x)(\mathcal{J},M)(F) = (\mathcal{J})(\langle E \rangle)(x(M)F)
\]

consists of all left module morphisms \( \mathcal{J} \to x(M)F : * \to E \), i.e. all left conic cells \( x \sim F : \mathcal{J} \sim M \).

Note. If \( \mathcal{J} \) in Proposition 1.2.10 is one-sided, we have the following.

1.4.5 Proposition. Given a right \([op.\ left]\) module \( \mathcal{J} \) and a composite module \( P \langle N \rangle Q \) as in

\[
E \xrightarrow{\mathcal{J}} * \xrightarrow{op.} * \xrightarrow{\mathcal{J}} E
\]

\[
X \xrightarrow{P(\langle N \rangle Q)} A
\]

\[
P \uparrow \begin{array}{l}\downarrow 1 \end{array} \quad Q
\]

\[
Y \xrightarrow{-N} B
\]

the identity

\[
[E,X] \xrightarrow{\langle \mathcal{J},P(\langle N \rangle Q) \rangle} \xrightarrow{op.} X \xrightarrow{P(\langle N \rangle Q)} [E,A]
\]

\[
\begin{array}{l}
[L,E] \xrightarrow{\langle \mathcal{J},N \rangle} \xrightarrow{op.} Y \xrightarrow{-N} B
\end{array}
\]

(i.e.

\[
\langle \mathcal{J},P(\langle N \rangle Q) \rangle = [E,P] \langle \mathcal{J},N \rangle [E, Q]
\]

holds).

Proof. The proof is a straightforward modification of that of Proposition 1.2.10. \( \square \)

1.4.6 Definition. Given a right \([op.\ left]\) conic cell and a cell as in

\[
E \xrightarrow{\mathcal{J}} * \xrightarrow{op.} * \xrightarrow{\mathcal{J}} E
\]

\[
G \downarrow \begin{array}{l}\theta \downarrow a \end{array}
\]

\[
X \xrightarrow{M} A
\]

\[
\begin{array}{l}
\uparrow \begin{array}{l}a \uparrow \theta \uparrow F \end{array}
\end{array}
\]

\[
Y \xrightarrow{-N} B
\]

their composite

\[
E \xrightarrow{\mathcal{J}} * \xrightarrow{op.} * \xrightarrow{\mathcal{J}} E
\]

\[
G \downarrow \begin{array}{l}\theta \circ \psi \downarrow a \end{array}
\]

\[
X \xrightarrow{-N} A
\]

\[
\begin{array}{l}
\uparrow \begin{array}{l}a \uparrow \theta \circ \psi \uparrow F \end{array}
\end{array}
\]

\[
Y \xrightarrow{-N} B
\]

is defined by the right \([op.\ left]\) module morphism

\[
\theta \circ \psi : \mathcal{J} \to G(\langle N \rangle)a \quad \text{op.} \quad \theta \circ \psi : \mathcal{J} \to x(\langle N \rangle)F
\]

given by the composition

\[
\begin{array}{l}
\mathcal{J} \xrightarrow{\theta} G(\langle M \rangle)a \xrightarrow{G(\psi)a} G(\langle N \rangle)a \quad \text{op.} \quad \mathcal{J} \xrightarrow{\theta} x(\langle M \rangle)F \xrightarrow{x(\psi)F} x(\langle N \rangle)F.
\end{array}
\]

1.4.7 Remark. By the identification stated in Remark 1.4.2(1), the composition in Definition 1.4.6 is regarded as a special case of Definition 1.2.12 where the codomain \([op.\ domain]\) of \( \mathcal{J} \) is the terminal category.

Note. The composition in Definition 1.4.6 yields the postcomposition module morphism \( \langle \mathcal{J},\psi \rangle : \langle \mathcal{J},M \rangle \to \langle \mathcal{J},N \rangle \) from the module of conic cells \( \mathcal{J} \to M \) to the module of conic cells \( \mathcal{J} \to N \). Here is the formal definition:
1.4.8 Definition.

- Given a right module \( J : E \to * \) and a module morphism \( \psi : M \to N : X \to A \), the postcomposition module morphism
  \[
  \langle J, \psi \rangle : \langle J, M \rangle \to \langle J, N \rangle : [E, X] \to A
  \]
  "postcomposition with \( \psi \)" is defined by
  \[
  (G) \langle J, \psi \rangle (a) = (J) (G \psi) (a)
  \]
  for each pair of a functor \( G : E \to X \) and an object \( a \in \| A \| \).

- Given a left module \( J : * \to E \) and a module morphism \( \psi : M \to N : X \to A \), the postcomposition module morphism
  \[
  \langle J, \psi \rangle : \langle J, M \rangle \to \langle J, N \rangle : X \to [E, A]
  \]
  "postcomposition with \( \psi \)" is defined by
  \[
  (x) \langle J, \psi \rangle (F) = (J) (x \psi) (F)
  \]
  for each pair of a functor \( F : E \to A \) and an object \( x \in \| X \| \).

1.4.9 Remark.

(1) The module morphism \( \langle J, \psi \rangle \) in Definition 1.4.8 is identified with a special case of the module morphism \( \langle J, \psi \rangle \) in Definition 1.2.14 where the codomain \([\text{op. domain}]\) of \( J \) is the terminal category.

(2) The module morphism \( \langle J, \psi \rangle : [E, X] \to A \) maps each right conic cell \( \theta \circ \psi : G \to a : J \to M \) to the right conic cell \( \theta \circ \psi : G \to a : J \to M \) defined in Definition 1.4.6, and the module morphism \( \langle J, \psi \rangle : X \to [E, A] \) maps each left conic cell \( \theta \circ \psi : x \to F : J \to M \) to the left conic cell \( \theta \circ \psi : x \to F : J \to M \) in Definition 1.4.10.

1.4.10 Definition. Given a pair of right \([\text{op. left}]\) conic cell and a cell as in

\[
\begin{array}{c|c|c|c}
E & J & * & \text{op.} \newcommand{\op}{\text{op.}} \\
G & \theta & a & \op \newcommand{\op}{\text{op.}} \\
X & M & A & \\
\parallel & \psi & Q & \\
Y & N & B & \\
\end{array}
\]

their composite \( \theta \circ \psi = \psi \circ \theta \) is the right conic cell

\[
\begin{array}{c|c|c|c}
E & J & * & \text{op.} \\
G \circ \psi & \theta \circ \psi & Q \circ a & \op \newcommand{\op}{\text{op.}} \\
Y & N & B & \\
\end{array}
\]

defined by the right \([\text{op. left}]\) module morphism

\[
\theta \circ \psi : J \to [G \circ \psi \circ a : J \to M] \to [Q \circ a : J \to N] \to [Q \circ F : J \to F]
\]
given by the composition

\[
J \overset{\theta}{\to} G(\langle M \rangle) a \xrightarrow{G(\psi) a} G(\langle N \rangle) Q a = [G \circ \psi \circ a : J \to M] \to [Q \circ F : J \to F]
\]

1.4.11 Remark. By the identification stated in Remark 1.4.2(1), the composition in Definition 1.4.10 is regarded as a special case of the composition in Definition 1.2.16 where the codomain \([\text{op. domain}]\) of \( J \) is the terminal category.

Note. The composition in Definition 1.4.10 yields the postcomposition cell \( \langle J, \psi \rangle : \langle J, M \rangle \to \langle J, N \rangle \) from the module of conic cells \( J \to M \) to the module of conic cells \( J \to N \); the postcomposition module morphisms defined in Definition 1.4.8 and the identities in Proposition 1.4.5 allow the following definition.
1.4.12 Definition. Given a right [op. left] module \( \mathcal{J} \) and a cell \( \psi \) as in
\[
\begin{align*}
\mathbf{E} \overset{\mathcal{J}}{\rightarrow} & \ast & \text{op.} & & \ast & \overset{\mathcal{J}}{\rightarrow} & \mathbf{E} \\
\mathbf{X} \overset{\mathcal{M}}{\rightarrow} & \mathbf{A} & \mathbf{Y} \overset{\mathcal{N}}{\rightarrow} & \mathbf{B} & \mathbf{P} \overset{\psi}{\rightarrow} & \mathbf{Q} & \mathbf{P} \overset{\psi}{\rightarrow} & \mathbf{Q} \\
\end{align*}
\]
, the postcomposition cell
\[
\begin{align*}
\langle \mathcal{E}, \mathbf{X} \rangle & \overset{\langle \mathcal{J}, \mathcal{M} \rangle}{\rightarrow} \mathbf{A} & \text{op.} & & \langle \mathcal{E}, \mathbf{A} \rangle \\
\langle \mathcal{E}, \mathbf{P} \rangle & \overset{\langle \mathcal{J}, \psi \rangle}{\rightarrow} & \langle \mathcal{E}, \mathbf{Q} \rangle \\
\langle \mathcal{E}, \mathbf{Y} \rangle & \overset{\langle \mathcal{J}, \mathcal{N} \rangle}{\rightarrow} \mathbf{B} & \text{op.} & & \langle \mathcal{E}, \mathbf{B} \rangle \\
\end{align*}
\]
is defined by the postcomposition module morphism
\[
\langle \mathcal{J}, \mathcal{M} \rangle \overset{\langle \mathcal{J}, \psi \rangle}{\rightarrow} \langle \mathcal{J}, \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathbf{Q} = \langle \mathcal{E}, \mathbf{P} \rangle \langle \mathcal{J}, \mathcal{N} \rangle \mathbf{Q} \quad \text{op.} \quad \langle \mathcal{J}, \mathcal{M} \rangle \overset{\langle \mathcal{J}, \psi \rangle}{\rightarrow} \langle \mathcal{J}, \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathbf{Q} = \mathbf{P} \langle \mathcal{J}, \mathcal{N} \rangle \langle \mathcal{E}, \mathbf{Q} \rangle
\]
(postcomposition with \( \psi : \mathcal{M} \rightarrow \mathcal{P} \langle \mathcal{N} \rangle \mathbf{Q} \)).

1.4.13 Remark.
(1) The postcomposition cell \( \langle \mathcal{J}, \psi \rangle \) in Definition 1.4.12 is identified with the postcomposition cell
\[
\begin{align*}
\langle \mathcal{J}, \mathcal{M} \rangle & \overset{\langle \mathcal{J}, \psi \rangle}{\rightarrow} \langle \mathcal{J}, \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathbf{Q} = \langle \mathcal{E}, \mathbf{P} \rangle \langle \mathcal{J}, \mathcal{N} \rangle \mathbf{Q} \quad \text{op.} \quad \langle \mathcal{J}, \mathcal{M} \rangle & \overset{\langle \mathcal{J}, \psi \rangle}{\rightarrow} \langle \mathcal{J}, \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathbf{Q} = \mathbf{P} \langle \mathcal{J}, \mathcal{N} \rangle \langle \mathcal{E}, \mathbf{Q} \rangle
\end{align*}
\]

(2) The cell \( \langle \mathcal{J}, \psi \rangle : \langle \mathcal{E}, \mathbf{P} \rangle \rightarrow \mathbf{Q} \) sends each right conic cell \( \theta : \mathbf{G} \Rightarrow \mathbf{a} : \mathcal{J} \Rightarrow \mathcal{M} \) to the right conic cell \( \theta \circ \psi : \mathbf{G} \circ \mathbf{P} \Rightarrow \mathbf{Q} \circ \mathbf{a} : \mathcal{J} \Rightarrow \mathcal{N} \) defined in Definition 1.4.10, and the cell \( \langle \mathcal{J}, \psi \rangle : \mathbf{P} \Rightarrow \langle \mathcal{E}, \mathbf{Q} \rangle \) sends each left conic cell \( \theta : \mathbf{x} \Rightarrow \mathbf{F} : \mathcal{J} \Rightarrow \mathcal{M} \) to the left conic cell \( \theta \circ \psi : \mathbf{x} \circ \mathbf{P} \Rightarrow \mathbf{Q} \circ \mathbf{F} : \mathcal{J} \Rightarrow \mathcal{N} \).
2 Slicing and Action

2.1 Module slicing

This section is about module and cell slicing. Right exponential transposition transforms a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ into a functor $[\mathcal{M} \downarrow] : \mathbf{A} \to [\mathbf{X}]$ from $\mathbf{A}$ to the category of right modules over $\mathbf{X}$, and left exponential transposition transforms it into a contravariant functor $[\mathcal{N} \uparrow] : \mathbf{X} \to [\mathbf{A}]^\ast$ from $\mathbf{X}$ to the category of left modules over $\mathbf{A}$. Right and left exponential transposition thus slice $\mathcal{M}$ into pieces of right modules $(\mathcal{M})a : \mathbf{X} \to \ast$ (the right slice of $\mathcal{M}$ at $a \in [\mathbf{A}]$) and into pieces of left modules $x(\mathcal{M}) : \ast \to \mathbf{A}$ (the left slice of $\mathcal{M}$ at $x \in \|\mathcal{X}\|$). If applied to a two-sided module cell $\xymatrix{ X \ar[r]^{\mathcal{M}} & A \ar[d]^p \ar[l]_\psi \ar[r]_Q \ar[d] \ar[l]_\psi & Y \ar[d] \ar[l]_\psi \ar[r]^{\mathcal{N}} & B }$, the device slices it into pieces of right conic cells $\xymatrix{ X \ar[r]^{(\mathcal{M})a} & \ast \ar[d]^p \ar[l]_\psi \ar[r]_Q \ar[d] \ar[l]_\psi & Y \ar[d] \ar[l]_\psi \ar[r]^{(\mathcal{N})a} & B }$ and into pieces of left conic cells $\xymatrix{ \ast \ar[r]^{x(\mathcal{M})a} & X \ar[d]^p \ar[l]_\psi \ar[r]_Q \ar[d] \ar[l]_\psi & \ast \ar[d] \ar[l]_\psi \ar[r]^{x(\mathcal{N})a} & A \ar[d] \ar[l]_\psi \ar[r]_p \ar[d] \ar[l]_\psi & Y \ar[d] \ar[l]_\psi \ar[r]^{(\mathcal{N})a} & B }$.

2.1.1 Definition. Given categories $\mathbf{X}$ and $\mathbf{A}$, the right [op. left] exponential transposition

$$\xymatrix{ [\mathbf{X} \times \mathbf{A}, \mathbf{Set}] \ar[r]^-\downarrow \ar_-\downarrow & [\mathbf{A}, [\mathbf{X}^\ast, \mathbf{Set}]] \ar_-\downarrow \ar_-\downarrow }$$

(see Preliminary 0.0.5) is denoted by

$$\xymatrix{ [\mathbf{X} : \mathbf{A}] \ar[r]^-\downarrow \ar_-\downarrow & [\mathbf{A}, [\mathbf{X} : ]] \ar_-\downarrow \ar_-\downarrow }$$

or

$$\xymatrix{ [\mathbf{X} : \mathbf{A}]^\ast \ar[r]^-\downarrow \ar_-\downarrow & [\mathbf{X}^\ast, [\mathbf{A}]] \ar_-\downarrow \ar_-\downarrow }$$

using the abbreviation in Definition 1.1.13 and Definition 1.1.3. The right [op. left] exponential transpose of a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$ is the covariant [op. contravariant] functor

$$\xymatrix{ [\mathcal{M} \downarrow] : \mathbf{A} \to [\mathbf{X}] \ar_-\downarrow \ar_-\downarrow & [\mathcal{N} \uparrow] : \mathbf{X} \to [\mathbf{A}]^\ast \ar_-\downarrow \ar_-\downarrow }$$

and the right [op. left] exponential transpose of a module morphism $\psi : \mathcal{M} \to \mathcal{N} : \mathbf{X} \to \mathbf{A}$ is the natural transformation

$$\xymatrix{ [\psi \downarrow] : [\mathcal{M} \downarrow] \to [\mathcal{N} \downarrow] : \mathbf{A} \to [\mathbf{X}] \ar_-\downarrow \ar_-\downarrow & [\psi \uparrow] : [\mathcal{N} \uparrow] \to [\mathcal{M} \uparrow] : \mathbf{X} \to [\mathbf{A}]^\ast \ar_-\downarrow \ar_-\downarrow }$$

; we adopt the following notation:

- the value of $\mathcal{M} \downarrow$ at $a \in \mathbf{A}$ is written $\mathcal{M} \downarrow a$, $(\mathcal{M})a$, or just $(\mathcal{M})a$, and the component of $\psi \downarrow$ at $a \in [\mathbf{A}]$ is written $\psi \downarrow a$ or $(\psi)a$.
- the value of $\mathcal{N} \uparrow$ at $x \in \mathbf{X}$ is written $x \mathcal{N} \uparrow$, $(x)\mathcal{N}$, or just $(x)\mathcal{N}$, and the component of $\psi \uparrow$ at $x \in [\mathbf{X}]$ is written $x \psi \uparrow$ or $x(\psi)$.

2.1.2 Remark.

(1) Given a module $\mathcal{M} : \mathbf{X} \to \mathbf{A}$,

- the right exponential transpose $[\mathcal{M} \downarrow] : \mathbf{A} \to [\mathbf{X}]$ sends each object $a \in \|\mathbf{A}\|$ to the right slice of $\mathcal{M}$ at $a$, i.e. the right module $(\mathcal{M})a : \mathbf{X} \to \ast$

  given by

  $$x((\mathcal{M})a) = x(\mathcal{M})a$$
for each \( x \in X \), and sends each \( A \rightarrow \text{arrow} f : a \rightarrow a' \) to the right module morphism

\[
\langle M \rangle f : \langle M \rangle a \rightarrow \langle M \rangle a' : X \rightarrow *
\]

which maps each \( M \rightarrow \text{arrow} m : x \sim a \) to the \( M \rightarrow \text{arrow} m \circ f : x \sim a' \) as indicated in

\[
\begin{array}{c}
x \sim^m \sim^a \sim^a' \\
\overrightarrow{f} \\
\overrightarrow{m} \circ (\langle M \rangle f)
\end{array}
\]

- the left exponential transpose \([\kappa, M] : X \rightarrow [: A]^\rightarrow\) sends each object \( x \in |X| \) to the left slice of \( M \) at \( x \), i.e. the left module

\[
x(\langle M \rangle) : * \rightarrow A
\]

given by

\[
\langle x(\langle M \rangle) \rangle a = x(\langle M \rangle) a
\]

for each \( a \in A \), and sends each \( X \rightarrow \text{arrow} f : x' \rightarrow x \) to the left module morphism

\[
f(\langle M \rangle) : x(\langle M \rangle) \rightarrow x'(\langle M \rangle) : * \rightarrow A
\]

which maps each \( M \rightarrow \text{arrow} m : x \sim a \) to the \( M \rightarrow \text{arrow} f \circ m : x' \sim a \) as indicated in

\[
\begin{array}{c}
x \sim^m \sim^a \sim^{x'} \\
\overrightarrow{f} \\
\overrightarrow{m}
\end{array}
\]

(2) Given a module morphism \( \psi : M \rightarrow \mathcal{N} : X \rightarrow A \),

- the component of the right exponential transpose \([\psi_\rightarrow] : [\mathcal{M}_\rightarrow] \rightarrow [\mathcal{N}_\rightarrow] : A \rightarrow [X_\rightarrow] \) at \( a \in |A| \) gives the right slice of \( \psi \) at \( a \), i.e. the right module morphism

\[
\langle \psi \rangle a : \langle M \rangle a \rightarrow \langle N \rangle a : X \rightarrow *
\]

given by

\[
x(\langle \psi \rangle a) = x(\langle \psi \rangle) a
\]

for each \( x \in |X| \).

- the component of the left exponential transpose \([\kappa, \psi] : [\kappa, \mathcal{N}] \rightarrow [\kappa, \mathcal{M}] : X \rightarrow [: A]^\sim\) at \( x \in |X| \) gives the left slice of \( \psi \) at \( x \), i.e. the left module morphism

\[
x(\langle \psi \rangle) : x(\langle M \rangle) \rightarrow x(\langle N \rangle) : * \rightarrow A
\]

given by

\[
\langle x(\langle \psi \rangle) \rangle a = x(\langle \psi \rangle) a
\]

for each \( a \in |A| \).

(3) Given a module \( \mathcal{M} : X \rightarrow A \), the evaluation of the right exponential transpose \( \mathcal{M}_\rightarrow \) at \( a \in A \) is the identified with the composition

\[
X \xrightarrow{\mathcal{M}_\rightarrow} A \xrightarrow{a} *
\]

, and the evaluation of the left exponential transpose \( \kappa, \mathcal{M} \) at \( x \in X \) is identified with the composition

\[
* \xrightarrow{x} X \xrightarrow{\mathcal{M}_\rightarrow} A
\]

(cf. Example 1.1.31(6)).

(4) For any module \( \mathcal{M} \),

\[
[\mathcal{M}_\rightarrow]^\sim = [\kappa, \mathcal{M}^\sim]
\]

; that is, the opposite of the right exponential transpose of \( \mathcal{M} \) is the left exponential transpose of the opposite module \( \mathcal{M}^\sim \).

(5) The diagonal functors in Example 1.1.31(8) and the right [op. left] exponential transposition
2.1. Module slicing

make the diagram

\[
\begin{array}{ccc}
[X] & \xrightarrow{!_E} & [X] \\
\downarrow & & \downarrow \\
[X] & \xrightarrow{\psi} & [E, [X]]
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
[X] & \xrightarrow{[E:A]} & [A] \\
\downarrow & & \downarrow \\
[X] & \xrightarrow{\psi E} & [E, [A]]
\end{array}
\]

commute (cf. Preliminary 0.0.7).

(6) The diagram

\[
\begin{array}{ccc}
[X] & \xrightarrow{\psi} & [A, [X]] \\
\downarrow & & \downarrow \\
[A] & \xrightarrow{\psi} & [A, [X]]
\end{array}
\quad \text{i.e.} \quad
\begin{array}{ccc}
[X] & \xrightarrow{\psi} & [A, [X]] \\
\downarrow & & \downarrow \\
[A] & \xrightarrow{\psi} & [A, [X]]
\end{array}
\]

commutes (see Preliminary 0.0.5 and Definition 1.1.36); that is, the right exponential transpose of a module \( M : X \to A \) is the same thing as the left exponential transpose of the opposite module \( M^\dashv : A^\dashv \to X^\dashv \); in particular, the right slice \( \langle M \rangle a \) of a module \( M : X \to A \) is the same thing as the left slice \( a \langle M^\dashv \rangle \) of the opposite module \( M^\dashv : A^\dashv \to X^\dashv \).

2.1.3 Proposition. For a module morphism \( \psi : M \to N : X \to A \), the following conditions are equivalent:

1. \( \psi \) is a module isomorphism;
2. the right exponential transpose \([\psi \, \mathcal{R}] : [M \, \mathcal{R}] \to [N \, \mathcal{R}] : A \to [X] :\) is a natural isomorphism;
3. each right slice \( \langle \psi \rangle a : \langle M \rangle a \to \langle N \rangle a : X \to \ast \) is a right module isomorphism;
4. the left exponential transpose \( [\mathcal{L} \, \psi] : [\mathcal{L} \, \mathcal{M}] : X \to [\mathcal{L} \, a]^- \) is a natural isomorphism;
5. each left slice \( x \langle \psi \rangle : x \langle M \rangle \to x \langle N \rangle : \ast \to A \) is a left module isomorphism.

Proof. Since the right and left exponential transpositions give isomorphisms \([A, [X]] \cong [X] : A \cong [X, [a]]^-\), the equivalence (1) \( \iff \) (2) \( \iff \) (4) holds. Since a natural transformation is iso iff each component is an isomorphism, the equivalences (2) \( \iff \) (3) and (4) \( \iff \) (5) hold. \( \square \)

2.1.4 Remark. Proposition 2.1.3 also follows from Proposition 1.1.6 and Proposition 1.1.16, noting that \( x \langle \psi \rangle a = x \langle \psi \rangle a = (x \langle \psi \rangle) a \).

2.1.5 Proposition. Let \( \psi : M \to N : X \to A \) be a module morphism.

- If \( B \) is an isomorphism-dense subcategory of \( A \), then \( \psi \) is an isomorphism if and only if its right slice \( \langle \psi \rangle b : \langle M \rangle b \to \langle N \rangle b : X \to \ast \) is a right module isomorphism for each \( b \in \|B\| \).
- If \( Y \) is an isomorphism-dense subcategory of \( X \), then \( \psi \) is an isomorphism if and only if its left slice \( y \langle \psi \rangle : y \langle M \rangle \to y \langle N \rangle : \ast \to A \) is a left module isomorphism for each \( y \in \|Y\| \).

Proof. By Proposition 2.1.3, \( \psi \) is an isomorphism iff its right exponential transpose \( \psi \, \mathcal{R} \) is a natural isomorphism, and by the lemma in Preliminary 0.0.11, this is the case iff the restriction of \( \psi \, \mathcal{R} \) to \( B \) is a natural isomorphism, and this is the case iff for each \( b \in \|B\| \), the right slice of \( \psi \) at \( b \) (i.e. the component of \( \psi \, \mathcal{R} \) at \( b \)) is an isomorphism. \( \square \)

2.1.6 Proposition. Given a module (or module morphism) \( M \) and a functor (or natural transformation) as in

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow & & \downarrow \\
E & \xrightarrow{\psi} & F
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E & \xrightarrow{G} & X \\
\downarrow & & \downarrow \\
A & \xrightarrow{\psi^M} & [A]^{-}
\end{array}
\]

, the right [op. left] exponential transpose of the composite \( \langle M \rangle F \) [op. \( \mathcal{L} \, G \langle M \rangle \)] is given by the composition

\[
\begin{array}{ccc}
[X] & \xrightarrow{\psi} & [A] \\
\downarrow & & \downarrow \\
[X] & \xrightarrow{\psi^M} & [A]^{-}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
[X] & \xrightarrow{\psi} & [A] \\
\downarrow & & \downarrow \\
[X] & \xrightarrow{\psi^M} & [A]^{-}
\end{array}
\]

; that is,

\[
[\langle (M) \, F \rangle] = [M] \cdot F \quad \text{op.} \quad [\mathcal{L} \, \langle G \langle M \rangle \rangle) = G \cdot [\mathcal{L} \, M].
\]
Proof. For any $e \in E$,

$$\left[\left(\left(M F\right) \mathcal{R} \right) \vdash e = \left(\left(M F\right) e = \left(M \mathcal{R}\right) \vdash (F \vdash e) = \left([M \mathcal{R}] \triangleright F\right) \vdash e\right)\right].$$

\[\square\]

2.1.7 Proposition. Given a module $M$ and a functor as in

$$\xymatrix{ E \ar[r]^G & X \ar[r]^{M} & A \ar[l]^\mathrm{op.} & X \ar[r]^{M} & A \ar[l]^F \ar[r]^E },$$

the right [op. left] exponential transpose of the composite module $G(M)$ [op. $(M)F$] is given by the composition

$$\xymatrix{ [E:] & [G] \ar[d] & [X:] \ar[r]^M & A \ar[l]^\mathrm{op.} & X \ar[r]^{\cdot M} & [A:] \ar[l]^\mathrm{op.} & [F] \ar[l]^\cdot \ar[d] \ar[r]^E & [E:] \ar[l]^\cdot },$$

that is,

$$\left((G(M)) \mathcal{R}\right) \vdash a = (G(M)) a = G((M) a) = \left[G:: (\langle M \rangle a) = \left[G:: (\langle M \rangle \vdash a) = \left[\langle G:: [M \mathcal{R}] \vdash a\right].

\[\square\]

2.1.8 Definition. Given a cell

$$\xymatrix{ X \ar[r]^{(M)a} & * \ar[l]^{\psi} \ar[l]_{\psi} \ar[l]_{Q} \ar[l]_P \ar[r] & Y \ar[r]_{\mathcal{N}} & B },$$

the right [op. left] slice of $\psi$ at

$$a \in \|A\| \quad \mathrm{op.} \quad x \in \|X\|$$
is the right [op. left] conic cell

$$\xymatrix{ X \ar[r]^{(M)a} & * \ar[l]^{\psi} \ar[l]_{Q} \ar[l]_P \ar[r] & Y \ar[r]_{\mathcal{N}} & B },$$
or the right [op. left] module cell

$$\xymatrix{ X \ar[r]^{(M)a} & * \ar[l]^{\psi} \ar[l]_{Q} \ar[l]_P \ar[r] & Y \ar[r]_{\mathcal{N}} & B },$$

(see Remark 1.4.2(2)) defined by the right [op. left] module

$$\xymatrix{ (\psi) a : (M) a \to (P\mathcal{N}) Q \ar[r] & X \ar[r] & * \ar[l]^{x(M)} \ar[r] & A } \quad \mathrm{op.} \quad x(\psi) : x(M) \to x(P\mathcal{N}) Q : * \to A,$$

i.e. the right [op. left] slice of the module morphism

$$\psi : M \to P\mathcal{N} Q : X \to A$$
at $a$ [op. $x$].

2.1.9 Proposition. Consider a cell as in Definition 2.1.8.

- If $\psi$ is fully faithful and $P$ is iso, then each right module cell $(\psi)a$ is iso.
- If $\psi$ is fully faithful and $Q$ is iso, then each left module cell $x(\psi)$ is iso.

Proof. The assertion is immediate from Proposition 2.1.3 on recalling the definition of fully faithfulness (Definition 1.2.23) and recalling Remark 1.2.22.

\[\square\]

2.1.10 Proposition. A cell is fully faithful if and only if each right [op. left] slice is fully faithful.

Proof. Recalling the definition of fully faithfulness (see Definition 1.2.23 and Remark 1.4.2(3)), this is immediate from Proposition 2.1.3.

\[\square\]
2.2 Module actions

In this section, we discuss the action of a module $\mathcal{M} : X \to A$ on a factor category $[E, A]$ and on a factor category $[E, X]$. The right action of $\mathcal{M}$ on $[E, A]$ is implemented by the functor $[\mathcal{M} \triangleright E] : [E, A] \to [X : E]$ that sends each functor $F : E \to A$ to the composite module $(\mathcal{M}) F : X \to E$, and the left action of $\mathcal{M}$ on $[E, X]$ is implemented by the contravariant functor $[E \sqcap \mathcal{M}] : [E, X] \to [E : A]^\to$ that sends each functor $G : E \to X$ to the composite module $G(\mathcal{M}) : E \to A$. Module action subsumes module slicing introduced in Section 2.1 as a special case (see Remark 2.2.2(2)).

2.2.1 Definition. Let $E$ be a category and $\mathcal{M} : X \to A$ be a module.

- The right action of $\mathcal{M}$ on the functor category $[E, A]$ is the covariant functor $[\mathcal{M} \triangleright E] : [E, A] \to [X : E]$ defined by $[\mathcal{M} \triangleright E] : F = (\mathcal{M}) F$ for $F \in [E, A]$.

- The left action of $\mathcal{M}$ on the functor category $[E, X]$ is the contravariant functor $[E \sqcap \mathcal{M}] : [E, X] \to [E : A]^\to$ defined by $G : [E \sqcap \mathcal{M}] = G(\mathcal{M})$ for $G \in [E, X]$.

2.2.2 Remark. (1) A module $\mathcal{M} : X \to A$ acts on a functor $F : E \to A$ by the composition $X \xymatrix@C=1.5ex{\ar[r]^-{\mathcal{M}} & A} \xymatrix@C=1.5ex{\ar[l]^-{\sigma_F} E}$ and yields the module $(\mathcal{M}) F : X \to E$, and acts on a natural transformation $\sigma : F \to F' : E \to A$ by the composition $X \xymatrix@C=1.5ex{\ar[r]^-{\mathcal{M}} & A} \xymatrix@C=1.5ex{\ar[l]^-{\sigma} E}$ and yields the module morphism $(\mathcal{M}) \sigma : (\mathcal{M}) F \to (\mathcal{M}) F' : X \to E$ which maps each $(\mathcal{M}) F$-arrow $m : x \to e$ to the $(\mathcal{M}) F'$-arrow $m \circ \sigma_e : x \to e$ as indicated in

\[
\begin{array}{c}
x \xymatrix@C=1.5ex{\ar[r]^-m & F' : e} \\
\xymatrix@C=1.5ex{\ar[u]^-{\sigma_e} m : (\mathcal{M}) \sigma} \\
\end{array}
\]

(cf. Example 1.1.31(3)).

- A module $\mathcal{M} : X \to A$ acts on a functor $G : E \to X$ by the composition $E \xymatrix@C=1.5ex{\ar[r]^-G & X} \xymatrix@C=1.5ex{\ar[l]^-{\mathcal{M}} A}$ and yields the module $G(\mathcal{M}) : E \to A$, and acts on a natural transformation $\tau : G' \to G : E \to X$ by the composition $E \xymatrix@C=1.5ex{\ar[r]^-\tau \circ G & X} \xymatrix@C=1.5ex{\ar[l]^-{\mathcal{M}} A}$ and yields the module morphism $\tau(\mathcal{M}) : G(\mathcal{M}) \to G'(\mathcal{M}) : E \to A$ which maps each $G(\mathcal{M})$-arrow $m : e \to a$ to the $G'(\mathcal{M})$-arrow $\tau_e \circ m : e \to a$ as indicated in

\[
\begin{array}{c}
e : G \xymatrix@C=1.5ex{\ar[r]^-m & a} \\
\xymatrix@C=1.5ex{\ar[u]^-{\tau_e} e : G'} \\
\end{array}
\]

(cf. Example 1.1.31(3)).
(2) By Remark 2.1.2(3), the right [op. left] exponential transpose

\[ [\mathcal{M}^*] : \mathcal{A} \to [X :] \quad \text{op.} \quad [^\wedge, \mathcal{M}] : X \to [^*: A] \]

of a module \( \mathcal{M} : X \to \mathcal{A} \) is identified with the right [op. left] action

\[ [\mathcal{M}^*] : [* , \mathcal{A}] \to [X :^*] \quad \text{op.} \quad [^\wedge, \mathcal{M}] : [* , X] \to [^*: A] \]

of \( \mathcal{M} \) on the functor category \([*, \mathcal{A}]\) [op. \([*, X]\)].

(3) For any category \( \mathcal{E} \) and any module \( \mathcal{M} : X \to \mathcal{A} \),

\[ [\mathcal{M} \times \mathcal{E}]^* \cong [\mathcal{E}^* \times \mathcal{M}]^* \]

; that is, the opposite of the right action of \( \mathcal{M} : X \to \mathcal{A} \) on \([E, A]\) is (identified with) the left action of the opposite module \( \mathcal{M}^* : A^* \to X^* \) on \([E^*, A^*]\).

2.2.3 Proposition. Given a category \( \mathcal{E} \) and a module \( \mathcal{M} : X \to \mathcal{A} \), the diagram

\[
\begin{array}{ccc}
[\mathcal{E}, X] & \xrightarrow{[\mathcal{E}, \mathcal{M}]} & [\mathcal{E}, X^*] \\
\downarrow{\mathcal{E} \times \mathcal{M}} & & \downarrow{\mathcal{E}, \mathcal{M}} \\
[\mathcal{X} : \mathcal{E}] & \xrightarrow{\mathcal{M} \times \mathcal{E}} & [\mathcal{X}, \mathcal{E}^*]
\end{array}
\]

commutes.

Proof. Immediate from Proposition 2.1.6. \( \square \)

2.2.4 Proposition. Given a category \( \mathcal{E} \) and a module \( \mathcal{M} : X \to \mathcal{A} \), the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{[\mathcal{I}, \mathcal{A}]} & [\mathcal{E}, X] \\
\downarrow{\mathcal{M} \times \mathcal{A}} & & \downarrow{\mathcal{E} \times \mathcal{M}} \\
[\mathcal{X} :] & \xrightarrow{\mathcal{M} \times \mathcal{X}} & [\mathcal{X}, \mathcal{E}^*]
\end{array}
\]

commutes.

Proof. The diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{[\mathcal{I}, \mathcal{A}]} & [\mathcal{E}, X] \\
\downarrow{\mathcal{M} \times \mathcal{A}} & & \downarrow{\mathcal{E} \times \mathcal{M}} \\
[\mathcal{X} :] & \xrightarrow{\mathcal{M} \times \mathcal{X}} & [\mathcal{X}, \mathcal{E}^*]
\end{array}
\]

commute by Proposition 2.2.3, and together with the commutative diagrams in Remark 2.1.2(5), yields the desired commutative diagram. \( \square \)

2.3 Yoneda functors and representations

The right Yoneda functor \([X^*] : X \to [X :]\) for a category \( X \) is defined by the right exponential transpose (see Section 2.1) of the hom-module \([X :]\), and the right generalized Yoneda functor \([X \times \mathcal{A}] : [\mathcal{A}, X] \to [\mathcal{A} : X] \) for a functor category \([\mathcal{A}, X]\) is defined by the right action (see Section 2.2) of the hom-module \([X :]\) on \([\mathcal{A}, X]\). The left Yoneda functor and the left general functor are defined dually.

The right Yoneda functor \([X^*] : X \to [X :]\) yields the representable right module\(^1\) \( X^* : X \to * \) for each object \( x \in X \) (a right module \( \mathcal{M} : X \to * \) is representable if there is an isomorphism \( \mathcal{M} \cong \langle X \rangle x \), representation of \( \mathcal{M} \), for some \( x \in \| X \| \)), and the left Yoneda functor \([^\wedge \mathcal{A}] : \mathcal{A} \to [^*: A] \) yields the representable left module \( \mathcal{A} : * \to \mathcal{A} \) for each object \( a \in \mathcal{A} \). Similarly, the right generalized Yoneda functor \([X \times \mathcal{A}] : [\mathcal{A}, X] \to [\mathcal{A} : X] \) yields the corepresentable module \( \langle X \rangle \mathcal{G} : X \to \mathcal{A} \) for each functor \( \mathcal{G} : \mathcal{A} \to X \), and the left generalized Yoneda functor \([^\wedge \mathcal{A}] : [\mathcal{A}, X] \to [^*: A] \)

\(^1\)A representable (one-sided) module is called a representable functor in many literatures.
yields the representable module $F(X) : X \to A$ for each functor $F : X \to A$ (a module $M : X \to A$ is corepresentable if there is an isomorphism $M \cong \langle X \rangle G$, corepresentation of $M$, for some $G : A \to X$ and representable if it has a representation $M \cong F(X)$ for some $F : X \to A$).

It is shown in Section 5.2 and Section 5.3 that the Yoneda functors and the generalized Yoneda functors are fully faithful. The Yoneda functors are thus frequently called the Yoneda embedding.

### 2.3.1 Definition.

- The right Yoneda functor for a category $X$ is the functor
  \[ [X^\to] : X \to [X :] \]
  given by the right exponential transpose of the hom-module $(X) : X \to X$; in short,
  \[ [X^\to] := [(X)^\to]. \]

- The left Yoneda functor for a category $A$ is the functor
  \[ [\wedge A] : A \to [: A]^\wedge \]
  given by the left exponential transpose of the hom-module $(A) : A \to A$; in short,
  \[ [\wedge A] := [\wedge (A)]. \]

### 2.3.2 Remark.

- The right Yoneda functor $X^\to$ sends each object $x \in \|X\|$ to the right module
  \[ (X)x : X \to \ast \]
  , called the representable right module of $x$, and sends each $X$-arrow $f : x \to a$ to the right module morphism
  \[ (X)f : (X)x \to (X)a : X \to \ast \]
  which maps each $X$-arrow $h : x' \to x$ to the $X$-arrow $h \circ f : x' \to a$ as indicated in
  \[
  \begin{array}{c}
  x' \\
  \downarrow h
  \end{array} \quad \begin{array}{c}
  x \\
  \downarrow f
  \end{array} \quad \begin{array}{c}
  \downarrow h \circ f
  \end{array} \quad \begin{array}{c}
  a
  \end{array}
  \]
  (cf. Remark 2.1.2(1)).

- The left Yoneda functor $\wedge A$ sends each object $a \in \|A\|$ to the left module
  \[ a(A) : \ast \to A \]
  , called the representable left module of $a$, and sends each $A$-arrow $f : x \to a$ to the left module morphism
  \[ f(A) : a(A) \to x(A) : \ast \to A \]
  which maps each $A$-arrow $h : a \to a'$ to the $A$-arrow $f \circ h : x \to a'$ as indicated in
  \[
  \begin{array}{c}
  a \\
  \downarrow f
  \end{array} \quad \begin{array}{c}
  a'
  \end{array} \quad \begin{array}{c}
  \downarrow f(A) \circ h
  \end{array}
  \]
  (cf. Remark 2.1.2(1)).

### 2.3.3 Definition.

- A representation of a right module $M : X \to \ast$ is a pair $(r, \phi)$ consisting of an object $r \in \|X\|$, “representing object”, and a right module isomorphism $\phi : M \cong \langle X \rangle r$. A right module is called representable if it has a representation.

- A representation\(^2\) of a left module $M : \ast \to A$ is a pair $(r, \phi)$ consisting of an object $r \in \|A\|$, “representing object”, and a left module isomorphism $\phi : M \cong r(A)$. A left module is called representable if it has a representation.

\(^2\)Some use the term “corepresentation” instead of “representation” for left modules. We followed [Rie16] (Definition 2.1.4).
2.3.4 Remark.
(1) A right module \( \mathcal{M} : \mathbf{X} \to * \) is representable if it is isomorphic to ‘the’ representable right module \( \mathcal{X} \mathbf{r} \) for some object \( \mathbf{r} \in \| \mathbf{X} \| \), and a left module \( \mathcal{M} : * \to \mathbf{A} \) is representable if it is isomorphic to ‘the’ representable left module \( \mathbf{r} \mathcal{A} \) for some object \( \mathbf{r} \in \| \mathbf{A} \| \).

(2) A representation \( (\mathbf{r}, \phi) \) of a right module \( \mathcal{M} : \mathbf{X} \to * \) [op. left module \( \mathcal{M} : * \to \mathbf{A} \)] is expressed by a fully faithful conic cell \( \mathbf{X} \xrightarrow{\phi} \mathbf{M} \xrightarrow{\mathbf{r}} \mathbf{A} \). \( \mathbf{X} \xrightarrow{i} \mathbf{M} \xrightarrow{\phi} \mathbf{X} \)  
\[ \mathbf{A} \xrightarrow{\mathbf{r}} \mathbf{M} \xrightarrow{\phi} \mathbf{A} \]

2.3.5 Definition.
- For a category \( \mathbf{X} \), \( \text{Rep}[\mathbf{X}] \) denotes the full subcategory of the category \( [\mathbf{X}] \) whose objects are representable right modules over \( \mathbf{X} \).
- For a category \( \mathbf{A} \), \( \text{Rep}[: \mathbf{A}] \) denotes the full subcategory of the category \( [: \mathbf{A}] \) whose objects are representable left modules over \( \mathbf{A} \).

2.3.6 Proposition.
- The image of \( \mathbf{X} \) under the right Yoneda functor \( [\mathbf{X}] \mathcal{X} : \mathbf{X} \to [\mathbf{X}] \) is isomorphism-dense in \( \text{Rep}[\mathbf{X}] \).
- The image of \( \mathbf{A}^{-} \) under the left Yoneda functor \( [\mathbf{A}] \mathcal{A} : \mathbf{A}^{-} \to [: \mathbf{A}] \) is isomorphism-dense in \( \text{Rep}[: \mathbf{A}] \).

Proof. Immediate from the definition of representable modules (Definition 2.3.3).

2.3.7 Definition. For a functor \( \mathcal{K} : \mathbf{D} \to \mathbf{E} \), the module
\( \mathcal{E} \mathcal{K} : \mathbf{E} \to \mathbf{D} \) \( \mathcal{K} \mathcal{E} : \mathbf{D} \to \mathbf{E} \)
given by the composition
\[ \mathbf{E} \xrightarrow{(\mathcal{E})} \mathbf{E} \xrightarrow{\mathcal{K}} \mathbf{D} \]  
\[ \mathbf{D} \xrightarrow{\mathcal{K}^{-1}} \mathbf{E} \xrightarrow{(\mathcal{E})^{-1}} \mathbf{E} \]
with the hom-module of \( \mathbf{E} \) is called the corepresentable [op. representable] module of \( \mathcal{K} \).

2.3.8 Remark. The composition in the above definition is a special case of Example 1.1.31(12):
- The corepresentable module \( \mathcal{K} \mathcal{E} : \mathbf{E} \to \mathbf{D} \) is defined by
\[ e \in \mathcal{K} \mathcal{E} \mathcal{K} \mathcal{E} d = e \mathcal{E} \mathcal{E} \mathcal{K} \mathcal{E} d \]
for \( e \in \mathbf{E} \) and \( d \in \mathbf{D} \).
- The representable module \( \mathcal{K} \mathcal{E} : \mathbf{D} \to \mathbf{E} \) is defined by
\[ d \mathcal{K} \mathcal{E} \mathcal{E} = \mathcal{E} \mathcal{E} \mathcal{K} \mathcal{E} d \]
for \( e \in \mathbf{E} \) and \( d \in \mathbf{D} \).
2.3.9 Proposition. Given a functor \( K : D \to E \), the right \([\text{op. left}]\) exponential transpose
\[
[(\langle E \rangle K)^\triangleright] : D \to [E:] \quad \text{op.} \quad [(\mathcal{K}(E))] : D \to [:E]^\triangleright
\]
of the corepresentable \([\text{op. representable}]\) module of \( K \) is given by the composition

\[
[E:] \xrightarrow{E^\triangleright} E \xrightarrow{K} D \quad \text{op.} \quad D \xrightarrow{K} E \xrightarrow{\mathcal{E}^\triangleright} [:E]^\triangleright
\]
of \( K \) and the right \([\text{op. left}]\) Yoneda functor for \( E \); that is,
\[
[(\langle E \rangle K)^\triangleright] = [E^\triangleright] \circ K \quad \text{op.} \quad [(\mathcal{K}(E))] = K \circ [\mathcal{E}. E].
\]

Proof. This is an instance of Proposition 2.1.6 where \( M \) is given by the hom-module of \( E \). \( \square \)

2.3.10 Definition. Let \( X \) and \( A \) be categories.

- The right generalized Yoneda functor for the functor category \([A, X] \) is the functor
  \[
  [X^\triangleright A] : [A, X] \to [X : A]
  \]
given by the right action of the hom-module \( \langle X \rangle : X \to X \) on the functor category \([A, X] \); in short,
  \[
  [X^\triangleright A] : = [\langle X \rangle A].
  \]

- The left generalized Yoneda functor for the functor category \([X, A] \) is the functor
  \[
  [X\mathcal{K} A] : [X, A] \to [X : A]^\triangleright
  \]
given by the left action of the hom-module \( \langle A \rangle : A \to A \) on the functor category \([X, A] \); in short,
  \[
  [X\mathcal{K} A] : = [X \langle A \rangle].
  \]

2.3.11 Remark.

(1) The right generalized Yoneda functor \( X^\triangleright A \) sends each functor \( G : A \to X \) to the corepresentable module \( \langle X \rangle G : X \to A \) of \( G \), and sends each natural transformation \( \tau : G \to F : A \to X \) to the module morphism \( \langle X \rangle \tau : (X) G \to (X) F \) which maps each \( (X) G \)-arrow \( h : x \sim a \) to the \((X) F\)-arrow \( h \circ \tau_a : x \sim a \) as indicated in

\[
\begin{array}{ccc}
  x & \xrightarrow[\tau_a]{h} & G \cdot a \\
  \downarrow & \Downarrow[\tau_a] & \downarrow \\
  F \cdot a & \xleftarrow[h \circ \tau_a]{\tau_a} & \end{array}
\]

(cf. Remark 2.2.2(1)).

(2) The left generalized Yoneda functor \( X\mathcal{K} A \) sends each functor \( F : X \to A \) to the representable module \( F \langle A \rangle : X \to A \) of \( F \), and sends each natural transformation \( \tau : G \to F : X \to A \) to the module morphism \( \tau F : F \langle A \rangle \to G \langle A \rangle \) which maps each \( F \langle A \rangle\)-arrow \( h : x \sim a \) to the \( G \langle A \rangle\)-arrow \( \tau_a \circ h : x \sim a \) as indicated in

\[
\begin{array}{c}
x \xrightarrow[F \cdot h]{F} \cdot a \\
\xrightarrow[\tau \cdot \tau_a \cdot \circ h]{\tau A \cdot \circ \tau_a \cdot \circ h} \\
x \xrightarrow[\tau \cdot \tau_a \cdot \circ h]{G} \cdot a
\end{array}
\]

(cf. Remark 2.2.2(1)).

2.3.12 Proposition. Given categories \( E, X, \) and \( A \), the diagram

\[
\begin{array}{ccc}
  X & \xrightarrow{[E, X]} & [E, X] \\
  \downarrow{X^\triangleright} & \quad \downarrow{X^\triangleright E} & \quad \downarrow{X^\triangleright E} \\
  [X] & \xrightarrow{[X, E]} & [X : E] \\
  \end{array} \quad \text{op.} \quad \begin{array}{ccc}
  A & \xrightarrow{[E, A]} & [E, A] \\
  \downarrow{\mathcal{K} A} & \quad \downarrow{\mathcal{K} A} & \quad \downarrow{\mathcal{K} A} \\
  \end{array}
\]

\[
\begin{array}{cc}
  \xrightarrow{\mathcal{E} \cdot \circ \mathcal{K} A} & \xrightarrow{\mathcal{E} \cdot \circ \mathcal{K} A} \\
  \xrightarrow{\mathcal{E} \cdot \circ \mathcal{K} A} & \xrightarrow{\mathcal{E} \cdot \circ \mathcal{K} A} \\
  \end{array}
\]

commutes.

Proof. This is a special case of Proposition 2.2.4 where $\mathcal{M}$ is given by the hom-module of $X$ [op. $A$].

2.3.13 Definition. Let $\mathcal{M} : X \to A$ be a module.

- A corepresentation of $\mathcal{M}$ is a pair $(R, \phi)$ consisting of a functor $R : A \to X$, “corepresenting functor”, and a module isomorphism $\phi : \mathcal{M} \cong (X)R$. A module is called corepresentable if it has a corepresentation.
- A representation of $\mathcal{M}$ is a pair $(R, \phi)$ consisting of a functor $R : X \to A$, “representing functor”, and a module isomorphism $\phi : \mathcal{M} \cong R(A)$. A module is called representable if it has a representation.

2.3.14 Remark.

1. A module $\mathcal{M} : X \to A$ is corepresentable if it is isomorphic to “the” corepresentable module $(X)R$ for some functor $R : A \to X$, and is representable if it is isomorphic to “the” representable module $R(A)$ for some functor $R : X \to A$.
2. A representation of a right module $\mathcal{M} : X \to \ast$ is identified with a corepresentation of a two-sided module $\mathcal{M} : X \to A$ where $A$ is the terminal category, and a representation of a left module $\mathcal{M} : \ast \to A$ is identified with a representation of a two-sided module $\mathcal{M} : X \to A$ where $X$ is the terminal category.
3. A corepresentation [op. representation] $(R, \phi)$ of a module $\mathcal{M} : X \to A$ is expressed by a fully faithful cell

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi} & A
\end{array}
\]

op.

4. Example 2.3.17 shows that not all modules are representable.

2.3.15 Proposition. Let $\mathcal{M} : X \to A$ be a module.

- A functor $R : A \to X$ and a module morphism $\phi : \mathcal{M} \to (X)R : X \to A$ form a corepresentation of $\mathcal{M}$ if and only if for every $a \in \|A\|$ the object $R : a$ and the right module morphism $(\phi)a : (\mathcal{M})a \to (X)(R : a) : X \to \ast$ form a representation of the right module $(X)a : X \to \ast$.
- A functor $R : X \to A$ and a module morphism $\phi : \mathcal{M} \to R(A) : X \to A$ form a representation of $\mathcal{M}$ if and only if for every $x \in \|X\|$ the object $x : R$ and the left module morphism $x(\phi) : x(\mathcal{M}) \to (x : R)(A) : \ast \to A$ form a representation of the left module $x(\mathcal{M}) : \ast \to A$.

Proof. By Proposition 2.1.3, $\phi : \mathcal{M} \to (X)R$ is iso iff

\[
(\phi)a : (\mathcal{M})a \to ((X)R)a = (X)(R : a)
\]
is iso for every $a \in \|A\|$.

2.3.16 Proposition. Let $\mathcal{M} : X \to A$ be a module.

- If $\mathcal{M}$ is corepresented by a functor $R : A \to X$, then the composition

\[
\begin{array}{ccc}
X & \xrightarrow{R} & A \\
\downarrow & & \downarrow \\
\mathcal{M} & \xleftarrow{\phi} & E
\end{array}
\]

with any functor $F : E \to A$ yields a corepresentation of the composite module $(\mathcal{M})F : X \to E$ by the composite functor $F \circ R : E \to X$.
- If $\mathcal{M}$ is represented by a functor $R : X \to A$, then the composition

\[
\begin{array}{ccc}
E & \xrightarrow{G} & X \\
\downarrow & & \downarrow \\
\mathcal{M} & \xleftarrow{R} & A
\end{array}
\]

with any functor $G : E \to X$ yields a representation of the composite module $G(\mathcal{M}) : E \to A$ by the composite functor $G \circ R : E \to A$. 
2.3. Yoneda functors and representations

Proof. Recall from Remark 2.3.14(3) that a corepresentation is expressed by a fully faithful cell. Now since the pasting composition

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & A \\
\downarrow & & \downarrow E \\
\xrightarrow{\phi} X & \xleftarrow{\mathcal{R}} & X \\
\end{array}
\]

yields

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}} & E \\
\downarrow & & \downarrow E \\
\xrightarrow{(\phi)\mathcal{R}} X & \xleftarrow{\mathcal{R}} & X \\
\end{array}
\]

, the assertion follows from Proposition 1.2.35.

2.3.17 Example.

(1) Let \(X\) and \(A\) be discrete categories. A correspondence \(\mathcal{R}\) from \([X]\) to \([A]\), i.e. a subset of \([X] \times [A]\), defines a module \(\mathcal{R} : X \rightarrow A\) by

\[
\begin{align*}
x(R)a = \begin{cases} 
\{0\} & \text{if } (x,a) \in \mathcal{R} \\
\emptyset & \text{otherwise.}
\end{cases}
\end{align*}
\]

\(\mathcal{R}\) is representable [op. corepresentable] if and only if \(\mathcal{R}\) is a function \([X] \rightarrow [A]\) [op. \([A] \rightarrow [X]\)]; \(\mathcal{R}\) is both corepresentable and representable if and only if \(\mathcal{R}\) is a one-to-one correspondence.

(2) Let \(X\) and \(A\) be thin categories, i.e. preordered sets. A correspondence \(\mathcal{R}\) from \([X]\) to \([A]\) defines a module \(\mathcal{R} : X \rightarrow A\) if and only if it satisfies the following conditions:

a) \((x,a) \in \mathcal{R}\) and \(a \leq b\) implies that \((x,b) \in \mathcal{R}\);

b) \((x,a) \in \mathcal{R}\) and \(y \leq x\) implies that \((y,a) \in \mathcal{R}\).

When this is the case,

- \(\mathcal{R}\) is corepresentable if and only if for every \(a \in [A]\) the set \(\{x \in [X] | (x,a) \in \mathcal{R}\}\) has a maximum element.

- \(\mathcal{R}\) is representable if and only if for every \(x \in [X]\) the set \(\{a \in [A] | (x,a) \in \mathcal{R}\}\) has a minimum element.

If \(\mathcal{R}\) is both corepresentable and representable, then a corepresenting functor \(G : A \rightarrow X\) and a representing functor \(F : X \rightarrow A\) form a Galois connection between \(X\) and \(A\), with

\[x \leq G \cdot a \Leftrightarrow (x,a) \in \mathcal{R} \Leftrightarrow x \cdot F \leq a.\]
3 Collages and Commas

3.1 Collages

This section introduces a collage, which is an alternative presentation of the concept of module. For a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \), a collage \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) is defined by a collage category \([\mathcal{M}]\) and two inclusions \( \mathbf{X} \xrightarrow{M_0} [\mathcal{M}] \xleftarrow{M_1} \mathbf{A} \). Collages \( \mathbf{X} \to \mathbf{A} \) and collage morphisms among them form the category \([\mathbf{X} \uparrow \mathbf{A}]\), and all collages and collage cells form the category \( \text{CLG} \).

We construct a module from a collage and a collage from a module (the arrows of a module \( \mathcal{M} \) become the arrows of the collage category \( [\mathcal{M}] \)), and establish the isomorphisms \([\mathbf{X} \uparrow \mathbf{A}] \cong [\mathbf{X} : \mathbf{A}]\) and \( \text{CLG} \cong \text{MOD} \); by these isomorphisms, we often identify a module with its corresponding collage and vice versa. At the end of the section, we show that the forgetful functor \( \mathcal{M} \mapsto [\mathcal{M}] : \text{MOD} \to \text{CAT} \) given by the collage category assignment is left adjoint to the embedding \( \mathbf{C} \mapsto (\mathbf{C}) : \text{CAT} \to \text{MOD} \) given by the hom-module assignment.

3.1.1 Definition. A (two-sided) collage \( \mathcal{M} \) from a category \( \mathbf{X} \) to a category \( \mathbf{A} \), written \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \), is defined by a category \([\mathcal{M}]\), “collage category”, satisfying the following conditions:

1. The inclusion of the coproduct category \( \mathbf{X} + \mathbf{A} \) is a wide full subcategory of \([\mathcal{M}]\);
2. \( a([\mathcal{M}])x = \emptyset \) if \( x \in \|X\| \) and \( a \in \|A\| \).

3.1.2 Remark.

1. The inclusion of the coproduct category \( \mathbf{X} + \mathbf{A} \) into the collage category \([\mathcal{M}]\) is denoted by
   \[ \mathcal{M} : \mathbf{X} + \mathbf{A} \to [\mathcal{M}] \]
   or by
   \[ \mathbf{X} \xrightarrow{M_0} [\mathcal{M}] \xleftarrow{M_1} \mathbf{A} \].

2. A right [op. left] collage is defined as a special case of a two-sided collage:
   - a right collage over a category \( \mathbf{X} \), written \( \mathcal{M} : \mathbf{X} \to * \), is defined as a collage from \( \mathbf{X} \) to the terminal category; the inclusion of \( \mathbf{X} \) into the collage category \([\mathcal{M}]\) is denoted by
     \[ \mathcal{M} : \mathbf{X} \to [\mathcal{M}] \].
   - a left collage over a category \( \mathbf{A} \), written \( \mathcal{M} : * \to \mathbf{A} \), is defined as a collage from the terminal category to \( \mathbf{A} \); the inclusion of \( \mathbf{A} \) into the collage category \([\mathcal{M}]\) is denoted by
     \[ \mathcal{M} : \mathbf{A} \to [\mathcal{M}] \].

3. A collage \( \mathcal{M} \) is called small (resp. locally small) if the collage category \([\mathcal{M}]\) is small (resp. locally small).

4. Any collage \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) defines the unique functor \( !_{\mathcal{M}} \) from the collage category \([\mathcal{M}]\) to the interval category \( \mathbf{2} \) making the diagram
   \[
   \begin{array}{ccc}
   \mathbf{X} & \xrightarrow{M_0} & [\mathcal{M}] & \xrightarrow{M_1} & \mathbf{A} \\
   \downarrow{!_{\mathcal{M}}} & & \downarrow{!_{\mathcal{M}}} & & \downarrow{!_{\mathcal{A}}} \\
   1 & \to & 2 & \leftarrow & 1
   \end{array}
   \]
   commute. Conversely, any functor \( H : \mathbf{C} \to \mathbf{2} \) defines a unique collage \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) with \([\mathcal{M}] = \mathbf{C}\) and \([!_{\mathcal{M}}] = H\).

3.1.3 Definition. Given a pair of collages \( \mathcal{M}, \mathcal{N} : \mathbf{X} \to \mathbf{A} \), a morphism \( \psi \) from \( \mathcal{M} \) to \( \mathcal{N} \), written \( \psi : \mathcal{M} \to \mathcal{N} : \mathbf{X} \to \mathbf{A} \), is defined by a functor \( [\psi] : [\mathcal{M}] \to [\mathcal{N}] \), “collage functor”, satisfying the following equivalent conditions:

1. \([\psi]\) is identity on \( \mathbf{X} + \mathbf{A} \);
3.1. Collages

(2) the triangle

\[ \begin{array}{c}
\mathcal{M} \\
\downarrow_{\psi} \\
\mathcal{N}
\end{array} \]

\[ \mathbf{X} + \mathbf{A} \]

commutes;

(3) the two triangles

\[ \begin{array}{c}
\mathcal{M}_0 \\
\downarrow_{\psi} \\
\mathcal{N}_0
\end{array} \]

\[ \begin{array}{c}
\mathcal{M}_1 \\
\downarrow_{\psi} \\
\mathcal{N}_1
\end{array} \]

\[ \mathbf{X} \quad \mathbf{A} \]

commute;

(4) the triangle

\[ \begin{array}{c}
\mathcal{M} \\
\downarrow_{\psi} \\
\mathcal{N}
\end{array} \]

\[ \mathbf{A} \quad \mathbf{X} \]

commutes.

3.1.4 Remark. Given a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \), all locally small collages \( \mathbf{X} \to \mathbf{A} \) and morphisms among them define the category \( [\mathbf{X} \uparrow \mathbf{A}] \) with the obvious identities and the composition. Indeed, noting the condition (2), we see that \( [\mathbf{X} \uparrow \mathbf{A}] \) is fully embedded in the coslice category under \( \mathbf{X} + \mathbf{A} \). The category \( [\mathbf{X} \uparrow] \) of right collages over \( \mathbf{X} \) and the category \( [\uparrow \mathbf{A}] \) of left collages over a category \( \mathbf{A} \) are defined by

\[ [\mathbf{X} \uparrow] = [\mathbf{X} \uparrow \ast] \quad [\uparrow \mathbf{A}] = [\ast \uparrow \mathbf{A}] \]

(recall Remark 3.1.2(2)).

3.1.5 Definition. Given a pair of collages \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) and \( \mathcal{N} : \mathbf{Y} \to \mathbf{B} \), and given a pair of functors \( \mathcal{P} : \mathbf{X} \to \mathbf{Y} \) and \( \mathcal{Q} : \mathbf{A} \to \mathbf{B} \), a collage cell \( \psi : \mathcal{P} \Rightarrow \mathcal{Q} : \mathcal{M} \to \mathcal{N} \) is defined by a functor \( \psi : [\mathcal{M}] \to [\mathcal{N}] \), "collage functor", satisfying the following equivalent conditions:

(1) \( [\psi] \) is identical to \( \mathcal{P} + \mathcal{Q} \) on \( \mathbf{X} + \mathbf{A} \);

(2) the diagram

\[ \begin{array}{c}
\mathcal{M}_0 \\
\downarrow \psi \\
\mathcal{N}_0
\end{array} \]

\[ \begin{array}{c}
\mathcal{M}_1 \\
\downarrow \psi \\
\mathcal{N}_1
\end{array} \]

\[ \mathbf{X} \quad \mathbf{A} \]

commutes;

(3) the triangle

\[ \begin{array}{c}
[\mathcal{M}] \\
\downarrow \psi \\
[\mathcal{N}]
\end{array} \]

\[ \mathbf{A} \quad \mathbf{X} \]

commutes.

3.1.6 Remark.

(1) All locally small collages and cells among them define the category \( \text{CLG} \) with the obvious identities and the composition. Indeed, there is an obvious isomorphism between \( \text{CLG} \) and the slice category \( \text{CAT}/\mathbf{2} \).

(2) There is an obvious forgetful functor \( [-] : \text{CLG} \to \text{CAT} \), which sends each collage \( \mathcal{M} \) to the collage category \( [\mathcal{M}] \) and sends each collage cell \( \psi \) to the collage functor \( [\psi] \).

(3) Given a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \), there is a canonical embedding \( [\mathbf{X} \uparrow \mathbf{A}] \to \text{CLG} \), identical on objects, defined by the arrow function \( \psi \mapsto (\psi : 1_{\mathbf{X}} \to 1_{\mathbf{A}}) \). The embedding is not, in general, full.
3.1.7 Definition. Given a parallel pair of collage cells $\psi : M \to N$ and $\varphi : M \to N$, a morphism from $\psi$ to $\varphi$, written $\tau : \psi \to \varphi : M \to N$, is defined by a natural transformation $[\tau] : \psi \to \varphi : [M] \to [N]$, “collage natural transformation”.

3.1.8 Remark. Given a pair of collages $M$ and $N$, all collage cells $M \to N$ and morphisms among them define the category $[M \uparrow N]$ with the obvious identities and the composition. Indeed, the assignment $\tau \mapsto [\tau]$ fully embeds $[M \uparrow N]$ in the functor category $[[M], [N]]$.

Note. In what follows we show a one-to-one correspondence between modules and collages. In Definition 3.1.9 and Definition 3.1.12, a module and a collage corresponding to each other are given the same name.

3.1.9 Definition.

(1) Given a module $M : X \to A$, the corresponding collage is defined by the collage category $[M]$ given in the following way:
   a) the objects of $[M]$ consist of all objects of the coproduct category $X + A$;
   b) the arrows of $[M]$ consist of all $X$-arrows, $A$-arrows and $M$-arrows:
      - $x([M]) y = x(X) y$ for $x, y \in |X|$;
      - $a([M]) b = a(A) b$ for $a, b \in |A|$;
      - $x([M]) a = x(M) a$ for $x \in |X|$ and $a \in |A|$;
      - $a([M]) x = \emptyset$ for $a \in |A|$ and $x \in |X|$.
   c) the composition law of $[M]$ is those of $X$, $A$, and $M$ (see Definition 1.1.19).

(2) Given a module morphism $\psi : M \to N : X \to A$, the corresponding collage morphism is defined by the collage functor $[\psi] : [M] \to [N]$ given in the following way:
   a) $[\psi]$ is the identity on $X + A$;
   b) $x([\psi]) a = x(\psi) a$ for $x \in |X|$ and $a \in |A|$.

(3) Given a module cell
   
   \[
   \begin{array}{c}
   X \\
   \psi \\
   \downarrow Q \\
   Y
   \end{array}
   \xrightarrow{\mathcal{M}}
   \begin{array}{c}
   A \\
   \downarrow Q \\
   B
   \end{array}
   \]

   , the corresponding collage cell
   
   \[
   \begin{array}{c}
   X \\
   \mathcal{M}_0 \\
   \downarrow \psi \\
   Y \\
   \mathcal{N}_0
   \end{array}
   \xrightarrow{\mathcal{M}_1} \begin{array}{c}
   A \\
   \mathcal{N}_1
   \end{array}
   \]

   is defined by the collage functor $[\psi] : [M] \to [N]$ given in the following way:
   a) $[\psi]$ is identical to $P + Q$ on $X + Y$;
   b) $x([\psi]) a = x(\psi) a$ for $x \in |X|$ and $a \in |A|$.

(4) Given a module cell morphism $\tau : \psi \to \varphi : M \to N$, the corresponding collage cell morphism is defined by the collage natural transformation $[\tau] : [\psi] \to [\varphi] : [M] \to [N]$ given by:
   - $[\tau]_x = [\tau_0]_x$ for $x \in |X|$;
   - $[\tau]_a = [\tau_1]_a$ for $a \in |A|$.

3.1.10 Remark.

(1) For a module morphism $\psi : M \to N$, the functoriality of $[\psi] : [M] \to [N]$ follows from the naturality of $\psi$ (see Remark 1.1.20(3)), and for a module cell $\psi : M \to N$, the functoriality of $[\psi] : [M] \to [N]$ follows from Proposition 1.2.3.

(2) For a right module $M : X \to \ast$, the corresponding right collage (see Remark 3.1.2(2)) is constructed as above by identifying $M$ with a two-sided module from $X$ to the terminal category. Dually, for a left module $M : \ast \to A$, the corresponding left collage is constructed by identifying $M$ with a two-sided module from the terminal category to $A$. 
3.1.11 Proposition. The module-to-collage correspondence given in Definition 3.1.9 is functorial and defines the following functors:

1. \([X : A] \downarrow \downarrow [X : A]\) for categories \(X\) and \(A\);
2. \(\text{MOD} \downarrow \downarrow \text{CLG}\);
3. \([M : N] \downarrow \downarrow [M : N]\) for modules \(M\) and \(N\).

\(\square\)

3.1.12 Definition. 

1. Given a collage \(M : X \rightarrow A\), the corresponding module is defined by the composition

\[
X \xrightarrow{M_0} [M] \xrightarrow{\{M\}} [M] \xrightarrow{M_1} A
\]

, where \(\{M\}\) is the hom-module of the collage category of \(M\).

2. Given a collage morphism \(\psi : M \rightarrow N : X \rightarrow A\), the corresponding module morphism is defined by the pasting composition

\[
\begin{array}{c}
\xymatrix{X \ar[r]^{M_0} \ar[d]_{\psi} & [M] \ar[r]^{\{M\}} & [M] \ar[r]^{M_1} & A \ar[d]_{\{\psi\}} \\
N \ar[r]_{N_0} & [N] \ar[r]_{\{N\}} & [N] \ar[r]_{N_1} & B
} \end{array}
\]

(see Remark 1.2.34), where \(\{\psi\}\) is the hom-cell of the collage functor of \(\psi\).

3. Given a collage cell

\[
\begin{array}{c}
\xymatrix{X \ar[r]^{M_0} \ar[d]_{\psi} & [M] \ar[r]^{M_1} & A \ar[d]_{\{\psi\}} \\
Y \ar[r]_{N_0} & [N] \ar[r]_{N_1} & B
} \end{array}
\]

, the corresponding module cell

\[
\begin{array}{c}
\xymatrix{X \ar[r]^{M} & A \ar[d]_{\{\psi\}} \\
Y \ar[r]_{\psi} & B
} \end{array}
\]

is defined by the pasting composition

\[
\begin{array}{c}
\xymatrix{X \ar[r]^{M_0} \ar[d]_{\psi} & [M] \ar[r]^{\{M\}} & [M] \ar[r]^{M_1} & A \ar[d]_{\{\psi\}} \\
Y \ar[r]_{N_0} & [N] \ar[r]_{\{N\}} & [N] \ar[r]_{N_1} & B
} \end{array}
\]

, where \(\{\psi\}\) is the hom-cell of the collage functor of \(\psi\).

4. Given a collage cell morphism \(\tau : \psi \rightarrow \varphi : M \rightarrow N\), the corresponding module cell morphism is defined by the pair of natural transformations

\[
\tau_0 : \psi_0 \rightarrow \varphi_0 \quad \text{and} \quad \tau_1 : \psi_1 \rightarrow \varphi_1
\]

, the restrictions of the collage natural transformation \([\tau] : [\psi] \rightarrow [\varphi] : [M] \rightarrow [N]\) to the domain and the codomain of the collage \(M\).

3.1.13 Proposition. The collage-to-module correspondence given in Definition 3.1.12 is functorial and defines the following functors:

1. \([X : A] \uparrow \uparrow [X : A]\) for categories \(X\) and \(A\);
2. \(\text{CLG} \uparrow \uparrow \text{MOD}\);
3. \([M : N] \uparrow \uparrow [M : N]\) for modules \(M\) and \(N\).

\(\square\)

3.1.14 Theorem. The corresponding functors

\[
\begin{array}{ccc}
[X : A] & \overset{\downarrow h}{\rightarrow} & [X : A] \\
\text{MOD} & \overset{\downarrow h}{\rightarrow} & \text{CLG} \\
[M : N] & \overset{\downarrow h}{\rightarrow} & [M : N]
\end{array}
\]
in Proposition 3.1.11 and Proposition 3.1.13 are isomorphisms inverse to each other.

Proof. Easily verified. □

3.1.15 Remark.
(1) As noted earlier, a module and a collage corresponding to each other are given the same name and freely identified with each other.
(2) A module \( \mathcal{M} : X \rightarrow A \) is recovered from its collage by the composition

\[
X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\psi} [\mathcal{M}] \xleftarrow{\mathcal{M}_1} A
\]

that is,

\[
\mathcal{M} = \mathcal{M}_0([\mathcal{M}]) \mathcal{M}_1
\]

this identity yields a cell

\[
\begin{array}{c}
X \xrightarrow{\mathcal{M}} A \\
\mathcal{M}_0 \downarrow \quad \mathcal{M}_1 \\
[\mathcal{M}] \xrightarrow{[\mathcal{M}]} [\mathcal{M}]
\end{array}
\]

called the unit cell of \( \mathcal{M} \).
(3) The composition of the isomorphism \( \text{MOD} \xrightarrow{\perp} \text{CLG} \) and the forgetful functor \( [-] : \text{CLG} \rightarrow \text{CAT} \) in Remark 3.1.6(2) yields the forgetful functor \( [-] : \text{MOD} \rightarrow \text{CAT} \), which sends each module \( \mathcal{M} \) to the collage category \( [\mathcal{M}] \) and sends each cell \( \psi \) to the collage functor \( [\psi] \).
(4) The composition of the isomorphism \( [\mathcal{M} : \mathcal{N}] \xrightarrow{\perp} [\mathcal{M} \uparrow \mathcal{N}] \) and the full embedding of \( [\mathcal{M} \uparrow \mathcal{N}] \) into the functor category \( [([\mathcal{M}],[\mathcal{N}])] \) described in Remark 3.1.8 yields the full embedding \( [-] : [\mathcal{M} : \mathcal{N}] \rightarrow [([\mathcal{M}],[\mathcal{N}])] \), which sends each cell \( \psi : \mathcal{M} \rightarrow \mathcal{N} \) to the collage functor \( [\psi] : [\mathcal{M}] \rightarrow [\mathcal{N}] \) and sends each cell morphism \( \tau : \psi \rightarrow \phi : \mathcal{M} \rightarrow \mathcal{N} \) to the collage natural transformation \( [\tau] : [\psi] \rightarrow [\phi] : [\mathcal{M}] \rightarrow [\mathcal{N}] \).

3.1.16 Theorem. There is a canonical adjunction between the forgetful functor \( [-] : \text{MOD} \rightarrow \text{CAT} \) (see Remark 3.1.15(3)) and the embedding \( [-] : \text{CAT} \rightarrow \text{MOD} \) (see Theorem 1.2.32), with the unit given by the family of unit cells \( 1_{\mathcal{M}} : \mathcal{M} \rightarrow ([\mathcal{M}]) \). Specifically, for each module \( \mathcal{M} : X \rightarrow A \) and each category \( E \),
(1) the adjunct of a cell

\[
\begin{array}{ccc}
\text{X} & \xrightarrow{\mathcal{M}} & A \\
\downarrow G \downarrow \psi & \quad & \downarrow F \\
\text{E} & \xrightarrow{[\mathcal{M}] TH} & [\mathcal{M}] \\
\end{array}
\]

is given by the functor \( H : [\mathcal{M}] \rightarrow E \) defined in the following way:

a) \( H \) is identical to \( G \circ F \) on \( X + A \);

b) \( x \{H\} \ a = x \{\psi\} \ a \) for \( x \in [\mathcal{M}] \) and \( a \in [A] \).
(2) the adjunct of a functor \( H : [\mathcal{M}] \rightarrow E \) is defined by the composition

\[
\begin{array}{c}
\text{X} \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{[\mathcal{M}]} [\mathcal{M}] \\
\downarrow H \downarrow \psi \quad \downarrow \mathcal{M}_1 \\
[\mathcal{M}] \xrightarrow{[\mathcal{M}]} [\mathcal{M}] \\
\text{E} \xrightarrow{H \downarrow \mathcal{M}} [\mathcal{M}] \\
\end{array}
\]

of the unit cell of \( \mathcal{M} \) and the hom-cell of \( H \).

Proof. It is easily verified that the correspondences are inverse to each other. It remains to prove that the family of the unit cells satisfies the naturality condition. For this we need to see that, given
3.1. Collages

Given a module \( M : X \to A \), the right \([\text{op. left}]\) collage envelope of \( M \) is the module

\[
\langle X \uparrow M \rangle : X \to [M] \quad \text{op.} \quad \langle M \downarrow A \rangle : [M] \to A
\]
given by the composition

\[
X \xrightarrow{M_0} [M] \xrightarrow{\langle M \rangle} [M] \quad \text{op.} \quad [M] \xrightarrow{\langle M \rangle} [M] \xrightarrow{M_1} A
\]

; that is, the right \([\text{op. left}]\) collage envelope of \( M \) is the representable \([\text{op. corepresentable}]\) module of the inclusion \( M_0 \) \([\text{op. } M_1] \); in short,

\[
\langle X \uparrow M \rangle := M_0 \langle [M] \rangle \quad \text{op.} \quad \langle M \downarrow A \rangle := \langle [M] \rangle M_1.
\]

3.1.19 Remark. The functor \([-] : \text{MOD} \to \text{CAT}\) is thus a left adjoint of the embedding \( \langle - \rangle : \text{CAT} \to \text{MOD} \).

Note. We use the following expansion of a module in Chapter 5 and Chapter 7.

3.1.18 Definition. Given a module \( \mathcal{M} : X \to A \), the right \([\text{op. left}]\) collage envelope of \( \mathcal{M} \) is

\[
\langle (X \uparrow \mathcal{M}) \rangle : X \to [\mathcal{M}] \quad \text{op.} \quad \langle \mathcal{M} \downarrow A \rangle : [\mathcal{M}] \to A
\]
given by the composition

\[
X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\langle [\mathcal{M}] \rangle} [\mathcal{M}] \quad \text{op.} \quad [\mathcal{M}] \xrightarrow{\langle [\mathcal{M}] \rangle} [\mathcal{M}] \xrightarrow{\mathcal{M}_1} A
\]

, i.e.

\[
X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xrightarrow{\langle [\mathcal{M}] \rangle} [\mathcal{M}] \xrightarrow{\mathcal{M}_1} A
\]

, yields \( \mathcal{M} \) (see Remark 3.1.15(2)). We thus have a cell (cf. Remark 1.2.2(5)),

\[
X \xrightarrow{\mathcal{M}} A \quad \text{op.} \quad X \xrightarrow{\mathcal{M}} A
\]

which embeds a module \( \mathcal{M} : X \to A \) in its right \([\text{op. left}]\) collage envelope.

(2) The composition

\[
X \xrightarrow{\langle (X \uparrow \mathcal{M}) \rangle} [\mathcal{M}] \xrightarrow{\mathcal{M}_0} X \quad \text{op.} \quad A \xrightarrow{\mathcal{M}_1} [\mathcal{M}] \xrightarrow{\langle [\mathcal{M}] \rangle} A
\]
, i.e.

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}_0} & \mathcal{M} \oplus \mathcal{M} \\
\downarrow & & \downarrow \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}
\]

, yields the hom-module of \(X\) [op. \(A\)].

### 3.2 Comma fibrations

In this section, we study right and left comma fibrations (or discrete fibrations and opfibrations in the common terminology). The term “comma fibration” is adopted to express the close tie with comma categories; in fact, we define a comma category in Section 3.3 and Section 3.4 as the domain of a comma fibration.

#### 3.2.1 Definition

A functor \(K : E \to D\) is a right [op. left] comma fibration\(^1\) if for every object \(e \in \|E\|\) and every \(D\)-arrow \(h : d \to K \cdot e\) [op. \(h : e \cdot K \to d\)], there is a unique \(E\)-arrow \(h^e\) as in

\[
\begin{array}{ccc}
e' & \xrightarrow{h^e} & e \\
\downarrow & & \downarrow \\
d & \xrightarrow{h} & K \cdot e \\
\end{array}
\]

—the lift of \(h\) at \(e\)—such that \(h^e \cdot K = h\).

#### 3.2.2 Remark

A functor \(K : E \to D\) is a left comma fibration, if its opposite \(K^- : E^- \to D^-\) is a right comma fibration.

#### 3.2.3 Proposition

For a right [op. left] comma fibration \(K : E \to D\), the lifting is functorial in the following sense:

1. for any object \(e \in \|E\|\), the lift of the identity \(e : K \to K \cdot e\) at \(e\) is the identity \(e \to e\);
2. for a composable pair of \(D\)-arrows \(f \to g \to \cdot\) and their lifts as in

\[
\begin{array}{ccc}
s' & \xrightarrow{f^s} & s \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{f} & \bullet \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
the inverse of \(g^t\). To prove (2), let \(k\) and \(h\) be such that \(k \circ g = h \circ g\). We have \(k^* \circ g^t = h^* \circ g^t\) by Proposition 3.2.3 and the uniqueness of the lifting, hence \(k^* = h^*\) by the monicity of \(g^t\), and thus \(k = k^*; K = h^*; K = h\), as required.

3.2.6 Remark. By Proposition 3.2.4, a right \([\text{op. left}]\) comma fibration reflects monomorphisms and epimorphisms.

3.2.7 Proposition. A functor \(K : E \to D\) is a right \([\text{op. left}]\) comma fibration if and only if the slice \([\text{op. coslice}]\) functor
\[
K/e : E/e \to D/(e; K) \quad \text{op.} \quad e\backslash K : e\backslash E \to (e; K)\backslash D
\]
(see Preliminary 0.0.8) is an isomorphism for every \(e \in \|E\|\).

Proof. By the uniqueness of the lifts, \(K/e\) is bijective on objects. \(K/e\) is faithful because \(K\) is (Proposition 3.2.4). To see that \(K/e\) is full, let \(h : s' \to e\) and \(g : s \to e\) be \(E\)-arrows and consider a \(D\)-arrow \(f : s' \to K \to s; K\) making the triangle
\[
\begin{array}{ccc}
s' & \xrightarrow{f} & s; K \\
K & \xrightarrow{h} & e; K \\
\end{array}
\]
commute. Then by Proposition 3.2.3(2), the lift of \(f\) at \(s\) makes the triangle
\[
\begin{array}{ccc}
s' & \xrightarrow{f^*} & s \\
h & \xrightarrow{\text{id}} & e \\
g & \xrightarrow{\text{id}} & g \\
\end{array}
\]
commute.

3.2.8 Proposition. Consider functors \(K\), \(L\), and \(H\) making the triangle
\[
\begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow H & & \downarrow L \\
C & \xrightarrow{L} & D \\
\end{array}
\]
commute. Then,
1. if \(K\) and \(L\) is a right \([\text{op. left}]\) comma fibration, so is \(H\).
2. if \(H\) and \(L\) is a right \([\text{op. left}]\) comma fibration, so is \(K\).

Proof. Assume that \(L\) is a right comma fibration. We need to show that \(K\) is a right comma fibration if and only if so is \(H\). For \(e \in \|E\|\), consider the commutative diagram
\[
\begin{array}{ccc}
E/e & \xrightarrow{K/e} & D/(e; K) \\
\downarrow H/e & & \downarrow L/(e; K) \\
C/(e; H) & & \\
\end{array}
\]
of the induced functors. Since \(L\) is a right comma fibration, \(L/(e; K)\) is an isomorphism by Proposition 3.2.7. Hence \(K/e\) is an isomorphism iff so is \(H/e\) for every \(e \in \|E\|\); that is, \(K\) is a right comma fibration if and only if so is \(H\).

3.2.9 Proposition. Comma fibrations are stable under pullback; that is, if \(K : E \to B\) is a right \([\text{op. left}]\) comma fibration, then its pullback
\[
\begin{array}{ccc}
H \times K & \xrightarrow{\text{pullback}} & E \\
\downarrow H \times K & & \downarrow K \\
C & \xrightarrow{\text{pullback}} & B \\
\end{array}
\]
(cf. Preliminary 0.0.9) along any functor \(H : C \to B\) is again a right \([\text{op. left}]\) comma fibration.

Proof. This follows from the observation that for any \(C\)-arrow \(f : c \to d\) the subcategory of \(H \times K\) above \(f\) is isomorphically mapped by \(!_H \times K\) to the subcategory of \(E\) above the \(B\)-arrow \(f; H\).
3.3. One-sided commas

3.2.10 Definition. The fibre of a right [op. left] comma fibration $\mathbf{K} : \mathbf{E} \to \mathbf{D}$ at $d \in \mathbb{D}$ is the subcategory of $\mathbf{E}$ consisting of all objects $e$ with $e \cdot \mathbf{K} = d$ and all $\mathbf{E}$-arrows $h$ with $h \cdot \mathbf{K} = 1_d$.

3.2.11 Proposition. Each fibre of a right [op. left] comma fibration is a discrete category.

Proof. Immediate from Proposition 3.2.3(1).

3.2.12 Remark. By Proposition 3.2.11, each fibre of a right [op. left] comma $\mathbf{K} : \mathbf{E} \to \mathbf{D}$ at each $d \in \mathbb{D}$ is seen just as the set of objects $e \in \mathbb{E}$ such that $e \cdot \mathbf{K} = d$.

3.2.13 Definition. A right [op. left] comma fibration is called fibre-small if every fibre of it is small.

3.2.14 Remark.

(1) Any isomorphism $\mathbf{K} : \mathbf{E} \to \mathbf{D}$, in particular any identity functor, is a fibre-small right [op. left] comma fibration with each fiber consists of a single object and the identity arrow.

(2) In Proposition 3.2.8,

a) if $\mathbf{K}$ and $\mathbf{L}$ is a fibre-small right [op. left] comma fibration, so is $\mathbf{H}$.

b) if $\mathbf{H}$ and $\mathbf{L}$ is a fibre-small right [op. left] comma fibration, so is $\mathbf{K}$.

(3) In Proposition 3.2.9, if $\mathbf{H}$ and $\mathbf{K}$ are fibre-small right [op. left] comma fibrations, so are $\mathbf{H} \times !_K$ and $!_H \times \mathbf{K}$.

Note. Remark 3.2.14(1) and (2) allow the following definition.

3.2.15 Definition. All locally small categories and fibre-small right [op. left] comma fibrations among them form a category $\mathcal{CFR}$ [op. $\mathcal{CFL}$].

3.2.16 Remark. $\mathcal{CFR}$ [op. $\mathcal{CFL}$] is a wide subcategory of $\mathcal{CAT}$, and closed under pullbacks by Remark 3.2.14(3).

3.3 One-sided commas

In Section 3.2, we saw that locally small categories and right comma fibrations among them constitute the category $\mathcal{CFR}$. Now we define a right comma over a category $\mathbf{X}$ as an object of the slice category of $\mathcal{CFR}$ over $\mathbf{X}$. The slice category $\mathcal{CFR}/\mathbf{X}$ is denoted by $[\mathbf{X} \downarrow]$, and a right comma over $\mathbf{X}$—an object of $[\mathbf{X} \downarrow]$—is written as $\mathbf{K} : \mathbf{X} \to \star$, which consists of a comma category $\mathbb{K}$ and a right comma fibration $\mathbf{K} : \mathbb{K} \to \mathbf{X}$. A comma is another alternative presentation of the concept of module, where arrows of a module are presented as objects of a category, i.e. a comma category. We transform a module into a comma and a comma into a module, and show that the transformation is reversible (up to isomorphism).

3.3.1 Definition.

- A right comma $\mathbf{K}$ over a base category $\mathbf{X}$, written $\mathbf{K} : \mathbf{X} \to \star$, consists of a category $\mathbb{K}$, “comma category”, and a right comma fibration $\mathbf{K} : \mathbb{K} \to \mathbf{X}$.

- A left comma $\mathbf{K}$ over a base category $\mathbf{A}$, written $\mathbf{K} : \star \to \mathbf{A}$, consists of a category $\mathbb{K}$, “comma category”, and a left comma fibration $\mathbf{K} : \mathbb{K} \to \mathbf{A}$.

3.3.2 Remark.

(1) A comma is thus defined by a comma fibration (or a discrete fibration in the common terminology). A comma category is defined as the domain of a comma fibration (i.e. discrete fibration); we shall see that this definition is consistent with the traditional definition of a comma category. Given a comma $\mathbf{K}$, we always use the same letter $\mathbb{K}$ to denote its comma fibration, and by default denote its comma category by $\mathbb{K}$.
3.3. One-sided commas

(2) A left comma over a category $A$ is the same thing as a right comma over the opposite category $A^\circ$.

(3) The fibre of a right comma $K : X \rightarrow \ast$ at $x \in \|X\|$, written as $x(K)$, is the fibre (see Definition 3.2.10) of the comma fibration $K : \|K\| \rightarrow X$ at $x$. The fibre of a left comma $K : \ast \rightarrow A$ at $a \in \|A\|$ is defined dually and written as $(K)a$. By Proposition 3.2.11, each fibre $x(Ka)$ [op. $(K)a$] is seen just as the set of objects $k \in \|K\|$ such that $K(k) = x$ [op. $K(k) = a$].

(4) A right [op. left] comma is called small (resp. locally small) if the base category is small (resp. locally small) and all its fibres are small.

(5) Right and left commas are referred to as one-sided commas to distinguish them from two-sided commas to be introduced in Section 3.4.

3.3.3 Proposition. If a right [op. left] comma is small (resp. locally small), so is its comma category.

Proof. By definition, if a right comma $K : X \rightarrow \ast$ is locally small, the base category $X$ is locally small. Since the comma fibration $K : \|K\| \rightarrow X$ is faithful (Proposition 3.2.4), the local smallness of $X$ implies the local smallness of the comma category $\|K\|$. If a right comma $K : X \rightarrow \ast$ is small, then $X$ is small and each fibre of $K : \|K\| \rightarrow X$ is small; hence so is $\|K\|$.

3.3.4 Definition. Given a pair of right commas $K,L : X \rightarrow \ast$ [op. left commas $K,L : \ast \rightarrow A$], a morphism $\psi$ from $K$ to $L$, written

$$\psi : K \rightarrow L : X \rightarrow \ast \quad \text{op.} \quad \psi : K \rightarrow L : \ast \rightarrow A$$

, is defined by a functor $[\psi] : \|K\| \rightarrow \|L\|$, “comma functor”, making the triangle

$$X \xleftarrow{\psi} K \xrightarrow{\|K\|} L \xrightarrow{\psi} A$$

commute.

3.3.5 Remark. (1) The fibre of a right [op. left] comma morphism

$$\psi : K \rightarrow L : X \rightarrow \ast \quad \text{op.} \quad \psi : K \rightarrow L : \ast \rightarrow A$$

at

$$x \in \|X\| \quad \text{op.} \quad a \in \|A\|$$

is the functor

$$x(\psi) : x(K) \rightarrow x(L) \quad \text{op.} \quad (\psi)a : (K)a \rightarrow (L)a$$

given by restricting the comma functor $[\psi] : \|K\| \rightarrow \|L\|$ to the fibre of $K$ at $x \in \|X\|$ [op. $a \in \|A\|$].

(2) The image of $k \in \|K\|$ under the comma functor $[\psi]$ is denoted by $\psi(k)$.

3.3.6 Proposition. A right [op. left] comma morphism $\psi : K \rightarrow L$ sends each $\|K\|$-arrow $h : s \rightarrow t$ to the lift of $K(h)$ at $\psi(t)$ [op. $\psi(s)$] as indicated in

$$\xymatrix{ \psi(s) \ar[r]_{\psi(h) : K(h) \rightarrow K(t)} & \psi(t) \ar@{-->}[l] \quad \text{op.} \quad \psi(s) \ar[r]_{\psi(h) : K(h) \rightarrow K(t)} & \psi(t) \ar@{-->}[l] }$$

Proof. By the definition of a comma morphism,

$$K(h) = L(\psi(h))$$

; that is, $\psi(h)$ is a lift of $K(h)$ along $L$. The assertion thus follows from the uniqueness of the lift. \qed
3.3.7 Remark. A right \([\text{op. left}]\) comma morphism \(\psi\) is thus determined by the object function of the comma functor \([\psi]\).

3.3.8 Proposition. For any right \([\text{op. left}]\) comma morphism \(\psi : \mathbb{K} \to \mathbb{L}\), the comma functor \([\psi] : [\mathbb{K}] \to [\mathbb{L}]\) is a right \([\text{op. left}]\) comma fibration; if \(\mathbb{K}\) and \(\mathbb{L}\) are locally small, then \([\psi]\) is fibre-small.

Proof. Immediate from Proposition 3.2.8(2) and Remark 3.2.14(2).

Note. The slice category and its forgetful functor in Preliminary 0.0.8 gives an example of a right comma.

3.3.9 Example. Let \(A\) be a category. For any object \(p \in \| A \|\),
- the slice category over \(p\) and its forgetful functor \(\Sigma_p : A/p \to A\) form a right comma over \(A\), and for any \(A\)-arrow \(k : p \to a\), the postcomposition functor \(A/k : A/p \to A/a\) form a right comma morphism \(\Sigma_p \to \Sigma_a\).
- the coslice category under \(p\) and its forgetful functor \(\Sigma^p : p\backslash A \to A\) form a left comma over \(A\), and for any \(A\)-arrow \(k : p \to a\), the precomposition functor \(k\backslash A : a\backslash A \to p\backslash A\) form a left comma morphism \(\Sigma^p \to \Sigma_a\).

3.3.10 Proposition. For any \(A\)-arrow \(k : p \to b\), the slice functor \(\Sigma^p/k : [A/b] / k \to A/p\) of the forgetful functor \(\Sigma^p : [A/b] \to A\) over \(k\) is an isomorphism and makes the triangle

\[
\begin{array}{c}
[A/b] / k \\
\Sigma^p/k \\
A/b \\
A/k \\
\end{array}
\]

commute.

Proof. Since the forgetful functor \(\Sigma^p : [A/b] \to A\) is a right comma fibration, the slice functor \(\Sigma^p/k : [A/b] / k \to A/p\) is an isomorphism by Proposition 3.2.7. The commutativity of the triangle may be verified using the commutative diagram

\[
\begin{array}{c}
\bullet \\
/ \\
/ \\
/ \\
b \\
/ \\
/ \\
/ \\
p \\
\end{array}
\]

, identifying objects and arrows of the slice categories \([A/b] / k\), \(A/p\), and \(A/b\), and figuring out the actions of \(\Sigma^p/k\), \(\Sigma^p\), and \(A/k\) on them.

3.3.11 Definition.
- Given a category \(X\), all locally small right commas over \(X\) and morphisms among them define the category \([X \downarrow]\).
- Given a category \(A\), all locally small left commas over \(A\) and morphisms among them define the category \([\downarrow A]\).

3.3.12 Remark.
(1) Noting Proposition 3.3.3 and Proposition 3.3.8, we see that the category \([X \downarrow]\) is the same thing as the slice category of \(\text{CFR}\) (see Definition 3.2.15) over \(X\) and the category \([\downarrow A]\) is the same thing as the slice category of \(\text{CFL}\) over \(A\).
(2) There is an obvious forgetful functor

\[
[\_] : [X \downarrow] \to \text{CAT} \quad \text{op.} \quad [\_] : [\downarrow A] \to \text{CAT}
\]

, which sends each right \([\text{op. left}]\) comma \(\mathbb{K}\) to the comma category \([\mathbb{K}]\) and sends each right \([\text{op. left}]\) comma morphism \(\psi\) to the comma functor \([\psi]\).
3.3.13 Definition.

- The comma category \([M]\) of a right module \(M : X \to \ast\) is defined in the following way:
  1. the objects of \([M]\) are all \(M\)-arrows \(m : x \to \ast\), to be precise, all pairs \((x, m)\) with \(x \in \|X\|\) and \(m \in x(M)\);
  2. an arrow of \([M]\) from \((m : x \to \ast)\) to \((n : y \to \ast)\) is an \(X\)-arrow \(g : x \to y\) making the triangle

\[
\begin{array}{c}
X \\
g \\
\left\downarrow m \right. \\
\left\downarrow \left. x \to \ast \right. \\
y \\
\right. \left. \downarrow n \right. \right) \end{array}
\]

commute.

- The comma category \([M]\) of a left module \(M : \ast \to A\) is defined in the following way:
  1. the objects of \([M]\) are all \(M\)-arrows \(m : \ast \to a\), to be precise, all triples \((m, a)\) with \(a \in \|A\|\), and \(m \in (M)a\);
  2. an arrow of \([M]\) from \((m : \ast \to a)\) to \((n : \ast \to b)\) is an \(A\)-arrow \(f : a \to b\) making the triangle

\[
\begin{array}{c}
\ast \\
\left\downarrow m \right. \\
\left\downarrow \left. \ast \to a \right. \\
f \\
\left\downarrow \left. n \to b \right. \right) \end{array}
\]

commute.

3.3.14 Remark.

1. The composition law of \([M]\) is that induced by the composition laws of \(X\) \([\text{op. } A]\).
2. For an object \(c \in \|C\|\), the slice category \(C/c\) is the same as the comma category of the representable right module \((C)c : C \to \ast\), and the coslice category \(c \setminus C\) is the same thing as the comma category of the representable left module \(c(C) : \ast \to C\).

3.3.15 Definition.

1. Given a right \([\text{op. left}]\) module

\[
M : X \to \ast \quad \text{op.} \quad M : \ast \to A
\]

, the right \([\text{op. left}]\) comma

\[
M^\dagger : X \to \ast \quad \text{op.} \quad M^\dagger : \ast \to A
\]

is defined by the comma category \([M]\) (Definition 3.3.13) and the right \([\text{op. left}]\) comma fibration

\[
\begin{array}{c}
M^\dagger : [M] \to X \\
\left\downarrow \left. \psi \right. \right) \\
\left\downarrow \left. \psi \right. \right) \\
\psi : M \to \mathcal{N} : X \to \ast
\end{array}
\]

given by the natural projection

\[
(x, m) \mapsto x \quad \text{op.} \quad (m, a) \mapsto a.
\]

2. Given a right \([\text{op. left}]\) module morphism

\[
\psi : M \to \mathcal{N} : X \to \ast \quad \text{op.} \quad \psi : M \to \mathcal{N} : \ast \to A
\]

, the right \([\text{op. left}]\) comma morphism

\[
\psi^\dagger : M^\dagger \to \mathcal{N}^\dagger : X \to \ast \quad \text{op.} \quad \psi^\dagger : M^\dagger \to \mathcal{N}^\dagger : \ast \to A
\]

is defined by the functor \([\psi] : [M] \to [\mathcal{N}]\), “comma functor”, sending each object \(m\) of \([M]\) (i.e. \(M\)-arrow \(m\)) to \(m^\dagger \psi\) such that the triangle

\[
\begin{array}{c}
X \\
\left\downarrow \left. \psi \right. \right) \\
\left\downarrow \left. \psi \right. \right) \\
\mathcal{N}^\dagger [\mathcal{N}]
\end{array}
\]

\[
\begin{array}{c}
\left\downarrow \left. \psi \right. \right) \\
\left\downarrow \left. \psi \right. \right) \\
A
\end{array}
\]

\[
\begin{array}{c}
\mathcal{N} \\
\left\downarrow \left. \psi \right. \right) \\
\left\downarrow \left. \psi \right. \right) \\
[\mathcal{N}]
\end{array}
\]

\[
\begin{array}{c}
\left\downarrow \left. \psi \right. \right) \\
\left\downarrow \left. \psi \right. \right) \\
\mathcal{N}^\dagger
\end{array}
\]

2 In the literature, the comma category \([M]\) of a left module \(M\) (i.e. a functor \(M : A \to \text{Set}\)) is called the category of elements.
commutes.

3.3.16 Remark. For a category $C$ and an object $c \in \|C\|$, the comma $(\langle C \rangle)_{i} : C \to *$ of the representable right module $(C)_{i} : C \to *$ is nothing but the forgetful functor $\Sigma_{c} : C/c \to C$ of the slice category $C/c$ (see Preliminary 0.0.8), and the comma $(c(C))_{i} : * \to C$ of the representable left module $c(C) : * \to C$ is just the forgetful functor $\Sigma^{c} : c\{C\} \to C$ of the coslice category $c\{C\}$ (cf. Remark 3.3.14(2)).

3.3.17 Proposition. The module-to-comma correspondence given in Definition 3.3.15 is functorial and defines the functor

$$[X] : \downarrow [X] \quad \text{op.} \quad [A] : \downarrow [A]$$

for any category $X$ [op. $A$].

3.3.18 Theorem. Given a functor

$$P : X \to Y \quad \text{op.} \quad Q : A \to B$$

and a right [op. left] module

$$N : Y \to * \quad \text{op.} \quad N : * \to B$$

we have a pullback diagram

$$\begin{array}{ccc}
\langle N \rangle & \to & N \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}$$

$; that is,

- the right comma $(P \langle N \rangle)_{i} : X \to *$ of the composite right module $P \langle N \rangle : X \to *$ is given by the pullback of the right comma $N^{i} : Y \to *$ along $P : X \to Y$.
- the left comma $(\langle N \rangle Q)_{i} : * \to A$ of the composite left module $(\langle N \rangle Q) : * \to A$ is given by the pullback of the left comma $N^{i} : * \to B$ along $Q : A \to B$.

Proof. Consider the pullback

$$\begin{array}{ccc}
P \times N^{i} \to [N] \\
\downarrow & & \downarrow N^{i} \\
X & \to & Y
\end{array}$$

(cf. Proposition 3.2.9) of the right comma fibration $N^{i}$ along $P$. Noting Preliminary 0.0.9 and recalling Definition 1.1.24 and Definition 3.3.13, we see that the identity

$$[P \langle N \rangle] = P \times N^{i}$$

holds. □

3.3.19 Definition.

(1) Given a right [op. left] comma

$$K : X \to * \quad \text{op.} \quad K : * \to A$$

the right [op. left] module

$$K^{i} : X \to * \quad \text{op.} \quad K^{i} : * \to A$$

is defined in the following way:

a) for an object $x \in \|X\|$ [op. $a \in \|A\|$], the set $x \langle K^{i} \rangle$ [op. $\langle K^{i} \rangle a$] is defined by

$$x \langle K^{i} \rangle = \|x(K)\| \quad \text{op.} \quad \langle K^{i} \rangle a = \|\langle K \rangle a\|$$

; that is, the set $x \langle K^{i} \rangle$ [op. $\langle K^{i} \rangle a$] consists of all objects of the fibre of $K$ at $x \in \|X\|$ [op. $a \in \|A\|$].
b) For an $X$-arrow $g : y \to x$ [op. $A$-arrow $f : a \to b$], the function
\[
g(\mathcal{K}) : x(\mathcal{K}) \to y(\mathcal{K})
\]
is defined by the assignment
\[
k \mapsto \text{dom}(g^k)
\]
\[
k \mapsto \text{cod}^f(k)
\]; that is, $g(\mathcal{K})$ [op. $\langle \mathcal{K} \rangle f$] is defined such that it maps each $k \in x(\mathcal{K})$ [op. $k \in \langle \mathcal{K} \rangle a$] to the domain [op. codomain] of the lift $g^k$ [op. $f^k$].

(2) Given a right [op. left] comma morphism
\[
\psi : \mathcal{K} \to \mathcal{L} : X \to *
\]
, the module morphism
\[
\psi^l : \mathcal{K} \to \mathcal{L} : X \to *
\]
is defined by
\[
\psi^l = (|x(\psi)| : |x(\mathcal{K})| \to |x(\mathcal{L})|)_{|x| |X|}
\]
and defines the functor
\[
\mathcal{K} \Rightarrow \mathcal{L}
\]
for any category $X$ [op. $A$].

3.3.20 Proposition. The comma-to-module correspondence given in Definition 3.4.25 is functorial and defines the functor
\[
[X \downarrow] \Rightarrow [X] \quad \text{op.} \quad [\downarrow A] \Rightarrow [: A]
\]
for any category $X$ [op. $A$].

3.3.21 Proposition. For each right [op. left] module $\mathcal{M}$, there is a canonical isomorphism
\[
\epsilon_\mathcal{M} : \langle \mathcal{M} \rangle \cong \mathcal{M}.
\]

3.3.22 Theorem. For each right [op. left] comma $\mathcal{K}$, there is a canonical isomorphism
\[
\eta_\mathcal{K} : \mathcal{K} \cong \langle \mathcal{K} \rangle.
\]
Hence $\epsilon_M$ is defined by the assignment $(x,m) \mapsto m$ and $\eta_K$ is given by the functor $[\eta_K]$ consisting of the object function $k \mapsto (x,k)$ and the arrow function $h \mapsto K(h)$. The bijectivity of the arrow function of $[\eta_K]$ follows from the faithfulness of $K$ (Proposition 3.2.4).

3.3.23 Remark. In Theorem 11.4.8, we will see that the corresponding functors

$$\begin{array}{ccc}
[X \downarrow] & \xrightarrow{\downarrow} & [X :] \\
[\downarrow A] & \xrightarrow{\downarrow} & [: A]
\end{array}$$

in Proposition 3.3.17 and Proposition 3.3.21 form an adjoint equivalence with the isomorphisms in Theorem 3.3.22.

### 3.4 Two-sided commas

This section deals with two-sided comma fibrations (or two-sided discrete fibrations in the common terminology), and somewhat analogous to Section 3.3, which dealt with one-sided comma fibrations. Again we define a comma category as the domain of a (two-sided) comma fibration.

Given categories $X$ and $A$, a two-sided comma $K : X \to A$ is defined by a comma category $[K]$ and a two-sided comma fibration $X \leftarrow [K] \stackrel{K_0}{\to} [K_1] \to A$. Two-sided commas $X \to A$ form the category $[X \downarrow A]$ with comma morphisms among them, while all two-sided commas and comma cells form the category $\text{COM}$. We construct a module $K^1 : X \to A$ from a comma $K : X \leftarrow [K] \to A$ and a comma $M^1 : X \leftarrow [M] \to A$ from a module $M : X \to A$ (the arrows of $M$ transform into the objects of the comma category $[M]$)\(^3\), and later in Section 11.2 and Section 11.3 establish the adjoint equivalences $[X \downarrow A] \simeq [X : A]$ and $\text{COM} \simeq \text{MOD}$. It is shown in Section 11.1 that the forgetful functor $M \mapsto [M] : \text{MOD} \to \text{CAT}$ given by the comma category assignment is right adjoint to the embedding $C \mapsto (C) : \text{CAT} \to \text{MOD}$ given by the hom-module assignment.

3.4.1 Definition. A (two-sided) comma $K$ from a category $X$ to a category $A$, written $K : X \to A$, consists of a category $[K]$, “comma category”, and a two-sided comma fibration\(^4\) $K : [K] \to X \times A$, i.e. a pair of functors $X \leftarrow [K] \xrightarrow{K_0} [K_1] \to A$, satisfying the following conditions:

1. for every object $k$ in $[K]$ and every $X$-arrow $g : y \to K_0(k)$, there is a unique $[K]$-arrow $g^k : s \to k$, “the lift of $g$ at $k$”, such that $K_0(g^k) = g$ and $K_1(g^k) = K_1(k)$;
2. for every object $k$ in $[K]$ and every $A$-arrow $f : K_1(k) \to b$, there is a unique $[K]$-arrow $f^k : k \to t$, “the lift of $f$ at $k$”, such that $K_1(f^k) = f$ and $K_0(f^k) = K_0(k)$;
3. for every $[K]$-arrow $h : s \to t$, the domain of the lift $K_0(h)^t$ equals the codomain of the lift $K_1(h)^s$ and the triangle

$$
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow & & \downarrow \\
K_0(h)^t & \xrightarrow{K_1(h)^s} & K_1(h)^s
\end{array}
$$

commutes.

3.4.2 Remark.

1. The fibre of a comma $K : X \to A$ at $(x,a) \in [X \times A]$, written $x(K)a$, is the subcategory of $[K]$ consisting of all objects $k$ with $K(k) = (x,a)$ and all arrows $h$ with $K(h) = 1_{(x,a)}$.
2. A comma $K : X \to A$ is called small (resp. locally small) if $X$ and $A$ are small (resp. locally small) and all its fibres are small.

\(^3\)Our construction of a comma category $[M]$ from a module $M : X \to A$ turns out to be the same thing as the usual construction of a comma category $(G \downarrow F)$ from a pair of functors $X \xrightarrow{G} C \xleftarrow{F} A$. (See Remark 3.4.20.)

\(^4\)A two-sided comma fibration is called a two-sided discrete fibration in the literature.
The associativity law

3.4.3 Definition. For a comma $K: X \to A$, the right slice of $K$ at $a \in |A|$, written $(K)a: X \to *$, is the right comma over $X$ defined by

1. the subcategory $((K)a)$ of $K$ consisting of all objects $k$ with $K_1(k) = a$ and all arrows $h$ with $K_1(h) = 1_a$, and
2. the left slice of $K$ at $x \in |X|$, written $x(K)$, is the left comma over $A$ defined by
   1. the subcategory $[x(K)]$ of $K$ consisting of all objects $k$ with $K_0(k) = x$ and all arrows $h$ with $K_0(h) = 1_x$, and
   2. the functor $K_1: [x(K)] \to A$, the restriction of $K_1: [K] \to X$ to $[x(K)]$.

3.4.4 Remark. The conditions (1) and (2) in Definition 3.4.1 precisely state respectively that

- for every $a \in |A|$, $K_0: (K)a \to X$ is a right comma fibration;
- for every $x \in |X|$, $K_1: [x(K)] \to A$ is a left comma fibration.

3.4.5 Proposition. Each fibre of a two-sided comma is a discrete category.

Proof. By Remark 3.4.4(2), this follows from Proposition 3.2.11.

3.4.6 Remark. By Proposition 3.4.5, each fibre $x(K)a$ of $K$ is seen just as the set of objects $k \in |K|$ such that $K(k) = (x,a)$.

3.4.7 Definition. Given a pair of commas $K, L: X \to A$, a morphism $\psi$ from $K$ to $L$, written $\psi: K \to L: X \to A$, is defined by a functor $[\psi]: [K] \to [L]$, “comma functor”, making the triangle

$$
\begin{array}{ccc}
|K| & \xrightarrow{\psi} & |L| \\
K & \downarrow & L \\
X \times A & \xrightarrow{\psi} & A
\end{array}
$$

commute or, equivalently, making the two triangles

$$
\begin{array}{ccc}
X & \xleftarrow{\psi_0} & [K] \\
& \downarrow \psi & \downarrow K_1 \\
[\psi] & \xleftarrow{\psi_1} & A
\end{array}
$$

commute.

3.4.8 Remark.

1. The fibre of a comma morphism $\psi: K \to L: X \to A$ at $(x,a) \in |X \times A|$ is the functor $x(\psi)a: x(K)a \to x(L)a$ given by restricting the comma functor $[\psi]: [K] \to [L]$ to the fibre of $K$ at $(x,a)$.
2. The image of $k \in |K|$ under the comma functor $[\psi]$ is denoted by $\psi(k)$.

3.4.9 Proposition. A comma morphism $\psi: K \to L: X \to A$ sends each $|K|$-arrow $h: s \to t$ to the composite of the lift of $K_1(h)$ at $\psi(s)$ and the lift of $K_0(h)$ at $\psi(t)$:

$$
\begin{array}{ccc}
\psi(h) & \xleftarrow{\psi(s)} & K_1(s) \\
\psi(\psi(s)) & \xleftarrow{K_1(s)} & K_1(t) \\
\psi(\psi(t)) & \xleftarrow{K_0(h)\psi(t)} & K_0(h)
\end{array}
$$
3.4. Two-sided commas

Proof. By the definition of a comma morphism, we have

\[ \mathbb{K}_0(h) = \mathbb{L}_0(\psi(h)) \quad \text{and} \quad \mathbb{K}_1(h) = \mathbb{L}_1(\psi(h)) \]

the assertion thus follows from the condition (3) in Definition 3.4.1.

3.4.10 Remark. A comma morphism \( \psi \) is thus determined by the object function of the comma functor \([\psi]\).

Note. A two-sided comma morphisms consists of pieces of right and left comma morphisms (cf. Definition 3.4.3):

3.4.11 Definition. The right \([\text{op. left}]\) slice of a comma morphism \( \psi : \mathbb{K} \to \mathbb{L} : X \to A \) at \( a \in \|A\| \) \([\text{op. } x \in \|X\|] \), written

\[ (\psi) a : \langle \mathbb{K} \rangle a \to \langle \mathbb{L} \rangle a : X \to * \quad \text{op.} \quad x(\psi) : x(\mathbb{K}) \to x(\mathbb{L}) : * \to A \]

is the right \([\text{op. left}]\) comma morphism over \( X \) \([\text{op. } A] \), defined by the comma functor

\[ [x(\psi)] : [x(\mathbb{K})] \to [x(\mathbb{L})] \]

given by the restriction of the comma functor \([\psi] : [\mathbb{K}] \to [\mathbb{L}]\).

3.4.12 Remark. The functor \([\psi] : [\mathbb{K}] \to [\mathbb{L}]\) forms a comma morphism \( \psi : \mathbb{K} \to \mathbb{L} \) if and only if it restricts to a right comma morphism \( \langle \mathbb{K} \rangle a \to \langle \mathbb{L} \rangle a \) for each \( a \in \|A\| \) and to a left comma morphism \( x(\mathbb{K}) \to x(\mathbb{L}) \) for each \( x \in \|X\| \), and satisfies the property in 3.4.9.

3.4.13 Definition. Given a pair of categories \( X \) and \( A \), all locally small commas \( X \to A \) and morphisms among them define the category \([X \downarrow A]\) with the obvious identities and the composition. Indeed, \([X \downarrow A]\) is fully embedded in the slice category over \( X \times A \).

3.4.14 Remark. The category \([X \downarrow ]\) \([\text{op. } [\downarrow A] \) in Definition 3.3.11 is seen as a special case of the category \([X \downarrow A]\) where \( A \) \([\text{op. } X]\) is the terminal category. Indeed, there is an obvious isomorphisms

\[ [X \downarrow] \cong [X \downarrow *] \quad \text{op.} \quad [\downarrow A] \cong [* \downarrow A] \]

by which

- a right comma over \( X \) is identified with a two-sided comma from \( X \) to the terminal category.
- a left comma over \( A \) is identified with a two-sided comma from the terminal category to \( A \).

3.4.15 Definition. Given a pair of commas \( \mathbb{K} : X \to A \) and \( \mathbb{L} : Y \to B \), and given a pair of functors \( P : X \to Y \) and \( Q : A \to B \), a comma cell \( \psi : P \sim Q : \mathbb{K} \to \mathbb{L} \) is defined by a functor \([\psi] : [\mathbb{K}] \to [\mathbb{L}]\), “comma functor”, making the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\mathbb{K}_0} & [\mathbb{K}] & \xrightarrow{\mathbb{K}_1} & A \\
\downarrow P & & [\psi] & & \downarrow Q \\
Y & \xleftarrow{\mathbb{L}_0} & [\mathbb{L}] & \xrightarrow{\mathbb{L}_1} & B
\end{array}
\]

commute.

3.4.16 Remark. The fibre of a comma cell \( \psi : P \sim Q : \mathbb{K} \to \mathbb{L} \) at \( (x, a) \in \|X \times A\| \), written as

\[ x(\psi) a : x(\mathbb{K}) a \to (x : P)(L : (Q : a)) \]

is the restriction of the comma functor \([\psi] : [\mathbb{K}] \to [\mathbb{L}]\) to the fibre of \( \mathbb{K} \) at \( (x, a) \).

3.4.17 Definition. All locally small commas and cells among them define the category \( \text{COM} \) with the obvious identities and the composition.
3.4.18 Remark.
(1) There is an obvious forgetful functor $[-]: \text{COM} \to \text{CAT}$, which sends each comma $\mathbb{K}$ to the comma category $[\mathbb{K}]$ and sends each comma cell $\psi$ to the comma functor $[\psi]$.
(2) Given a pair of categories $X$ and $A$, there is a canonical embedding $[X \downarrow A] \hookrightarrow \text{COM}$, identical on objects, defined by the arrow function $\psi \mapsto (\psi: 1_X \rightsquigarrow 1_A)$. The embedding is not, in general, full.

3.4.19 Definition. The comma category $[\mathcal{M}]$ of a module $\mathcal{M}: X \to A$ is defined in the following way:\(^5\)
(1) the objects of $[\mathcal{M}]$ are all $\mathcal{M}$-arrows $m: x \rightsquigarrow a$, to be precise, all triples $(x, m, a)$ with $x \in [X]_1$, $a \in [A]_0$, and $m \in x(\mathcal{M})a$;
(2) an arrow of $[\mathcal{M}]$ from $(m: x \rightsquigarrow a)$ to $(n: y \rightsquigarrow b)$ is a pair $(g, f)$ consisting of an $X$-arrow $g: x \to y$ and an $A$-arrow $f: a \to b$ making the square
\[
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
g \downarrow & & \downarrow f \\
y & \xleftarrow{n} & b
\end{array}
\]
commute;
(3) the composition law of $[\mathcal{M}]$ is that induced by the composition laws of $X$ and $A$.

3.4.20 Remark.
(1) Traditionally, a comma category is defined for a pair of functors $X \xrightarrow{G} C \xleftarrow{F} A$ and written as $(G \downarrow F)$. Note that
\[(G \downarrow F) = [G(C)F]\]
; that is, the comma category $(G \downarrow F)$ is the same thing as the comma category $[G(C)F]$ of the composite module $G(C)F$ (see Example 1.1.31(12)). Conversely, for a module $\mathcal{M}: X \to A$,
\[[\mathcal{M}] = (\mathcal{M}_0 \downarrow \mathcal{M}_1)\]
, where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the inclusions $X \xrightarrow{\mathcal{M}_0} [\mathcal{M}] \xleftarrow{\mathcal{M}_1} A$ in Remark 3.1.2(1).
(2) The comma category $[\mathcal{M}]$ is identified with the full subcategory of the functor category $[2, [\mathcal{M}]]$ consisting of all sections of the functor $[!_\mathcal{M}]: [\mathcal{M}] \to 2$ (see Remark 3.1.2(4)), giving a canonical inclusion $[\mathcal{M}] \hookrightarrow [2, [\mathcal{M}]]$.
(3) The arrow category of a category $C$ is the same thing as the comma category of the hom-module $\langle C \rangle : C \to C$.

3.4.21 Definition.
(1) Given a module $\mathcal{M}: X \to A$, the comma $\mathcal{M}^\dagger: X \to A$ is defined by the comma category $[\mathcal{M}]$ and the pair of functors $\mathcal{M}_0^\dagger: [\mathcal{M}] \to X$ and $\mathcal{M}_1^\dagger: [\mathcal{M}] \to A$ given by the natural projections natural projections $(x, m, a) \mapsto x$ and $(x, m, a) \mapsto a$.
$\mathcal{M}_0^\dagger$ and $\mathcal{M}_1^\dagger$ are the unique functors making the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}_0^\dagger} & [\mathcal{M}] \\
\downarrow & & \downarrow \mathcal{M}_0 \\
[\mathcal{M}] & \xrightarrow{[0, [\mathcal{M}]]} & [2, [\mathcal{M}]] \\
\downarrow & & \downarrow [1, [\mathcal{M}]] \\
[\mathcal{M}] & \xrightarrow{[1, [\mathcal{M}]]} & [1, [\mathcal{M}]]
\end{array}
\]
commute, where $[\mathcal{M}] \hookrightarrow [2, [\mathcal{M}]]$ is the inclusion described in Remark 3.4.20(2) and $[0, [\mathcal{M}]]$ and $[1, [\mathcal{M}]]$ are the evaluations at $0 \in 2$ and $1 \in 2$.

\(^5\)This construction of the comma category is taken from [Ell06].
Given a module morphism $\psi : \mathcal{M} \to \mathcal{N} : X \to A$, the comma morphism $\psi^i : \mathcal{M}^i \to \mathcal{N}^i : X \to A$
is defined by the functor $[\psi] : [\mathcal{M}] \to [\mathcal{N}]$, “comma functor”, sending each object $m$ of $[\mathcal{M}]$ (i.e. $\mathcal{M}$-arrow $m$) to $m : \psi$ such that the diagram

\[
\begin{array}{c}
\mathcal{M}^i \downarrow [\psi] \\
\mathcal{N}^i
\end{array}
\]

commutes ($[\psi]$ is the unique functor making the diagram

\[
\begin{array}{ccc}
[\mathcal{M}] & \xrightarrow{\psi} & [2, [\mathcal{M}]] \\
\downarrow & & \downarrow \\
[\mathcal{N}] & \xrightarrow{\psi} & [2, [\mathcal{N}]]
\end{array}
\]

commute, i.e. the reflection of the functor $[2, [\psi]]$, the postcomposition with the collage functor $[\psi] : [\mathcal{M}] \to [\mathcal{N}]$ (see Definition 3.1.9(2)), along the inclusions described in Remark 3.4.20(2)).

Given a module cell

\[
\begin{array}{c}
X \xleftarrow{\mathcal{M}} A \\
\mathcal{P} \downarrow \psi \\
Y \xleftarrow{\mathcal{N}} B
\end{array}
\]

, the comma cell $\psi^i : \mathcal{P} \to \mathcal{Q} : \mathcal{M}^i \to \mathcal{N}^i$ is defined by the functor $[\psi] : [\mathcal{M}] \to [\mathcal{N}]$, “comma functor”, sending each object $m$ of $[\mathcal{M}]$ (i.e. $\mathcal{M}$-arrow $m$) to $m : \psi$ such that the diagram

\[
\begin{array}{c}
\mathcal{M}^i \downarrow [\psi] \\
\mathcal{N}^i
\end{array}
\]

commutes ($[\psi]$ is the unique functor making the diagram

\[
\begin{array}{ccc}
[\mathcal{M}] & \xrightarrow{\psi} & [2, [\mathcal{M}]] \\
\downarrow & & \downarrow \\
[\mathcal{N}] & \xrightarrow{\psi} & [2, [\mathcal{N}]]
\end{array}
\]

commute, i.e. the reflection of the functor $[2, [\psi]]$, the postcomposition with the collage functor $[\psi] : [\mathcal{M}] \to [\mathcal{N}]$ (see Definition 3.1.9(3)), along the inclusions described in Remark 3.4.20(2)).

**3.4.22 Remark.** Given an arrow

\[
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
g \downarrow & & \downarrow f \\
y & \xrightarrow{n} & b
\end{array}
\]

of the comma category $[\mathcal{M}]$, the decomposition diagram

\[
\begin{array}{ccc}
x & \xrightarrow{m} & a \\
\downarrow m \circ f \circ g & & \downarrow f \\
x & \xrightarrow{n} & b \\
g \downarrow & & \downarrow 1 \\
y & \xrightarrow{n} & b
\end{array}
\]

illustrates the way the comma $\mathcal{M}^i : X \to A$ satisfies the condition (3) in Definition 3.4.1; note that the upper square forms the lift of the $A$-arrow $f$ at $m$ and the lower square forms the lift of the $X$-arrow $g$ at $n$.

**3.4.23 Proposition.** The module-to-comma correspondence given in Definition 3.4.21 is functorial and defines the following functors:

1. $[X : A] \xrightarrow{i} [X \downarrow A]$ for categories $X$ and $A$;
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(2) \( \text{MOD} \downarrow \text{COM} \).

\[ \]

3.4.24 Remark.

(1) The composition of the functor \( \text{MOD} \downarrow \text{COM} \) and the forgetful functor \([-]\) : \( \text{COM} \rightarrow \text{CAT} \) in Remark 3.4.16(1) yields the functor \([-]\) : \( \text{MOD} \downarrow \text{CAT} \), which sends each module \( \mathcal{M} \) to the comma category \([\mathcal{M}]\) and sends each cell \( \psi \) to the comma functor \([\psi]\).

(2) The canonical isomorphisms in Remark 1.1.14(4) and Remark 3.4.14 make the diagram

\[
\begin{array}{c}
\text{X} \\
\downarrow x
\end{array}
\begin{array}{c}
\text{MOD} \\
\downarrow z
\end{array}
\begin{array}{c}
\text{COM} \\
\downarrow z
\end{array}
\begin{array}{c}
\text{A} \\
\downarrow z
\end{array}
\begin{array}{c}
\text{op.} \\
\downarrow z
\end{array}
\begin{array}{c}
\text{Y}
\end{array}
\]

commute; the functor \( \text{X} : \downarrow \text{MOD} \rightarrow \text{COM} \rightarrow \text{op.} \rightarrow \text{A} \rightarrow \text{Mod} \rightarrow \text{COM} \) in Proposition 3.3.17 is thus seen as a special instance of the functor \( \text{X} : \mathcal{A} \rightarrow \mathcal{A} \downarrow \mathcal{A} \) where \( \mathcal{A} \) op. is the terminal category.

(3) The functors in Proposition 3.4.23 yield the functors \( \text{X} : \mathcal{A} \rightarrow \mathcal{A} \downarrow \mathcal{A} \) and \( \text{CLG} \downarrow \text{COM} \) via composition with the isomorphisms in Theorem 3.1.14, as shown in the commutative diagrams:

\[
\begin{array}{c}
\text{X} \downarrow \mathcal{A} \\
\mathcal{B}
\end{array}
\begin{array}{c}
\text{MOD} \downarrow \text{COM}
\end{array}
\begin{array}{c}
\text{CLG} \downarrow \text{COM}
\end{array}
\begin{array}{c}
\mathcal{A}
\end{array}
\]

The functor \( \text{X} \downarrow \mathcal{A} \) sends a collage \( \mathcal{M} : \mathcal{X} \rightarrow \mathcal{A} \) to the comma \( \mathcal{M} \downarrow \mathcal{A} \downarrow \mathcal{A} \) and sends a collage morphism \( \psi : \mathcal{M} \rightarrow \mathcal{N} : \mathcal{X} \rightarrow \mathcal{A} \) to the comma morphism \( \psi : \mathcal{M} \rightarrow \mathcal{N} : \mathcal{X} \rightarrow \mathcal{A} \):

\[
\begin{array}{c}
\mathcal{X} \downarrow \mathcal{A} \\
\mathcal{M}
\end{array}
\begin{array}{c}
\mathcal{N}
\end{array}
\]

; similarly, the functor \( \text{CLG} \downarrow \text{COM} \) sends a collage cell \( \psi : \mathcal{P} \rightarrow \mathcal{Q} : \mathcal{M} \rightarrow \mathcal{N} \) to the comma cell \( \psi : \mathcal{P} \rightarrow \mathcal{Q} : \mathcal{M} \rightarrow \mathcal{N} \):

\[
\begin{array}{c}
\mathcal{X} \downarrow \mathcal{A} \\
\mathcal{P}
\end{array}
\begin{array}{c}
\mathcal{N}
\end{array}
\]

(4) The composition of the functor \( \text{CLG} \downarrow \text{COM} \) and the forgetful functor \([-]\) : \( \text{COM} \rightarrow \text{CAT} \) in Remark 3.4.16(1) yields the functor \([\_]\) : \( \text{CLG} \rightarrow \text{CAT} \), which sends each collage \( \mathcal{M} \) to the comma category \([\mathcal{M}]\) and sends each collage cell \( \psi \) to the comma functor \([\psi]\).

3.4.25 Definition.

(1) Given a comma \( \mathcal{K} : \mathcal{X} \rightarrow \mathcal{A} \), the module \( \mathcal{K} : \mathcal{X} \rightarrow \mathcal{A} \) is defined in the following way:

a) for a pair of objects \( x \in \mathcal{X} \) and \( a \in \mathcal{A} \), the set \( x(\mathcal{K})a \) is defined by

\[
x(\mathcal{K})a = \|x(\mathcal{K})a\|
\]

; that is, the set \( x(\mathcal{K})a \) consists of all objects of the fibre of \( \mathcal{K} \) at \( (x,a) \).

b) for an \( \mathcal{X} \)-arrow \( g : y \rightarrow x \) and an object \( a \in \mathcal{A} \), the function \( g(x(\mathcal{K})a : x(\mathcal{K})a \rightarrow y(\mathcal{K})a \) is defined by the assignment

\[
k \mapsto \text{dom} (g^k)
\]

; that is, \( g(x(\mathcal{K})a \) is defined such that it maps each \( k \in \|x(\mathcal{K})a\| \) to the domain of the lift \( g^k \).

c) for an object \( x \in \mathcal{X} \) and an \( \mathcal{A} \)-arrow \( f : a \rightarrow b \), the function \( x(x(\mathcal{K})f : x(\mathcal{K})a \rightarrow x(\mathcal{K})b \) is defined by the assignment

\[
k \mapsto \text{cod} (f^k)
\]
Given a comma morphism \( \psi : K \to L : X \to A \), the module morphism \( \psi^! : K^! \to L^! : X \to A \) is defined by
\[
\psi^! = (\|x(\psi)\| : \|x(K)\| \to \|x(L)\|)_{(x,a) \in |X \times A|}
\]
, where \( \|x(\psi)\| \) is the object function of the fibre of \( \psi \) at \( (x,a) \in \|X \times A\| \).

### 3.4.26 Remark.
(1) Comparing Definition 3.4.25(1) and Definition 3.3.19(1), we see that
\[
\left(\{K^!\}\right) a = \left(\{K\}\right)^\dagger \quad \text{and} \quad x\{K\} = \{x\{K\}\}^\dagger
\]
; that is, the right slice \( \left(\{K^!\}\right) a \) of the module \( K^! \) at \( a \in \|A\| \) is the right module \( \left(\{K\}\right)^\dagger \) of the right slice \( \{K\} a \) (see Definition 3.4.3) of \( K \) at \( a \in \|A\| \), and the left slice \( x\{K\} \) of the module \( K^! \) at \( x \in \|X\| \) is the left module \( \{x\{K\}\}^\dagger \) of the left slice \( x\{K\} \) of \( K \) at \( x \in \|A\| \).

(2) Comparing Definition 3.4.25(2) and Definition 3.3.19(2), we see that
\[
\left(\{\psi^!\}\right) a = \left(\{\psi\}\right)^\dagger \quad \text{and} \quad x\{\psi\} = \{x\{\psi\}\}^\dagger
\]
; that is, the right slice \( \left(\{\psi^!\}\right) a \) of the module morphism \( \psi^! \) at \( a \in \|A\| \) is the right module morphism \( \left(\{\psi\}\right)^\dagger \) of the right slice \( \{\psi\} a \) (see Definition 3.4.11) of \( \psi \) at \( a \in \|A\| \), and the left slice \( x\{\psi\} \) of the module morphism \( \psi^! \) at \( x \in \|X\| \) is the left module morphism \( \{x\{\psi\}\}^\dagger \) of the left slice \( x\{\psi\} \) of \( \psi \) at \( x \in \|A\| \).

### 3.4.27 Proposition.
(1) \( K^! \) defined in Definition 3.4.25(1) is indeed a module.
(2) \( \psi^! \) defined in Definition 3.4.25(2) is indeed a module morphism.

Proof.
(1) Since \( \left(\{K^!\}\right) a = \left(\{K\}\right)^\dagger \) and \( x\{K\} = \{x\{K\}\}^\dagger \) by Remark 3.4.26(1), the functorialities of the right slice \( \left(\{K^!\}\right) a : X^\dagger \to \textbf{Set} \) and the left slice \( x\{K^!\} : A \to \textbf{Set} \) follows from Proposition 3.3.20(1). By the bifunctor lemma (see [ML98] p37 Proposition 1), the proof is complete if we show that the square
\[
\begin{array}{ccc}
x\{K^!\} a & \xrightarrow{g(x^! a)} & y\{K^!\} a \\
x\{K^!\} f & \downarrow & y\{K^!\} f \\
x\{K^!\} b & \xrightarrow{g(x^! b)} & y\{K^!\} b
\end{array}
\]
commutes for any \( X \)-arrow \( g : y \to x \) and any \( A \)-arrow \( f : a \to b \). Given an object \( k \in \|x(K^!)a\| \),
consider the diagram
\[
\begin{array}{c}
k \xleftarrow{g^k} s \\
\xrightarrow{\ell^k} \varepsilon^k \circ f^k \xrightarrow{\ell^s} f^s \\
t \xrightarrow{\varepsilon^t} \bullet \\
\xleftarrow{s^t} y
\end{array}
\]
, where \(s\) is domain of the lift \(g^k\) and \(t\) is the codomain of the lift \(f^k\). \(x(\mathbb{K}!)(f \circ g(\mathbb{K}!))b\) maps \(k\) to the domain of \(g^t\), and \(g(\mathbb{K}!)(a \circ y(\mathbb{K}!))f\) maps \(k\) to the codomain of \(f^s\). But they are equal by the condition (3) in Definition 3.4.1.

(2) By [ML98] p38 Proposition 2, it suffices to show that the right slice \(\langle \psi^! \rangle a\) is a right module morphism for each \(a \in [A]\) and the left slice \(x(\psi^!)\) a left module morphism for each \(x \in [X]\). But since \(\langle \psi^! \rangle a = \langle (\psi) a \rangle^!\) and \(x(\psi^!) = \langle x(\psi) \rangle^!\) by Remark 3.4.26(2), this is the case by Proposition 3.3.20(2).

\[\square\]

3.4.28 Proposition. The comma-to-module correspondence given in Definition 3.4.25 is functorial and defines the following functors:

1. \([X \downarrow A] \xrightarrow{\dashv} [X : A]\) for categories \(X\) and \(A\);
2. \(\text{COM} \xrightarrow{\dashv} \text{MOD}\).

\[\square\]

3.4.29 Remark.

1. The canonical isomorphisms in Remark 3.4.14 and Remark 1.1.14(4) make the diagram

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{\dashv} & [X : A] \\
\downarrow z & \downarrow \uparrow \downarrow z & \downarrow \uparrow \downarrow z \\
[X \downarrow *] & \xrightarrow{\dashv} & [X : *] \\
\end{array}
\]

commute; the functor \([X \downarrow A] \xrightarrow{\dashv} [X : A]\) in Proposition 3.3.21 is thus seen as a special instance of the functor \([X \downarrow A] \xrightarrow{\dashv} [X : A]\) where \(A\) [op. \(X\)] is the terminal category.

2. The functors in Proposition 3.4.28 yield the functors \([X \downarrow A] \xrightarrow{\dashv} [X \uparrow A]\) and \(\text{COM} \xrightarrow{\dashv} \text{CLG}\) via composition with the isomorphisms in Theorem 3.1.14, as shown in the commutative diagrams:

\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{\dashv} & [X \uparrow A] \\
\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
[X \downarrow A] & \xrightarrow{\dashv} & [X : A] \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{COM} & \xrightarrow{\dashv} & \text{CLG} \\
\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
\text{COM} & \xrightarrow{\dashv} & \text{MOD} \\
\end{array}
\]

The functor \([X \downarrow A] \xrightarrow{\dashv} [X \uparrow A]\) sends a comma \(\mathbb{K} : X \rightarrow A\) to the collage \(\mathbb{K}! : X \rightarrow A\) and sends a comma morphism \(\psi : \mathbb{K} \rightarrow \mathbb{L} : X \rightarrow A\) to the collage morphism \(\psi^! : \mathbb{K}! \rightarrow \mathbb{L}! : X \rightarrow A\):

\[
\begin{array}{c}
X \xleftarrow{[\mathbb{K}]} \xrightarrow{[\psi]} A \\
\downarrow L_0 \downarrow L_1 \\
[\mathbb{L}] \xleftarrow{[\psi]} \xrightarrow{L_1} \mathbb{L} \\
\end{array}
\]

, and similarly the functor \(\text{COM} \xrightarrow{\dashv} \text{CLG}\) sends a comma cell \(\psi : P \rightarrow Q : \mathbb{K} \rightarrow \mathbb{L}\) to the collage cell \(\psi^! : P \rightarrow Q : \mathbb{K}! \rightarrow \mathbb{L}!:\n
\[
\begin{array}{c}
X \xleftarrow{[\mathbb{K}]} \xrightarrow{[\psi]} A \\
\downarrow L_0 \downarrow L_1 \\
[\mathbb{L}] \xleftarrow{[\psi]} \xrightarrow{L_1} \mathbb{L} \\
\end{array}
\]
(the collage category of $K'$ is denoted by $[K]$ and called the collage category of $K$, and the collage functor of $\psi'$ is denoted by $[\psi]$ and called the collage functor of $\psi$).

(3) The composition of the functor $\text{COM} \xrightarrow{\sim} \text{CLG}$ and the forgetful functor $[-]: \text{CLG} \to \text{CAT}$ in Remark 3.1.6(2) yields the functor $[-]: \text{COM} \to \text{CAT}$, which sends each comma $K$ to the collage category $[K]$ and sends each comma cell $\psi$ to the collage functor $[\psi]$.

3.4.30 Theorem.

(1) For each module $M: X \to A$, there is a canonical isomorphism

$$\epsilon_M: \{M^i\}^\uparrow \cong M.$$

(2) For each comma $K: X \to A$, there is a canonical isomorphism

$$\eta_K: K \cong \{K^i\}^\downarrow.$$

Proof. By Definition 3.4.21 and Definition 3.4.25,

(1) the module $\{M^i\}^\uparrow$ is obtained from $M$ by changing each element $m \in x(M)a$ to the triple $(x, m, a)$.

(2) the comma $\{K^i\}^\downarrow$ is obtained from $K$ by changing each object $k \in \|x(K)a\|$ to the triple $(x, k, a)$ and changing each $[K]$-arrow $h: s \to t$ to the pair $(K_0(h), K_1(h))$.

Hence $\epsilon_M$ is defined by the assignment $(x, m, a) \mapsto m$ and $\eta_K$ is given by the functor $[\eta_K]$ consisting of the object function $k \mapsto (x, k, a)$ and the arrow function $h \mapsto (K_0(h), K_1(h))$. The bijectivity of the arrow function of $[\eta_K]$ follows from the condition (3) in Definition 3.4.1.

3.4.31 Remark. In Theorem 11.2.10 and Theorem 11.3.12, we will see that the corresponding functors

$$\quad [X \downarrow A] \xrightarrow{\sim} [X : A] \quad \text{and} \quad \text{COM} \xrightarrow{\sim} \text{MOD}$$

in Proposition 3.4.23 and Proposition 3.4.28 form adjoint equivalences with the isomorphisms in Theorem 3.4.30.
4 Frames and Orbits

4.1 Cylindrical frames

A cylindrical frame (see Definition 4.1.1) of an endomodule abstracts the notion of a natural transformation. Ordinary and extraordinary cylinders are defined respectively in Section 4.3 and Section 4.4 as instances of cylindrical frames. In Section 13.3, we will see that the set of frames of a small endomodule \( \mathcal{M} \) gives an end of \( \mathcal{M} \) in the category \( \text{Set} \).

4.1.1 Definition. A (cylindrical) frame \( \alpha \) of an endomodule \( \mathcal{M} : E \to E \) is a family of \( \alpha \)-arrows \( \alpha_e : e \leadsto e \), one for each object \( e \in \|E\| \), that is natural in the sense that the square

\[
\begin{array}{ccc}
  e & \overset{\alpha_e}{\longrightarrow} & e \\
  h \downarrow & & \downarrow h \\
  e' & \overset{\alpha_{e'}}{\longrightarrow} & e'
\end{array}
\]

commutes for every \( E \)-arrow \( h : e \to e' \). The \( \alpha \)-arrow \( \alpha_e \) is called the component of \( \alpha \) at \( e \). The set of frames of an endomodule \( \mathcal{M} : E \to E \) is denoted by \( \prod E \mathcal{M} \) and called the frame-set of \( \mathcal{M} \).

4.1.2 Remark.  
(1) If \( \mathcal{M} \) is small, so is the frame-set \( \prod E \mathcal{M} \).  
(2) If \( E \) is discrete, then the frame-set \( \prod E \mathcal{M} \) is just the cartesian product \( \prod_{e \in \|E\|} e(\mathcal{M}) e \).  
(3) The naturality of a frame of an endomodule \( \mathcal{M} \) is also expressed in the extraordinary\(^1\) form by regarding \( \mathcal{M} \) as a one-sided module (cf. Remark 1.1.14(3)): if \( \mathcal{M} : E \to E \) is seen as a left module \( \mathcal{M} : * \to E^\times E \) [op. right module \( \mathcal{M} : E \times E^\to * \)], the naturality of a frame \( \alpha \) of \( \mathcal{M} \) is expressed by the commutativity of the square

\[
\begin{array}{ccc}
  * & \overset{\alpha_e}{\longrightarrow} & (e, e) \\
  \downarrow \alpha_{e'} & & \downarrow \alpha_{(e', h)} \\
 (e', e') & \overset{\alpha_{e', h}}{\longrightarrow} & (e, e)
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \) (cf. Remark 1.1.20(4)).  
(4) A frame of an endomodule \( \mathcal{M} : E \to E \) is the same thing as a frame of the opposite endomodule \( \mathcal{M}^\to : E^\to \to E^\to \); that is,

\[
\prod E \mathcal{M} = \prod E \mathcal{M}^\to.
\]

4.1.3 Example. Let \( E \) and \( C \) be categories.  
(1) A natural transformation \( \alpha : G \to F : E \to C \) is the same thing as a frame \( \alpha \) of the composite endomodule \( G(C)F : E \to E \). (Conversely, a frame \( \alpha \) of an endomodule \( \mathcal{M} : E \to E \) is the same thing as a natural transformation \( \alpha : \mathcal{M}_0 \to \mathcal{M}_1 : E \to [\mathcal{M}] \) (see Remark 3.1.2(1)).  
(2) Given a bifunctor \( L : E^\times E \to C \) and an object \( c \in \|C\| \),  
- an extraordinary natural transformation \( \alpha \) from \( c \) to \( L \) is the same thing as a cylindrical frame \( \alpha \) of the composite left module \( c(C)L : * \to E^\times E \) (note that a left module \( * \to E^\times E \) is the same thing as an endomodule \( E \to E \)).  
- an extraordinary natural transformation \( \alpha \) from \( L \) to \( c \) is the same thing as a cylindrical frame \( \alpha \) of the composite right module \( L(C)c : E^\times E \to * \) (note that a right module \( E^\times E \to * \) is the same thing as an endomodule \( E^\to \to E^\to \)).

\(^1\)The term “extraordinary (natural)” is borrowed from [Kel05]. An alternative term “extranatural” seems more common in the other literature.
4.1.4 Theorem. Let \( \mathcal{M} : E \times D \rightarrow E \times D \) be an endomodule. A family of \( \mathcal{M} \)-arrows \( \alpha_{(e,d)} : (e,d) \sim (e,d) \) indexed by the objects of \( E \times D \) is a frame of \( \mathcal{M} \) if and only if for each \( e \in |E| \), \( (\alpha_{(e,d)})_{d \in |D|} \) is a frame of the endomodule \( [e \times D] \langle \mathcal{M} \rangle [e \times D] : D \rightarrow D \) (see Example 1.1.31(7)) and for each \( d \in |D| \), \( (\alpha_{(e,d)})_{e \in |E|} \) is a frame of the endomodule \( [E \times d] \langle \mathcal{M} \rangle [E \times d] : E \rightarrow E \).

\[ \alpha \circ \psi = \psi \circ \alpha \]

Proof. See [ML98] p38 Proposition 2.

4.1.5 Definition. If \( \psi : \mathcal{M} \rightarrow \mathcal{N} : E \rightarrow E \) is a module morphism and \( \alpha \) is a frame of \( \mathcal{M} \), then their composite \( \alpha \circ \psi = \psi \circ \alpha \) is the frame of \( \mathcal{N} \) defined by

\[ [\alpha \circ \psi]_e = \alpha_e \cdot e(\psi) e \]

for \( e \in |E| \); the function

\[ \Pi_E \psi : \Pi_E \mathcal{M} \rightarrow \Pi_E \mathcal{N} \]

is defined by

\[ \alpha \cdot \Pi_E \psi = \alpha \circ \psi \]

4.1.6 Remark. The composite \( \alpha \circ \psi \) so defined does form a frame of \( \mathcal{N} \). Indeed, for each \( E \)-arrow \( h : e \rightarrow e' \), the commutativity of the square

\[ \begin{array}{ccc}
    e & \sim_{\alpha} & e \\
    h \downarrow & & \downarrow h \\
    e' & \sim_{\alpha} & e'
\end{array} \]

follows from the commutativity of the square

\[ \begin{array}{ccc}
    e & \sim_{\alpha_e} & e \\
    h \downarrow & & \downarrow h \\
    e' & \sim_{\alpha_{e'}} & e'
\end{array} \]

by the naturality of \( \psi \).

4.1.7 Proposition. The assignment \( \psi \mapsto \Pi_E \psi \) is functorial and defines a functor

\[ \Pi_E : [E : E] \rightarrow \text{Set} \]

, i.e. a (locally small) left module

\[ \Pi_E : \ast \rightarrow [E : E] \]

for \( E \) small.

4.1.8 Definition. If \( K : D \rightarrow E \) is a functor and \( \alpha \) is a frame of an endomodule \( \mathcal{M} : E \rightarrow E \), then their composite \( K \circ \alpha = \alpha \circ K \) is the frame of the endomodule \( K \langle \mathcal{M} \rangle K : D \rightarrow D \) defined by

\[ [K \circ \alpha]_d = \alpha_{(K,d)} \]

for \( d \in |D| \); the function

\[ \Pi_K \mathcal{M} : \Pi_E \mathcal{M} \rightarrow \Pi_D \langle \mathcal{M} \rangle K \]

is defined by

\[ \alpha \cdot \Pi_K \mathcal{M} = K \circ \alpha \]

4.1.9 Remark. The composite \( K \circ \alpha \) so defined does form a frame of \( \langle \mathcal{M} \rangle K \). Indeed, by the naturality of \( \alpha \), the square

\[ \begin{array}{ccc}
    d & \sim_{\alpha_{(K,d)}} & K \cdot d \\
    h \downarrow & & \downarrow h \\
    d' & \sim_{\alpha_{(K,d')}} & K \cdot d'
\end{array} \]

commutes for every \( D \)-arrow \( h : d \rightarrow d' \).
4.1.10 Proposition. Given functors \( C \xrightarrow{G} D \xrightarrow{K} E \) and a frame \( \alpha \) of an endomodule \( M : E \to E \), the associative law
\[
[G \circ K] \circ \alpha = G \circ [K \circ \alpha]
\]
holds.

\textit{Proof.} For any \( c \in \|C\| \),
\[
[[G \circ K] \circ \alpha]_c = \alpha_{(K; G \cdot c)} = [K \circ \alpha]_{(G \cdot c)} = [G \circ [K \circ \alpha]]_c.
\]

4.1.11 Proposition. The function \( \prod_K M \) is natural in \( M \); that is, for every module morphism \( \psi : M \to N : E \to E \), the square
\[
\begin{array}{ccc}
\prod_E M & \xrightarrow{\prod_K M} & \prod_D K(\langle M \rangle) K \\
\downarrow_{\prod_E \psi} & & \downarrow_{\prod_D K(\psi) K} \\
\prod_E N & \xrightarrow{\prod_K N} & \prod_D K(\langle N \rangle) K
\end{array}
\]
commutes.

\textit{Proof.} For any frame \( \alpha \) of \( M \),
\[
\alpha : \prod_E \psi : \prod_K N = K \circ [\alpha \circ \psi]
\]
and
\[
\alpha : \prod_K M : \prod_D K(\langle \psi \rangle) K = [K \circ \alpha] \circ K(\psi) K
\]
; hence we need to verify that
\[
K \circ [\alpha \circ \psi] = [K \circ \alpha] \circ K(\psi) K
\]
; but for any \( d \in \|D\| \),
\[
[K \circ [\alpha \circ \psi]]_d = [\alpha \circ \psi]_{(K \cdot d)} = \alpha_{(K \cdot d)} \cdot (d \cdot K(\langle \psi \rangle) K) d = [[K \circ \alpha] \circ K(\psi) K]_d.
\]

4.1.12 Proposition. Given a functor \( K : D \to E \) between small categories \( D \) and \( E \), the family of functions \( \prod_K M : \prod_E M \to \prod_D K(\langle M \rangle) K \), one for each endomodule \( M : E \to E \), defines a left module cell
\[
\begin{array}{ccc}
\ast & \xrightarrow{\prod_E} & [E : E] \\
\downarrow & & \downarrow \prod_K & \downarrow \prod_\langle K \rangle K \\
\ast & \xrightarrow{\prod_D} & [D : D]
\end{array}
\]
; moreover, the assignment \( K \mapsto \prod_K \) is contravariant functorial.

\textit{Proof.} The first assertion is immediate from Proposition 4.1.11. The functoriality of the assignment is easily verified using Proposition 4.1.10.

4.2 Conical frames

A conical frame (see Definition 4.2.1) of a one-sided module abstracts the notion of a cone. Just like a cone is a special instance of a natural transformation, a conical frame is a special instance of a cylindrical frame. A cone along a module is defined in Section 4.6 as an instance of a conical frame. The set of frames of a module may be seen as a categorical generalization of the notion of a cartesian product; we will see in Section 8.5 that the frame-set of a small module \( M \) gives a limit of \( M \) in the category \( \text{Set} \).
4.2.1 Definition. A (conical) frame $\alpha$ of a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is a family of $\mathcal{M}$-arrows $\alpha_e$, one for each object $e \in \mathcal{E}$, that is natural in the sense that the triangle
\[
\begin{array}{c}
\star \\
\Downarrow h
\end{array}
\begin{array}{c}
e \\
\Downarrow h
\end{array}
\begin{array}{c}
e' \\
\Downarrow h
\end{array}
\end{array}
\begin{array}{c}
\alpha_e \\
\alpha_e'
\end{array}
\]
commutes for every $E$-arrow $h : e \to e'$ [op. $h : e' \to e$]. The $\mathcal{M}$-arrow $\alpha_e$ is called the component of $\alpha$ at $e$. The set of frames of a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is denoted by $\Pi_E \mathcal{M}$ [op. $\Pi_E \mathcal{M}$] and called the frame-set of $\mathcal{M}$.

4.2.2 Remark. (1) Recall from Remark 1.1.4(3) that a right module over a category $E$ is the same thing as a left module over the opposite category $E^\text{op}$, and note that a frame of a right module $\mathcal{M} : E \to *$ is the same thing as a frame of a left module $\mathcal{M} : * \to E^\text{op}$; $\Pi_E \mathcal{M}$ denotes the frame-set of a right module $\mathcal{M} : * \to E^\text{op}$.

(2) If a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is small, so is the frame-set $\Pi_E \mathcal{M}$ [op. $\Pi_E \mathcal{M}$].

(3) If $E$ is discrete, then the frame-set of a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is just the cartesian product $\prod_{e \in \mathcal{E}} \{\mathcal{M}\} e$ [op. $\Pi_E \mathcal{M}$]. A frame-set is thus seen as a categorical generalization of the notion of a cartesian product.

4.2.3 Example. Given a functor $L : E \to C$ and an object $c \in \mathcal{C}$,
\begin{itemize}
  \item a cone $\alpha$ from $c$ to $L$ is the same thing as a frame $\alpha$ of the composite left module $c(C)L : * \to E$.
  \item a cone $\alpha$ from $L$ to $c$ is the same thing as a frame $\alpha$ of the composite right module $L(C)c : E \to *$.
\end{itemize}

\textbf{Note.} The diagonal functor $[!E : E] : [E : [E : E]] \to [E : E]$ (see Example 1.1.31(8)) embeds the category of left modules $* \to E$ into the category of endomodules $E \to E$. A left module $* \to E$ may thus be seen as a special instance of an endomodule $E \to E$, and

4.2.4 Proposition. A frame of a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is the same thing as a frame of the endomodule $[!E] \{\mathcal{M}\} : E \to E$ [op. $\{\mathcal{M}\}[!E] : E \to E$]; that is,
\[
\Pi_E \mathcal{M} = \Pi_E [!E] \{\mathcal{M}\} \quad \text{op.} \quad \Pi_E \mathcal{M} = \Pi_E \{\mathcal{M}\}[!E].
\]

\textbf{Proof.} Consider a family of $\mathcal{M}$-arrows $\alpha_e : * \to e$ indexed by the objects $e \in \mathcal{E}$. Since the functor $!E : E \to *$ sends each $E$-arrow $h : e \to e'$ to the identity $* \to *$, the triangle
\[
\begin{array}{c}
\star \\
\Downarrow h
\end{array}
\begin{array}{c}
e \\
\Downarrow h
\end{array}
\begin{array}{c}
e' \\
\Downarrow h
\end{array}
\end{array}
\begin{array}{c}
\alpha_e \\
\alpha_e'
\end{array}
\]
commutes iff the square
\[
\begin{array}{c}
e \\
\Downarrow h
\end{array}
\begin{array}{c}
e' \\
\Downarrow h
\end{array}
\end{array}
\begin{array}{c}
h \cdot !E \\
\Downarrow h
\end{array}
\]
in $[!E] \{\mathcal{M}\}$ commutes. Hence $\alpha$ forms a frame of $\mathcal{M}$ iff it forms a frame of $[!E] \{\mathcal{M}\}$.

\textbf{Note.} Since the endomodule $[!E] \{\mathcal{M}\} : E \to E$ is the same thing as the left module $\{\mathcal{M}\}[!E \times E] : * \to E \times E$ (see Example 1.1.31(9)), we have the following variation of Proposition 4.2.4.

4.2.5 Proposition. A frame of a left module $\mathcal{M} : * \to E$ [op. right module $\mathcal{M} : E \to *$] is the same thing as a cylindrical frame of the left module $\{\mathcal{M}\}[!E \times E] : * \to E \times E$ [op. $[!E \times E] \{\mathcal{M}\} : E \times E \to *$]; that is,
\[
\Pi_E \mathcal{M} = \Pi_E \{\mathcal{M}\}[!E \times E] \quad \text{op.} \quad \Pi_E \mathcal{M} = \Pi_E [!E \times E] \{\mathcal{M}\}.
\]
4.2.6 Definition. For a left module morphism $\psi : \mathcal{M} \to \mathcal{N}$ and a frame $\alpha$ of $\mathcal{M}$, their composite $\alpha \circ \psi = \psi \circ \alpha$ is the frame of $\mathcal{N}$ defined by $[\alpha \circ \psi]_e = \alpha_e \cdot \langle \psi \rangle e$ for any frame $\alpha$ of $\mathcal{M}$.

4.2.7 Remark. The composite $\alpha \circ \psi$ so defined does form a frame of $\mathcal{N}$. In fact, we have the following.

4.2.8 Proposition. The diagram

\[
\begin{array}{ccc}
\Pi \mathcal{E} \mathcal{M} & \longrightarrow & \Pi \mathcal{E} \langle \mathcal{M} \rangle \\
\Pi \mathcal{E} \psi & \downarrow & \Pi \mathcal{E} \langle \psi \rangle \downarrow \\
\Pi \mathcal{E} \mathcal{N} & \longrightarrow & \Pi \mathcal{E} \langle \mathcal{N} \rangle
\end{array}
\]

(see Proposition 4.2.4 for the identities on the top and bottom) commutes; that is,

\[
\Pi \mathcal{E} \Psi = \Pi \mathcal{E} \langle \mathcal{M} \rangle [\mathcal{E}].
\]

4.2.9 Proposition. The assignment $\Psi \mapsto \Pi \mathcal{E} \Psi$ is functorial and defines a functor

\[
\Pi \mathcal{E} : [: \mathcal{E}] \to \text{Set} \quad \text{op.} \quad \Pi \mathcal{E} : [\mathcal{E}] \to \text{Set}
\]

, i.e. a (locally small) left module

\[
\Pi \mathcal{E} : * \to [: \mathcal{E}] \quad \text{op.} \quad \Pi \mathcal{E} : * \to [\mathcal{E}]
\]

for $\mathcal{E}$ small; in fact, $\Pi \mathcal{E}$ is obtained from the left module $\Pi \mathcal{E}$ in Proposition 4.1.7 by the composition

\[
* \Rightarrow \Pi \mathcal{E} \mathcal{E} \Rightarrow \mathcal{E} \Rightarrow \mathbb{E} \Rightarrow [\mathcal{E}].
\]

Proof. The second assertion is immediate from Proposition 4.2.8. The first assertion follows from the second. \qed
4.2.10 Definition. If \( K : D \to E \) is a functor and \( \alpha \) is a frame of a left module \( M : * \to E \) [op. right module \( M : E \to * \)], then their composite \( K \circ \alpha = \alpha \circ K \) is the frame of the left module \( (M)K : * \to D \) [op. right module \( K(M) : D \to * \)] defined by
\[
[K \circ \alpha]_d = \alpha(K \cdot d)
\]
for \( d \in [D] \); the function
\[
\prod K M : \prod E M \to \prod D (M) K \quad \text{op.} \quad \prod K M : \prod E M \to \prod D K (M)
\]
is defined by
\[
\alpha : \prod K M = K \circ \alpha \quad \text{op.} \quad \alpha : \prod K M = K \circ \alpha.
\]

4.2.11 Remark. The composite \( K \circ \alpha \) so defined does form a frame of \( (M)K \) [op. \( K(M) \)]. In fact, we have the following.

4.2.12 Proposition. For any functor \( K : D \to E \) and any left module \( M : * \to E \) [op. right module \( M : E \to * \)], the diagram
\[
\begin{array}{ccc}
\prod E M & \longrightarrow & \prod E ![E](M) \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\prod D (M) K & = & \prod D K ([E](M)) K
\end{array}
\]
commutes; that is,
\[
\prod K M = \prod K ![E](M) \quad \text{op.} \quad \prod K M = \prod K (M) ![E].
\]

Proof. By Proposition 4.2.4, \( \prod E M = \prod E ![E](M) \) and \( \prod D (M) K = \prod D ![D](M) K \), but since the composition \( D \xrightarrow{K} E \xrightarrow{[E]} * \) yields \( D \xrightarrow{[D]} * \),
\[
[!D](M) K = [K \circ ![E](M) K = K([!E](M)) K
\]
; hence the identities on the top and bottom of the diagram hold. It is now clear that for any frame \( \alpha \in \prod E M = \prod E ![E](M) \), the composition \( K \circ \alpha \) in Definition 4.2.10 and Definition 4.1.8 yields the same frame. \( \square \)

4.2.13 Proposition. Given a functor \( K : D \to E \) between small categories \( D \) and \( E \), the family of functions
\[
\prod K M : \prod E M \to \prod D (M) K \quad \text{op.} \quad \prod K M : \prod E M \to \prod D K (M)
\]
, one for each left module \( M : * \to E \) [op. right module \( M : E \to * \)], defines a left module cell
\[
\begin{array}{ccc}
* \xrightarrow{\prod E} [E] & \longrightarrow & \prod E [E] \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
* \xrightarrow{\prod D} [D] & = & \prod D [D]
\end{array}
\]
; in fact, the cell \( \prod K \) is obtained from the cell \( \prod K \) in Proposition 4.1.12 by the pasting composition
\[
\begin{array}{ccc}
* \xrightarrow{\prod E} [E] & \longrightarrow & \prod E [E] \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
* \xrightarrow{\prod D} [D] & = & \prod D [D]
\end{array}
\]
; moreover, the assignment \( K \mapsto \prod K \) [op. \( K \mapsto \prod K \)] is contravariant functorial.

Proof. The first assertion follows from the second, which is immediate from Proposition 4.2.12. The functoriality of the assignment \( K \mapsto \prod K \) is now reduced to that in Proposition 4.1.12 by virtue of Corollary 1.2.39. \( \square \)
4.3 Cylinders

This section introduces a cylinder as an instance of a cylindrical frame. For a category $\mathcal{E}$ and a module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$, a two-sided cylinder

$$
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar@{-->}[r]^{\alpha} \ar@{-->}[d]_{\mathcal{M}} & \mathcal{A} \\
\mathcal{E} \ar[r]^{\mathcal{F}} & \mathcal{F} 
}
\end{array}
\end{array}
$$

(a natural transformation $\mathcal{G} \to \mathcal{F}$ spanning $\mathcal{M}$, so to speak) is defined by a frame $\alpha$ of the composite endomodule $\mathcal{G}(\mathcal{M}) \mathcal{F} : \mathcal{E} \to \mathcal{E}$, while a right (one-sided) cylinder

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^\alpha & \mathcal{A} \\
\mathcal{E} \ar[r]^\mathcal{F} & \mathcal{F}
}
\end{array}
$$

is defined by a frame $\alpha$ of the composite endomodule $\mathcal{G}(\mathcal{M}) : \mathcal{A} \to \mathcal{A}$, and a left (one-sided) cylinder

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^\alpha & \mathcal{A} \\
\mathcal{E} \ar[r]^\mathcal{F} & \mathcal{F}
}
\end{array}
$$

is defined by a frame $\alpha$ of the composite endomodule $(\mathcal{M}) \mathcal{F} : \mathcal{X} \to \mathcal{X}$.

Two-sided cylinders between a category $\mathcal{E}$ and a module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ form the module $(\mathcal{E}, \mathcal{M}) : [\mathcal{E}, \mathcal{X}] \to [\mathcal{E}, \mathcal{A}]$, and a cell

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^{\mathcal{M}} & \mathcal{A} \\
\mathcal{P} \ar[u]^\psi & \mathcal{Q} \\
\mathcal{Y} \ar[u]_{\psi} & \mathcal{B}
}
\end{array}
$$

yields the postcomposition cell

$$
\begin{array}{c}
\xymatrix{
[\mathcal{E}, \mathcal{X}] \ar[r]^{[\mathcal{E}, \mathcal{M}]} & [\mathcal{E}, \mathcal{A}] \\
[\mathcal{E}, \mathcal{P}] \ar[u]^\psi & [\mathcal{E}, \mathcal{Q}] \\
[\mathcal{E}, \mathcal{Y}] \ar[u]_{\psi} & [\mathcal{E}, \mathcal{B}]
}
\end{array}
$$

that sends each cylinder $\xymatrix{
\mathcal{E} \ar[r]^{\mathcal{F}} & \mathcal{F} \\
\mathcal{G} \ar[u]^\alpha \ar[r] & \mathcal{M} \ar[u]^\alpha
}$ to its composite $\xymatrix{
\mathcal{G} \diamond \mathcal{F} \ar[r]^{\alpha \diamond \mathcal{F}} & \mathcal{M} \ar[r]^{\mathcal{M} \alpha} & \mathcal{A} \\
\mathcal{E} \ar[u]^\psi \ar[r] & \mathcal{B}
}$ with $\psi$. In Section 6.5, a lift is defined by a universal arrow of the module $(\mathcal{E}, \mathcal{M})$.

4.3.1 Definition. Let $\mathcal{E}$ be a category and $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ be a module. Given a pair of functors $\mathcal{G} : \mathcal{E} \to \mathcal{X}$ and $\mathcal{F} : \mathcal{E} \to \mathcal{A}$, a (two-sided) cylinder $\alpha$ from $\mathcal{G}$ to $\mathcal{F}$ along $\mathcal{M}$, or diagrammatically as

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^\alpha & \mathcal{A} \\
\mathcal{E} \ar[r]^\mathcal{F} & \mathcal{F}
}
\end{array}
$$

is defined by a frame $\alpha$ of the composite endomodule $\mathcal{G}(\mathcal{M}) \mathcal{F} : \mathcal{E} \to \mathcal{E}$.

4.3.2 Remark.

(1) The naturality of a cylinder $\alpha : \mathcal{G} \to \mathcal{F} : \mathcal{E} \to \mathcal{M}$ is expressed by the commutativity of the square

$$
\begin{array}{c}
\xymatrix{
\mathcal{E} \ar[r]^\mathcal{F} \ar[d]^\mathcal{G} & \mathcal{F} \ar[d]^\mathcal{F} \\
\mathcal{F} \ar[r]^\mathcal{F} & \mathcal{F}
}
\end{array}
$$

for each $\mathcal{E}$-arrow $h : e \to e'$.

(2) By Example 4.1.3(1), a natural transformation $\alpha : \mathcal{G} \to \mathcal{F} : \mathcal{E} \to \mathcal{C}$ is the same thing as a cylinder $\alpha : \mathcal{G} \to \mathcal{F} : \mathcal{E} \to (\mathcal{C})$ along the hom-module of $\mathcal{C}$. Conversely, since $\mathcal{M} = [\mathcal{M}_0]([\mathcal{M}]) [\mathcal{M}_1]$ for any module $\mathcal{M} : \mathcal{X} \to \mathcal{A}$ (see Remark 3.1.15(2)), a cylinder

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^\alpha & \mathcal{A} \\
\mathcal{E} \ar[r]^\mathcal{F} & \mathcal{F}
}
\end{array}
$$

from $\mathcal{G}$ to $\mathcal{F}$ along $\mathcal{M}$ is the same thing as a natural transformation

$$
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^{\mathcal{M}} \ar[r] & [\mathcal{M}] \ar[r]_{\mathcal{M}_1} & \mathcal{A}
}
\end{array}
$$
from $G \circ \mathcal{M}_0$ to $\mathcal{M}_1 \circ F$ in the collage category $[\mathcal{M}]$.

(3) Since

$$[G \circ \mathcal{K}(E)] F = G(\mathcal{K}(E)) F \quad \text{op.} \quad G(\mathcal{E}) [K \circ F] = G(\mathcal{E} K) F$$

a natural transformation

![Diagram](image)

is the same thing as a cylinder

![Diagram](image)

from $G$ to $F$ along the representable module $K(E)$ [op. corepresentable module $(E) K$].

4.3.3 Definition. Let $\mathcal{M} : X \to A$ be a module.

- Given a functor $G : A \to X$, a right cylinder $\alpha$ from $G$ to $\mathcal{M}$, written $\alpha : G \rightsquigarrow \mathcal{M}$ or diagrammatically as $X \xrightarrow{\alpha} \mathcal{M} \xrightarrow{G} A$, is defined by a frame $\alpha$ of the composite endomodule $G(\mathcal{M}) : A \to A$.

- Given a functor $F : X \to A$, a left cylinder $\alpha$ from $\mathcal{M}$ to $F$, written $\mathcal{M} \rightsquigarrow F$ or diagrammatically as $X \xleftarrow{\alpha} \mathcal{M} \xleftarrow{F} A$, is defined by a frame $\alpha$ of the composite endomodule $\mathcal{M} F : X \to X$.

4.3.4 Remark.

(1) The naturality of a right cylinder $\alpha : G \rightsquigarrow \mathcal{M}$ is expressed by the commutativity of the square

$$\begin{array}{ccc}
a : G & \xrightarrow{\alpha} & a \\
\| & G \downarrow & \|_f \\
\alpha : G & \xrightarrow{\alpha} & a'
\end{array}$$

for each $A$-arrow $f : a \to a'$, and the naturality of a left cylinder $\alpha : \mathcal{M} \rightsquigarrow F$ is expressed by the commutativity of the square

$$\begin{array}{ccc}
x & \xrightarrow{\alpha x} & F x \\
\| & \|_{F f} \\
x' & \xleftarrow{\alpha x'} & F x'
\end{array}$$

for each $X$-arrow $f : x \to x'$.

(2) An arrow $m : x \rightsquigarrow \ast$ of the right module $\mathcal{M} : X \to \ast$ is identified with a right cylinder $X \xrightarrow{m} \ast$

(where $\ast$ denotes the terminal category), and an arrow $m : \ast \rightsquigarrow a$ of the left module $\mathcal{M} : \ast \to A$

is identified with a left cylinder $\ast \xleftarrow{m} A$.

(3) Given a pair of functors $X \xrightarrow{G \circ F} A$, since

$$[G \circ F]\langle A \rangle [1_A] = G(\mathcal{F}(A))$$

a natural transformation $\epsilon : G \circ F \to 1_A : A \to A$ is the same thing as a right cylinder $X \xrightarrow{\epsilon} \mathcal{M} \xrightarrow{G \circ F} A$ from $G$ to the representable module $\mathcal{F}(A)$, and since

$$[1_X]\langle X \rangle [G \circ F] = (\langle X \rangle G) F$$

a natural transformation $\eta : 1_X \to G \circ F : X \to X$ is the same thing as a left cylinder $X \xleftarrow{\eta} \mathcal{M} \xleftarrow{G \circ F} A$

from the corepresentable module $\langle X \rangle G$ to $F$. 


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(4) By Remark 4.1.2(4) and Remark 1.1.37(3),

$\prod X (\mathcal{M}) F = \prod X F (\mathcal{M}^\prime)$

; hence a left cylinder $\alpha : \mathcal{M} \rightsquigarrow F$ is the same thing as a right cylinder $\alpha : F \rightsquigarrow \mathcal{M}^\prime$.

(5) A right cylinder $\xrightarrow{\alpha \ M} A$ from $G$ to $\mathcal{M}$ can be depicted as a two-sided cylinder $\xrightarrow{\alpha} A$ from $G$ to the identity $1_{\mathcal{A}}$ along $\mathcal{M}$, and a left cylinder $\xrightarrow{\alpha \ M} A$ from $\mathcal{M}$ to $F$ can be depicted as a two-sided cylinder $\xrightarrow{\alpha} A$ from the identity $1_{\mathcal{X}}$ to $F$ along $\mathcal{M}$.

(6) Since $G(\mathcal{M}) F = G(\mathcal{M}) F = (G(\mathcal{M})) F$, the following right, two-sided, and left cylinders are the same thing:

$$\begin{align*}
\xrightarrow{\alpha \ M} E & = \xrightarrow{\alpha \ M} E \\
\xrightarrow{\alpha \ M} A & = \xrightarrow{\alpha \ M} A
\end{align*}$$

Note. The following definition is an instance of Definition 4.1.5 where $\psi : \mathcal{M} \to \mathcal{N} : E \to E$ is given by the module morphism $\sigma (\mathcal{M}) \tau : G(\mathcal{M}) F \to (G(\mathcal{M})) F : E \to E$ (cf. Example 1.1.31(3)).

4.3.5 Definition. Given a cylinder $\alpha : \mathcal{M} \to F$ and natural transformations $\tau : G' \to G$ and $\sigma : F \to F'$ as in

$$\begin{align*}
G' & \xrightarrow{\tau \ M} G \\
\xrightarrow{\alpha \ M} E & \xrightarrow{\sigma \ M} F'
\end{align*}$$

, their composite is the cylinder

$$\begin{align*}
\xrightarrow{\tau \ M} G & \xrightarrow{\alpha \ M} E \\
\xrightarrow{\sigma \ M} F' & \xrightarrow{\alpha \ M} A
\end{align*}$$

defined by

$$\tau \circ \alpha \circ \sigma = \alpha \circ \tau (\mathcal{M}) \sigma$$

, where the composition on the right-hand side of the identity is that defined in Definition 4.1.5.

4.3.6 Remark.

(1) Recalling Definition 4.1.5 and noting Example 1.1.31(3), we see that each component of the composite is given by

$$[\tau \circ \alpha \circ \sigma]_e = [\alpha \circ \tau (\mathcal{M}) \sigma]_e = \alpha_e \cdot e (\tau (\mathcal{M}) \sigma) e = \tau_e \circ \alpha_e \circ \sigma_e.$$

(2) If $\mathcal{M}$ is given by the hom-module of a category $\mathcal{C}$, then the composition is just the usual composition of natural transformations.

(3) As a special case (see Remark 4.3.4(5)), the composition of a right [op. left] cylinder and a natural transformation as in

$$\begin{align*}
\xrightarrow{\tau \ M} G & \xrightarrow{\alpha \ M} E \\
\xrightarrow{\alpha \ M} A & \xrightarrow{\sigma \ M} F'
\end{align*}$$

yields the right [op. left] cylinder

$$\begin{align*}
\xrightarrow{\tau \circ \alpha \ M} A & \xrightarrow{\alpha \ M} A \\
\xrightarrow{\alpha \ M} A & \xrightarrow{\alpha \ M} A
\end{align*}$$

defined by

$$\tau \circ \alpha = \alpha \circ \tau (\mathcal{M}) \quad \text{op.} \quad \alpha \circ \sigma = \alpha \circ (\mathcal{M}) \sigma$$
The composition in the module for a pair of functors \( E \rightarrow M \); the functoriality in Proposition 4.1.7 and Remark 1.1.27 allow the following definition.

### 4.3.7 Definition

Given a category \( E \) and a module \( M : X \rightarrow A \), the module \( (E,M) : [E,X] \rightarrow [E,A] \) of cylinders \( E \sim M \) is defined by

\[
(G)(E,M)(F) = \prod_E G(M) F
\]

for \( G \in [E,X] \) and \( F \in [E,A] \).

### 4.3.8 Remark

1. For a pair of functors \( G : E \rightarrow X \) and \( F : E \rightarrow A \), the set

\[
(G)(E,M)(F) = \prod_E G(M) F
\]

consists of all frames of the composite endomodule \( G(M)F : E \rightarrow E \), i.e. all cylinders \( G \sim F : E \sim M \).

2. The composition in the module \( (E,M) \) is given by

\[
\begin{array}{c}
\tau \downarrow \\
\downarrow \sigma \\
G' \tau \ast \sigma
\end{array}
\]

in \( (E,M) \) is given by

\[
\tau \ast \sigma = \alpha : \prod_E \tau(M) \sigma = \alpha : \tau(M) \sigma
\]

\((\alpha : \prod_E \tau(M) \sigma)\) is the image of \( \alpha \in \prod_E G(M) F \) under the function \( \prod_E \tau(M) \sigma : \prod_E G(M) F \rightarrow \prod_E G'(M) F' \).

3. Noting Remark 4.3.6(2), we see that the module of cylinders \( E \sim C \) (i.e. natural transformations \( E \sim C \)) is the same thing as the hom-module of the functor category \([E,C] \):

\[
(E,C) = (E,C).
\]

4. If \( E \) is small and \( M \) is locally small, then the module \( (E,M) \) is locally small.

### 4.3.9 Proposition

Given a category \( E \) and a composite module \( P(N)Q \) as in

\[
\begin{array}{c}
X \quad P(N)Q \\
\downarrow \quad 1 \\
Y \quad \downarrow Q
\end{array}
\]

the identity

\[
\begin{array}{c}
[E,X] \quad [E,P(N)Q] \\
\downarrow \quad 1 \\
[E,Y] \quad [E,N] \quad [E,Q]
\end{array}
\]

(i.e. \( (E,P(N)Q) = [E,P]([E,N] [E,Q]) \)) holds.
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Proof. For any $G \in [E, X]$ and $F \in [E, A]$, 
\[ (G) \langle E, P \langle N \rangle Q \rangle (F) = \prod_E G(P \langle N \rangle Q) F \]
\[ = \prod_E [G \circ P] \langle N \rangle [Q \circ F] \]
\[ = (G \circ P) \langle E, N \rangle (Q \circ F) \]
\[ = (G \cdot [E, P]) \langle E, N \rangle ([E, Q] \cdot F) \]
\[ = (G) \langle [E, P] \langle E, N \rangle [E, Q] \rangle (F). \]

4.3.10 Remark. The cell $\langle E, P \langle N \rangle Q \rangle \xrightarrow{1} \langle E, N \rangle$ above sends each cylinder

\[
\begin{array}{c}
X \\
\xrightarrow{\alpha}
\end{array}
\begin{array}{c}
E \\
\xrightarrow{\psi}
\end{array}
\begin{array}{c}
\psi \cdot F \\
\xrightarrow{P}
\end{array}
\begin{array}{c}
A \\
\xrightarrow{\cdot}
\end{array}
\]

\[
\begin{array}{c}
Y \\
\xrightarrow{\alpha}
\end{array}
\begin{array}{c}
\psi \cdot E \\
\xrightarrow{G \cdot P}
\end{array}
\begin{array}{c}
\psi \cdot Q \cdot F \\
\xrightarrow{Q \cdot F}
\end{array}
\begin{array}{c}
B \\
\xrightarrow{\cdot}
\end{array}
\]

defined by the same frame.

Note. The following definition is an instance of Definition 4.1.5 where $\psi : M \to N : E \to E$ is given by the module morphism $G(\psi) F : G(M) F \to G(N) F : E \to E$ (cf. Example 1.1.31(2)).

4.3.11 Definition. If $\alpha : E \to M$ is a cylinder and $\psi : M \to N$ is a module morphism as in

\[
\begin{array}{c}
X \\
\xrightarrow{\alpha}
\end{array}
\begin{array}{c}
\psi \cdot M \\
\xrightarrow{G \cdot \psi}
\end{array}
\begin{array}{c}
\psi \cdot F \\
\xrightarrow{\psi \cdot F}
\end{array}
\begin{array}{c}
A \\
\xrightarrow{\cdot}
\end{array}
\]

then their composite $\alpha \circ \psi = \psi \circ \alpha$ is the cylinder

\[
\begin{array}{c}
X \\
\xrightarrow{\alpha \circ \psi}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot M \\
\xrightarrow{G \circ \psi \cdot \alpha}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot F \\
\xrightarrow{\alpha \circ \psi \cdot F}
\end{array}
\begin{array}{c}
A \\
\xrightarrow{\cdot}
\end{array}
\]

defined by

$\alpha \circ \psi = \alpha \circ \psi \circ G(\psi) F$

where the composition on the right-hand side of the identity is that defined in Definition 4.1.5.

4.3.12 Remark.

(1) Recalling Definition 4.1.5 and noting Example 1.1.31(2), we see that each component of the composite is given by

$[\alpha \circ \psi]_e = [\alpha \circ \psi \cdot G(\psi) F]_e = \alpha_e \cdot (e \cdot G(\psi) F) e = \alpha_e \cdot (e \cdot G \psi) (F \cdot e)$.

(2) As a special case (see Remark 4.3.4(5)), the composition of a right [op. left] cylinder $\alpha$ and a module morphism $\psi$ as in

\[
\begin{array}{c}
X \\
\xrightarrow{\alpha \circ \psi}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot M \\
\xrightarrow{G \circ \alpha \circ \psi}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot F \\
\xrightarrow{\alpha \circ \psi \cdot F}
\end{array}
\begin{array}{c}
A \\
\xrightarrow{\cdot}
\end{array}
\]

yields the right [op. left] cylinder

\[
\begin{array}{c}
X \\
\xrightarrow{\alpha \circ \psi}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot M \\
\xrightarrow{G \circ \alpha \circ \psi}
\end{array}
\begin{array}{c}
\alpha \circ \psi \cdot F \\
\xrightarrow{\alpha \circ \psi \cdot F}
\end{array}
\begin{array}{c}
A \\
\xrightarrow{\cdot}
\end{array}
\]

defined by

$\alpha \circ \psi = \alpha \circ \psi \circ G(\psi) \quad \text{op.} \quad \psi \circ \alpha = \alpha \circ (\psi) F$
so that
\[ [\alpha \circ \psi]_a = \alpha_x \cdot (a : G)(\psi) a \quad \text{op.} \quad [\psi \circ \alpha]_a = \alpha_x \cdot x(\psi)(F \cdot x) \]
for each \( a \in [A] \) [op. \( x \in [X] \)].

Note. For any category \( E \) and any module morphism \( \psi : M \to N \), the composition in Definition 4.3.11 yields the postcomposition module morphism \( \langle E, \psi \rangle : \langle E, M \rangle \to \langle E, N \rangle \) from the module of cylinders \( E \rightsquigarrow M \) to the module of cylinders \( E \rightsquigarrow N \). Here is the formal definition:

4.3.13 Definition. Given a category \( E \) and a module morphism \( \psi : M \to N : X \to A \), the postcomposition module morphism
\[ \langle E, \psi \rangle : \langle E, M \rangle \to \langle E, N \rangle : [E, X] \to [E, A] \]
“postcomposition with \( \psi \),” is defined by
\[ (G)(E, \psi)(F) = \prod_E G(\psi) F \]
for each pair of functors \( G : E \to X \) and \( F : E \to A \).

4.3.14 Remark.
(1) Replacing \( \psi : M \to N \) by \( G(\psi)F : G(M)F \to G(N)F \) in Definition 4.1.5, we have a function
\[ \prod_E G(\psi)F : \prod_E G(M)F \to \prod_E G(N)F \]
or
\[ (G)(E, \psi)(F) : (G)(E, M)(F) \to (G)(E, N)(F) \]
in the notation introduced in Definition 4.3.7. Since \( G(\psi)F \) is natural in \( G \) and \( F \) by Remark 1.1.27, so is \( (G)(E, \psi)(F) = \prod_E G(\psi) F \) by Proposition 4.1.7. Hence \( \langle E, \psi \rangle \) so defined does form a module morphism \( \langle E, M \rangle \to \langle E, N \rangle \).

(2) The module morphism \( \langle E, \psi \rangle \) maps each cylinder \( \alpha : G \rightsquigarrow F : E \rightsquigarrow M \) to
\[ \alpha : \prod_E G(\psi)F = \alpha \circ G(\psi)F \]
; that is, to the cylinder \( \alpha \circ \psi : G \rightsquigarrow F : E \rightsquigarrow N \) defined in Definition 4.3.11.

(3) The assignment \( \psi \mapsto \langle E, \psi \rangle \) is functorial; indeed, the functor
\[ \langle E, \cdot \rangle : [X : A] \to [[E, X] : [E, A]] \]
is defined by
\[ (G)(E, M)(F) = \prod_E G(M)F \]
for \( G \in [E, X] \), \( F \in [E, A] \), and \( M \in [X : A] \).

Note. By Remark 1.2.2(3), the following definition is a special case of Definition 4.3.11, and vice versa.

4.3.15 Definition. Given a cylinder \( \alpha \) and a cell \( \psi \) as in
\[ \begin{array}{ccc}
G & F \\
\alpha \\
\downarrow E \\
X & \rightsquigarrow M & \to A \\
\downarrow \psi & \downarrow Q \\
Y & \rightsquigarrow N & \to B
\end{array} \]
, their composite \( \alpha \circ \psi = \psi \circ \alpha \) is the cylinder
\[ \begin{array}{ccc}
G & F \\
\alpha \circ \psi \\
\downarrow E \\
Y & \rightsquigarrow N & \to B
\end{array} \]
defined by
\[ \alpha \circ \psi = \alpha \circ G(\psi)F \]
, where the composition on the right-hand side of the identity is that defined in Definition 4.1.5.
4.3.16 Remark.  
(1) Each component of the composite cylinder \( \alpha \circ \psi \) is given as in Remark 4.3.12(1):
\[
[\alpha \circ \psi]_a = \alpha_a \cdot (e : G)(\psi)(F \cdot e).
\]
(2) If a cell is given by the hom-cell of a functor \( H \) as in
\[
\begin{diagram}
G & \xrightarrow{\alpha} & F \\
\node{C} \arrow{e} \node{(C)} \arrow{w} \node{C} \\
\node{(H)} \arrow{e} \node{B} \arrow{w} \node{B}
\end{diagram}
\]
then the composition \( \alpha \circ (H) \) is just the usual composition of a natural transformation and a functor; that is,
\[
\alpha \circ (H) = \alpha \circ H.
\]
(3) As a special case (see Remark 4.3.4(5)), the composition of a right \( \text{op. left} \) cylinder \( \alpha \) and a cell \( \psi \) as in
\[
\begin{diagram}
X & \xrightarrow{\alpha} & A \\
\node{Y} \arrow{e} \node{\alpha \circ \psi} \arrow{w} \node{B}
\end{diagram}
\]
yields the cylinder
\[
\begin{diagram}
G \circ \psi & \xrightarrow{\alpha} & A \\
\node{Y} \arrow{e} \node{\alpha \circ \psi} \arrow{w} \node{B}
\end{diagram}
\]
defined by
\[
\alpha \circ \psi = \alpha \circ G(\psi)
\]
so that
\[
[\alpha \circ \psi]_a = \alpha_a \cdot (a : G)(\psi)a
\]
for each \( a \in \|A\| \) \( \text{op.} x \in \|X\| \).

4.3.17 Definition. Given a category \( E \) and a cell
\[
\begin{diagram}
X & \xrightarrow{\alpha} & A \\
\node{Y} \arrow{e} \node{\alpha \circ \psi} \arrow{w} \node{B}
\end{diagram}
\]
, the postcomposition cell
\[
\begin{diagram}
E & \xrightarrow{(E,M)} & E \\
\node{P} \arrow{e} \node{(E,P)} \arrow{w} \node{Q}
\end{diagram}
\]
is defined by the postcomposition module morphism
\[
(E,M) \xrightarrow{(E,\psi)} (E,P \{N\} Q) = [E,P] (E,N) [E,Q]
\]
(postcomposition with \( \psi : M \rightarrow P (N) Q \)).

4.3.18 Remark.  
(1) The cell \( (E,\psi) \) sends each cylinder \( \alpha : G \sim F : E \sim M \) to the cylinder \( \alpha \circ \psi : G \circ \psi \sim Q \circ \psi \circ F : E \sim N \) defined in Definition 4.3.15.
(2) If \( \psi \) is given by the hom-cell of a functor \( H : C \to B \), the postcomposition cell

\[
\begin{align*}
\left[ E, C \right] & \xrightarrow{(E,C)} \left[ E, C \right] \\
\left[ E, H \right] & \xrightarrow{(E,H)} \left[ E, H \right] \\
\left[ E, B \right] & \xrightarrow{(E,B)} \left[ E, B \right]
\end{align*}
\]

sends each cylinder \( \alpha : E \sim (C) \) to the cylinder \( \alpha \circ (H) : E \sim (B) \); that is, sends each natural transformation \( \alpha : E \to C \) to the natural transformation \( \alpha \circ H : E \to B \), the usual composite of a natural transformation and a functor (see Remark 4.3.16(2)). Hence the postcomposition cell \( \langle E, (H) \rangle \) is the same thing as the hom-cell

\[
\begin{align*}
\left[ E, C \right] & \xrightarrow{(E,C)} \left[ E, C \right] \\
\left[ E, H \right] & \xrightarrow{(E,H)} \left[ E, H \right] \\
\left[ E, B \right] & \xrightarrow{(E,B)} \left[ E, B \right]
\end{align*}
\]

of the postcomposition functor \( [E, H] \); that is,

\[ \langle E, (H) \rangle = \langle E, H \rangle. \]

4.3.19 Proposition. If a cell \( \psi \) is fully faithful, so is the postcomposition cell \( \langle E, \psi \rangle \) for any category \( E \).

Proof. Since the postcomposition operation \( \langle E, - \rangle \) is functorial (see Remark 4.3.14(3)), it preserves isomorphisms. \hfill \Box

Note. The following is a special case of Proposition 4.3.19 where \( \psi \) is given by the hom-cell of a functor.

4.3.20 Proposition. If a functor \( H \) is fully faithful, so is the postcomposition functor \( [E, H] \) for any category \( E \).

Proof. Since a functor is fully faithful iff its hom-cell is fully faithful (see Proposition 1.2.31), and since \( \langle E, (H) \rangle = \langle E, H \rangle \) (see Remark 4.3.18(2)), the assertion follows from Proposition 4.3.19. \hfill \Box

Note. The following is analogous to Proposition 1.2.27. The proofs given for them are almost identical.

4.3.21 Proposition. The assignment \( \psi \mapsto \langle E, \psi \rangle \) of the postcomposition cell is functorial.

Proof. Clearly, the assignment \( \psi \mapsto \langle E, \psi \rangle \) preserves the identities. To verify that it preserves the composition, let \( \psi \) and \( \varphi \) be a composable pair of cells and consider the cells \( \langle E, \psi \rangle, \langle E, \varphi \rangle \), and \( \langle E, \psi \circ \varphi \rangle \) depicted in

\[
\begin{align*}
\text{\( X \rightarrow M \rightarrow A \)} & \quad \text{\( [E, X] \rightarrow [E, A] \)} \\
\text{\( P \downarrow \quad \psi \downarrow \quad \varphi \downarrow \)} & \quad \text{\( [E, P] \rightarrow [E, \psi] \rightarrow [E, \varphi] \rightarrow [E, Q] \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( Y \rightarrow N \rightarrow B \)} & \quad \text{\( [E, Y] \rightarrow [E, B] \)} \\
\text{\( P' \downarrow \quad \psi \downarrow \quad \varphi' \downarrow \)} & \quad \text{\( [E, P'] \rightarrow [E, \psi] \rightarrow [E, \varphi'] \rightarrow [E, Q'] \)}
\end{align*}
\]

\[
\begin{align*}
\text{\( Z \rightarrow L \rightarrow C \)} & \quad \text{\( [E, Z] \rightarrow [E, C] \)} \\
\text{\( P' \downarrow \quad \psi \circ \varphi \downarrow \quad \varphi' \downarrow \)} & \quad \text{\( [E, P' \circ \varphi] \rightarrow [E, \psi \circ \varphi] \rightarrow [E, Q' \circ \varphi] \rightarrow [E, Q] \)}
\end{align*}
\]

; we need to verify that the composition of the cells \( \langle E, \psi \rangle \) and \( \langle E, \varphi \rangle \) yields the cell \( \langle E, \psi \circ \varphi \rangle \). First note that \( [E, P' \circ \varphi] = [E, P] \circ [E, P'] \) and \( [E, Q' \circ \varphi] = [E, Q'] \circ [E, Q] \) by the
functoriality of the operation \([E, -]\). The cell \(\langle E, \psi \rangle \circ \langle E, \varphi \rangle\) is defined by the module morphism \(\langle E, \psi \rangle \circ [E, P] \circ \langle E, \varphi \rangle [E, Q]\) and the cell \(\langle E, \psi \circ \varphi \rangle\) is defined by the module morphism \((E, \psi \circ P(\varphi) Q)\). But by the functoriality of the operation \((E, -)\) (see Remark 4.3.14(3)) and Proposition 4.3.9,

\[
\langle E, \psi \circ P(\varphi) Q\rangle = \langle E, \psi \rangle \circ \langle E, P(\varphi) Q\rangle = \langle E, \psi \rangle \circ [E, P] \circ \langle E, \varphi \rangle [E, Q].
\]

\[\Box\]

**4.3.22 Remark.**

(1) Given a small category \(E\), the assignment \(\psi \mapsto \langle E, \psi \rangle\) gives the functor

\[
\langle E, - \rangle : \text{MOD} \to \text{MOD}
\]

, extending the functor \(\langle E, - \rangle\) in Remark 4.3.14(3) as shown in the commutative diagram

\[
\begin{array}{ccc}
[X : A] & \xrightarrow{\langle E, - \rangle} & [[[E, X] : [E, A]]] \\
\downarrow & & \downarrow \\
\text{MOD} & \xrightarrow{\langle E, - \rangle} & \text{MOD}
\end{array}
\]

, where \(\uparrow\) denotes the canonical embedding in Remark 1.2.19(2).

(2) For \(E\) small, the identity \(\langle E, \langle H \rangle \rangle = \langle E, H \rangle\) in Remark 4.3.18(2) is now expressed by the commutativity of

\[
\begin{array}{ccc}
\text{CAT} & \xrightarrow{\langle E, - \rangle} & \text{CAT} \\
\downarrow & \downarrow \langle-\rangle & \downarrow \langle-\rangle \\
\text{MOD} & \xrightarrow{\langle E, - \rangle} & \text{MOD}
\end{array}
\]

**4.3.23 Example.** The postcomposition

\[
\begin{array}{ccc}
[E, X] & \xrightarrow{\langle E, M \rangle} & [E, A] \\
[E, M_0] & \downarrow \langle E, M_1 \rangle & \downarrow [E, M_1] \\
[E, [M]] & \xrightarrow{(E, - \circ \cdot)} & \text{MOD}
\end{array}
\]

with the unit cell in Remark 3.1.15(2) sends each cylinder \(\alpha : G \to F : E \to M\) to the natural transformation \(\alpha : G \circ M_0 \to M_1 \circ F : E \to [M]\) in Remark 4.3.2(2).

**Note.** Given a category \(E\), a cell \(\psi : M \to N\) yields the postcomposition cell \(\langle E, \psi \rangle : \langle E, M \rangle \to \langle E, N \rangle\) as in Definition 4.3.17, and a cell morphism \(\tau : \psi \to \varphi : M \to N\) yields the postcomposition cell morphism \(\langle E, \tau \rangle : \langle E, \psi \rangle \to \langle E, \varphi \rangle\) as in:

**4.3.24 Definition.** Given a category \(E\) and a cell morphisms \(\tau : \psi \to \varphi : M \to N\), the postcomposition cell morphism

\[
\langle E, \tau \rangle : \langle E, \psi \rangle \to \langle E, \varphi \rangle : \langle E, M \rangle \to \langle E, N \rangle
\]

, “postcomposition with \(\tau\)”, is defined by the pair of postcomposition natural transformations

\[
\begin{array}{ccc}
[E, \tau_0] : [E, \psi_0] & \to [E, \varphi_0] \\
[E, \tau_1] : [E, \psi_1] & \to [E, \varphi_1]
\end{array}
\]

(see Preliminary 0.0.2(1)), where \(\tau_0 : \psi_0 \to \varphi_0\) and \(\tau_1 : \psi_1 \to \varphi_1\) are the left and right components of \(\tau\).

**4.3.25 Remark.**

(1) \(\langle E, \tau \rangle\) so defined does form a cell morphism. Indeed, given a cylinder \(\alpha : G \sim F : E \to M\), the commutativity of

\[
\begin{array}{ccc}
G : [E, \psi_0] & \circ \to & [E, \psi_1] : F \\
G : [E, \tau_0] & \downarrow & [E, \tau_1] : F
\end{array}
\]

\[
\begin{array}{ccc}
G : [E, \varphi_0] & \circ \to & [E, \varphi_1] : F
\end{array}
\]
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(i.e.
\[ G \circ \psi_0 \sim \sim \psi_1 \delta F \]
\[ G \circ \varphi_0 \sim \sim \varphi_1 \delta F \]
)
follows from the commutativity of
\[ e \cdot G \circ \psi_0 \sim \sim \psi_1 \cdot F \cdot e \]
\[ e \cdot G \circ \varphi_0 \sim \sim \varphi_1 \cdot F \cdot e \]
for each object \( e \in \| E \| \).

(2) The assignment \( \tau \mapsto (E, \tau) \) defines a functor
\[ (E, -) : [M : N] \rightarrow [(E, M) : (E, N)] \]
; indeed, by the definition of the cell morphism \((E, \tau)\), the functoriality of \((E, -)\) is reduced to that of \([E, -] : [M_i, N_i] \rightarrow [[E, M_i], [E, N_i]]\) for \( i = 0, 1 \).

Note. The following definition is an instance of Definition 4.1.8 where \( M : E \rightarrow E \) is given by the composite endomodule \( G(M) F : E \rightarrow E \).

4.3.26 Definition. If \( K \) is a functor and \( \alpha \) is a cylinder as in

\[ \begin{array}{ccc}
D & \overset{\delta \alpha}{\rightarrow} & E \\
\downarrow G \downarrow & \overset{\alpha}{\rightarrow} & \downarrow F \\
X & \overset{\alpha \cdot M}{\rightarrow} & A
\end{array} \]

, then their composite \( K \circ \alpha = \alpha \cdot K \) is the cylinder

\[ \begin{array}{ccc}
D & \overset{K \circ \alpha}{\rightarrow} & E \\
\downarrow K \circ G \downarrow & \overset{K \circ \delta \alpha}{\rightarrow} & \downarrow K \circ F \downarrow \\
X & \overset{K \circ \alpha \cdot M}{\rightarrow} & A
\end{array} \]

defined by
\[ [K \circ \alpha]_d = \alpha(K \cdot d) \]
for each \( d \in \| D \| \).

4.3.27 Remark.
(1) If \( D \) is a subcategory of \( E \), then the composition of the inclusion \( D \overset{D}{\rightarrow} E \) with \( \alpha \) yields a cylinder \( D \circ \alpha \), the restriction of \( \alpha \) to \( D \).

(2) If \( M \) is given by the hom-module of a category, then the composition \( K \circ \alpha \) is just the usual composition of a functor and a natural transformation.

(3) As a special case (recall Remark 4.3.4(5)), if \( K \) is a functor and \( \alpha \) is a right \([\text{op. left}]\) cylinder as in

\[ \begin{array}{ccc}
X & \overset{\alpha \cdot M}{\rightarrow} & A \\
\downarrow G \downarrow & \overset{\alpha}{\rightarrow} & \downarrow F \\
E & \overset{K}{\rightarrow} & A
\end{array} \]

, then their composite \( K \circ \alpha = \alpha \circ K \) is the two-sided cylinder

\[ \begin{array}{ccc}
X & \overset{K \circ \alpha}{\rightarrow} & A \\
\downarrow K \circ G \downarrow & \overset{K \circ \alpha}{\rightarrow} & \downarrow K \circ F \downarrow \\
E & \overset{K \circ \alpha}{\rightarrow} & A
\end{array} \]

defined by
\[ [K \circ \alpha]_e = \alpha(K \cdot e) \]
for each \( e \in \| E \| \).
Note. Given a functor $K : D \to E$, the composition in Definition 4.3.26 yields the precomposition cell $(K, M) : (E, M) \to (D, M)$ from the module of cylinders $E \rightsquigarrow M$ to the module of cylinders $D \rightsquigarrow M$. Here is the formal definition:

**4.3.28 Definition.** Given a functor $K : D \to E$ and a module $M : X \to A$, the precomposition cell

\[
\begin{align*}
[E, X] & \xrightarrow{(E, M)} [E, A] \\
[K, X] & \downarrow \quad \downarrow (K, A) \\
[D, X] & \xrightarrow{(D, M)} [D, A]
\end{align*}
\]

, “precomposition with $K$”, is defined by

\[(G \langle K, M \rangle (F) = \prod_K G(M) F\]

for each pair of functors $G : E \to X$ and $F : E \to A$.

**4.3.29 Remark.**

1. Replacing $M : E \to E$ by $G(M) F : E \to E$ in Definition 4.1.8, we have a function

\[\prod_K G(M) F : \prod_E G(M) F \to \prod_D K(G(M) F) K = \prod_D [K \circ G] (M) [F \circ K]\]

or

\[(G \langle K, M \rangle (F) : (G \langle E, M \rangle (F) \to (K \circ G) \langle D, M \rangle (F \circ K)\]

in the notation introduced in Definition 4.3.7, natural in $G$ and $F$ by Proposition 4.1.11 and Remark 1.1.27. Hence $(K, M)$ so defined does form a cell $(E, M) \to (D, M)$.

2. The cell $(K, M)$ sends each cylinder $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$ to

\[\alpha : \prod_K G(M) F = K \circ \alpha\]

; that is, to the cylinder $K \circ \alpha : K \circ G \rightsquigarrow F \circ K : D \rightsquigarrow M$ defined in Definition 4.3.26.

3. If $M$ is given by the hom-module of a category $C$, then the precomposition cell

\[
\begin{align*}
[E, C] & \xrightarrow{(E, C)} [E, C] \\
[K, C] & \downarrow \quad \downarrow (K, C) \\
[D, C] & \xrightarrow{(D, C)} [D, C]
\end{align*}
\]

sends each cylinder $\alpha : E \rightsquigarrow (C)$ to the cylinder $K \circ \alpha : D \rightsquigarrow (C)$; that is, sends each natural transformation $\alpha : E \to C$ to the natural transformation $K \circ \alpha : D \to C$, the usual composite of a functor and a natural transformation. Hence the precomposition cell $(K, \langle C \rangle)$ is the same thing as the hom-cell

\[
\begin{align*}
[E, C] & \xrightarrow{(E, C)} [E, C] \\
[K, C] & \downarrow \quad \downarrow (K, C) \\
[D, C] & \xrightarrow{(D, C)} [D, C]
\end{align*}
\]

of the precomposition functor $[K, C]$; that is,

\[\langle K, \langle C \rangle \rangle = \langle K, C \rangle\].

**4.3.30 Proposition.** Given a module $M$, the assignment $K \mapsto (K, M)$ defines the contravariant functor

\[\langle -, M \rangle : \mathbf{Cat}^\circ \to \mathbf{MOD}\]

**Proof.** We note that the precomposition cell $(K, M)$, depicted as

\[
\begin{align*}
* & \xrightarrow{(K, M)} [E, X] \times [E, A] \\
\downarrow & \quad \downarrow (K, M) \\
* & \xrightarrow{(D, M)} [D, X] \times [D, A]
\end{align*}
\]
(see Remark 1.2.5(2)), is obtained from the left module cell in Proposition 4.1.12 by the pasting composition

\[
\begin{array}{c}
\ast \rightarrow \prod_{\mathcal{E}} \left[ E : E \right] \overset{\left(\mathcal{M}, \ast\right)}{-} \left[ E, X \right] \times \left[ E, A \right] \\
\downarrow \prod_{K} \left[ K, K \right] \quad \downarrow \left( K, X \right) \times \left[ K, A \right]
\end{array}
\]

\[
\ast \rightarrow \prod_{\mathcal{D}} \left[ D : D \right] \overset{\left(\mathcal{M}, \ast\right)}{-} \left[ D, X \right] \times \left[ D, A \right]
\]

\((-\mathcal{M})^\ast\) denotes the functor given by the assignment \((G, F) \mapsto G\mathcal{M}F\). The functoriality of the assignment \(K \mapsto (K, \mathcal{M})\) is thus reduced to that of the assignment \(K \mapsto \prod_{K}\) (see Proposition 4.1.12) by virtue of Corollary 1.2.39.

\[\square\]

4.3.31 Example. Let \(E\) be a category and \(\mathcal{M} : X \to A\) be a module. Given an object \(e \in \mathcal{E}\), precomposition with the functor \(e : \ast \to E\) yields the cell

\[
\begin{array}{c}
\left[ E, X \right] \rightarrow \left[ E, A \right] \\
\left[ \left[ \left[ e \right], X \right] \right] \rightarrow \left[ \left[ \left[ e \right], A \right] \right]
\end{array}
\]

\(\ast\rightarrow \mathcal{M}\rightarrow \ast\rightarrow A\)

, “evaluation at \(e\)”, which sends each cylinder \(\alpha : G \rightarrow F : E \rightarrow \mathcal{M}\) to the \(\mathcal{M}\)-arrow \(\alpha_e : e \circ G \rightarrow F \circ e\), the component of \(\alpha\) at \(e\).

4.3.32 Remark. If \(\mathcal{M}\) in the example above is given by the hom-module of a category \(C\), we have a cell

\[
\begin{array}{c}
\left[ E, C \right] \rightarrow \left[ E, C \right] \\
\left[ \left[ \left[ e \right], C \right] \right] \rightarrow \left[ \left[ \left[ e \right], C \right] \right]
\end{array}
\]

\(\ast\rightarrow \left(\mathcal{M}, C\right)\rightarrow \ast\rightarrow C\)

; by Remark 4.3.29(3), this cell is the same thing as the hom-cell

\[
\begin{array}{c}
\left[ E, C \right] \rightarrow \left[ E, C \right] \\
\left[ \left[ \left[ e \right], C \right] \right] \rightarrow \left[ \left[ \left[ e \right], C \right] \right]
\end{array}
\]

\(\ast\rightarrow \left(\mathcal{M}, C\right)\rightarrow \ast\rightarrow C\)

of the precomposition functor \([e, C] : [E, C] \to C\) — evaluation at \(e\) (see Preliminary 0.0.6(1)).

4.3.33 Theorem. There is a functor

\[
\langle - , - \rangle : \text{Cat}^\ast \times \text{MOD} \to \text{MOD}
\]

such that for each small category \(E\),

\[
\langle E, - \rangle : \text{MOD} \to \text{MOD}
\]

coincides with the functor in Remark 4.3.22(1), and for each locally small module \(\mathcal{M}\),

\[
\langle - , \mathcal{M} \rangle : \text{Cat}^\ast \to \text{MOD}
\]

coincides with the functor in Proposition 4.3.30.

Proof. By the bifunctor lemma (see [ML98] p37 Proposition 1), it suffices to show that the square

\[
\begin{array}{c}
\langle E, \mathcal{M} \rangle \rightarrow \langle E, \mathcal{N} \rangle \\
\downarrow \left( K, \mathcal{M} \right) \quad \downarrow \left( K, \mathcal{N} \right)
\end{array}
\]

\[
\langle D, \mathcal{M} \rangle \rightarrow \langle D, \mathcal{N} \rangle
\]

commutes for any functor \(K : D \to E\) and any cell \(\psi : \mathcal{M} \to \mathcal{N}\); that is,

\[
\alpha : \langle E, \psi \rangle : \langle K, \mathcal{N} \rangle = \alpha : \langle K, \mathcal{M} \rangle : \langle D, \psi \rangle
\]

for any cylinder \(\alpha : G \rightarrow F : E \rightarrow \mathcal{M}\). But by Remark 4.3.18(1) and Remark 4.3.29(2),

\[
\alpha : \langle E, \psi \rangle : \langle K, \mathcal{N} \rangle = K \circ (\alpha \circ \psi) = (K \circ \alpha) \circ \psi = \alpha : \langle K, \mathcal{M} \rangle : \langle D, \psi \rangle
\]
Note. Any endomodule $\mathcal{M} : \mathbf{E} \to \mathbf{E}$ yields the endomodule $(\mathbf{D}, \mathcal{M}) : [\mathbf{D}, \mathcal{M}] : [\mathbf{D}, \mathbf{E}] \to [\mathbf{D}, \mathbf{E}]$ of cylinders $\mathbf{D} \Rightarrow \mathcal{M}$ (see Definition 4.3.7) with any category $\mathbf{D}$, and given a frame $\alpha$ of $\mathcal{M}$, the composition in Definition 4.1.8 yields, as defined below, the postcomposition frame $[\mathbf{D}, \alpha]$ of the endomodule $(\mathbf{D}, \mathcal{M})$.

4.3.34 Definition. Let $\mathbf{D}$ be a category and $\mathcal{M} : \mathbf{E} \to \mathbf{E}$ be an endomodule. For any frame $\alpha$ of $\mathcal{M}$, the postcomposition frame $[\mathbf{D}, \alpha]$ of the endomodule $(\mathbf{D}, \mathcal{M}) : [\mathbf{D}, \mathcal{M}] : [\mathbf{D}, \mathbf{E}] \to [\mathbf{D}, \mathbf{E}]$ is defined by

\[ [\mathbf{D}, \alpha]_K = \mathbf{K} \circ \alpha : \mathbf{K} \Rightarrow \mathbf{K} : \mathbf{D} \Rightarrow \mathcal{M} \]

for $\mathbf{K}$ a functor $\mathbf{D} \Rightarrow \mathbf{E}$.

4.3.35 Remark.

1. $[\mathbf{D}, \alpha]$ so defined does form a frame of $(\mathbf{D}, \mathcal{M})$. Indeed, given a natural transformation $\tau : \mathbf{K} \Rightarrow \mathbf{K}' : \mathbf{D} \Rightarrow \mathbf{E}$, the commutativity of the square

\[
\begin{array}{ccc}
\mathbf{K} & \xrightarrow{\mathbf{D}, \alpha}_k = \mathbf{K} \circ \alpha & \mathbf{K} \\
\mathbf{\tau} & \downarrow & \mathbf{\tau} \\
\mathbf{K}' & \xrightarrow{\mathbf{D}, \alpha}_{\mathbf{K}' \circ \alpha} & \mathbf{K}'
\end{array}
\]

is reduced to the commutativity of the square

\[
\begin{array}{ccc}
\mathbf{d} : \mathbf{K} & \xrightarrow{\mathbf{[K, \alpha]}_{\mathbf{K}}} \mathbf{K} \circ \mathbf{d} & \mathbf{K}' \\
\mathbf{d} \circ \mathbf{\tau} & \downarrow & \mathbf{d} \circ \mathbf{\tau} \\
\mathbf{d} : \mathbf{K}' & \xrightarrow{\mathbf{[K', \alpha]}_{\mathbf{K}' \circ \alpha}} \mathbf{K}' \circ \mathbf{d}
\end{array}
\]

for each object $\mathbf{d} \in \parallel \mathbf{D} \parallel$, i.e., the naturality of $\alpha$.

2. If $\mathbf{D}$ is a category and $\alpha$ is a frame of a composite endomodule $\mathbf{G} \langle \mathcal{M} \rangle : \mathbf{E} \Rightarrow \mathbf{E}$, i.e., a cylinder

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathbf{G}} \mathbf{E} & \xrightarrow{\mathbf{F}} \mathbf{A} \\
\mathbf{\alpha} & \mathcal{M} & \mathbf{A}
\end{array}
\]

, then $[\mathbf{D}, \alpha]$ is a frame of the endomodule $(\mathbf{D}, \mathbf{G} \langle \mathcal{M} \rangle F) = [\mathbf{D}, \mathbf{G}] (\mathbf{D}, \mathcal{M}) [\mathbf{D}, \mathbf{F}]$ (see Proposition 4.3.9), i.e., a cylinder

\[
\begin{array}{ccc}
[\mathbf{D}, \mathbf{G}] & \xrightarrow{[\mathbf{D}, \alpha]} & [\mathbf{D}, \mathbf{F}] \\
[\mathbf{D}, \mathbf{X}] & \mathcal{M} & [\mathbf{D}, \mathbf{A}]
\end{array}
\]

; the component of $[\mathbf{D}, \alpha]$ at a functor $\mathbf{K} : \mathbf{D} \Rightarrow \mathbf{E}$ is the cylinder $\mathbf{K} \circ \alpha$ defined in Definition 4.3.26. If $\mathcal{M}$ is given by the hom-module of a category $\mathbf{C}$, then $[\mathbf{D}, \alpha]$ is the same thing as the postcomposition natural transformation

\[ [\mathbf{D}, \alpha] : [\mathbf{D}, \mathbf{G}] \Rightarrow [\mathbf{D}, \mathbf{F}] : [\mathbf{D}, \mathbf{E}] \Rightarrow [\mathbf{D}, \mathbf{C}] \]

(see Preliminary 0.0.2(1)).

3. If $\alpha$ is a frame of a composite endomodule

\[ \mathbf{G} \langle \mathcal{M} \rangle : \mathbf{A} \Rightarrow \mathbf{A} \]  

, i.e., a right [op. left] cylinder

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathbf{G}} \mathbf{A} & \xrightarrow{\mathbf{\alpha}} \mathbf{A} \\
\mathcal{M} & \mathbf{A}
\end{array}
\]

, then, for any category $\mathbf{E}$, $[\mathbf{E}, \alpha]$ is a frame of the endomodule

\[ \langle \mathbf{E}, \mathbf{G} \langle \mathcal{M} \rangle \rangle = \langle \mathbf{E}, \mathbf{G} \rangle \langle \mathcal{E}, \mathcal{M} \rangle \]  

, i.e., a right [op. left] cylinder

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathbf{G}} \mathbf{A} & \xrightarrow{\mathbf{\alpha}} \mathbf{A} \\
\mathcal{M} & \mathbf{A}
\end{array}
\]

, then, for any category $\mathbf{E}$, $[\mathbf{E}, \alpha]$ is a frame of the endomodule

\[ \langle \mathbf{E}, \mathbf{G} \langle \mathcal{M} \rangle \rangle = \langle \mathbf{E}, \mathbf{G} \rangle \langle \mathcal{E}, \mathcal{M} \rangle \]  

, i.e., a right [op. left] cylinder

\[
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\mathbf{G}} \mathbf{A} & \xrightarrow{\mathbf{\alpha}} \mathbf{A} \\
\mathcal{M} & \mathbf{A}
\end{array}
\]
(see Proposition 4.3.9), i.e. a right [op. left] cylinder

\[
\begin{array}{c}
\mathcal{E} \times \mathcal{E} \\
\mathcal{E} \times \mathcal{A}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E} \times \mathcal{E} \\
\mathcal{E} \times \mathcal{A}
\end{array}
\]

; the component of \([\mathcal{E}, \alpha]\) at a functor \(K : \mathcal{E} \to \mathcal{A}\) [op. \(K : \mathcal{E} \to \mathcal{X}\)] is the cylinder \(K \circ \alpha\) defined in Remark 4.3.27(3).

## 4.4 Extraordinary cylinders

This section introduces an extraordinary cylinder as another instance of a cylindrical frame. An ordinary cylinder \(E \times E\) (an extraordinary natural transformation \(x \to F\) spanning a \(X \to \mathcal{E}\), so to speak) is defined by a cylindrical frame \(\alpha\) of the composite left module \(x(\mathcal{M}) F : * \to E^{-} \times E\) (a left module \(* \to E^{-} \times E\) is the same thing as an endomodule \(E \to E\)).

Extraordinary cylinders between a category \(E\) and a module \(M : X \to A\) form the module \((E, M) : X \to [E^{-} \times E, A]\), and a cell

\[
\begin{array}{c}
X \to \mathcal{M} \to A \\
\mathcal{P} \downarrow \psi \downarrow \mathcal{Q}
\end{array}
\]

yields the postcomposition cell

\[
\begin{array}{c}
X \to \mathcal{E} \times \mathcal{E} \\
\mathcal{P} \downarrow \mathcal{E} \mathcal{Q} \\
Y \to \mathcal{E} \times \mathcal{E}
\end{array}
\]

that sends each extraordinary cylinder \(* \to E^{-} \times E\) to its composite \(* \to E^{-} \times E\) with \(\psi\). In

\[
\begin{array}{c}
X \to \mathcal{E} \times \mathcal{E} \\
\mathcal{P} \downarrow \mathcal{E} \mathcal{Q} \\
Y \to \mathcal{E} \times \mathcal{E}
\end{array}
\]

Section 13.1, an end is defined by a universal arrow of the module \((E, M)\).

The section is largely analogous to Section 4.3.

### 4.4.1 Definition. Let \(E\) be a category and \(M : X \to A\) be a module.

- Given an object \(x \in \|X\|\) and a bifunctor \(F : E^{-} \times E \to A\), an (extraordinary) cylinder \(\alpha\) from \(x\) to \(F\) along \(M\), written \(\alpha : x \to F : E \to M\) or diagrammatically as \(* \to E^{-} \times E\), is defined by a \(x \downarrow \alpha \downarrow \mathcal{F}\).

- Given an object \(a \in \|A\|\) and a bifunctor \(G : E^{-} \times E \to X\), an (extraordinary) cylinder \(\alpha\) from \(G\) to \(a\) along \(M\), written \(\alpha : G \to a : E^{-} \to M\) or diagrammatically as \(E^{-} \times E \to \_\), is defined by a \(G \downarrow \alpha \downarrow a\).

Cylindrical frame \(\alpha\) of the composite left module \(x(\mathcal{M}) F : * \to E^{-} \times E\) (note that a left module \(* \to E^{-} \times E\) is the same thing as an endomodule \(E \to E\)).

Cylindrical frame \(\alpha\) of the composite right module \(G(\mathcal{M}) a : E^{-} \times E \to \_\) (note that a right module \(E^{-} \times E \to \_\) is the same thing as an endomodule \(E^{-} \to E^{-}\)).
4.4.2 Remark.
(1) The naturality of an extraordinary cylinder \( \alpha : x \to F : E \leadsto M \) [op. \( \alpha : G \leadsto a : E^\ast \leadsto M \)] is expressed by the commutativity of the square

\[
\begin{array}{ccc}
\alpha & \downarrow & \alpha' \\
F(e,e) & \downarrow & F(e',e') \\
E & \downarrow & E \\
\end{array}
\]

for each \( E \)-arrow \( h : e \to e' \) (cf. Remark 4.1.2(3)).

(2) Any of the cylinders we saw in Section 4.3 has the naturality expressed in the “ordinary” form, and is called ordinary to distinguish it from extraordinary cylinders. However, the adjectives “ordinary” and “extraordinary” are often omitted when the context makes it clear which type of cylinder is being talked about. In fact, some bicylinders (see Section 4.7) have both ordinary and extraordinary naturalities.

(3) Noting Example 4.1.3(2), we see that an extraordinary natural transformation in a category \( C \) is the same thing as an extraordinary cylinder along the hom-module of \( C \). Conversely, an extraordinary cylinder along a module \( M : X \to A \) is the same thing as an extraordinary natural transformation in the collage category \([M]\) (cf. Remark 4.3.2(2)). Given a bifunctor \( L : E^\ast \times E \to C \) and an object \( c \in \|C\| \), an extraordinary natural transformation \( \alpha \) from \( c \) to \( L \) [op. from \( L \) to \( c \)] is denoted by \( \alpha : c \sim L : E \to C \) [op. \( \alpha : L \leadsto c : E^\ast \to C \)].

Note. The following definition is an instance of Definition 4.1.5 where \( \psi : M \to N : E \to E \) is given by the module morphism \( g(M) \sigma : x(M) F \to x'(M) F' : \ast \to E^\ast \times E \) [op. \( \tau(M) f : G(M) a \to G'(M) a' : E^\ast \times E \to \ast ]

4.4.3 Definition. Let \( E \) be a category and \( M : X \to A \) be a module. Given a category arrow, a cylinder, and a natural transformation, all as in

\[
\begin{array}{ccc}
\alpha & \downarrow & \alpha' \\
X & \downarrow & X \\
M & \downarrow & M \\
\end{array}
\]

their composite is the cylinder

\[
\begin{array}{ccc}
\alpha & \downarrow & \alpha' \\
X & \downarrow & X \\
M & \downarrow & M \\
\end{array}
\]

defined by

\[
g \circ \alpha \circ \sigma = \alpha \circ g(M) \sigma \quad \text{op.} \quad \tau \circ \alpha \circ f = \alpha \circ \tau(M) f
\]

where the composition on the right-hand side of the identity is that defined in Definition 4.1.5.

4.4.4 Remark. Each component of the composite \( f \circ \alpha \circ \tau \) [op. \( \tau \circ \alpha \circ f \)] is given by

\[
[g \circ \alpha \circ \sigma]_e = [\alpha \circ g(M) \sigma]_e = \alpha_e \cdot (g(M) \sigma)(e,e) = g \circ \alpha_e \circ \sigma_e(e,e)
\]

op.

\[
[\tau \circ \alpha \circ f]_e = [\alpha \circ \tau(M) f]_e = \alpha_e \cdot (\tau(M) f)(e,e) = \tau(e,e) \circ \alpha_e \circ f
\]

(cf. Remark 4.3.6(1)).

Note. The composition in Definition 4.4.3 yields the module of extraordinary cylinders \( E \leadsto M \); the functorialities in Proposition 4.1.7 and Remark 1.1.25 allow the following definition.

4.4.5 Definition. Given a category \( E \) and a module \( M : X \to A \),

- the module

\[
(E,M) : X \to [E^\ast \times E,A]
\]
of extraordinary cylinders \( E \xrightarrow{\sim} M \) is defined by

\[
(x) \langle E, M \rangle (F) = \prod_{E^x} \langle M \rangle F
\]

for \( x \in X \) and \( F \in [E^x \times E, A] \).

- the module

\[
\langle E^-, M \rangle : [E^-, X] \xrightarrow{\sim} A
\]

of extraordinary cylinders \( E^- \xrightarrow{\sim} M \) is defined by

\[
(G) \langle E^-, M \rangle (a) = \prod_{E^-} G \langle M \rangle a
\]

for \( a \in A \) and \( G \in [E^- \times E, X] \).

4.4.6 Remark.

(1) We use the same notation \( \langle E, M \rangle \) for the module of extraordinary cylinders and the module of ordinary cylinders (see Definition 4.3.7), and let the context determine which is meant.

(2) For an object \( x \in \| X \| \) and a bifunctor \( F : E^x \times E \to A \), the set

\[
(x) \langle E, M \rangle (F) = \prod_{E^x} \langle M \rangle F
\]

consists of all cylindrical frames of the composite left module \( x \langle M \rangle F : \ast \to E^x \times E \), i.e. all cylinders \( x \ast F : E \xrightarrow{\sim} M \), and for an object \( a \in \| A \| \) and a bifunctor \( G : E^x \times E \to X \), the set

\[
(G) \langle E^-, M \rangle (a) = \prod_{E^-} G \langle M \rangle a
\]

consists of all cylindrical frames of the composite right module \( G \langle M \rangle a : E^- \times E \to \ast \), i.e. all cylinders \( G \ast a : E^- \xrightarrow{\sim} M \).

(3) The composition in the module \( \langle E, M \rangle \) [op. \( \langle E^-, M \rangle \)] is that defined in Definition 4.4.3; indeed, by definition, the composition

\[
\begin{array}{c}
\xymatrix{ x \ar[r]^-{\alpha} & \ar[d]^{\sigma} F \ar[d]_f \\
\ar[r]_{\alpha} & g \quad \ar[r]^-{\alpha} & x'/ \ar[d]^{\tau} \\
g \ar[r]_{\alpha} & \ar[r]^-{\tau} \ar[r]^-{\alpha} & \ar[r]^-{f} F' \\
\end{array}
\]

in \( \langle E, M \rangle \) [op. \( \langle E^-, M \rangle \)] is given by

\[
g \circ \alpha \circ \tau = \alpha : \prod_{E^x} \langle M \rangle \sigma = \alpha \circ g \langle M \rangle \sigma \quad \text{op.} \quad \tau \circ \alpha \circ \sigma = \alpha : \prod_{E^-} \tau \langle M \rangle f = \alpha \circ \tau \langle M \rangle f
\]

\( (\alpha : \prod_{E^x} \langle M \rangle \sigma) \) is the image of \( \alpha \in \prod_{E^x} \langle M \rangle F \) under the function \( \prod_{E^x} \langle M \rangle \sigma : \prod_{E^x} \langle M \rangle F \to \prod_{E^x} \langle M \rangle F' \).

(4) The module \( \langle E, C \rangle \) [op. \( \langle E^-, C \rangle \)] of extraordinary natural transformations \( E \to C \) [op. \( E^- \to C \)] is defined as a special case of Definition 4.4.5 where \( M \) is given by the hom-module of a category \( C \) (see Remark 4.4.2.3); that is,

\[
\langle E, C \rangle := \langle E, \langle C \rangle \rangle \quad \text{op.} \quad \langle E^-, C \rangle := \langle E^-, \langle C \rangle \rangle
\]

; for an object \( c \in \| C \| \) and a bifunctor \( L : E^x \times E \to C \), the set \( \langle c \rangle \langle E, C \rangle \langle L \rangle \) [op. \( \langle L \rangle \langle E^-, C \rangle \langle c \rangle \)] consists of all extraordinary natural transformations \( c \ast L : E \to C \) [op. \( L \ast c : E^- \to C \)].

4.4.7 Proposition. Given a category \( E \) and a composite module \( P \langle N \rangle Q \) as in

\[
\begin{array}{c}
X \ar[r]^-{P} & \ar[d]_{\sim} A \ar[d]_1 \\
\ar[r]_{\sim} & Y \ar[r]^-{Q} & B
\end{array}
\]

the identity

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^-{P} & \ar[d]_{\sim} [E^x \times E, A] \ar[d]_1 \\
\ar[r]_{\sim} & [E^x \times E, Y] \ar[r]^-{Q} & B
\end{array}
\]

for \( x \in X \) and \( F \in [E^x \times E, A] \).
(i.e. 
\[ \langle E, P \langle \mathcal{N} \rangle Q \rangle = P \langle E, \mathcal{N} \rangle [E^\times E, Q] \text{ op.} \quad \langle E^\times P \langle \mathcal{N} \rangle Q \rangle = [E^\times E, P] \langle E^\times, \mathcal{N} \rangle Q \]

) holds.

**Proof.** For any \( x \in X \) and \( F \in [E^\times E, A] \),
\[
\langle x \rangle \langle E, P \langle \mathcal{N} \rangle Q \rangle (F) = \prod_{E^x \langle P \langle \mathcal{N} \rangle Q \rangle} F
\]
\[= \prod_{E} \langle x \rangle \langle E, \mathcal{N} \rangle [Q \delta F]
\]
\[= (x : P) \langle E, \mathcal{N} \rangle (Q \delta F)
\]
\[= (x : P) \langle E, \mathcal{N} \rangle ([E^\times E, Q] : F)
\]
\[= (x : P \langle \mathcal{N} \rangle [E^\times E, Q]) (F).
\]

\[ \square \]

### 4.4.8 Remark

The cell
\[
\begin{array}{cccc}
X & \xrightarrow{E^\times E, A} & \xrightarrow{E^\times E, X} & \xrightarrow{E^\times E, A} \\
\downarrow P & & & \\
Y & \xrightarrow{E^\times E, B} & \xrightarrow{E^\times E, Y} & \xrightarrow{E^\times E, B}
\end{array}
\]

sends each cylinder
\[
\begin{array}{cccc}
\ast & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} \\
\uparrow x & \xrightarrow{\alpha} & \xrightarrow{\alpha} & \xrightarrow{\alpha}
\end{array}
\]

\[
\begin{array}{cccc}
X & \xrightarrow{\mathcal{M} \mathcal{N}} & \xrightarrow{\mathcal{N}} & \xrightarrow{\mathcal{N}} \\
\uparrow & & \uparrow & \uparrow
\end{array}
\]

\[
\begin{array}{cccc}
\ast & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} \\
\downarrow x : P & \xrightarrow{\alpha} & \xrightarrow{Q \delta F} & \xrightarrow{Q \delta a}
\end{array}
\]

\[
\begin{array}{cccc}
Y & \xrightarrow{\mathcal{N} \mathcal{N}} & \xrightarrow{\mathcal{N}} & \xrightarrow{\mathcal{N}} \\
\uparrow & & \uparrow & \uparrow
\end{array}
\]

defined by the same frame.

**Note.** The following definition is an instance of Definition 4.1.5 where \( \psi : M \rightarrow \mathcal{N} : E \rightarrow E \) is given by the module morphism \( x(\psi) F : x(M) F \rightarrow x(\mathcal{N}) F : * \rightarrow E^\times E \) [op. \( G(\psi) a : G(M) a \rightarrow G(\mathcal{N}) a : E^\times E \rightarrow * \)].

### 4.4.9 Definition

Given a cylinder \( \alpha \) and a module morphism \( \psi \) as in
\[
\begin{array}{cccc}
\ast & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} \\
\uparrow x & \xrightarrow{\alpha} & \xrightarrow{M} & \xrightarrow{\alpha}
\end{array}
\]

\[
\begin{array}{cccc}
X & \xrightarrow{\mathcal{M} \mathcal{N}} & \xrightarrow{\mathcal{N}} & \xrightarrow{\mathcal{N}} \\
\uparrow & & \uparrow & \uparrow
\end{array}
\]

\[
\begin{array}{cccc}
\ast & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} & \xrightarrow{E^\times E} \\
\uparrow x : \alpha \psi & \xrightarrow{\alpha \psi} & \xrightarrow{\alpha \psi} & \xrightarrow{\alpha \psi}
\end{array}
\]

\[
\begin{array}{cccc}
X & \xrightarrow{\mathcal{N} \mathcal{N}} & \xrightarrow{\mathcal{N}} & \xrightarrow{\mathcal{N}} \\
\uparrow & & \uparrow & \uparrow
\end{array}
\]

defined by
\[
\alpha \psi = \psi \alpha \psi \text{ op.} \quad \alpha \psi = \alpha \psi G(\psi) a
\]

where the composition on the right-hand side of the identity is that defined in Definition 4.1.5.

### 4.4.10 Remark

Each component of the composite \( \alpha \psi : x \rightarrow F \) [op. \( \alpha \psi : G \rightarrow a \)] is given by
\[
[\alpha \psi]_e = [\alpha \psi x(\psi) F]_e = \alpha \psi \psi \chi (\psi) F (e, e) = \alpha \psi \psi \chi (\psi) (F (e, e))
\]

\[
[\alpha \psi]_e = [\alpha \psi G(\psi) a]_e = \alpha \psi \psi \chi (e, e) (G(\psi) a) = \alpha \psi \psi \chi (G(e, e)) (\psi) (a)
\]
The assignment \( \psi : \mathcal{M} \to \mathcal{N} \), the composition in Definition 4.4.9 yields the postcomposition module morphism \( (\mathcal{E}, \psi) : (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}, \mathcal{N}) \) from the module of extraordinary cylinders \( \mathcal{E} \to \mathcal{M} \) to the module of extraordinary cylinders \( \mathcal{E} \to \mathcal{N} \). Here is the formal definition:

**4.4.11 Definition.** Given a category \( \mathcal{E} \) and a module morphism \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \),
- the postcomposition module morphism
  \[
  (\mathcal{E}, \psi) : (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}, \mathcal{N}) : \mathcal{X} \to [\mathcal{E}^\ast \times \mathcal{E}, \mathcal{A}]
  \]
  “postcomposition with \( \psi \)”, is defined by
  \[
  (x) (\mathcal{E}, \psi) (F) = \prod_{\mathcal{E}} x (\psi) F
  \]
  for each pair of an object \( x \in \| \mathcal{X} \| \) and a bifunctor \( F : \mathcal{E}^\ast \times \mathcal{E} \to \mathcal{A} \).
- the postcomposition module morphism
  \[
  (\mathcal{E}^\ast, \psi) : (\mathcal{E}^\ast, \mathcal{M}) \to (\mathcal{E}^\ast, \mathcal{N}) : [\mathcal{E}^\ast \times \mathcal{E}, \mathcal{X}] \to \mathcal{A}
  \]
  “postcomposition with \( \psi \)”, is defined by
  \[
  (G) (\mathcal{E}^\ast, \psi) (a) = \prod_{\mathcal{E}^\ast} G (\psi) a
  \]
  for each pair of an object \( a \in \| \mathcal{A} \| \) and a bifunctor \( G : \mathcal{E}^\ast \times \mathcal{E} \to \mathcal{X} \).

**4.4.12 Remark.**
1. Replacing \( \psi : \mathcal{M} \to \mathcal{N} \) by \( x (\psi) F : x (\mathcal{M}) F \to x (\mathcal{N}) F \) in Definition 4.1.5, we have a function
   \[
   \prod_{\mathcal{E}} x (\psi) F \to \prod_{\mathcal{E}} x (\mathcal{N}) F
   \]
   in the notation introduced in Definition 4.4.5. Since \( x (\psi) F \) is natural in \( x \) and \( F \) by Remark 1.2.25, so is \( (x) (\mathcal{E}, \psi) (F) = \prod_{\mathcal{E}} x (\psi) F \) by Proposition 4.1.7. Hence \( (\mathcal{E}, \psi) \) so defined does form a module morphism \( (\mathcal{E}, \mathcal{M}) \to (\mathcal{E}, \mathcal{N}) \).
2. The module morphism \( (\mathcal{E}, \psi) [\text{op. } (\mathcal{E}^\ast, \psi)] \) maps each cylinder
   \[
   \alpha : x \to F : \mathcal{E} \to \mathcal{M}
   \]
   to
   \[
   \alpha : \prod_{\mathcal{E}} x (\psi) F = \alpha \circ x (\psi) F \quad \text{op. } \alpha : \prod_{\mathcal{E}^\ast} G (\psi) a = \alpha \circ G (\psi) a
   \]
   ; that is, to the cylinder
   \[
   \alpha \circ \psi : x \to F : \mathcal{E} \to \mathcal{N}
   \]
   defined in Definition 4.4.9.
3. The assignment \( \psi \mapsto (\mathcal{E}, \psi) \) is functorial; indeed, the functor
   \[
   (\mathcal{E}, -) : [\mathcal{X} : \mathcal{A}] \to [\mathcal{X} : [\mathcal{E}^\ast \times \mathcal{E}, \mathcal{A}]]
   \]
   is defined by
   \[
   (x) (\mathcal{E}, M) (F) = \prod_{\mathcal{E}} x (\psi) M F
   \]
   for \( x \in \mathcal{X} \), \( F \in [\mathcal{E}^\ast \times \mathcal{E}, \mathcal{A}] \), and \( M \in [\mathcal{X} : \mathcal{A}] \), and the functor
   \[
   (\mathcal{E}^\ast, -) : [\mathcal{X} : \mathcal{A}] \to [[\mathcal{E}^\ast \times \mathcal{E}, \mathcal{X}] : \mathcal{A}]
   \]
   is defined by
   \[
   (G) (\mathcal{E}^\ast, M) (a) = \prod_{\mathcal{E}^\ast} G (\psi) a
   \]
   for \( a \in \mathcal{A} \), \( G \in [\mathcal{E}^\ast \times \mathcal{E}, \mathcal{X}] \), and \( M \in [\mathcal{X} : \mathcal{A}] \).

**Note.** By Remark 1.2.2(3), the following definition is a special case of Definition 4.4.9, and vice versa.
4.4.13 Definition. Given a cylinder \( \alpha \) and a cell \( \psi \) as in
\[
\begin{array}{c c c}
\times & E^\times E & \alpha \\
\downarrow & \downarrow F & \downarrow a \\
X & \sim M & \rightarrow A \\
\psi & \downarrow Q & \downarrow Q \\
Y & \sim N & \rightarrow B
\end{array}
\]
their composite \( \alpha \circ \psi = \psi \circ \alpha \) is the cylinder
\[
\begin{array}{c c c}
\times : P & \alpha \circ \psi & \downarrow Q \circ F \\
\downarrow & \downarrow P & \downarrow \psi \\
Y & \sim N & \rightarrow B \\
\end{array}
\]
defined by
\[
\alpha \circ \psi = \alpha \circ x(\psi) F \\
\alpha \circ \psi = \alpha \circ \psi \circ G(\psi) a
\]
where the composition on the right-hand side of the identity is that defined in Definition 4.4.15 (\( \psi \) is the module morphism \( \mathcal{M} \rightarrow \mathcal{P}(\mathcal{N}) \)).

4.4.14 Remark.
(1) Each component of the composite cylinder \( \alpha \circ \psi \) is given as in Remark 4.4.10:
\[
[\alpha \circ \psi]_e = \alpha e \cdot (x) \cdot (\psi) (F(e, e)) \quad \text{op.} \quad [\alpha \circ \psi]_{\text{op.}} = \alpha e \cdot (G(e, e)) (\psi)(a).
\]
(2) If a cell is given by the hom-cell of a functor \( H \), then the composition \( \alpha \circ \psi \) is just the usual composition of an extraordinary natural transformation (recall Remark 4.4.2(3)) and a functor; that is,
\[
\alpha \circ (H) = \alpha \circ H.
\]

Note. For any category \( E \) and any cell \( \psi : \mathcal{M} \rightarrow \mathcal{N} \), the composition in Definition 4.4.13 yields the postcomposition cell \( \langle E, \psi \rangle : \langle E, \mathcal{M} \rangle \rightarrow \langle E, \mathcal{N} \rangle \) from the module of extraordinary cylinders \( E \sim \mathcal{M} \) to the module of extraordinary cylinders \( E \sim \mathcal{N} \); the postcomposition module morphism in Definition 4.4.11 and the identity in Proposition 4.4.7 allow the following definition.

4.4.15 Definition. Given a category \( E \) and a cell \( \begin{array}{c c c}
X & \sim M & \rightarrow A \\
\psi & \downarrow Q & \downarrow Q \\
Y & \sim N & \rightarrow B
\end{array} \), the postcomposition cell
\[
\begin{array}{c c c}
X & \sim E^\times E, A & \downarrow \downarrow [E^\times E, A] \\
\psi & \downarrow \downarrow [E^\times E, Q] & \downarrow \downarrow Q \\
Y & \sim E^\times E, B & \downarrow \downarrow [E^\times E, B]
\end{array}
\]
is defined by the postcomposition module morphism
\[
\langle E, \mathcal{M} \rangle \xrightarrow{(E, \psi)} \langle E, \mathcal{P}(\mathcal{N}) \rangle = \mathcal{P}(\langle E, \mathcal{N} \rangle \langle E^\times E, Q \rangle)
\]
and
\[
\langle E^\times E, \mathcal{M} \rangle \xrightarrow{(E^\times E, \psi)} \langle E^\times E, \mathcal{P}(\mathcal{N}) \rangle = \langle E^\times E, \mathcal{P} \langle E^\times E, \mathcal{N} \rangle \rangle.
\]

4.4.16 Remark. The cell \( \langle E, \psi \rangle : \mathcal{P} \rightarrow \langle E^\times E, \mathcal{N} \rangle \) sends each cylinder \( \alpha : x \rightarrow F : E \sim \mathcal{M} \) to the cylinder \( \alpha \circ \psi : x^\times P \sim Q \circ F : E \sim \mathcal{N} \) defined in Definition 4.4.13, and the cell \( \langle E^\times E, \psi \rangle : \langle E^\times E, \mathcal{P} \rangle \sim Q : \langle E^\times E, \mathcal{M} \rangle \rightarrow \langle E^\times E, \mathcal{N} \rangle \) sends each cylinder \( \alpha : G \sim a : E^\times \mathcal{M} \) to the cylinder \( \alpha \circ \psi : G \circ P \circ Q \sim \psi a : E^\times \sim \mathcal{N} \).

4.4.17 Proposition. The assignment \( \psi \mapsto \langle E, \psi \rangle \) [op. \( \psi \mapsto \langle E^\times E, \psi \rangle \)] is functorial.

Proof. The functoriality is verified similarly to Proposition 4.3.21 using Remark 4.4.12(3) and Proposition 4.4.7 in place of Remark 4.3.14(3) and Proposition 4.3.9.
4.5 Weighted cylinders

This section introduces a special sort of cylinder called a weighted cylinder (see Definition 4.5.1). A cone to be introduced in Section 4.6 is seen as a special instance of a weighted cylinder, and so is a wedge, which will be introduced in Section 4.8.

Weighted cylinders between a functor $K : E \to D$ and a module $M : X \to A$ form the module $(K \ast M) : [D, X] \to [E, A]$, and a cell

\[
\begin{array}{ccc}
X \xrightarrow{M} A \\
\psi \downarrow & \downarrow & \downarrow \psi \\
Y \xrightarrow{N} B
\end{array}
\]

yields the postcomposition cell

\[
\begin{array}{ccc}
[D, X] \xrightarrow{(K \ast M)} [E, A] \\
[D, P] \downarrow & \downarrow & \downarrow [E, Q] \\
[D, Y] \xrightarrow{(K \ast N)} [E, B]
\end{array}
\]

that sends each weighted cylinder $D \xrightarrow{K} E$ to its composite $D \xrightarrow{K} E$ with $\psi$. In Section 12.6, a cylindrical extension is defined by a universal arrow of the module $(K \ast M)$. If $M : X \to A$ is replaced by the hom-module $(C) : C \to C$ of a category $C$, we have the module $(K \ast C) : [D, C] \to [E, C]$ of $K$-weighted natural transformations in $C$. A universal arrow of the module $(K \ast C)$ is precisely a Kan extension (see Section 12.7).

4.5.1 Definition.
- A cylinder $D \xrightarrow{K} E$ from $K \circ G$ to $F$ along $M$ is said to be right weighted by $K$ (or right $\xrightarrow{K}$-weighted).
- A cylinder $E \xrightarrow{K} D$ from $G \circ K$ to $F$ along $M$ is said to be left weighted by $K$ (or left $\xrightarrow{K}$-weighted).

4.5.2 Remark.
(1) Since $[K \circ G](\mathcal{M}) F = K(\mathcal{G}(\mathcal{M}) F)$ op. $G(\mathcal{M})[F \circ K] = (G(\mathcal{M}) F) K$
Given a right \([\text{op. left}]\) \(K\)-weighted cylinder

\[
\begin{array}{ccc}
D \xleftarrow{K} E & \xrightarrow{\alpha} & F \\
G \downarrow & \alpha & \downarrow F \\
X \xrightarrow{\otimes M} A & \xrightarrow{\otimes M} & A
\end{array}
\]

is the same thing as a right \([\text{op. left}]\) cylinder

\[
\begin{array}{ccc}
D \xleftarrow{K} E & \xrightarrow{\alpha} & F \\
G \downarrow & \alpha & \downarrow F \\
X \xrightarrow{\otimes M} A & \xrightarrow{\otimes M} & A
\end{array}
\]

(2) Conversely, a right \([\text{op. left}]\) cylinder

\[
\begin{array}{ccc}
X \xleftarrow{\alpha} A & \xrightarrow{\otimes M} & A \\
G \downarrow & \alpha & \downarrow \circ \\
X \xrightarrow{\otimes M} A & \xrightarrow{\otimes M} & A
\end{array}
\]

may be depicted as a right \([\text{op. left}]\) weighted cylinder

\[
\begin{array}{ccc}
X \xleftarrow{G} A & \xrightarrow{\otimes M} & A \\
\alpha \downarrow & \otimes M & \downarrow \circ \\
X \xrightarrow{\otimes M} A & \xrightarrow{\otimes M} & A
\end{array}
\]

(3) A cylinder

\[
\begin{array}{ccc}
X \xleftarrow{\alpha} A & \xrightarrow{\otimes M} & A \\
\end{array}
\]

is regarded to be weighted by the identity \(E \rightarrow E\) and sometimes depicted as

\[
\begin{array}{ccc}
E \xleftarrow{\alpha} E & \xrightarrow{\otimes M} & E \\
\alpha \downarrow & \otimes M & \downarrow \circ \\
X \xrightarrow{\otimes M} A & \xrightarrow{\otimes M} & A
\end{array}
\]

Note. The module defined in Definition 4.3.7 for ordinary cylinders yields the following modules for weighted cylinders.

4.5.3 Definition. Given a functor \(K : E \rightarrow D\) and a module \(M : X \rightarrow A\), the module

\[
\langle K \circ M \rangle : [D, X] \rightarrow [E, A] \quad \text{op.} \quad \langle K \circ M \rangle : [E, X] \rightarrow [D, A]
\]

of right \([\text{op. left}]\) \(K\)-weighted cylinders along \(M\) is defined by the composition

\[
[D, X] \xrightarrow{\langle K \circ M \rangle} [E, X] \xrightarrow{\langle E, M \rangle} [E, A] \quad \text{op.} \quad [E, X] \xrightarrow{\langle E, M \rangle} [E, A] \xrightarrow{\langle K \circ A \rangle} [D, A].
\]

4.5.4 Remark.

(1) For any pair of functors \(G : D \rightarrow X\) and \(F : E \rightarrow A\), the set

\[
(G) \langle K \circ M \rangle (F) = (G) \langle [K, X] \langle E, M \rangle \rangle (F) = (K \circ G) \langle E, M \rangle (F)
\]

consists of all cylinders \(K \circ G \rightarrow F : E \rightarrow M\) right weighted by \(K\), and for any pair of functors \(G : E \rightarrow X\) and \(F : D \rightarrow A\), the set

\[
(G) \langle K \circ M \rangle (F) = (G) \langle [E, M] \langle K, A \rangle \rangle (F) = (G) \langle E, M \rangle (F \circ K)
\]

consists of all cylinders \(G \circ F \rightarrow K : E \rightarrow M\) left weighted by \(K\).

(2) Given a right \([\text{op. left}]\) \(K\)-weighted cylinder \(\alpha\) and natural transformations \(\tau\) and \(\sigma\) as in

\[
\begin{array}{ccc}
X \xleftarrow{\alpha} A & \xrightarrow{\otimes M} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
D \xleftarrow{K} E & \xrightarrow{\otimes M} & D \\
\end{array}
\]

, their composite in the module \(\langle K \circ M \rangle\) \([\text{op.} \langle K \circ M \rangle]\) is the right \([\text{op. left}]\) \(K\)-weighted cylinder

\[
\begin{array}{ccc}
D \xleftarrow{K} E & \xrightarrow{\otimes M} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
X \xleftarrow{\alpha} A & \xrightarrow{\otimes M} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
X \xleftarrow{\alpha} A & \xrightarrow{\otimes M} & A \\
\end{array}
\]
given by the composition
\[ \tau \circ \alpha \circ \sigma = [K \circ \tau] \circ \alpha \circ \sigma \quad \text{op.} \quad \tau \circ \alpha \circ \sigma = \tau \circ \alpha \circ [K \circ \sigma] \]
defined in Definition 4.3.5 with \( K \circ \tau \) [op. \( K \circ \sigma \)] the usual composition of a functor and a natural transformation.

(3) If, as a special case, \( K \) is given by the identity \( E \to E \) (see Remark 4.5.2(3)), we have
\[ (1_E \circ \text{M}) = (E, \text{M}) = (1_E \circ \text{M}) \]
immediately from the definition.

**Note.** The postcomposition module morphism defined in Definition 4.3.13 for ordinary cylinders yields the following postcomposition module morphisms for weighted cylinders.

**4.5.5 Definition.** Given a functor \( K : E \to D \) and a module morphism \( \psi : M \to N : X \to A \), the postcomposition module morphism
\[ (K \circ \psi) : (K \circ \text{M}) \to (K \circ \text{N}) : [D, X] \to [E, A] \]
op.
\[ (K \circ \psi) : (K \circ \text{M}) \to (K \circ \text{N}) : [E, X] \to [D, A] \]
, “postcomposition with \( \psi \)”, is defined by the composition
\[
\begin{align*}
[D, X] \xrightarrow{[K, X]} [E, X] & \xrightarrow{(E, \psi)} [E, A] \\
\xrightarrow{(E, \text{M})} & \xrightarrow{(E, N)} [E, A]
\end{align*}
\]op.
\[
\begin{align*}
[E, X] & \xrightarrow{(E, \psi)} [E, A] \\
\xrightarrow{(E, \text{M})} & \xrightarrow{(E, N)} [E, A] \xrightarrow{[K, A]} [D, A] .
\end{align*}
\]

**4.5.6 Remark.**
(1) The module morphism \( (K \circ \psi) \) [op. \( (K \circ \psi) \)] maps each right [op. left] \( K \)-weighted cylinder
\[
\begin{array}{ccc}
\text{D} \xleftarrow{K} E & \text{op.} & \text{E} \xrightarrow{K} D \\
\text{G} \downarrow \alpha & \text{op.} & \text{G} \downarrow \alpha \\
\text{X} \xrightarrow{\text{M}} A & \text{op.} & \text{X} \xrightarrow{\text{M}} A
\end{array}
\]
to the right [op. left] \( K \)-weighted cylinder
\[
\begin{array}{ccc}
\text{D} \xleftarrow{K} E & \text{op.} & \text{E} \xrightarrow{K} D \\
\text{G} \downarrow \alpha \circ \psi & \text{op.} & \text{G} \downarrow \alpha \circ \psi \\
\text{X} \xrightarrow{\text{M}} A & \text{op.} & \text{X} \xrightarrow{\text{M}} A
\end{array}
\]
given by the composition defined in Definition 4.3.11.

(2) The assignment \( \psi \mapsto (K \circ \psi) \) [op. \( \psi \mapsto (K \circ \psi) \)] is functorial; indeed, the functor
\[ (K \circ -) : [X : A] \to [[D, X] : [E, A]] \quad \text{op.} \quad (K \circ -) : [X : A] \to [[E, X] : [D, A]] \]
is defined by
\[ (K \circ \text{M}) = [K, X] (E, \text{M}) \quad \text{op.} \quad (K \circ \text{M}) = (E, \text{M}) [K, A] .
\]
**Note.** The postcomposition cell defined in Definition 4.3.17 for ordinary cylinders yields the following postcomposition cells for weighted cylinders.
4.5.7 Definition. Given a functor $K: E \to D$ and a cell $X \xrightarrow{\phi} A$, the postcomposition cell $Y \xrightarrow{\psi} B$ as shown in the commutative diagram

\[
\begin{align*}
[D, X] & \xrightarrow{(K, M)} [E, A] \\
[D, P] & \xrightarrow{(K, \psi)} [E, Q] \\
[D, Y] & \xrightarrow{(K, \psi)} [E, B]
\end{align*}
\]

is defined by the pasting composition

\[
\begin{align*}
[D, X] & \xrightarrow{[K, X]} [E, X] \xrightarrow{[E, M]} [E, A] \\
[D, P] & \xrightarrow{(E, \psi)} [E, Q] \\
[D, Y] & \xrightarrow{(K, \psi)} [E, B]
\end{align*}
\]

op.

\[
\begin{align*}
[E, X] & \xrightarrow{(K, M)} [D, A] \\
[E, P] & \xrightarrow{(E, \psi)} [D, Q] \\
[E, Y] & \xrightarrow{(K, \psi)} [D, B]
\end{align*}
\]

4.5.8 Remark.

(1) The cell $\langle K \uplus \psi \rangle$ [op. $\langle K \uplus \psi \rangle$] sends each right [op. left] $K$-weighted cylinder

\[
\begin{align*}
D & \xrightarrow{K} E \\
X & \xrightarrow{M} A
\end{align*}
\]

to the right [op. left] $K$-weighted cylinder

\[
\begin{align*}
D & \xrightarrow{K} E \\
X & \xrightarrow{M} A
\end{align*}
\]

\[
\begin{align*}
E & \xrightarrow{K} D \\
X & \xrightarrow{M} A
\end{align*}
\]

op.

given by the composition as in Definition 4.3.15.

(2) If, as a special case, $K$ is given by the identity $E \to E$ (see Remark 4.5.2(3)), we have

\[
\langle 1_E \uplus \psi \rangle = \langle E, \psi \rangle = \langle 1_E \uplus \psi \rangle
\]

immediately from the definition.

4.5.9 Proposition. The assignment $\psi \mapsto \langle K \uplus \psi \rangle$ [op. $\psi \mapsto \langle K \uplus \psi \rangle$] is functorial.

Proof. Since the cell $\langle K \uplus \psi \rangle$ is obtained from the cell $\langle E, \psi \rangle$ by the pasting composition as in Definition 4.5.7, the functoriality of the assignment $\psi \mapsto \langle K \uplus \psi \rangle$ is reduced to that of the assignment $\psi \mapsto \langle E, \psi \rangle$ (see Proposition 4.3.21) by virtue of Corollary 1.2.39.

4.5.10 Remark. Given a functor $K: E \to D$ with $E$ small, the assignment $\psi \mapsto \langle K \uplus \psi \rangle$ [op. $\psi \mapsto \langle K \uplus \psi \rangle$] gives the functor

\[
\langle K \uplus - \rangle: \text{MOD} \to \text{MOD}
\]

op.

\[
\langle K \uplus - \rangle: \text{MOD} \to \text{MOD}
\]

, extending the functor in Remark 4.5.6(2) as shown in the commutative diagram

\[
\begin{align*}
[X: A] & \xrightarrow{(K, \uplus -)} [[D, X]: [E, A]] \\
\text{MOD} & \xrightarrow{(K, \uplus -)} \text{MOD}
\end{align*}
\]

, where $\uplus$ denotes the canonical embedding in Remark 1.2.19(2).

4.5.11 Note. The following is a special case of Definition 4.5.1 where $M$ is given by the hom-module of a category.
4.5.12 Definition.
- A natural transformation $\begin{array}{c} \Delta \xrightarrow{K} \mathcal{E} \\ \alpha \Downarrow L \end{array}$ from $K \circ J$ to $L$ is said to be right weighted by $K$ (or right $K$-weighted).
- A natural transformation $\begin{array}{c} \mathcal{E} \xrightarrow{K} \Delta \\ \alpha \Downarrow J \end{array}$ from $L$ to $J \circ K$ is said to be left weighted by $K$ (or left $K$-weighted).

4.5.13 Remark.
(1) A weighted natural transformation in a category $\mathcal{C}$ is just a special instance of a weighted cylinder in Definition 4.5.1 where $\mathcal{M}$ is the hom-module of $\mathcal{C}$. Conversely, a weighted cylinder along a module $\mathcal{M}$ is the same thing as a weighted natural transformation in the collage category $[\mathcal{M}]$ (cf. Remark 4.3.2(2)).
(2) A natural transformation $\alpha : L \to J : \mathcal{E} \to \mathcal{C}$ is regarded to be weighted by the identity $\mathcal{E} \to \mathcal{E}$ and sometimes depicted as $\begin{array}{c} \mathcal{E} \xrightarrow{1} \mathcal{E} \end{array}$, $\begin{array}{c} \mathcal{C} \xrightarrow{\alpha} \mathcal{C} \end{array}$.

4.5.14 Definition. Given a pair of weighted natural transformations

$$\begin{array}{c} \mathcal{E} \xrightarrow{K} \mathcal{E}' \xrightarrow{J} \mathcal{E}'' \\ \alpha \Downarrow L \quad \beta \Downarrow L' \end{array} \quad \text{op.} \quad \begin{array}{c} \mathcal{E}'' \xrightarrow{J} \mathcal{E}' \xrightarrow{K} \mathcal{E} \\ \beta \Downarrow L' \quad \alpha \Downarrow L \end{array}$$

, their composite (pasting composite) is the weighted natural transformation

$$\begin{array}{c} \mathcal{E} \xrightarrow{\alpha \circ \beta} \mathcal{E}'' \\ \alpha \Downarrow L \quad \beta \Downarrow L' \end{array} \quad \text{op.} \quad \begin{array}{c} \mathcal{E}'' \xrightarrow{\beta \circ \alpha} \mathcal{E} \\ \beta \Downarrow L' \quad \alpha \Downarrow L \end{array}$$

defined by

$$[\alpha \circ \beta]_e = \alpha_{(e : J)} \circ \beta_e$$

or componentwise by

$$[\alpha \circ \beta] = \beta \circ \alpha$$

for $e \in |\mathcal{E}''|$.

4.5.15 Proposition. The composition in Definition 4.5.14 is associative, and the identity natural transformations $\begin{array}{c} \mathcal{E} \xrightarrow{1} \mathcal{E} \end{array}$ act as identities.

4.5.16 Remark. For any category $\mathcal{C}$, the category $\mathcal{C}^\op$ is defined as follows:
- objects are functors whose codomain is $\mathcal{C}$;
- arrows between functors $L : \mathcal{E} \to \mathcal{C}$ and $J : \mathcal{D} \to \mathcal{C}$ are pairs $(K, \alpha)$ consisting of a functor $K : \mathcal{E} \to \mathcal{D}$ and a $K$-weighted natural transformation $\alpha : K \circ J \to L : \mathcal{E} \to \mathcal{C}$ [op. $\alpha : L \to J \circ K : \mathcal{E} \to \mathcal{C}$].

Note. The following definition is a special case of Definition 4.5.3 where $\mathcal{M}$ is given by the hom-module of a category $\mathcal{C}$.

4.5.17 Definition. Given a functor $K : \mathcal{E} \to \mathcal{D}$ and a category $\mathcal{C}$, the module

$$\langle K \uparrow \mathcal{C} \rangle : [\mathcal{D}, \mathcal{C}] \to [\mathcal{E}, \mathcal{C}]$$

op. $$\langle K \circ \mathcal{C} \rangle : [\mathcal{E}, \mathcal{C}] \to [\mathcal{D}, \mathcal{C}]$$
The identity is given by the composition

\[ [D, C] \xrightarrow{[K,C]} [E, C] \xrightarrow{[E,C]} [E, C] \text{ op.} \quad [E, C] \xrightarrow{[E,C]} [E, C] \xrightarrow{[K,C]} [D, C] \]

i.e. by the representable [op. corepresentable] module of the precomposition functor \([K,C]\).

4.5.18 Remark.
(1) A \((K \otimes C)\)-arrow \(\alpha : J \to L\) is a right \(K\)-weighted natural transformation \(\alpha : K \otimes J \to L : E \to C\), and a \((K \triangleright C)\)-arrow \(\alpha : L \to J\) is a left \(K\)-weighted natural transformation \(\alpha : L \to J \triangleright K : E \to C\).
(2) Given a right [op. left] \(K\)-weighted natural transformation \(\alpha\) and natural transformations \(\tau\) and \(\sigma\) as in

\[
\begin{array}{ccc}
D & \xrightarrow{K} & E \\
\downarrow J & \xrightarrow{\alpha} & \downarrow L \\
C & \xleftarrow{\otimes \mathcal{L}} & L',
\end{array}
\quad \text{op.}
\begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow L & \xrightarrow{\alpha} & \downarrow J \\
C & \xleftarrow{\otimes \mathcal{L}} & J',
\end{array}
\]

their composite in the module \((K \otimes C)\) [op. \((K \triangleright C)\)] is the right [op. left] \(K\)-weighted natural transformation

\[
\begin{array}{ccc}
D & \xrightarrow{K} & E \\
\downarrow J & \xrightarrow{\tau \circ \alpha \circ \sigma} & \downarrow L' \\
C & \xleftarrow{\otimes \mathcal{L}} & L',
\end{array}
\quad \text{op.}
\begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow L & \xrightarrow{\tau \circ \alpha \circ \sigma} & \downarrow J' \\
C & \xleftarrow{\otimes \mathcal{L}} & J',
\end{array}
\]

given by the pasting composition; that is,

\[
\tau \circ \alpha \circ \sigma = [K \circ \tau] \circ \alpha \circ \sigma \quad \text{op.} \quad \tau \circ \alpha \circ \sigma = \tau \circ \alpha \circ [K \circ \sigma]
\]
(cf. Remark 4.5.4(2)).
(3) The identity \((K \otimes C) = (K \otimes \{C\})\) [op. \((K \triangleright C) = (K \triangleright \{C\})\)] follows from the identity \((E, C) = \{E, \{C\}\})\) (see Remark 4.3.8(3)).

Note. The following definition is a special case of Definition 4.5.7 where \(\Psi\) is given by the hom-cell of a functor \(H : C \to B\).

4.5.19 Definition. Given functors \(K : E \to D\) and \(H : C \to B\), the postcomposition cell

\[
\begin{array}{ccc}
[D, C] & \xrightarrow{[K,C]} & [E, C] \\
\downarrow [D,H] & \xleftarrow{[K,H]} & \downarrow [E,H] \\
[D, B] & \xrightarrow{[K,B]} & [E, B]
\end{array}
\]

is defined by the pasting composition

\[
\begin{array}{ccc}
[D, C] & \xrightarrow{[K,C]} & [E, C] \\
\downarrow [D,H] & \xleftarrow{[K,H]} & \downarrow [E,H] \\
[D, B] & \xrightarrow{[K,B]} & [E, B]
\end{array}
\quad \text{op.}
\begin{array}{ccc}
[E, C] & \xrightarrow{[E,C]} & [E, C] \\
\downarrow [E,H] & \xleftarrow{[E,H]} & \downarrow [E,H] \\
[E, B] & \xrightarrow{[E,B]} & [E, B]
\end{array}
\quad \text{op.}
\begin{array}{ccc}
[E, C] & \xrightarrow{[E,C]} & [E, C] \\
\downarrow [E,H] & \xleftarrow{[E,H]} & \downarrow [E,H] \\
[E, B] & \xrightarrow{[E,B]} & [E, B]
\end{array}
\quad \text{op.}
\begin{array}{ccc}
[D, C] & \xrightarrow{[K,C]} & [D, C] \\
\downarrow [D,H] & \xleftarrow{[K,H]} & \downarrow [D,H] \\
[D, B] & \xrightarrow{[K,B]} & [D, B]
\end{array}
\]

4.5.20 Remark.
(1) The cell \((K \otimes H)\) [op. \((K \triangleright H)\)] sends each right [op. left] \(K\)-weighted natural transformation

\[
\begin{array}{ccc}
D & \xrightarrow{K} & E \\
\downarrow G & \xleftarrow{\alpha} & \downarrow F \\
C & \xleftarrow{\otimes \mathcal{L}} & C
\end{array}
\quad \text{op.}
\begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow G & \xleftarrow{\alpha} & \downarrow F \\
C & \xleftarrow{\otimes \mathcal{L}} & C
\end{array}
\]

to the right [op. left] \(K\)-weighted natural transformation

\[
\begin{array}{ccc}
D & \xrightarrow{K} & E \\
\downarrow G \circ H & \xleftarrow{\alpha \circ H} & \downarrow H \circ F \\
B & \xleftarrow{\otimes \mathcal{L}} & B
\end{array}
\quad \text{op.}
\begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow G \circ H & \xleftarrow{\alpha \circ H} & \downarrow H \circ F \\
B & \xleftarrow{\otimes \mathcal{L}} & B
\end{array}
\]

given by the usual composition of a natural transformation \(\alpha\) and a functor \(H\) (see Remark 4.3.16(2)).
(2) The identity 
\[
\langle K \circ H \rangle = \langle K \circ (H) \rangle \quad \text{op.} \quad \langle K \triangleright H \rangle = \langle K \triangleright (H) \rangle
\]
follows from the identity \( \langle E, H \rangle = \langle E, (H) \rangle \) (see Remark 4.3.18(2)).

### 4.6 Cones

This section introduces a cone as an instance of a conical frame. A cone is defined by a frame \( \alpha \) of the composite left module \( x(M)F : \ast \to E \).

Cones between a category \( E \) and a module \( M : X \to A \) form the module \( \langle \ast E, M \rangle : X \to [E, A] \), and a cell

\[
\begin{array}{c}
X - \mathcal{M} - A \\
\downarrow \psi \quad \downarrow \alpha \\
Y - \mathcal{N} - B
\end{array}
\]

yields the postcomposition cell

\[
\begin{array}{c}
X - \langle \ast E, M \rangle [E, A] \\
\downarrow \psi \quad \downarrow \alpha \\
Y - \langle \ast E, N \rangle [E, B]
\end{array}
\]

that sends each cone \( \ast \longleftarrow E \) to its composite \( \ast \longleftarrow E \) with \( \psi \). In Section 8.1, a limit is defined by a universal arrow of the module \( \langle \ast E, M \rangle \).

This section is largely analogous to Section 4.3 and Section 4.4; in fact, cones are regarded as a special sort of ordinary and extraordinary cylinders.

#### 4.6.1 Definition.

- Let \( M : \ast \to A \) be a left module. Given a functor \( F : E \to A \), a cone \( \alpha \) to \( F \) along \( M \), written \( \alpha : \ast \to F : \ast E \to M \), is defined by a frame \( \alpha \) of the composite left module \( \langle M \rangle F : \ast \to E \).
- Let \( M : X \to \ast \) be a right module. Given a functor \( G : E \to X \), a cone \( \alpha \) from \( G \) along \( M \), written \( \alpha : G \triangleright \ast : \ast E \triangleright M \), is defined by a frame \( \alpha \) of the composite right module \( G(M) : E \to \ast \).

#### 4.6.2 Remark. By Proposition 4.2.4,

- a frame \( \alpha \) of the left module \( \langle M \rangle F \) is the same thing as a frame of the endomodule \( \langle \ast E \rangle [M] F : E \to E \).
- a frame \( \alpha \) of the right module \( G(M) \) is the same thing as a frame of the endomodule \( G(M) \rangle \ast E \to E \).

#### 4.6.3 Definition. Let \( E \) be a category and \( M : X \to A \) be a module.

- Given an object \( x \in \|X\| \) and a functor \( F : E \to A \), a cone \( \alpha \) from \( x \) to \( F \) along \( M \), written \( \alpha : x \to F : \ast E \to M \), is defined by a frame \( \alpha \) of the composite left module \( x(M)F : \ast \to E \).
- Given an object \( a \in \|A\| \) and a functor \( G : E \to X \), a cone \( \alpha \) from \( G \) to \( a \) along \( M \), written \( \alpha : G \triangleright a : E \triangleright M \), is defined by a frame \( \alpha \) of the composite right module \( G(M) a : E \to \ast \).
4.6.4 Remark.  

(1) By Proposition 4.2.4,

- a frame $\alpha$ of the left module $x(\mathcal{M})F$ is the same thing as a frame of the endomodule $[\iota_E(x)(\mathcal{M})F] = [\iota_E \cdot x]F(\mathcal{M})$. Hence a cone $\alpha : x \sim F : *E \sim \mathcal{M}$ is the same thing as a cylinder $\alpha : [\iota_E \cdot x]F : E \sim \mathcal{M}$ right weighted by the unique functor $E \to *$, and thus depicted as

\[ \begin{array}{c}
\begin{array}{c}
E \\
\downarrow \alpha \\
F \\
\downarrow \delta \\
G \\
\downarrow \gamma \\
A \\
\end{array}
\end{array}
\]

- a frame $\alpha$ of the right module $G(\mathcal{M})a$ is the same thing as the endomodule $\langle G(\mathcal{M})a \rangle[\iota_E] = G(\mathcal{M})[\iota_E \cdot [\iota_E \cdot E]]$. Hence a cone $\alpha : G \sim a : E^* \sim \mathcal{M}$ is the same thing as a cylinder $\alpha : G \sim a \circ \iota_E : E \sim \mathcal{M}$ left weighted by the unique functor $E \to *$, and thus depicted as

\[ \begin{array}{c}
\begin{array}{c}
E \\
\downarrow \alpha \\
F \\
\downarrow \delta \\
G \\
\downarrow \gamma \\
A \\
\end{array}
\end{array}
\]

(2) By Proposition 4.2.5,

- a frame $\alpha$ of the right module $G(\mathcal{M})a$ is the same thing as a new cylinder of the left module $(x(\mathcal{M})F)[\iota_E \cdot E] = x(\mathcal{M})[\iota_E \cdot [\iota_E \cdot E]]$. Hence a cone $\alpha : x \sim F : *E \sim \mathcal{M}$ is the same thing as an extraordinary cylinder $\alpha : x \sim F \circ \gamma : E \sim \mathcal{M}$; a cone is thus regarded as an extraordinary cylinder $\alpha : x \sim F : E \sim \mathcal{M}$ with $F : E^* \times E \to A$ dummy in the first variable.

- a frame $\alpha$ of the right module $G(\mathcal{M})a$ is the same thing as the extraordinary cylinder $\alpha : [\iota_E \cdot E] \circ G \sim a : E \sim \mathcal{M}$; a cone is thus regarded as an extraordinary cylinder $\alpha : G \sim a : E \sim \mathcal{M}$ with $G : E^* \times E \to X$ dummy in the first variable.

(3) A cone defined in Definition 4.6.1 is a special instance of a cone defined in Definition 4.6.3 where $X$ [op. $A$] is the terminal category. Conversely, a cone

\[ \begin{array}{c}
\begin{array}{c}
E \\
\downarrow \alpha \\
F \\
\downarrow \delta \\
G \\
\downarrow \gamma \\
A \\
\end{array}
\end{array}
\]

along a module $\mathcal{M} : X \to A$ is the same thing as a cone

\[ \begin{array}{c}
\begin{array}{c}
\Delta x \\
\downarrow \alpha \\
X \\
\downarrow \gamma \\
\mathcal{M} \\
\downarrow \delta \\
A \\
\end{array}
\end{array}
\]

along the left module $x(\mathcal{M}) : * \to A$ [op. right module $\langle \mathcal{M} \rangle a : X \to *$].

(4) If $E$ is discrete, a cone $\alpha : x \sim F : *E \sim \mathcal{M}$ is called discrete as well.

- Given an object $x \in |X|$ and a family of objects $a_i \in |A|$ indexed by some set $\mathcal{I}$, a discrete cone $\alpha$ from $x$ to $(a_i)_{i \in \mathcal{I}}$ along $\mathcal{M}$ is defined by a family of $\mathcal{M}$-arrows $\alpha_i : x \sim a_i$ indexed by $\mathcal{I}$; conversely, a family of $\mathcal{M}$-arrows $\alpha_i : x \sim a_i$ indexed by $\mathcal{I}$ defines a discrete cone from $x$ to $(a_i)_{i \in \mathcal{I}}$ along $\mathcal{M}$.

- Given an object $a \in |A|$ and a family of objects $x_i \in |X|$ indexed by some set $\mathcal{I}$, a discrete cone $\alpha$ from $(x_i)_{i \in \mathcal{I}}$ to $a$ along $\mathcal{M}$ is defined by a family of $\mathcal{M}$-arrows $\alpha_i : x_i \sim a_i$ indexed by $\mathcal{I}$; conversely, a family of $\mathcal{M}$-arrows $\alpha_i : x_i \sim a_i$ indexed by $\mathcal{I}$ defines a discrete cone from $(x_i)_{i \in \mathcal{I}}$ to $a$ along $\mathcal{M}$.

(5) Recall from Remark 1.1.37(4) that the composite right module $G(\mathcal{M})a : E \to *$ is the same thing as the composite left module $a(\mathcal{M}^\circ) : * \to E^*$. Hence (see Remark 4.2.2(1)) a cone $\alpha : G \sim a : E^* \sim \mathcal{M}^\circ$ is the same thing as a cone $\alpha : a \sim G : *E^* \sim \mathcal{M}^\circ$. Dually, a cone $\alpha : x \sim F : *E^* \sim \mathcal{M}$ is the same thing as a cone $\alpha : F \sim x : E^* \sim \mathcal{M}^\circ$.

Note. The following is a special case of Definition 4.6.2 where $\psi : \mathcal{M} \to \mathcal{N}$ is given by the module morphism $g(\mathcal{M}) \sigma : x(\mathcal{M})F \to x'(\mathcal{M})F'$ [op. $\tau(\mathcal{M})f : G(\mathcal{M})a \to G'(\mathcal{M})a'$].
4.6.5 Definition. Let $E$ be a category and $M : X \to A$ be a module. Given a category arrow, a cone, and a natural transformation, all as in

$$\begin{align*}
\[x'] \xrightarrow{\varphi} \mathcal{E}' & \xrightarrow{g} F' \\
X - \mathcal{M} & = A
\end{align*}$$

and

$$\begin{align*}
\[x] \xrightarrow{\varphi'} \mathcal{E}' & \xrightarrow{g} F' \\
X - \mathcal{M} & = A
\end{align*}$$

their composite is the cone

$$\begin{align*}
\[x'] \xrightarrow{\varphi} \mathcal{E}' & \xrightarrow{g} F' \\
X - \mathcal{M} & = A
\end{align*}$$

defined by

$$g \circ \alpha \circ \sigma = \alpha \circ g(\mathcal{M}) \circ \sigma$$

and

$$\tau \circ \alpha \circ f = \alpha \circ \tau(\mathcal{M}) \circ f$$

where the composition on the right-hand side of the identity is that defined in Definition 4.2.6.

4.6.6 Remark. Each component of the composite $g \circ \alpha \circ \sigma$ [op. $\tau \circ \alpha \circ f$] is given by

$$[g \circ \alpha \circ \sigma]_e = [\alpha \circ g(\mathcal{M}) \circ \sigma]_e = \alpha_e \cdot (g(\mathcal{M}) \circ \sigma) = g \circ \alpha_e \circ \sigma_e$$

and

$$[\tau \circ \alpha \circ f]_e = [\alpha \circ \tau(\mathcal{M}) \circ f]_e = \alpha_e \cdot (\tau(\mathcal{M}) \circ f) = \tau_e \circ \alpha_e \circ f$$

(cf. Remark 4.4.4).

Note. The composition in Definition 4.6.5 yields the module of cones $\ast \mathcal{E} \rightsquigarrow \mathcal{M}$; the functorialities in Proposition 4.2.9 and Remark 1.1.25 allow the following definition.

4.6.7 Definition. Given a category $E$ and a module $M : X \to A$,

- the module

$$\langle \ast \mathcal{E}, \mathcal{M} \rangle : X \to [E, A]$$

of cones $\ast \mathcal{E} \rightsquigarrow \mathcal{M}$ is defined by

$$(x) \langle \ast \mathcal{E}, \mathcal{M} \rangle (F) = \prod_{E \cdot x} \langle \mathcal{M} \rangle F$$

for $x \in X$ and $F \in [E, A]$.

- the module

$$\langle \mathcal{E} \ast, \mathcal{M} \rangle : [E, X] \to A$$

of cones $\mathcal{E} \ast \rightsquigarrow \mathcal{M}$ is defined by

$$(G) \langle \mathcal{E} \ast, \mathcal{M} \rangle (a) = \prod_{E \cdot G(\mathcal{M}) a}$$

for $a \in A$ and $G \in [E, X]$.

4.6.8 Remark.

1. For an object $x \in \|X\|$ and a functor $F : E \to A$, the set

$$(x) \langle \ast \mathcal{E}, \mathcal{M} \rangle (F) = \prod_{E \cdot x} \langle \mathcal{M} \rangle F$$

consists of all frames of the composite left module $x \langle \mathcal{M} \rangle F : \ast \to E$, i.e. all cones $x \rightsquigarrow F : \ast \mathcal{E} \rightsquigarrow \mathcal{M}$, and for an object $a \in \|A\|$ and a functor $G : E \to X$, the set

$$(G) \langle \mathcal{E} \ast, \mathcal{M} \rangle (a) = \prod_{E \cdot G(\mathcal{M}) a}$$

consists of all frames of the composite right module $G(\mathcal{M}) a : E \to \ast$, i.e. all cones $G \rightsquigarrow a : \mathcal{E} \ast \rightsquigarrow \mathcal{M}$.

2. The composition in the module $\langle \ast \mathcal{E}, \mathcal{M} \rangle$ [op. $\langle \mathcal{E} \ast, \mathcal{M} \rangle$] is that defined in Definition 4.6.5; indeed, by definition, the composition

$$\xymatrix{ x \ar[r]^-{\alpha} & F \\
\x' \ar[r]^-{\alpha'} \ar@{.>}[u]|{g} \ar@{.>}[ru]|{\sigma} & F' \\
G \ar[r]^-{\alpha} & x \\
G' \ar[r]^-{\alpha'} \ar@{.>}[u]|{\tau} \ar@{.>}[ru]|{\tau(\mathcal{M})} & X' \ar[u]|{\tau(\mathcal{M})} \ar@{.>}[ru]|{\tau(\mathcal{M})} }$$
in \(\{*,\mathcal{M}\}\) [op. \(\langle E^*,\mathcal{M}\rangle\)] is given by
\[
g \circ \alpha \circ \sigma = \alpha \cdot \Pi_E \mathcal{M} \sigma = \alpha \cdot \Pi_E (\alpha \circ \mathcal{M}) \sigma \quad \text{op.} \quad \tau \circ \alpha \circ \mathcal{M} = \alpha \cdot \Pi_E \tau (\mathcal{M}) f = \alpha \cdot \tau (\mathcal{M}) f
\]
\(\alpha \cdot \Pi_E \mathcal{M} \sigma\) is the image of \(\alpha \in \Pi_E \mathcal{M} \mathcal{F}\) under the function \(\Pi_E \mathcal{M} : \Pi_E \mathcal{M} \mathcal{F} \rightarrow \Pi_E \mathcal{M} \mathcal{F}'\).

(3) If \(E\) is small and \(\mathcal{M}\) is locally small, then the module \(\{*,\mathcal{M}\} [\text{op.} \langle E^*,\mathcal{M}\rangle]\) is locally small.

(4) The identity
\[
\Pi_E \mathcal{M} \mathcal{F} = \Pi_E \mathcal{M} \mathcal{F} [\text{op.} \Pi_E \mathcal{M} \mathcal{G}\] holds by Remark 4.2.2(1) and Remark 1.1.37(4), giving canonical isomorphism
\[
\langle *,\mathcal{M}\rangle \cong \langle *,\mathcal{M}\rangle [\text{op.} \langle *,\mathcal{M}\rangle \cong \langle *,\mathcal{M}\rangle]
\]
as indicated by the identity
\[
\begin{array}{c}
\mathbb{E}, A] \xrightarrow{\text{[E,M]^*}} X^- \quad \text{op.} \quad A^{-\langle*,\mathcal{M}\rangle} \xrightarrow{\text{[E,M]^*}} X^- \\
\mathbb{E}^-, A^- \xrightarrow{\text{[E,M]^*}} X^- \quad \text{op.} \quad A^{-\langle*,\mathcal{M}\rangle} \xrightarrow{\text{[E,M]^*}} X^-
\end{array}
\]

4.6.9 Proposition. Given a category \(E\) and a composite module \(P\langle N \rangle \mathcal{Q}\) as in
\[
\begin{array}{c}
\mathbb{X} \xrightarrow{P\langle N \rangle \mathcal{Q}} \mathbb{A} \\
\mathbb{P} \downarrow \quad 1 \downarrow \mathbb{Q}
\end{array}
\]
the identity
\[
\begin{array}{c}
\mathbb{X} \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{E}, A] \quad \text{op.} \quad [E, X] \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{A} \\
\mathbb{P} \downarrow \quad 1 \downarrow \mathbb{Q}
\end{array}
\]
(i.e.,
\[
\langle *,P\langle N \rangle \mathcal{Q}\rangle = P\langle *,N \rangle [E, \mathcal{Q}] \quad \text{op.} \quad \langle *,P\langle N \rangle \mathcal{Q}\rangle = [E, P] \langle *,N \rangle \mathcal{Q}
\]
) holds.

Proof. For any \(x \in \mathbb{X}\) and \(F \in [\mathbb{E}, \mathbb{A}],\)
\[
(x) \langle E,P\langle N \rangle \mathcal{Q}\rangle (F) = \Pi_E x \langle P\langle N \rangle \mathcal{Q}\rangle F
\]
\[
= \Pi_E (x: P): [\mathbb{Q}, F] \mathcal{M}
\]
\[
= (x: P) \langle E,N \rangle ([E, \mathcal{Q}] [F])
\]
\[
= (x: P) \langle E,N \rangle \langle E, \mathcal{Q}\rangle [F]
\]
\[
= (x) \langle P \langle E,N \rangle \mathcal{Q}\rangle [E, \mathcal{Q}] (F).
\]

4.6.10 Remark. The cell
\[
\begin{array}{c}
\mathbb{X} \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{E}, A] \quad \text{op.} \quad [E, X] \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{A} \\
\mathbb{P} \downarrow \quad 1 \downarrow \mathbb{Q}
\end{array}
\]

sends each cone
\[
\begin{array}{c}
\mathbb{X} \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{A} \quad \text{op.} \quad \mathbb{E} \xrightarrow{\langle E,P\langle N \rangle \mathcal{Q}\rangle} \mathbb{A}
\end{array}
\]
to the cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{1} & E \\
\downarrow \alpha & \downarrow \vdash F & \downarrow \alpha \\
X - \mathcal{M} & \xrightarrow{\mathcal{P}} & \mathcal{A}
\end{array}
\quad \text{op.}
\quad \begin{array}{ccc}
E & \xrightarrow{1} & \ast \\
\downarrow \alpha & \downarrow \vdash F & \downarrow \alpha \\
\mathcal{Y} - \mathcal{N} & \xrightarrow{\mathcal{P}} & \mathcal{B}
\end{array}
\]

defined by the same frame.

**Note.** The following definition is an instance of Definition 4.2.6 where \( \psi : \mathcal{M} \rightarrow \mathcal{N} \) is given by the module morphism \( \mathbf{x}(\psi) : \mathbf{x}(\mathcal{M}) \psi \rightarrow \mathbf{x}(\mathcal{N}) \psi \) [op. \( \mathbf{G}(\psi) \mathbf{a} : \mathbf{G}(\mathcal{M}) \mathbf{a} \rightarrow \mathbf{G}(\mathcal{N}) \mathbf{a} \)].

**4.6.11 Definition.** Given a cone \( \alpha \) and a module morphism \( \psi \) as in

\[
\begin{array}{ccc}
\ast & \xleftarrow{1} & E \\
\downarrow \alpha & \downarrow \vdash F & \downarrow \alpha \\
X - \mathcal{M} & \xrightarrow{\mathcal{P}} & \mathcal{A}
\end{array}
\quad \text{op.}
\quad \begin{array}{ccc}
E & \xrightarrow{1} & \ast \\
\downarrow \alpha & \downarrow \vdash F & \downarrow \alpha \\
\mathcal{Y} - \mathcal{N} & \xrightarrow{\mathcal{P}} & \mathcal{B}
\end{array}
\]

their composite \( \alpha \circ \psi = \psi \circ \alpha \) is the cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{1} & E \\
\downarrow \alpha \circ \psi & \downarrow \vdash F & \downarrow \alpha \circ \psi \\
X - \mathcal{N} & \xrightarrow{\mathcal{P}} & \mathcal{A}
\end{array}
\quad \text{op.}
\quad \begin{array}{ccc}
E & \xrightarrow{1} & \ast \\
\downarrow \alpha \circ \psi & \downarrow \vdash F & \downarrow \alpha \circ \psi \\
\mathcal{Y} - \mathcal{N} & \xrightarrow{\mathcal{P}} & \mathcal{B}
\end{array}
\]

defined by

\[
\alpha \circ \psi = \alpha \circ \mathbf{x}(\psi) \mathbf{F} \quad \text{op.} \quad \alpha \circ \psi = \alpha \circ \mathbf{G}(\psi) \mathbf{a}
\]

where the composition on the right-hand side of the identity is that defined in Definition 4.2.6.

**4.6.12 Remark.** Each component of the composite \( \alpha \circ \psi : \mathbf{x} \rightsquigarrow \mathbf{F} \) [op. \( \alpha \circ \psi : \mathbf{G} \rightsquigarrow \mathbf{a} \)] is given by

\[
[\alpha \circ \psi]_\mathbf{e} = [\alpha \circ \mathbf{x}(\psi) \mathbf{F}]_\mathbf{e} = \alpha_\mathbf{e} \circ (\mathbf{x}(\psi) \mathbf{F}) \mathbf{e} = \alpha_\mathbf{e} \circ (\mathbf{x} \circ (\psi) \mathbf{F} : \mathbf{F} : \mathbf{e})
\]

op.

\[
[\alpha \circ \psi]_\mathbf{a} = [\alpha \circ \mathbf{G}(\psi) \mathbf{a}]_\mathbf{a} = \alpha_\mathbf{a} \circ (\mathbf{G}(\psi) \mathbf{a}) = \alpha_\mathbf{a} \circ (\mathbf{e} \circ \mathbf{G} \circ (\psi) \mathbf{a})
\]

(cf. Remark 4.4.10).

**Note.** For any category \( \mathbf{E} \) and any module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{N} \), the composition in Definition 4.6.11 yields the postcomposition module morphism \( (\ast \mathbf{E}, \psi) : (\ast \mathbf{E}, \mathcal{M}) \rightarrow (\ast \mathbf{E}, \mathcal{N}) \) from the module of cones \( \ast \mathbf{E} \rightsquigarrow \mathcal{M} \) to the module of cones \( \ast \mathbf{E} \rightsquigarrow \mathcal{N} \). Here is the formal definition:

**4.6.13 Definition.** Given a category \( \mathbf{E} \) and a module morphism \( \psi : \mathcal{M} \rightarrow \mathcal{N} : \mathbf{x} \rightarrow \mathbf{A} \),

- the postcomposition module morphism \( (\ast \mathbf{E}, \psi) : (\ast \mathbf{E}, \mathcal{M}) \rightarrow (\ast \mathbf{E}, \mathcal{N}) : \mathbf{x} \rightarrow [\mathbf{E}, \mathbf{A}] \)

“postcomposition with \( \psi \),” is defined by

\[
(\mathbf{x}) (\ast \mathbf{E}, \psi) (\mathbf{F}) = \prod_{\mathbf{E}} \mathbf{x}(\psi) \mathbf{F}
\]

for each pair of an object \( \mathbf{x} \in \| \mathbf{X} \| \) and a functor \( \mathbf{F} : \mathbf{E} \rightarrow \mathbf{A} \).

- the postcomposition module morphism \( (\mathbf{E} \ast, \psi) : (\mathbf{E} \ast, \mathcal{M}) \rightarrow (\mathbf{E} \ast, \mathcal{N}) : [\mathbf{E}, \mathbf{X}] \rightarrow \mathbf{A} \)

“postcomposition with \( \psi \),” is defined by

\[
(\mathbf{G}) (\mathbf{E} \ast, \psi) (\mathbf{a}) = \prod_{\mathbf{E}} \mathbf{G} \circ (\psi) \mathbf{a}
\]

for each pair of an object \( \mathbf{a} \in \| \mathbf{A} \| \) and a functor \( \mathbf{G} : \mathbf{E} \rightarrow \mathbf{X} \).

**4.6.14 Remark.**

(1) Replacing \( \psi : \mathcal{M} \rightarrow \mathcal{N} \) by \( \mathbf{x}(\psi) \mathbf{F} : \mathbf{x}(\mathcal{M}) \mathbf{F} \rightarrow \mathbf{x}(\mathcal{N}) \mathbf{F} \) in Definition 4.2.6, we have a function

\[
\prod_{\mathbf{E}} \mathbf{x}(\psi) \mathbf{F} : \prod_{\mathbf{E}} \mathbf{x}(\mathcal{M}) \mathbf{F} \rightarrow \prod_{\mathbf{E}} \mathbf{x}(\mathcal{N}) \mathbf{F}
\]
or
\[(x) \langle *E, \psi \rangle (F) : (x) \langle *E, M \rangle (F) \rightarrow (x) \langle *E, N \rangle (F)\]
in the notation introduced in Definition 4.6.7. Since \(x \langle \psi \rangle F\) is natural in \(x\) and \(F\) by Remark 1.1.25, so is \( (x) \langle *E, \psi \rangle (F) = \prod_E x(\psi) F\) by Proposition 4.2.9. Hence \(\langle *E, \psi \rangle\) so defined does form a module morphism \(\langle *E, M \rangle \rightarrow \langle *E, N \rangle\).

(2) The module morphism \(\langle *E, \psi \rangle\) maps each cone
\[\alpha : x \rightharpoonup F : *E \rightharpoonup M\] to
\[\alpha \cdot \prod_E x(\psi) F = \alpha \cdot x(\psi) F\] that is, to the cone
\[\alpha \cdot \psi : x \rightharpoonup F : *E \rightharpoonup N\] defined in Definition 4.6.11.

(3) The assignments \(\psi \mapsto \langle *E, \psi \rangle\) and \(\psi \mapsto \langle E^+, \psi \rangle\) are functorial; indeed, the functor
\[\langle *E, - \rangle : [X : A] \rightarrow [X : [E, A]]\]
is defined by
\[(x) \langle *E, M \rangle (F) = \prod_E x(M) F\]
for \(x \in X, F \in [E, A]\), and \(M \in [X : A]\), and the functor
\[\langle E^+, - \rangle : [X : A] \rightarrow [[E, X] : A]\]
is defined by
\[(G) \langle E^+, M \rangle (a) = \prod_E G(M) a\]
for \(a \in A, G \in [E, X]\), and \(M \in [X : A]\).

Note. By Remark 1.2.2(3), the following definition is a special case of Definition 4.6.11, and vice versa.

4.6.15 Definition. Given a cone \(\alpha\) and cell \(\psi\) as in
\[
\begin{array}{ccc}
* & \xymatrix@+1pc{\leftarrow \ar[r]^l & E} & \op. & \begin{array}{c}
\begin{array}{ccc}
\xymatrix@+1pc{E & \ar[r]^l & *} \\
\xymatrix@+1pc{X & \ar[r]_{\sim M} & A}
\end{array}
\end{array} \\
\xymatrix@+1pc{Y & \ar[r]_{\sim N} & B}
\end{array}
\]
their composite \(\alpha \cdot \psi = \psi \cdot \alpha\) is the cone
\[
\begin{array}{ccc}
* & \xymatrix@+1pc{\leftarrow \ar[r]^l & E} & \op. & \begin{array}{c}
\begin{array}{ccc}
\xymatrix@+1pc{E & \ar[r]^l & *} \\
\xymatrix@+1pc{Y & \ar[r]_{\sim N} & B}
\end{array}
\end{array} \\
\xymatrix@+1pc{X & \ar[r]_{\sim M} & A}
\end{array}
\]
defined by
\[\alpha \cdot \psi = \alpha \cdot x(\psi) F\] and
\[\alpha \cdot \psi = \alpha \cdot G(\psi) a\]
where the composition on the right-hand side of the identity is that defined in Definition 4.2.6 (\(\psi\) is the module morphism \(M \rightarrow P(\langle N \rangle Q)\)).

4.6.16 Remark. Each component of the composite cone \(\alpha \cdot \psi\) is given as in Remark 4.6.12:
\[\alpha \cdot \psi_e = \alpha \cdot (x) \langle \psi \rangle (F \cdot e)\] and
\[\alpha \cdot \psi_e = \alpha \cdot (e : G \langle \psi \rangle a)\].

Note. For any category \(E\) and any cell \(\psi : M \rightarrow N\), the composition in Definition 4.6.15 yields the postcomposition cell \(\langle *E, \psi \rangle : \langle *E, M \rangle \rightarrow \langle *E, N \rangle\) from the module of cone \(*E \rightharpoonup M\) to the module of cones \(*E \rightharpoonup N\); the postcomposition module morphism in Definition 4.6.13 and the identity in Proposition 4.6.9 allow the following definition.
4.6.17 Definition. Given a category $E$ and a cell $X \xrightarrow{\mathcal{M}} A$, the postcomposition cell

$$
X \xrightarrow{\psi} Y \xrightarrow{\mathcal{N}} B
$$

is defined by the postcomposition module morphism

$$
\langle \ast E, \mathcal{M} \rangle \xrightarrow{\langle \ast E, \psi \rangle} \langle \ast E, \mathcal{P} \rangle Q = P \langle \ast E, \mathcal{N} \rangle Q
$$

(4.6.18 Remark). The cell $\langle \ast E, \psi \rangle$ sends each cone $\alpha : x \sim F : \ast E \sim \mathcal{M}$ to the cone $\alpha \circ \psi : x : P \sim Q \circ F : \ast E \sim \mathcal{N}$ defined in Definition 4.6.15, and the cell $\langle \ast E, \psi \rangle$ sends each cone $\alpha : G \sim a : \ast E \sim \mathcal{M}$ to the cone $\alpha \circ \psi : G \circ P \sim Q \circ a : \ast E \sim \mathcal{N}$.

4.6.19 Proposition. If a cell $\psi$ is fully faithful, so is the postcomposition cell $\langle \ast E, \psi \rangle$ [op. $\langle \ast E, \psi \rangle$] for any category $E$.

Proof. Since the postcomposition operation $(\ast E, \sim)$ is functorial (see Remark 4.6.14(3)), it preserves isomorphisms. \qed

4.6.20 Proposition. The assignment $\psi \mapsto \langle \ast E, \psi \rangle$ [op. $\psi \mapsto \langle \ast E, \psi \rangle$] of the postcomposition cell is functorial.

Proof. The functoriality is verified similarly to Proposition 4.3.21 using Remark 4.6.14(3) and Proposition 4.6.9 in place of Remark 4.3.14(3) and Proposition 4.3.9. \qed

4.6.21 Remark.

1. In fact, the functoriality of the assignment $\psi \mapsto \langle \ast E, \psi \rangle$ [op. $\psi \mapsto \langle \ast E, \psi \rangle$] is reduced to that of the assignment $\psi \mapsto \langle E, \psi \rangle$ by Corollary 4.6.23 below.

2. Given a small category $E$, the assignment $\psi \mapsto \langle \ast E, \psi \rangle$ [op. $\psi \mapsto \langle \ast E, \psi \rangle$] gives the functor

$$
\langle \ast E, \sim \rangle : \text{MOD} \to \text{MOD} \quad \text{op.} \quad \langle \ast E, \sim \rangle : \text{MOD} \to \text{MOD}
$$

, extending the functor $\langle \ast E, \sim \rangle$ [op. $\langle \ast E, \sim \rangle$] in Remark 4.6.14(3) as shown in the commutative diagram

$$
\begin{array}{ccc}
\text{MOD} & \xrightarrow{\langle \ast E, \sim \rangle} & \text{MOD} \\
\downarrow & \downarrow & \downarrow \\
\text{MOD} & \xrightarrow{\langle \ast E, \sim \rangle} & \text{MOD}
\end{array}
$$

, where $\leftrightarrow$ denotes the canonical embedding in Remark 1.2.19(2).

Note. Recall from Remark 4.6.4(1) that a cone $\ast E \sim \mathcal{M}$ is the same thing as a cylinder $E \sim \mathcal{M}$ right weighted by the functor $!_E$. The module of cones defined earlier is thus also obtained as an instance of the module of weighted cylinders. We state this fact formally in the following.

4.6.22 Theorem.

1. For a category $E$ and a module $\mathcal{M} : X \to A$, the module

$$
\langle \ast E, \mathcal{M} \rangle : X \to [E, A] \quad \text{op.} \quad \langle \ast E, \mathcal{M} \rangle : [E, X] \to A
$$

...
of cones defined in Definition 4.6.7 is obtained from the module \((E, M) : [E, X] \to [E, A]\) of cylinders defined in Definition 4.3.7 by the composition

\[
X \xrightarrow{[E, X]} [E, X] \xrightarrow{\cdot (E, M)} [E, A] \quad \text{op.} \quad [E, X] \xrightarrow{\cdot (E, M)} [E, A] \xrightarrow{[E, A]} A
\]

; that is,

\[
\langle *E, M \rangle = \langle !E \circ M \rangle \quad \text{op.} \quad \langle E^*, M \rangle = \langle !E^* \circ \rho \rangle M
\]

(see Definition 4.5.3).

(2) For a category \(E\) and a module morphism \(\psi : M \to N : X \to A\), the postcomposition module morphism

\[
\langle *E, \psi \rangle : \langle *E, M \rangle \to \langle *E, N \rangle \quad \text{op.} \quad \langle E^*, \psi \rangle : \langle E^*, M \rangle \to \langle E^*, N \rangle
\]

defined in Definition 4.6.13 is obtained from the postcomposition module morphism \((E, \psi) : (E, M) \to (E, N)\) defined in Definition 4.3.13 by the composition

\[
X \xrightarrow{[E, X]} [E, X] \xrightarrow{\cdot (E, \psi)} [E, A] \quad \text{op.} \quad [E, X] \xrightarrow{\cdot (E, \psi)} [E, A] \xrightarrow{[E, A]} A
\]

; that is,

\[
\langle *E, \psi \rangle = \langle !E \circ \psi \rangle \quad \text{op.} \quad \langle E^*, \psi \rangle = \langle !E^* \circ \rho \rangle \psi
\]

(see Definition 4.5.5).

Proof. For any \(x \in X\) and \(F \in [E, A]\),

\[
(x) \langle *E, M \rangle (F) = \prod_E x (M) F
\]

\[= \prod_E [!E] (x (M) F)\]

\[= \prod_E [!E \circ x] (M) F\]

\[= [!E \circ x] (E, M) (F)\]

\[= (x) ([!E, X]) (E, M) (F)\]

\[= (x) ([!E, X]) (E, M) (F)\]

\[\text{\((\ast 1)\) by Proposition 4.2.4.}\]

4.6.23 Corollary. The postcomposition cell

\[
\begin{array}{ccc}
X \xrightarrow{\langle *E, M \rangle} & [E, A] & \quad \text{op.} \quad [E, X] \xrightarrow{\langle E^*, M \rangle} A \\
\rho \downarrow & \langle *E, \psi \rangle \downarrow [E, Q] & \quad \text{op.} \quad \langle E^*, \psi \rangle \downarrow Q \\
Y \xrightarrow{\langle *E, Q \rangle} & [E, B] & \quad \text{op.} \quad [E, Y] \xrightarrow{\langle E^*, Q \rangle} B
\end{array}
\]

defined in Definition 4.6.17 is obtained from the postcomposition cell defined in Definition 4.3.17 by the pasting composition

\[
\begin{array}{ccc}
X \xrightarrow{[E, X]} & [E, X] \xrightarrow{\cdot (E, M)} [E, A] & \quad \text{op.} \quad [E, X] \xrightarrow{\cdot (E, M)} [E, A] \xrightarrow{[E, A]} A \\
\rho \downarrow & \langle E, \psi \rangle \downarrow [E, Q] & \quad \text{op.} \quad \langle E, \psi \rangle \downarrow [E, Q] \downarrow Q \\
Y \xrightarrow{[E, Y]} & [E, Y] \xrightarrow{\cdot (E, N)} [E, B] & \quad \text{op.} \quad [E, Y] \xrightarrow{\cdot (E, N)} [E, B] \xrightarrow{[E, B]} B
\end{array}
\]

; that is,

\[
\langle *E, \psi \rangle = \langle !E \circ \psi \rangle \quad \text{op.} \quad \langle E^*, \psi \rangle = \langle !E^* \circ \rho \rangle \psi
\]

(see Definition 4.5.7).

Proof. Immediate from Theorem 4.6.22, recalling the definition of the postcomposition cell \((E, \psi)\) and the definition of pasting composition. \(\square\)

4.6.24 Remark. By Theorem 4.6.22 and Corollary 4.6.23, the compositions in Definition 4.6.5, Definition 4.6.11, and Definition 4.6.15 are in fact the same thing as those in Definition 4.3.5, Defini-
4.6. Cones

4.6.25 Definition. Given a category \( E \) and a cell morphisms \( \tau : \psi \to \varphi : M \to N \),

- the postcomposition cell morphism

\[
\langle \ast E, \tau \rangle : \langle \ast E, \psi \rangle \to \langle \ast E, \varphi \rangle : \langle \ast E, M \rangle \to \langle \ast E, N \rangle
\]

"postcomposition with \( \tau \)" is defined by the pair of natural transformations

\[
\tau_0 : \psi_0 \to \varphi_0 \quad \text{and} \quad [E, \tau_1] : [E, \psi_1] \to [E, \varphi_1]
\]

, the left component of \( \tau \) and the postcomposition with the right component of \( \tau \) (see Preliminary 0.0.2(1)).

- the postcomposition cell morphism

\[
\langle E \ast, \tau \rangle : \langle E \ast, \psi \rangle \to \langle E \ast, \varphi \rangle : \langle E \ast, M \rangle \to \langle E \ast, N \rangle
\]

"postcomposition with \( \tau \)" is defined by the pair of natural transformations

\[
[E, \tau_0] : [E, \psi_0] \to [E, \varphi_0] \quad \text{and} \quad \tau_1 : \psi_1 \to \varphi_1
\]

, the postcomposition with the left component of \( \tau \) (see Preliminary 0.0.2(1)) and the right component of \( \tau \).

4.6.26 Remark. (1) \( \langle \ast E, \tau \rangle \) [op. \( \langle E \ast, \tau \rangle \)] so defined does form a cell morphism. Indeed, given a cone \( \alpha : x \Rightarrow F : \ast E \Rightarrow M \) [op. \( \alpha : G \Rightarrow a : E \ast \Rightarrow M \)], the commutativity of

\[
\begin{align*}
x : \psi_0 & \Rightarrow (E, \psi)_F \quad \text{op.} \quad G : [E, \psi_0] \Rightarrow (E \ast, \psi)_a \\
x : \psi_0 & \Rightarrow (E, \psi)_F \quad \text{op.} \quad G : [E, \psi_0] \Rightarrow (E \ast, \psi)_a
\end{align*}
\]

(i.e.

\[
\begin{align*}
x : \psi_0 & \Rightarrow \psi_1 \circ \varphi F \quad \text{op.} \quad G : \psi_0 \Rightarrow \psi_1 \circ \varphi a \\
x : \psi_0 & \Rightarrow \psi_1 \circ \varphi F \quad \text{op.} \quad G : \psi_0 \Rightarrow \psi_1 \circ \varphi a
\end{align*}
\]

(2) The assignment \( \tau \mapsto \langle \ast E, \tau \rangle \) defines a functor

\[
\langle \ast E, - \rangle : [M : N] \to [(\ast E, M) : (\ast E, N)]
\]

; indeed, by the definition of the cell morphism \( \langle \ast E, \tau \rangle \), the functoriality of \( \langle \ast E, - \rangle \) is reduced to that of \([E, -] : [M_1, N_1] \to [[E, M_1], [E, N_1]]\). Dually, the assignment \( \tau \mapsto \langle E \ast, \tau \rangle \) defines a functor

\[
\langle E \ast, - \rangle : [M : N] \to [(E \ast, M) : (E \ast, N)].
\]

\textbf{Note.} The following definition is an instance of Definition 4.2.10 where \( M \) is given by the composite module \( x(M) F : \ast \Rightarrow E \) [op. \( G(M) a : E \Rightarrow \ast \)].
4.6.27 Definition. If $K$ is a functor and $\alpha$ is a cone as in
\[
\begin{array}{ccc}
\ast & \overset{\ast}{\leftarrow} & E \\ x \downarrow & \alpha & \downarrow \phi \\
X - \overset{\ast}{\rightarrow} - A
\end{array}
\] \hspace{1cm}
\begin{array}{ccc}
D & \overset{\ast}{\leftarrow} & \ast \\ \downarrow & \alpha & \downarrow \alpha \\
X - \overset{\ast}{\rightarrow} - A
\end{array}
\]
then their composite $K \circ \alpha = \alpha \circ K$ is the cone
\[
\begin{array}{ccc}
\ast & \overset{\ast}{\leftarrow} & D \\ x \downarrow & K \circ \alpha & \downarrow \phi \circ K \\
X - \overset{\ast}{\rightarrow} - A
\end{array}
\] \hspace{1cm}
\begin{array}{ccc}
D & \overset{\ast}{\leftarrow} & \ast \\ \downarrow & K \circ \alpha & \downarrow \alpha \\
X - \overset{\ast}{\rightarrow} - A
\end{array}
\]
defined by
\[
[K \circ \alpha]_d = \alpha_{(K \cdot d)}
\]
for each $d \in \|D\|$.

Note. Given a functor $K : D \to E$, the composition in Definition 4.6.27 yields the precomposition cell $(K, \mathcal{M}) : (\ast, \mathcal{E}) \to (\ast, \mathcal{D})$ from the module of cones $\ast \mathcal{E} \rightsquigarrow \mathcal{M}$ to the module of cones $\ast \mathcal{D} \rightsquigarrow \mathcal{M}$. Here is the formal definition:

4.6.28 Definition. Given a functor $K : D \to E$ and a module $\mathcal{M} : X \rightsquigarrow A$, the precomposition cell
\[
\begin{array}{ccc}
X & \overset{\ast}{\leftarrow} [E, A] \\ \downarrow & \overset{\ast}{\leftarrow} \mathcal{M} \\
\ast & \overset{\ast}{\leftarrow} \mathcal{M}
\end{array}
\] \hspace{1cm}
\begin{array}{ccc}
[E, X] & \overset{\ast}{\leftarrow} A \\ \downarrow & \mathcal{M} \\
\ast & \overset{\ast}{\leftarrow} \mathcal{M}
\end{array}
\]
“precomposition with $K$”, is defined by
\[
(x) = (\ast, \mathcal{M}) (F) = \prod_{\ast \mathcal{K} \mathcal{X}} \mathcal{M} (F) \quad \overset{\ast}{\leftarrow} \mathcal{E} \overset{\ast}{\leftarrow} \mathcal{A}
\] \hspace{1cm}
\begin{array}{ccc}
D, X & \overset{\ast}{\leftarrow} [E, A] \\ \downarrow & \mathcal{D} \overset{\ast}{\leftarrow} \mathcal{M} \\
\ast & \overset{\ast}{\leftarrow} \mathcal{M}
\end{array}
\]
for each object $x \in \|X\|$ [op. $a \in \|A\|$] and each functor $F : \mathcal{E} \to \mathcal{A}$ [op. $G : \mathcal{E} \to X$].

4.6.29 Remark.
(1) We will see in Proposition 4.6.31 that the cell $(\ast, \mathcal{M}) [\text{op. } (K \ast, \mathcal{M})]$ is in fact obtained from the cell $(K, \mathcal{M})$ in Definition 4.3.28.
(2) The cell $(\ast, \mathcal{M}) [\text{op. } (K \ast, \mathcal{M})]$ sends each cone
\[
\alpha : x \rightsquigarrow F : \ast \mathcal{E} \rightsquigarrow \mathcal{M}
\] \hspace{1cm}
\begin{array}{ccc}
\alpha : G \rightsquigarrow a : \ast \mathcal{E} \rightsquigarrow \mathcal{M}
\end{array}
\]

\[
\alpha : \prod_{\ast \mathcal{K} \mathcal{X}} \mathcal{M} (F) = K \circ \alpha
\] \hspace{1cm}
\begin{array}{ccc}
\alpha : \prod_{K \ast \mathcal{G}} \mathcal{M} (a) = K \circ \alpha
\end{array}
\]
; that is, to the cone
\[
K \circ \alpha : x \rightsquigarrow F \circ K : \ast \mathcal{D} \rightsquigarrow \mathcal{M}
\] \hspace{1cm}
\begin{array}{ccc}
K \circ \alpha : K \circ \alpha \mathcal{G} \rightsquigarrow a : \ast \mathcal{D} \rightsquigarrow \mathcal{M}
\end{array}
\]
defined in Definition 4.6.27.

4.6.30 Example.
(1) Let $\mathcal{M} : X \rightsquigarrow A$ be a module and $\mathcal{E}$ be a category. Given a subcategory $\mathcal{D}$ of $\mathcal{E}$, precomposition with the inclusion $\mathcal{D} \to \mathcal{E}$ yields the cell
\[
\begin{array}{ccc}
\ast & \overset{\ast}{\leftarrow} [E, A] \\ \downarrow & \overset{\ast}{\leftarrow} \mathcal{M} \\
\ast & \overset{\ast}{\leftarrow} \mathcal{M}
\end{array}
\] \hspace{1cm}
\begin{array}{ccc}
\mathcal{D} & \overset{\ast}{\leftarrow} [E, A] \\ \downarrow & \mathcal{D} \overset{\ast}{\leftarrow} \mathcal{M} \\
\mathcal{D} & \overset{\ast}{\leftarrow} \mathcal{M}
\end{array}
\]
“restriction to $\mathcal{D}$”, which sends each cone $\alpha : x \rightsquigarrow F : \ast \mathcal{E} \rightsquigarrow \mathcal{M}$ [op. $\alpha : G \rightsquigarrow a : \ast \mathcal{E} \rightsquigarrow \mathcal{M}$] to the cone $\mathcal{D} \circ \alpha : x \rightsquigarrow F \circ \mathcal{D} : \ast \mathcal{D} \rightsquigarrow \mathcal{M}$ [op. $\mathcal{D} \circ \alpha : \mathcal{D} \circ \mathcal{G} \rightsquigarrow a : \ast \mathcal{D} \rightsquigarrow \mathcal{M}$], $\alpha$ restricted to $\mathcal{D}$ (cf. Preliminary 0.0.3).
(2) Let \( \mathcal{M} : X \to A \) be a module and \( E \) be a category. Given an object \( e \in \|E\| \), precomposition with the functor \( e : \ast \to E \) yields the cell

\[
\begin{array}{c}
X \xrightarrow{\langle \ast, E, \mathcal{M} \rangle} \{E, X\} \xrightarrow{\langle \ast, E \rangle} \{E, A\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} \{E, X\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} \{E, A\} \\
\quad \downarrow 1 \\
X \xrightarrow{\langle \ast, E, \mathcal{M} \rangle} \{E, X\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} \{E, A\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} \{E, X\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} \{E, A\} \\
\end{array}
\]

, “evaluation at \( e \)”, which sends each cone \( \alpha : x \to F : \ast \to \mathcal{M} \) [op. \( \alpha : G \to a : E \to \mathcal{M} \)] to the \( \mathcal{M} \)-arrow \( \alpha_e : x \to F \circ e \) [op. \( \alpha_e : e : G \to a \)], the component of \( \alpha \) at \( e \) (cf. Preliminary 0.0.6(1)).

4.6.31 Proposition. The precomposition cell \( \langle \ast, K, \mathcal{M} \rangle \) of \( \mathcal{M} \) is obtained from the precomposition cell \( \langle K, \mathcal{M} \rangle \) in Definition 4.3.28 by the pasting composition

\[
\begin{array}{c}
X \xrightarrow{[1_E, X]} \{E, X\} \xrightarrow{\langle E, \mathcal{M} \rangle} \{E, A\} \xrightarrow{\langle \ast, E \rangle} \{E, A\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} A \\
\quad \downarrow 1 \\
X \xrightarrow{[1_D, X]} \{D, X\} \xrightarrow{\langle D, \mathcal{M} \rangle} \{D, A\} \xrightarrow{\langle \ast, D \rangle} \{D, A\} \xrightarrow{\langle \ast, \mathcal{M} \rangle} A \\
\end{array}
\]

Proof. We need to verify that \( \langle \ast, K, \mathcal{M} \rangle = [1_E, X] \langle K, \mathcal{M} \rangle \). But for any object \( x \in \|X\| \) and any functor \( F : E \to A \),

\[
\begin{align*}
(x) \langle \ast, K, \mathcal{M} \rangle (F) &= \prod_{x} x(\mathcal{M} : F) \\
&= \prod_{K} [1_E, x](\mathcal{M} : F) \\
&= \prod_{K} \{[1_E, x] \delta x \} \mathcal{M} (F) \\
&= ([1_E, x] \delta x) \{K, \mathcal{M} \} (F) \\
&= (x) \{[1_E, x] \delta x \} \mathcal{M} (F)
\end{align*}
\]

(\( \ast_1 \) by Proposition 4.2.12).

4.6.32 Proposition. If \( \mathcal{M} \) is a locally small module, then the assignment

\[
K \mapsto \langle \ast, K, \mathcal{M} \rangle \quad \text{op.} \quad K \mapsto \langle K, \mathcal{M} \rangle
\]

defines the contravariant functor

\[
\langle \ast, \_ \rangle : \text{Cat}^\to \to \text{Mod} \quad \text{op.} \quad \langle \_ \rangle : \text{Cat}^\to \to \text{Mod}.
\]

Proof. Since the cell \( \langle \ast, K, \mathcal{M} \rangle \) is obtained from the cell \( \langle K, \mathcal{M} \rangle \) by the pasting composition in Proposition 4.6.31, the functoriality of the assignment \( K \mapsto \langle \ast, K, \mathcal{M} \rangle \) is reduced to that of the assignment \( K \mapsto \langle K, \mathcal{M} \rangle \) (see Proposition 4.3.30) by virtue of Corollary 1.2.39.

4.6.33 Theorem. There is a functor

\[
\langle \ast, \_ \rangle : \text{Cat}^\to \times \text{Mod} \to \text{Mod} \quad \text{op.} \quad \langle \_ , \_ \rangle : \text{Cat}^\to \times \text{Mod} \to \text{Mod}
\]

such that for each small category \( E \),

\[
\langle \ast, E, \_ \rangle : \text{Mod} \to \text{Mod} \quad \text{op.} \quad \langle E, \_ \rangle : \text{Mod} \to \text{Mod}
\]

coincides with the functor in Remark 4.6.21(2), and for each locally small module \( \mathcal{M} \),

\[
\langle \ast, \_ \rangle : \text{Cat}^\to \to \text{Mod} \quad \text{op.} \quad \langle \_ , \_ \rangle : \text{Cat}^\to \to \text{Mod}
\]

coincides with the functor in Proposition 4.6.32.

Proof. Similar to the proof of Theorem 4.3.33.

Note. In Remark 4.6.4(2), we saw that a cone \( \ast E \to \mathcal{M} \) is a special instance of an extraordinary cylinder \( E \to \mathcal{M} \), and in the following, we will see that the module of cones is obtained from the module of extraordinary cylinders.
4.6.34 Theorem.

(1) For a category $\mathbf{E}$ and a module $\mathcal{M} : X \rightarrow \mathbf{A}$, the module

$$\langle \ast \mathbf{E}, \mathcal{M} \rangle : X \rightarrow [\mathbf{E}, \mathbf{A}] \quad \text{op.} \quad \langle \mathbf{E}^*, \mathcal{M} \rangle : [\mathbf{E}, \mathbf{X}] \rightarrow [\mathbf{E}, \mathbf{A}]$$

of cones defined in Definition 4.6.7 is obtained from the module $\langle \mathbf{E}, \mathcal{M} \rangle : [\mathbf{E}, \mathbf{X}] \rightarrow [\mathbf{E}, \mathbf{A}]$ of extraordinary cylinders defined in Definition 4.4.5 by the composition

$$X \xrightarrow{(\mathbf{E}, \mathcal{M})} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}, \mathbf{A}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{X}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{X}] \xrightarrow{([\mathbf{E}^\ast, \mathcal{M}]} \mathbf{A}$$

; that is,

$$\langle \ast \mathbf{E}, \mathcal{M} \rangle = \langle \mathbf{E}, \mathcal{M} \rangle [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}] \quad \text{op.} \quad \langle \mathbf{E}^*, \mathcal{M} \rangle = [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{X}] (\mathbf{E}^*, \mathcal{M}) .$$

(2) For a category $\mathbf{E}$ and a module morphism $\psi : \mathcal{M} : \mathcal{N} : X \rightarrow \mathbf{A}$, the postcomposition module morphism

$$\langle \ast \mathbf{E}, \psi \rangle : \langle \ast \mathbf{E}, \mathcal{M} \rangle \rightarrow \langle \ast \mathbf{E}, \mathcal{N} \rangle \quad \text{op.} \quad \langle \mathbf{E}^*, \psi \rangle : \langle \mathbf{E}^*, \mathcal{M} \rangle \rightarrow \langle \mathbf{E}^*, \mathcal{N} \rangle$$

defined in Definition 4.6.13 is obtained from the postcomposition module morphism $\langle \mathbf{E}, \psi \rangle : \langle \mathbf{E}, \mathcal{M} \rangle \rightarrow \langle \mathbf{E}, \mathcal{N} \rangle$ defined in Definition 4.4.11 by the composition

$$X \xrightarrow{(\mathbf{E}, \mathcal{M})} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}, \mathbf{A}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{X}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{X}] \xrightarrow{([\mathbf{E}^\ast, \mathcal{M}]} \mathbf{A}$$

; that is,

$$\langle \ast \mathbf{E}, \psi \rangle = \langle \mathbf{E}, \psi \rangle [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}] \quad \text{op.} \quad \langle \mathbf{E}^*, \psi \rangle = [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{X}] (\mathbf{E}^*, \psi) .$$

Proof. For any $x \in X$ and $F \in [\mathbf{E}, \mathbf{A}]$,

$$(x) \langle \ast \mathbf{E}, \mathcal{M} \rangle (F) = \prod_{\mathbf{E}} x(M) F$$

$$\quad = \prod_{\mathbf{E}} (x(M) F) [\mathbf{E}^\ast \times \mathbf{E}]$$

$$\quad = \prod_{\mathbf{E}} x(M) \big[ F^\ast [\mathbf{E}^\ast \times \mathbf{E}] \big]$$

$$\quad = (x) \langle \mathbf{E}, \mathcal{M} \rangle (F) [\mathbf{E}^\ast \times \mathbf{E}]$$

$$\quad = (x) \langle \mathbf{E}, \mathcal{M} \rangle (\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}) : F$$

$$\quad = (x) \langle \mathbf{E}, \mathcal{M} \rangle ((\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}) : F)$$

$$(\ast \mathbf{E}, \mathcal{M}) \quad \text{op.} \quad \langle \mathbf{E}, \mathcal{M} \rangle \quad \text{op.} \quad \langle \mathbf{E}^*, \mathcal{M} \rangle .$$

(1) by Proposition 4.2.5). \hfill \Box

4.6.35 Corollary. The postcomposition cell

$$\begin{array}{c}
\mathbf{X} \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}, \mathbf{A}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{X}] \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}, \mathbf{A}] \\
\mathbf{Y} \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}, \mathbf{B}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{Y}] \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}, \mathbf{B}] \\
\end{array}$$

defined in Definition 4.6.17 is obtained from the postcomposition cell defined in Definition 4.4.15 by the pasting composition

$$\begin{array}{c}
\mathbf{X} \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{A}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}, \mathbf{A}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{X}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{X}] \xrightarrow{([\mathbf{E}^\ast, \mathcal{M}]} \mathbf{A} \\
\mathbf{Y} \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{B}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}, \mathbf{B}] \quad \text{op.} \quad [\mathbf{E}, \mathbf{Y}] \xrightarrow{[\mathbf{E}^\ast \times \mathbf{E}, \mathcal{M}]} [\mathbf{E}^\ast \times \mathbf{E}, \mathbf{Y}] \xrightarrow{([\mathbf{E}^\ast, \mathcal{M}]} \mathbf{B} \\
\end{array}$$

Proof. Immediate from Theorem 4.6.34, recalling the definition of the postcomposition cell $\langle \mathbf{E}, \psi \rangle$ and the definition of pasting composition. \hfill \Box
4.7 Bicylinders

A cylinder between bifunctors is called a bicylinder. Bicylinders are classified into three types: ordinary, extraordinary, and complex. A complex bicylinder is used when defining a parameterized end in Section 13.2.

**Note.** Presented below are representative instances of bicylinders.

### 4.7.1 Definition

Given a pair of categories $E$ and $D$, and a module $M : X \to A$, the following types of bicylinders $E \times D \Rightarrow M$ are considered: ordinary

$$
\begin{array}{ccc}
G \quad E \times D \quad F \\
\xrightarrow{\alpha} \quad X \quad \Rightarrow \quad M \\
\xrightarrow{\beta} \quad A
\end{array}
$$

, extraordinary

$$
\begin{array}{cc}
\star & [E \times D]^{-} \times [E \times D] \\
\times & \downarrow \alpha \downarrow F \\
X \quad \Rightarrow \quad M \\
\xrightarrow{\alpha} \quad A
\end{array}
\quad \text{op.} \quad \begin{array}{cc}
\star & [E \times D]^{-} \times [E \times D] \\
\times & \downarrow \alpha \downarrow a \\
X \quad \Rightarrow \quad M \\
\xrightarrow{\alpha} \quad A
\end{array}
$$

, and complex

$$
\begin{array}{cc}
E \quad E \times D^{-} \times D \\
\downarrow \alpha \downarrow F \\
X \quad \Rightarrow \quad M \\
\xrightarrow{\alpha} \quad A
\end{array}
\quad \text{op.} \quad \begin{array}{cc}
E \quad E \times D^{-} \times D \\
\downarrow \alpha \downarrow F \\
X \quad \Rightarrow \quad M \\
\xrightarrow{\alpha} \quad A
\end{array}
$$

### 4.7.2 Remark

1. The complex bicylinder in Definition 4.7.1 is defined by a cylindrical frame $\alpha$ of the composite module

$$G(M) F : E \to E \times D^{-} \times D \quad \text{op.} \quad G(M) F : E \times D^{-} \times D \to E$$

(regarded as an endomodule $E \times D \to E \times D$ [op. $E \times D^{-} \to E \times D^{-}$]). The module of complex bicylinders,

$$(E \times D, M) : [E, X] \to [E \times D^{-} \times D, A] \quad \text{op.} \quad (E \times D^{-}, M) : [E \times D^{-} \times D, X] \to [E, A]$$

is defined by

$$(G)(E \times D, M) (F) = \prod_{E \times D} G(M) F \quad \text{op.} \quad (G)(E \times D^{-}, M) (F) = \prod_{E \times D} G(M) F$$

(cf. Definition 4.3.7).

2. Each of the bicylinders in Definition 4.7.1 is thus given by a cylindrical frame of an endomodule $E \times D \Rightarrow E \times D$. The same notation $(E \times D, M)$ is used to denote the following modules of bicylinders:

<table>
<thead>
<tr>
<th>Module</th>
<th>Bicylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(E \times D, M) : [E \times D, X] \to [E \times D, A]$</td>
<td>Ordinary</td>
</tr>
<tr>
<td>$(E \times D, M) : X \to [(E \times D)^{-} \times [E \times D], A]$</td>
<td>Extraordinary (left)</td>
</tr>
<tr>
<td>$(E^{-} \times D^{-}, M) : [E \times D^{-}] \times [E \times D], X] \to A$</td>
<td>Extraordinary (right)</td>
</tr>
<tr>
<td>$(E \times D^{-}, M) : [E, X] \to [E \times D^{-} \times D, A]$</td>
<td>Complex (left)</td>
</tr>
<tr>
<td>$(E \times D^{-}, M) : [D \times D^{-} \times D, X] \to [E, A]$</td>
<td>Complex (right)</td>
</tr>
</tbody>
</table>

3. An ordinary bicylinder along the hom-module of a category is just a natural transformation between bifunctors.

4. A cone over a bifunctor is called a bicone. A bicone is seen as a special instance of a bicylinder (cf. Remark 4.6.4(1)).

### 4.7.3 Proposition

*Given an extraordinary natural transformation $\nu$ and a complex bicylinder as in

$$
\begin{array}{cc}
\star & E^{-} \times E \\
\times & \downarrow \nu \downarrow R \\
X \quad \Rightarrow \quad \tilde{X} \\
\xrightarrow{\beta} \quad \Rightarrow \quad M \\
\xrightarrow{\alpha} \quad A
\end{array}
\quad \text{op.} \quad \begin{array}{cc}
E^{-} \times E \times D^{-} \times D \\
\downarrow \mu \downarrow \nu \downarrow a \\
X \quad \Rightarrow \quad \tilde{X} \\
\xrightarrow{\alpha} \quad \Rightarrow \quad A \\
\xrightarrow{\beta} \quad A
\end{array}
$$

*
To show the latter, thus yields a right natural in Proof. By Theorem 4.1.4, it suffices to show that holds. 4.7.6 Remark. The second assertion is immediate from the definition of the compositionνe for each d = ∈ |E| and natural in e for each d ∈ |D|. The former is the case because μe.d is natural in d for each e ∈ |E|. To show the latter, let h: e → e’ be an E-arrow and consider the diagram

\[ \begin{array}{c}
\xymatrix{ X \ar[r]^{\nu_e} \ar[d]_{\mu_{e.d}} & R(e,e) \ar[d]_{R(e,h)} \ar[r]^{\mu_{e.d}} & F(e,e,d,d) \ar[d]_{F(e,h,d,d)} \\
R(e’,e’) \ar[r]_{\mu_{e’,d}} & R(e,e’) \ar[r]_{F(h,e’,d,d)} & F(e,e’,d,d) }
\end{array} \]

; moreover for an X-arrow f: y → x [op. A-arrow f: a → b], the associative law

\[ f \circ \mu = \mu \circ \nu \]

holds. 4.7.4 Remark. The assignment

\[
\nu \mapsto \nu \circ \mu \quad \text{op.} \quad \mu \mapsto \mu \circ \nu
\]

thus yields a right [op. left] module morphism

\[
(E,X)(R) \rightarrow (E \times D,M)(F): X \rightarrow * \quad \text{op.} \quad (R)(E,A) \rightarrow (G)(E \times D,M): * \rightarrow A.
\]

4.7.5 Definition. The right and left exponential transposes of a bicylinder

\[ \begin{array}{c}
\xymatrix{ X \ar[r]^{\alpha} \ar[d]_{\alpha \downarrow} & E \ar[r]^{F} \ar[d]_{\alpha \downarrow} & A \\
D \ar[r]_{\alpha \downarrow} & (E,M) \ar[r]_{\alpha \downarrow} & (D,M) \ar[r]_{\alpha \downarrow} & (E,A) \ar[r]_{\alpha \downarrow} & [D, A] \ar[d]_{\alpha \downarrow} \\
\end{array} \]

are the cylinders

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{\alpha \downarrow} \ar[d]_{\alpha \downarrow} & E \ar[r]^{F} \ar[d]_{\alpha \downarrow} & A \\
D \ar[r]_{\alpha \downarrow} & (E,M) \ar[r]_{\alpha \downarrow} & (D,M) \ar[r]_{\alpha \downarrow} & (E,A) \ar[r]_{\alpha \downarrow} & [D, A] \ar[d]_{\alpha \downarrow} \\
\end{array} \]

whose component at d ∈ |D| and at e ∈ |E| are the cylinders

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{\alpha \downarrow} \ar[d]_{\alpha \downarrow} & E \ar[r]^{F} \ar[d]_{\alpha \downarrow} & A \\
D \ar[r]_{\alpha \downarrow} & (E,M) \ar[r]_{\alpha \downarrow} & (D,M) \ar[r]_{\alpha \downarrow} & (E,A) \ar[r]_{\alpha \downarrow} & [D, A] \ar[d]_{\alpha \downarrow} \\
\end{array} \]

— the right slice of α at d and the left slice of α at e—defined by

\[ [\alpha \downarrow d]_e = \alpha_{(e,d)} = [e \downarrow \alpha]_d. \]

4.7.6 Remark. By Theorem 4.1.4, α ⇸ and \( \kappa \alpha \) and all of their components do form cylinders, and
the right and left exponential transpositions form the iso cells

\[ [E \times D, X] \xrightarrow{(E \times D, M)} \alpha \quad \text{and} \quad [E \times D, X] \xrightarrow{(E \times D, M)}\]

\[ [D, [E, X]] \xrightarrow{(\alpha, (E, M), D)} [D, [E, A]] \quad \text{or} \quad [E, [D, X]] \xrightarrow{(E, (D, M), A)} [E, [D, A]]\]

, natural in \(E, D\), and \(M\).

**4.7.7 Definition.** The right and left exponential transposes of an extraordinary bicylinder

\[
\begin{align*}
\star & \quad [E \times D]^* \times [E \times D] & \text{op.} & \quad [E \times D]^* \times [E \times D] \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M} \rightarrow A & \quad \xrightarrow{\alpha} & \quad X - \underleftarrow{M} \rightarrow A
\end{align*}
\]

are the extraordinary cylinders

\[
\begin{align*}
\star & \quad D^* \times D & \quad \star \quad E^* \times E & \quad \text{op.} & \quad D^* \times D & \quad \star \quad E^* \times E & \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]

whose component at \(d \in |D|\) and at \(e \in |E|\) are the extraordinary cylinders

\[
\begin{align*}
\star & \quad E^* \times E & \quad \star \quad D^* \times D & \quad \text{op.} & \quad E^* \times E & \quad \star \quad D^* \times D & \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]

—the right slice of \(\alpha\) at \(d\) and the left slice of \(\alpha\) at \(e\)—defined by

\[
[\alpha \times d]_e = \alpha_{(e, d)} = [e \times \alpha]_d.
\]

**4.7.8 Remark.** The right and left exponential transposition for extraordinary bicylinders form the iso cells

\[
\begin{align*}
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \text{op.} & \quad [[E \times D]^* \times [E \times D], A]\left( \alpha, (E, M), D \right) & \quad \text{or} \quad [[E \times D]^* \times [E \times D], A]\left( \alpha, (E, M), D \right)\left( D^* \times D, X \right) & \quad \text{or} \quad [[E \times D]^* \times [E \times D], A]\left( D^* \times D, X \right)\left( D^*, (E^*, M) \right) & \quad \text{or} \quad [[E \times D]^* \times [E \times D], A]\left( D^*, (E^*, M) \right)\left( E^* \times D, X \right)
\end{align*}
\]

, natural in \(E, D\), and \(M\).

**4.7.9 Definition.** The right and left exponential transposes of a complex bicylinder

\[
\begin{align*}
\star & \quad D^* \times D & \quad \text{op.} & \quad E \times D^* \times D & \quad E \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]

are the extraordinary and ordinary cylinders

\[
\begin{align*}
\star & \quad D^* \times D & \quad \text{op.} & \quad D^* \times D & \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]

whose component at \(d \in |D|\) and at \(e \in |E|\) are the ordinary and extraordinary cylinders

\[
\begin{align*}
\star & \quad D^* \times D & \quad \text{op.} & \quad D^* \times D & \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]

\[
\begin{align*}
\star & \quad D^* \times D & \quad \text{op.} & \quad D^* \times D & \quad \star \\
\downarrow \alpha & \quad \downarrow F & \quad \downarrow \alpha & \quad \downarrow F \\
X - \underleftarrow{M, \alpha} \rightarrow A & \quad \xrightarrow{\alpha, \alpha} & \quad X - \underleftarrow{M, \alpha} \rightarrow A
\end{align*}
\]
the right slice of \(\alpha\) at \(d\) and the left slice of \(\alpha\) at \(e\)—defined by
\[
\left[\alpha \triangleright d\right]_e = \left[\alpha \triangleleft e\right]_d.
\]

4.7.10 Remark. The right and left exponential transpositions form the iso cells
\[
\begin{align*}
\left[\mathbf{E}, \mathbf{X}\right] & \xrightarrow{\left(E,\left(D,\mathbf{M}\right)\right)} \left[\mathbf{E} \times D^\times D, \mathbf{A}\right] \quad \text{op.} \quad \left[\mathbf{E} \times D^\times D, \mathbf{X}\right] & \xrightarrow{\left(E,\left(D,\mathbf{M}\right)\right)} \left[\mathbf{E}, \mathbf{A}\right] \\
\left[\mathbf{E}, \mathbf{X}\right] & \xrightarrow{\left(D^\times D,\mathbf{M}\right)} \left[D^\times D, [\mathbf{E}, \mathbf{A}]\right] \quad \text{op.} \quad \left[D^\times D, [\mathbf{E}, \mathbf{X}]\right] & \xrightarrow{\left(D,\left(E,\mathbf{M}\right)\right)} \left[\mathbf{E}, \mathbf{A}\right]
\end{align*}
\]
and
\[
\begin{align*}
\left[\mathbf{E}, \mathbf{X}\right] & \xrightarrow{\left(E,\left(D,\mathbf{M}\right)\right)} \left[\mathbf{E} \times D^\times D, \mathbf{A}\right] \quad \text{op.} \quad \left[\mathbf{E} \times D^\times D, \mathbf{X}\right] & \xrightarrow{\left(E,\left(D,\mathbf{M}\right)\right)} \left[\mathbf{E}, \mathbf{A}\right] \\
\left[\mathbf{E}, \mathbf{X}\right] & \xrightarrow{\left(D^\times D,\mathbf{M}\right)} \left[E, [\mathbf{D}^\times D, \mathbf{A}]\right] \quad \text{op.} \quad \left[E, [\mathbf{D}^\times D, \mathbf{X}]\right] & \xrightarrow{\left(D,\left(E,\mathbf{M}\right)\right)} \left[\mathbf{E}, \mathbf{A}\right]
\end{align*}
\]
, natural in \(\mathbf{E}, \mathbf{D},\) and \(\mathbf{M}\).

4.7.11 Definition. The twist transpose of an extraordinary cylinder
\[
\begin{align*}
\begin{array}{c}
\left[\mathbf{E}, \mathbf{X}\right] \xrightarrow{\left(E,\mathbf{M}\right)} [\mathbf{E}, \mathbf{A}]
\end{array}
\end{align*}
\]
is the ordinary cylinder
\[
\begin{align*}
\begin{array}{c}
\left[\mathbf{E}, \mathbf{X}\right] \xrightarrow{\left(E,\mathbf{M}\right)} [\mathbf{D}^\times D, \mathbf{A}]
\end{array}
\end{align*}
\]
whose component at \(e \in \left[\mathbf{E}\right]\) is the extraordinary cylinder
\[
\begin{align*}
\begin{array}{c}
\left[\mathbf{E}, \mathbf{X}\right] \xrightarrow{\left(E,\mathbf{M}\right)} [\mathbf{D}^\times D, \mathbf{A}]
\end{array}
\end{align*}
\]
—outer slice of \(\alpha\) at \(e\) —defined by
\[
\left[\alpha \triangleright e\right]_d = \left[\alpha \triangleleft e\right]_d
\]
for each \(d \in \left[\mathbf{D}\right]\).

4.7.12 Remark.
(1) The twist transposition \(\alpha \mapsto \alpha^\top\) forms the iso cell
\[
\begin{align*}
\begin{array}{c}
\left[\mathbf{E}, \mathbf{X}\right] \xrightarrow{\left(D,\left(E,\mathbf{M}\right)\right)} [\mathbf{D}^\times D, [\mathbf{E}, \mathbf{A}]]
\end{array}
\end{align*}
\]
.. , natural in \(\mathbf{E}, \mathbf{D},\) and \(\mathbf{M}\), making the diagram
\[
\begin{align*}
\begin{array}{c}
\left[\mathbf{E} \times \mathbf{D}, \mathbf{M}\right] \xrightarrow{\left(D^\times D, \mathbf{M}\right)} [\mathbf{D}^\times D, [\mathbf{E}, \mathbf{M}]]
\end{array}
\end{align*}
\]
commute, where \(\triangleright\) and \(\triangleleft\) are the cells in Remark 4.7.10.

(2) The diagram
\[
\begin{align*}
\begin{array}{c}
\left(D, \left(E, \mathbf{M}\right)\right) \xrightarrow{\left(D,\left(E,\mathbf{M}\right)\right)} \left(E, \left(D, \mathbf{M}\right)\right)
\end{array}
\end{align*}
\]
\[
\begin{align*}
\begin{array}{c}
\left(D^\times D, \mathbf{M}\right) \xrightarrow{\left(D^\times D, \mathbf{M}\right)} \left(E, \left(D^\times D, \mathbf{M}\right)\right)
\end{array}
\end{align*}
\]

commutes, where \( \langle e, \mathcal{M} \rangle \) and \( \langle e, \{D, \mathcal{M}\} \rangle \) are evaluations at \( e \in \|\mathbf{E}\| \) (see Example 4.3.31). Hence
the postcomposition
\[
\begin{array}{c}
\ast & \text{D}^- \times \text{D} & \text{op.} & \text{D}^- \times \text{D} & \ast \\
\mathcal{G} \downarrow & \alpha & \downarrow \mathcal{F} & \mathcal{G} \downarrow & \alpha & \downarrow \mathcal{F} \\
\langle \mathbf{E}, \mathbf{X} \rangle - \leftarrow \langle (\mathbf{E}, \mathcal{M}) \rangle - \leftarrow \langle \mathbf{E}, \mathbf{A} \rangle \\
\langle e, \mathbf{X} \rangle - \leftarrow \langle (e, \mathcal{M}) \rangle - \leftarrow \langle e, \mathbf{A} \rangle \\
\mathbf{X} - \leftarrow \mathcal{M} - \leftarrow \mathbf{A}
\end{array}
\]

of \( \alpha \) with the evaluation \( \langle e, \mathcal{M} \rangle \) yields the same extraordinary cylinder as the one given by the component of \( \alpha^* \) at \( e \), i.e. the slice of \( \alpha \) at \( e \) (cf. Preliminary 0.0.6(3)).
(3) Since \( \langle \alpha, \mathcal{X} \rangle = \kappa, \alpha \), the left slice of a complex bicylinder \( \alpha : \mathbf{E} \times \mathbf{D} \rightarrow \mathcal{M} \) at \( e \in \|\mathbf{E}\| \) coincides with the slice of the extraordinary cylinder \( \alpha : \mathbf{D} \rightarrow \langle (\mathbf{E}, \mathcal{M}) \rangle \) at \( e \), both being given by the component of the ordinary cylinder \( \kappa, \alpha : \mathbf{E} \rightarrow \langle \mathbf{D}, \mathcal{M} \rangle \) at \( e \).

## 4.8 Wedges

A combination of a cone and a cylinder is called a wedge\(^2\); a wedge is to a bicylinder what a cone is to a cylinder. A wedge is used when defining a parameterized limit in Section 8.3.

### 4.8.1 Definition

Let \( \mathbf{E} \) and \( \mathbf{D} \) be categories and \( \mathcal{M} : \mathbf{X} \rightarrow \mathbf{A} \) be a module.

- Given a functor \( \mathcal{G} : \mathbf{E} \rightarrow \mathbf{X} \) and a bifunctor \( \mathcal{F} : \mathbf{E} \times \mathbf{D} \rightarrow \mathbf{A} \), a wedge \( \alpha \) from \( \mathcal{G} \) to \( \mathcal{F} \) along \( \mathcal{M} \), written \( \alpha : \mathcal{G} \Rightarrow \mathcal{F} : \mathbf{E} \times \mathcal{D} \Rightarrow \mathcal{M} \), is a cylinder
\[
\begin{array}{c}
\mathbf{E} \\
\mathcal{G} \downarrow & \alpha & \downarrow \mathcal{F} \\
\mathbf{E} \times \mathcal{D} \rightarrow \mathbf{E}
\end{array}
\]

- Given a functor \( \mathcal{F} : \mathbf{E} \rightarrow \mathbf{A} \) and a bifunctor \( \mathcal{G} : \mathbf{E} \times \mathbf{D} \rightarrow \mathbf{X} \), a wedge \( \alpha \) from \( \mathcal{F} \) to \( \mathcal{G} \) along \( \mathcal{M} \), written \( \alpha : \mathcal{F} \Rightarrow \mathcal{G} : \mathbf{E} \times \mathcal{D} \Rightarrow \mathcal{M} \), is a cylinder
\[
\begin{array}{c}
\mathbf{E} \times \mathcal{D} \\
\mathcal{F} \downarrow & \alpha & \downarrow \mathcal{G} \\
\mathbf{X} \rightarrow \mathcal{M} \rightarrow \mathbf{A}
\end{array}
\]

### 4.8.2 Remark

(1) Just like a cone is a special instance of a cylinder, a wedge is a special instance of a bicylinder.
(2) A wedge \( \alpha : \mathcal{G} \Rightarrow \mathcal{F} : \mathbf{E} \times \mathcal{D} \Rightarrow \mathcal{M} \) is also defined by a bicylinder
\[
\begin{array}{c}
\mathbf{E} \\
\downarrow \mathcal{G} & \mathcal{G} \downarrow \\
\mathbf{E} \times \mathcal{D} \rightarrow \mathbf{E} \times \mathcal{D} \\
\alpha & \downarrow \mathcal{F} \\
\mathbf{X} \rightarrow \mathcal{M} \rightarrow \mathbf{A}
\end{array}
\]

with a dummy variable varying over \( \mathbf{D}^- \) introduced to \( \mathcal{F} \).
(3) A wedge also arises as a combination of a cone and an extraordinary cylinder. For example, given an object \( x \in \|\mathbf{X}\| \) and a bifunctor \( \mathcal{F} : \mathbf{E}^\mathbf{x} \times \mathbf{E} \times \mathbf{D} \rightarrow \mathbf{A} \), an extraordinary wedge
\[
\alpha : \mathbf{x} \Rightarrow \mathcal{F} : \mathbf{E} \times \mathbf{D} \Rightarrow \mathcal{M}
\]

\(^2\)This terminology differs from the literature, which uses the term “wedge” to refer to what this book calls an extraordinary cylinder.
is defined by one of the following bicylinders:

\[
\begin{array}{c}
\text{D} & \text{E}^\times \times \text{E} \times \text{D} & \ast & \text{E}^\times \times \text{D} \times \times \text{E} \times \text{D} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\ast & \text{F} & \times & \text{E} \times \times \text{E} \times \text{D} \\
X \to \to \to \overline{\text{M}} = \to \to \to \text{A} & \text{X} \to \to \to \overline{\text{M}} = \to \to \to \text{A}
\end{array}
\]

**Note.** The module of wedges \( \text{E} \times \times \text{D} \to \text{M} \) is defined below as an instance of the module \( (\text{K} \times \text{M}) \)—the module of right \( \text{K} \)-weighted cylinders along \( \text{M} \)—defined in Definition 4.5.3.

### 4.8.3 Definition

Let \( \text{E} \) and \( \text{D} \) be categories and \( \text{M} : \text{X} \to \text{A} \) be a module.

- The module

\[
\langle \text{E} \times \times \text{D}, \text{M} \rangle : [\text{E}, \text{X}] \to [\text{E} \times \text{D}, \text{A}]
\]

of wedges \( \text{E} \times \times \text{D} \to \text{M} \) is defined by the composition

\[
[\text{E}, \text{X}] \xrightarrow{[\text{E}, \text{D}, \text{X}]} [\text{E} \times \text{D}, \text{X}] \xrightarrow{[\text{E}, \text{D}, \text{M}]} [\text{E} \times \text{D}, \text{A}]
\]

; that is, \( \langle \text{E} \times \times \text{D}, \text{M} \rangle := \langle \text{E} \times \text{D}^\times \text{M} \rangle \).

- The module

\[
\langle \text{E} \times \text{D}^\ast, \text{M} \rangle : [\text{E} \times \text{D}, \text{X}] \to [\text{E}, \text{A}]
\]

of wedges \( \text{E} \times \text{D}^\ast \to \text{M} \) is defined by the composition

\[
[\text{E} \times \text{D}, \text{X}] \xrightarrow{[\text{E}, \text{D}^\ast, \text{M}]} [\text{E} \times \text{D}, \text{A}] \xrightarrow{[\text{E}, \text{D}^\ast, \text{A}]} [\text{E}, \text{A}]
\]

; that is, \( \langle \text{E} \times \text{D}^\ast, \text{M} \rangle := \langle \text{E} \times \text{D}^\ast \text{M} \rangle \).

### 4.8.4 Remark

1. For a functor \( \text{G} : \text{E} \to \text{X} \) and a bifunctor \( \text{F} : \text{E} \times \text{D} \to \text{A} \), the set \( (\text{G}) \langle \text{E} \times \times \text{D}, \text{M} \rangle (\text{F}) \) consists of all wedges \( \text{G} \sim \text{F} : \text{E} \times \times \text{D} \to \text{M} \), and for a functor \( \text{F} : \text{E} \to \text{A} \) and a bifunctor \( \text{G} : \text{E} \times \text{D} \to \text{X} \), the set \( (\text{G}) \langle \text{E} \times \text{D}^\ast, \text{M} \rangle (\text{F}) \) consists of all wedges \( \text{G} \sim \text{F} : \text{E} \times \times \text{D}^\ast \to \text{M} \).
2. The module \( \langle \text{E} \times \times \text{D}, \text{M} \rangle \) is alternatively defined by the composition

\[
[\text{E}, \text{X}] \xrightarrow{[\text{E}, \text{D}, \text{M}]} [\text{E} \times \text{D}, \text{A}] \xrightarrow{[\text{E}, \text{D}^\times, \text{D}, \text{A}]} [\text{E} \times \text{D}, \text{A}]
\]

(see Remark 4.8.2(2)).
3. The module of ordinary wedges

\[
\langle \text{E} \times \times \text{D}, \text{M} \rangle : \text{X} \to [\text{E} \times \times \text{E} \times \text{D}, \text{A}]
\]

(see Remark 4.8.2(3)) is defined by the composition

\[
\text{X} \xrightarrow{[\text{D}, \text{X}]} [\text{D}, \text{X}] \xrightarrow{[\text{E}, \text{D}, \text{M}]} [\text{E} \times \times \text{E} \times \text{D}, \text{A}]
\]

, or by the composition

\[
\text{X} \xrightarrow{[\text{D}, \text{X}]} [\text{D}, \text{X}] \xrightarrow{[\text{E}, \text{D}, \text{M}]} [\text{E} \times \times \text{E} \times \text{D}, \text{A}] \xrightarrow{[\text{E}, \text{D}, \times, \text{D}, \text{A}]} [\text{E} \times \times \text{E} \times \text{D}, \text{A}]
\]

### 4.8.5 Definition

The right and left exponential transposes of a wedge

\[
\begin{array}{c}
\text{E} \xrightarrow{\text{E}^\times !} \text{E} \times \text{D} & \text{op.} & \text{E} \times \text{D} \xrightarrow{\text{E}^\times !} \text{E} \\
\uparrow & \uparrow & \uparrow \\
\text{X} \to \to \to \overline{\text{M}} = \to \to \to \text{A} & \text{X} \to \to \to \overline{\text{M}} = \to \to \to \text{A}
\end{array}
\]
are the cone and cylinder

\[
\begin{align*}
\ast & \hookrightarrow \overset{1}{D} & \quad \text{op.} & \quad \overset{1}{D} \rightarrow \ast \\
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]
whose component at \(d \in \vert D \vert\) and at \(e \in \vert E \vert\) are the cylinder and cone

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

---the right slice of \(\alpha\) at \(d\) and the left slice of \(\alpha\) at \(e\)—defined by

\[
\begin{align*}
[\alpha \leftarrow d]_e = \alpha_{(e,d)} = [e \times \alpha]_d.
\end{align*}
\]

### 4.8.6 Remark
The right and left exponential transpositions of wedges \(E \times \ast D \to \mathcal{M}\) [op. \(E \times D \ast \to \mathcal{M}\)] form the iso cells

\[
\begin{align*}
\overset{\ast \leftarrow 1}{D} & \quad \overset{1}{\rightarrow} \quad \overset{\ast \rightarrow 1}{D} & \quad \text{op.} & \quad \overset{1}{D} \rightarrow \ast \quad \overset{1}{\rightarrow} \quad \overset{1}{\rightarrow} \\
\overset{E}{\overset{\alpha \leftarrow}{\leftarrow}} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

and

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

, natural in \(E, D,\) and \(\mathcal{M}\). In fact, these iso cells are obtained from the iso cells in Remark 4.7.6 by pasting a commutative diagram of diagonal functors (see Preliminary 0.0.7) as shown below:

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{E}{\alpha \leftarrow}}{\leftarrow}
\end{align*}
\]

Note. The following definition is analogous to Definition 4.7.11.

### 4.8.7 Definition
The twist transpose of a cone

\[
\begin{align*}
\ast & \hookrightarrow \overset{1}{D} & \quad \text{op.} & \quad \overset{1}{D} \rightarrow \ast \\
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

is the cylinder

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow} & \quad \overset{G}{\alpha \leftarrow} & \quad \overset{F}{\leftarrow}
\end{align*}
\]

whose component at \(e \in \vert E \vert\) is the cone

\[
\begin{align*}
\ast & \hookrightarrow \overset{1}{D} & \quad \text{op.} & \quad \overset{1}{D} \rightarrow \ast \\
\overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow} & \quad \overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow} & \quad \overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow} & \quad \overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow} & \quad \overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow}
\end{align*}
\]

\[
\begin{align*}
\overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow} & \quad \overset{\overset{G}{\alpha \leftarrow}}{\leftarrow} & \quad \overset{\overset{F}{\leftarrow}}{\leftarrow}
\end{align*}
\]
the slice of \( \alpha \) at \( e \)—defined by

\[
[\alpha^\top_d]_d = [\alpha_d]_e
\]

for each \( d \in \|D\| \).

4.8.8 Remark.

1. The twist transposition \( \alpha \mapsto \alpha^\top \) forms the iso cell

\[
\begin{align*}
\quad [E, X] & \xrightarrow{\{0, \{D(E, M)\}\} \otimes \tau} [D, \{E, A\}] \quad \text{op.} & [D, [E, X]] & \xrightarrow{\{D(E, M)\} \otimes \tau} [E, A] \\
\begin{array}{c}
\quad \uparrow_{\{E, \langle D, M\rangle\}} \\
\{E, X\}
\end{array} & \xrightarrow{\{D, \{E, M\}\} \otimes \tau} [E, \{D, A\}] & \begin{array}{c}
\quad \uparrow_{\{E, \langle D, M\rangle\}} \\
\{D, X\}
\end{array}
\]

\]

, natural in \( E, D, \) and \( M, \) making the diagram

\[
\begin{align*}
\quad [E \times \{D, M\}] & \xrightarrow{} [E, [\{D, M\}\}] \quad \text{op.} & [E \times [D, M]] & \xrightarrow{} [D, \{E, M\}] \\
\quad \big| & \xrightarrow{} \big| & \quad \big| & \xrightarrow{} \big|
\end{align*}
\]

commute, where \( \tau \) and \( \kappa \) are the cells in Remark 4.8.6.

2. The diagram

\[
\begin{align*}
\quad \{*, \{E, M\}\} & \xrightarrow{\tau} \{E, \{D, M\}\} \quad \text{op.} & \{D*, \{E, M\}\} & \xrightarrow{\tau} \{E, \{D, M\}\} \\
\quad \{\{D, \{e, M\}\\} \xrightarrow{\{e, D, M\}\}} \quad \text{op.} & \quad \{\{D*, \{e, M\}\} \xrightarrow{\{e, D, M\}\}}
\end{align*}
\]

commutes, where \( \{e, M\} \) and \( \{e, \{D, M\}\} \) [op. \( \{e, \{D, M\}\}\)] are evaluations at \( e \in \|E\| \) (see Example 4.3.31); the postcomposition

\[
\begin{align*}
\quad * & \xrightarrow{l} D \quad \text{op.} & D & \xrightarrow{l} * \\
\quad \big| & \xrightarrow{\alpha} \big| & \quad \big| & \xrightarrow{\alpha} \big|
\end{align*}
\]

of \( \alpha \) with the evaluation \( \{e, M\} \) thus yields the same cone as the one given by the component of \( \alpha^\top \) at \( e \), i.e. the slice of \( \alpha \) at \( e \) (cf. Preliminary 0.0.6(3)).

3. Since \( \alpha^\top = \kappa \alpha \), the left slice of a wedge \( \alpha : E \times \{D \rightharpoonup M\} \) at \( e \in \|E\| \) coincides with the slice of the cone \( \alpha^\top : \{D \rightharpoonup E, M\} \) at \( e \), both being given by the component of the cylinder \( \kappa \alpha : E \rightharpoonup \{D, M\} \) at \( e \).

4.9 Cones and wedges in categories

A cone in a category \( C \) is the same thing as a cone along the hom-module \( \{C\} \). Various concepts introduced for a cone along a module are thus carried over to a cone in a category. The story is the same for a wedge.

If a module \( M \) and a cell \( \psi \) in Section 4.6 are replaced by the hom-module of a category \( C \) and by the hom-cell of a functor \( H : C \rightharpoonup B \), we have the module \( \{E, C\} : C \rightharpoonup [E, C] \) and the postcomposition cell

\[
\begin{align*}
\quad C & \xrightarrow{\{E, C\}} [E, C] \quad \text{op.} & \quad B & \xrightarrow{\{E, B\}} [E, B] \\
\quad \big| & \xrightarrow{\{E, H\}} \big| & \quad \big| & \xrightarrow{\{E, H\}} \big|
\end{align*}
\]

that sends each cone \( \bullet \xrightarrow{\alpha} E \) to its composite \( \bullet \xrightarrow{e} E \) with \( H \). In Section 8.1, a limit in

\[
\begin{align*}
\quad C & \xrightarrow{\{E, C\}} [E, C] \quad \text{op.} & \quad B & \xrightarrow{\{E, B\}} [E, B] \\
\quad \big| & \xrightarrow{\{E, H\}} \big| & \quad \big| & \xrightarrow{\{E, H\}} \big|
\end{align*}
\]
4.9. Cones and wedges in categories

4.9.1 Definition. Given a functor \( L : E \to C \) and an object \( c \in \| C \| \),
- a cone from \( c \) to \( L \) is denoted by \( \alpha : c \to L : *E \to C \).
- a cone from \( L \) to \( c \) is denoted by \( \alpha : L \to c : E* \to C \).

4.9.2 Remark.
(1) By Example 4.2.3, a cone in a category \( C \) is just a special instance of a cone in Definition 4.6.3
where \( M \) is the hom-module of \( C \). Conversely, a cone along a module \( M \) is the same thing as
a cone in the collage category \( [M] \) (cf. Remark 4.3.2(2)).

(2) By Remark 4.6.4(1), a cone

\[
\alpha : c \to L : *E \to C \quad \text{op.} \quad \alpha : L \to c : E* \to C
\]

is the same thing as a natural transformation

\[
\begin{array}{ccc}
* \xrightarrow{l} & E & \xrightarrow{\alpha} & L \\
\downarrow & \downarrow & \downarrow \\
C - & \xrightarrow{\alpha} & (C) - & C
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E \xrightarrow{l} & * & \xrightarrow{\alpha} & c \\
\downarrow & \downarrow & \downarrow \\
C - & \xrightarrow{\alpha} & (C) - & C
\end{array}
\]

weighted by the unique functor \( E \to * \).

Note. The module of cones in a category is defined below as a special case of Definition 4.6.7 where
\( M \) is given by the hom-module of a category.

4.9.3 Definition. Given categories \( E \) and \( C \), the module

\( \langle *E, C \rangle : C \to [E, C] \quad \text{op.} \quad \langle E*, C \rangle : [E, C] \to C \)

of cones \( *E \to C \) [\( \text{op.} \ E* \to C \)] is defined by

\( \langle *E, C \rangle (L) = \prod_{E} \{C, L \} \quad \text{op.} \quad \langle E*, C \rangle (c) = \prod_{E} L \{C, c \} \)

for \( c \in C \) and \( L \in [E, C] \); that is,

\( \langle *E, C \rangle := \langle *E, \{C, \rangle \} \quad \text{op.} \quad \langle E*, C \rangle := \langle E*, \{C, \rangle \} \).

4.9.4 Remark.
(1) For an object \( c \in \| C \| \) and a functor \( L : E \to C \), the set \( \langle c \rangle \langle *E, C \rangle (L) \) [\( \text{op.} \ (L) \langle E*, C \rangle (c) \)]
consists of all cones \( c \to L : *E \to C \) [\( \text{op.} \ \ell : L \to c : E* \to C \)].

(2) By applying Theorem 4.6.22 to the hom-module \( \langle C \rangle \) and noting Remark 4.5.18(3), we have

\( \langle *E, C \rangle = \langle \langle E, C \rangle \rangle \quad \text{op.} \quad \langle E*, C \rangle = \langle \langle E, C \rangle \rangle \)

that is, the module \( \langle *E, C \rangle \) [\( \text{op.} \ \langle E*, C \rangle \)] is given by the composition

\[
C \xrightarrow{\langle *E, C \rangle} \{E, C \} \xrightarrow{\langle E, C \rangle} \{E, C \} \quad \text{op.} \quad \{E, C \} \xrightarrow{\langle E, C \rangle} \{E, C \} \xrightarrow{\langle E, C \rangle} \{E, C \}
\]

i.e. by the representable [\( \text{op.} \ \text{corepresentable} \)] module of the diagonal functor \( \langle \{E, C \} \rangle \).

Note. The following definition is a special case of Definition 4.6.17 where \( \psi \) is given by the hom-cell
of a functor.

4.9.5 Definition. Given a category \( E \) and a functor \( H : C \to B \), the postcomposition cell

\[
\begin{array}{ccc}
C - & \xrightarrow{\langle *E, C \rangle} & [E, C] \\
\downarrow & \downarrow & \downarrow \\
B - & \xrightarrow{\langle *E, B \rangle} & [E, B]
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
[E, C] & \xrightarrow{(E*, C)} & C \\
\downarrow & \downarrow & \downarrow \\
[H] & \xrightarrow{(E*, H)} & [E, H]
\end{array}
\]

is defined by the postcomposition module morphism

\[
\langle *E, \{C, \rangle \} \xrightarrow{(\langle E, H \rangle \}) \langle *E, H (B) \rangle \} \to \{H \langle *E, \{B, \rangle \} \} \to \{E, H \}
\]
4.9. Cones and wedges in categories

By replacing module of a category.

\[
\langle E*, \langle C \rangle \rangle \xrightarrow{\langle E^*, (H) \rangle} \langle E^*, H(B) H \rangle = [E, H] \langle E^*, \langle B \rangle \rangle [H]
\]

(postcomposition with the hom-cell of \(H\)); that is,

\[
\langle *E, H \rangle := \langle *E, (H) \rangle \quad \text{op.} \quad \langle E^*, H \rangle := \langle E^*, (H) \rangle.
\]

4.9.6 Remark.

1. The cell \(\langle *E, H \rangle\) sends each cone \(\alpha : c \leadsto \_ : *E \rightarrow C\) to the cone \(\alpha \circ H : c \circ H \leadsto \_ : *E \circ H \rightarrow B\), the usual composite of a cone and a functor, and the cell \(\langle E^*, H \rangle\) sends each cone \(\alpha : L \rightarrow c : E^* \rightarrow C\) to the cone \(\alpha \circ H : L \circ H \rightarrow H \circ c : E^* \rightarrow B\).

2. By replacing \(\psi\) in Corollary 4.6.23 with the hom-cell \(H\) and noting Remark 4.5.20(2), we have

\[
\langle *E, H \rangle = \langle |E | \uparrow H \rangle \quad \text{op.} \quad \langle E^*, H \rangle = \langle |E^* | \uparrow H \rangle
\]

; that is, the cell \(\langle *E, H \rangle \) [op. \(\langle E^*, H \rangle\)] is obtained from the hom-cell of the postcomposition functor \([E, H]\) by the pasting composition

\[
\begin{align*}
\text{C} & \xrightarrow{|E | \cup C} \text{[E, C]} \xrightarrow{\langle E, C \rangle} \text{[E, C]} \quad \text{op.} \quad \text{[E, C]} \xrightarrow{\langle E, C \rangle} \text{[E, C]} \xrightarrow{|E | \cup C} \text{C} . \\
\text{B} & \xrightarrow{|E | \cup B} \text{[E, B]} \xrightarrow{\langle E, B \rangle} \text{[E, B]} \quad \text{op.} \quad \text{[E, B]} \xrightarrow{\langle E, B \rangle} \text{[E, B]} \xrightarrow{|E | \cup B} \text{B}
\end{align*}
\]

Note. The following is a special case of Proposition 4.6.19 where \(\psi\) is given by the hom-cell of a functor.

4.9.7 Proposition. If a functor \(H : C \rightarrow B\) is fully faithful, so is the postcomposition cell \(\langle *E, H \rangle \) [op. \(\langle E^*, H \rangle\)] for any category \(E\).

Proof. Since \(\langle *E, H \rangle = \langle *E, (H) \rangle\) by definition, and since \(H\) is fully faithful iff the hom-cell \(\langle H \rangle\) is fully faithful (Proposition 1.2.31), the assertion follows from Proposition 4.6.19.

Note. The following definition is a special case of Definition 4.6.25 where \(\tau : \psi \rightarrow \varphi\) is given by the hom-cell morphism (see Definition 1.3.5) of a natural transformation.

4.9.8 Definition. Given a category \(E\) and a natural transformation \(\tau : G \rightarrow F : C \rightarrow B\),

- the postcomposition cell morphism

\[
\langle *E, \tau \rangle : \langle *E, G \rangle \rightarrow \langle *E, F \rangle : \langle *E, C \rangle \rightarrow \langle *E, B \rangle
\]

, “postcomposition with \(\tau\)”, is defined by the pair of natural transformations

\[
\tau : G \rightarrow F \quad \text{and} \quad [E, \tau] : [E, G] \rightarrow [E, F].
\]

- the postcomposition cell morphism

\[
\langle E^*, \tau \rangle : \langle E^*, G \rangle \rightarrow \langle E^*, F \rangle : \langle E^*, C \rangle \rightarrow \langle E^*, B \rangle
\]

, “postcomposition with \(\tau\)”, is defined by the pair of natural transformations

\[
[E, \tau] : [E, G] \rightarrow [E, F] \quad \text{and} \quad \tau : G \rightarrow F.
\]

4.9.9 Remark. The assignment \(\tau \mapsto \langle *E, \tau \rangle\) defines a functor

\[
\langle *E, \_ \rangle : [C, B] \rightarrow [\langle *E, C \rangle : \langle *E, B \rangle] \quad \text{op.} \quad \langle E^*, \_ \rangle : [C, B] \rightarrow [\langle E^*, C \rangle : \langle E^*, B \rangle].
\]

; indeed, by the definition of \(\langle *E, \tau \rangle\) [op. \(\langle E^*, \tau \rangle\)], the functoriality is reduced to that of \([E, \_] : [C, B] \rightarrow [[E, C], [E, B]]\).

Note. The following definition is a special case of Definition 4.6.28 where \(M\) is given by the hom-module of a category.
4.9.10 Definition. Given a functor \( K : D \to E \) and a category \( C \), the precomposition cell

\[
\begin{array}{ccc}
C \overset{(\ast, \mathcal{C})}{\to} [E, C] & \overset{\text{op.}}{\to} & [E, C] \overset{(\ast, \mathcal{C})}{\to} C \\
\downarrow^{(\ast, K, \mathcal{C})} & \downarrow^{(K, \mathcal{C})} & \downarrow^{(K, \mathcal{C})} \\
[D, C] & \overset{\text{op.}}{\to} & [D, C] \overset{(\ast, \mathcal{C})}{\to} C
\end{array}
\]

is called a cone over \( K \), because precomposition with \( K \) is the identity on \( C \). For each pair of an object \( e \in \| E \| \) and a functor \( E \to C \); that is, 

\[
\ast (K, e) := (K, e) \circ (\ast, C).
\]

4.9.11 Remark.

(1) The cell \( \ast (K, C) \) sends each cone \( \alpha : x \to \ast : E \to C \) to the cone \( K \circ \alpha : x \to \ast : D \to C \) to the cone \( K \circ \alpha : x \to \ast : E \to C \) for each pair of an object \( e \in \| E \| \) and a functor \( E \to C \); that is,

\[
\ast (K, e) := (K, e) \circ (\ast, C).
\]

(2) Replacing \( \mathcal{M} \) in Proposition 4.6.31 with the hom-module \( C \) and noting that \( \langle K, \langle C \rangle \rangle = \langle K, C \rangle \) for each pair of an object \( e \in \| E \| \) and a functor \( E \to C \); that is,

\[
\ast (K, e) := (K, e) \circ (\ast, C).
\]

4.9.12 Example.

(1) Let \( C \) and \( E \) be categories. Given a subcategory \( D \) of \( E \), precomposition with the inclusion \( D \to E \) yields the cell

\[
\begin{array}{ccc}
C \overset{(\ast, \mathcal{C})}{\to} [E, C] & \overset{\text{op.}}{\to} & [E, C] \overset{(\ast, \mathcal{C})}{\to} C \\
\downarrow^{(\ast, D, \mathcal{C})} & \downarrow^{(D, \mathcal{C})} & \downarrow^{(D, \mathcal{C})} \\
[D, C] & \overset{\text{op.}}{\to} & [D, C] \overset{(\ast, \mathcal{C})}{\to} C
\end{array}
\]

(2) Let \( C \) and \( E \) be categories. Given an object \( e \in \| E \| \), precomposition with the functor \( e : \ast \to E \) yields the cell

\[
\begin{array}{ccc}
C \overset{(\ast, \mathcal{C})}{\to} [E, C] & \overset{\text{op.}}{\to} & [E, C] \overset{(\ast, \mathcal{C})}{\to} C \\
\downarrow^{(\ast, e, \mathcal{C})} & \downarrow^{(e, \mathcal{C})} & \downarrow^{(e, \mathcal{C})} \\
[D, C] & \overset{\text{op.}}{\to} & [D, C] \overset{(\ast, \mathcal{C})}{\to} C
\end{array}
\]

Note. Recall from Remark 4.7.2(4) that a cone over a bifunctor is called a bicone.

4.9.13 Definition. The right and left exponential transposes of a bicone

\[
\begin{array}{ccc}
\ast \overset{\ast}{\to} E \times D & \overset{\text{op.}}{\to} & E \times D \overset{\ast}{\to} \ast \\
\downarrow^{\alpha} & \downarrow^{\alpha} & \downarrow^{\alpha} \\
C \overset{(\ast, \mathcal{C})}{\to} [E, C] & \overset{\text{op.}}{\to} & [E, C] \overset{(\ast, \mathcal{C})}{\to} C
\end{array}
\]

are the cones

\[
\begin{array}{ccc}
\ast \overset{\ast}{\to} D & \overset{\text{op.}}{\to} & D \overset{\ast}{\to} \ast \\
\downarrow^{\alpha} & \downarrow^{\alpha} & \downarrow^{\alpha} \\
C \overset{(\ast, \mathcal{C})}{\to} [E, C] & \overset{\text{op.}}{\to} & [E, C] \overset{(\ast, \mathcal{C})}{\to} C
\end{array}
\]
whose component at \( d \in \|D\| \) and at \( e \in \|E\| \) are the cones
\[
\begin{array}{ccc}
C \overset{\text{\( \ast \)}}{\rightarrow} E & C \overset{\text{\( \ast \)}}{\rightarrow} D & C \overset{\text{\( \ast \)}}{\rightarrow} E \\
C \overset{\text{\( \alpha \)}}{\rightarrow} C & C \overset{\text{\( \alpha \)}}{\rightarrow} C & C \overset{\text{\( \alpha \)}}{\rightarrow} C \\
E \overset{\text{\( \text{op.} \)}}{\rightarrow} D & E \overset{\text{\( \text{op.} \)}}{\rightarrow} D & D \overset{\text{\( \text{op.} \)}}{\rightarrow} D
\end{array}
\]
— the right slice of \( \alpha \) at \( d \) and the left slice of \( \alpha \) at \( e \) — defined by
\[
[\alpha \cdot d]_e = \alpha(e, d) = [e \cdot \alpha]_d.
\]

4.9.14 Remark. The right and left exponential transposition for bicones form the isos cells
\[
\begin{array}{ccc}
C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] \\
C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] & C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] & C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] \\
C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C]
\end{array}
\]
and
\[
\begin{array}{ccc}
C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] \\
C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] & C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] & C \overset{\text{\( \ast \)}}{\rightarrow} [D, [E, C]] \\
C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C] & C \overset{\text{\( \ast \)}}{\rightarrow} [[E \times D], C]
\end{array}
\]
, natural in \( E, D, \) and \( C. \)

Note. A wedge in a category is defined below as a special case of Definition 4.8.1 where \( \mathcal{M} \) is given by the hom-module of a category.

4.9.15 Definition. Given a functor \( J : E \rightarrow C \) and a bifunctor \( L : E \times D \rightarrow C, \)
- a wedge \( \alpha \) from \( J \) to \( L \), written \( \alpha : J \cdot \alpha : E \times D \rightarrow C, \) is a natural transformation
  \[
  E \overset{\text{\( \text{op.} \)}}{\rightarrow} E \times D \overset{\text{\( J \)}}{\rightarrow} C \overset{\text{\( \alpha \)}}{\rightarrow} C
  \]
right weighted by the projection \( E \times D \rightarrow E. \)
- a wedge \( \alpha \) from \( L \) to \( J \), written \( \alpha : L \cdot \alpha : E \times D \rightarrow C, \) is a natural transformation
  \[
  E \times D \overset{\text{\( \text{op.} \)}}{\rightarrow} E \overset{\text{\( \text{op.} \)}}{\rightarrow} C \overset{\text{\( \alpha \)}}{\rightarrow} C
  \]
left weighted by the projection \( E \times D \rightarrow E. \)

4.9.16 Remark. A wedge in a category \( C \) defined above is just a special instance of a wedge in Definition 4.8.1 where \( \mathcal{M} \) is the hom-module of \( C. \) Conversely, a wedge along a module \( \mathcal{M} \) is the same thing as a wedge in the collage category \( \left\| \mathcal{M} \right\| \) (cf. Remark 4.3.2(2)).

Note. Recall from Definition 4.8.3 that the module of wedges \( E \times D \rightarrow \mathcal{M} \) is defined as an instance of the module of right weighted cylinders across \( \mathcal{M}. \) As a special case, the module of wedges \( E \times D \rightarrow C \) is defined below as an instance of the module of weighted natural transformations in \( C \) (see Definition 4.5.17).

4.9.17 Definition. Given categories \( E, D, \) and \( C, \) the module
\[
\langle E \times D, C \rangle : [E, C] \rightarrow [E \times D, C] \quad \text{op.} \quad \langle E \times D, C \rangle : [E, C] \rightarrow [E \times D, C]
\]
of wedges \( E \times D \rightarrow C \) [op. \( E \times D \rightarrow C \)] is defined by
\[
\langle E \times D, C \rangle = \langle E \times D, C \rangle \quad \text{op.} \quad \langle E \times D, C \rangle = \langle E \times D, C \rangle
\]
; that is,
\[
\langle E \times D, C \rangle := \langle E \times D, \{C\} \rangle \quad \text{op.} \quad \langle E \times D, C \rangle := \langle E \times D, \{C\} \rangle
\].
4.9.18 Remark. (1) For a functor \( F : E \to C \) and a bifunctor \( L : E \times D \to C \), the set \((F)(E \times D, C)(L)[\text{op.}]
(L)(E \times D*, C)(F)\) consists of all wedges \( F \Rightarrow L : E \times D \to C \) [op. \( L \Rightarrow F : E \times D* \to C \)].

(2) By Definition 4.5.17, the module \((E \times D, C)[\text{op.} (E \times D*, C)]\) is given by the composition

\[
[E, C] \underbrace{[E \times D, C]}_{\text{op.}} \to [E \times D, C] \xrightarrow{[E \times D, C]} [E, C]
\]

\(, i.e.\) by the representable [op. corepresentable] module of the diagonal functor \([E \times 1, D, C]\).

Note. The exponential transpositions of a wedge in a category is defined as a special case of Definition 4.8.5 where \( \mathcal{M} \) is given by the hom-module of a category.

4.9.19 Definition. The right and left exponential transposes of a wedge

\[
E \xrightarrow{E \times 1} E \times D \quad \text{op.} \quad E \times D \xleftarrow{E \times 1} E
\]

are the cone and cylinder

\[
\begin{array}{ccc}
\ast & \xleftarrow{1} & D \\
\downarrow{\alpha} & \downarrow{\ell} & \downarrow{\eta} \\
[E, C] & \xrightarrow{\ell} & [E, C]
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
D & \xrightarrow{1} & \ast \\
\downarrow{\alpha} & \downarrow{\ell} & \downarrow{\eta} \\
[E, C] & \xrightarrow{\ell} & [E, C]
\end{array}
\]

\(\text{whose component at } d \in |D| \text{ and at } e \in |E| \text{ are the cylinder and cone}

\[
\begin{array}{ccc}
& \ast & \xleftarrow{1} D \\
\downarrow{e \cdot j} & \downarrow{\alpha \cdot e} & \downarrow{\ell} \\
C & \xrightarrow{\ell} & C
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
D & \xrightarrow{1} & \ast \\
\downarrow{e \cdot j} & \downarrow{\alpha \cdot e} & \downarrow{\ell} \\
C & \xrightarrow{\ell} & C
\end{array}
\]

—\(\text{the right slice of } \alpha \text{ at } d \text{ and the left slice of } \alpha \text{ at } e\)—defined by

\[
[\alpha \cdot d]_e = \alpha_{(e, d)} = [e \cdot \alpha]_d.
\]

4.9.20 Remark. The right and left exponential transpositions of wedges \( E \times D \to C \) [op. \( E \times D* \to C \)] form the iso cells

\[
\begin{array}{ccc}
E, C & \xrightarrow{\ell} & [E, D, C] \\
\downarrow{\ell} & \downarrow{\ell} & \downarrow{\ell} \\
E, C & \xrightarrow{\ell} & [E, D, C]
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E \times D, C & \xrightarrow{\ell} & [E, D, C] \\
\downarrow{\ell} & \downarrow{\ell} & \downarrow{\ell} \\
E \times D, C & \xrightarrow{\ell} & [E, D, C]
\end{array}
\]

and

\[
\begin{array}{ccc}
E, C & \xrightarrow{\ell} & [E, D, C] \\
\downarrow{\ell} & \downarrow{\ell} & \downarrow{\ell} \\
E, C & \xrightarrow{\ell} & [E, D, C]
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E \times D, C & \xrightarrow{\ell} & [E, D, C] \\
\downarrow{\ell} & \downarrow{\ell} & \downarrow{\ell} \\
E \times D, C & \xrightarrow{\ell} & [E, D, C]
\end{array}
\]

, natural in \( E, D, \) and \( C \). In fact, these iso cells are obtained from the homs of the functors \( [E \times D, C] \Rightarrow [D, [E, C]] \) and \( [E \times D, C] \Rightarrow [E, [D, C]] \) by pasting a commutative diagram of diagonal functors (see Preliminary 0.0.7) as shown below:
4.9. Cones and wedges in categories

4.9.21 Definition. The twist transpose of a cone

\[
\begin{array}{ccc}
\star & \xleftarrow{\downarrow} & D \\
\uparrow{J} & \alpha & \downarrow{L}
\end{array}
\]

\[
[E, C] \xrightarrow{[\star, D, C]} [E \times D, C] \xrightarrow{[(E \times D, C)]} [E, C]
\]

is the cylinder

\[
\begin{array}{ccc}
J & \uparrow{\alpha^*} & L^\tau \\
E & \xleftarrow{\downarrow} & (D^*, C)
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\downarrow{\alpha^*}} & \star \\
\uparrow{L^\tau} & \downarrow{\alpha^*} & \uparrow{J}
\end{array}
\]

whose component at \( e \in |E| \) is the cone

\[
\begin{array}{ccc}
\star & \xleftarrow{\downarrow{J(e)}} & D \\
\uparrow{\alpha^*_e} & \alpha^*_e & \downarrow{L^\tau(e)}
\end{array}
\]

\[
C \xrightarrow{\downarrow{(C)}} C
\]

—the slice of \( \alpha \) at \( e \)—defined by

\[
[\alpha^*_e]_d = [\alpha_d]_e
\]

for each \( d \in |D| \).

4.9.22 Remark.

1. The twist transposition \( \alpha \mapsto \alpha^* \) forms the iso cell

\[
\begin{array}{ccc}
[E, C] & \xrightarrow{[\star, D, C]} & [D, [E, A]] \\
\uparrow{\tau} & \xrightarrow{\downarrow{1}} & \downarrow{1}
\end{array}
\]

\[
\begin{array}{ccc}
[D, [E, C]] & \xrightarrow{[(D, \star, C)]} & [E, \star, C] \\
\uparrow{\tau} & \xrightarrow{\downarrow{1}} & \downarrow{1}
\end{array}
\]

, natural in \( E, D, \) and \( C \), making the diagram

\[
\begin{array}{ccc}
[E \times \star, D, C] & \xrightarrow{\downarrow{\tau}} & [\star, D, [E, C]] \\
\uparrow{\tau} & \xrightarrow{\downarrow{1}} & \downarrow{1}
\end{array}
\]

\[
\begin{array}{ccc}
[E \times \star, D, C] & \xrightarrow{\downarrow{\tau}} & [D, \star, [E, C]] \\
\uparrow{\tau} & \xrightarrow{\downarrow{1}} & \downarrow{1}
\end{array}
\]

commute, where \( \tau \) and \( \downarrow{1} \) are the cells in Remark 4.9.20.

2. The diagram

\[
\begin{array}{ccc}
\langle \star, D, (E, C) \rangle & \xrightarrow{\downarrow{\tau}} & \langle \star, D, (E, C) \rangle \\
\langle D, (E, C) \rangle & \xrightarrow{\downarrow{\tau}} & \langle D, (E, C) \rangle
\end{array}
\]

\[
\begin{array}{ccc}
\langle \star, D, (E, C) \rangle & \xrightarrow{\downarrow{\tau}} & \langle \star, D, (E, C) \rangle \\
\langle D, (E, C) \rangle & \xrightarrow{\downarrow{\tau}} & \langle D, (E, C) \rangle
\end{array}
\]

commutes, where \( \langle e, C \rangle \) and \( \langle e, \star, D, C \rangle \) [op. \( \langle e, \star, D, C \rangle \)] are evaluations at \( e \in |E| \) (see Remark 4.3.32); the postcomposition

\[
\begin{array}{ccc}
\star & \xleftarrow{\downarrow{1}} & D \\
\uparrow{\alpha} & \alpha & \downarrow{J}
\end{array}
\]

\[
\begin{array}{ccc}
E, C \xrightarrow{[\star, (E, C)]} [E, C] \\
\uparrow{\downarrow{(e, C)}} & \downarrow{(e, C)} & \uparrow{\downarrow{(e, C)}}
\end{array}
\]

\[
\begin{array}{ccc}
C \xrightarrow{\downarrow{(C)}} C \\
\uparrow{\downarrow{(e, C)}} & \downarrow{(e, C)} & \uparrow{\downarrow{(e, C)}}
\end{array}
\]

Note. The twist transpose of a cone in a functor category is defined as a special case of Definition 4.8.7 where \( \mathcal{M} \) is given by the hom-module of a category.
of $\alpha$ with the evaluation $(e, C)$ thus yields the same cone as the one given by the component of $\alpha^\top$ at e, i.e. the slice of $\alpha$ at $e$ (cf. Preliminary 0.0.6(3)).

(3) Since $[\alpha \triangleright] = \kappa \alpha$, the left slice of a wedge $\alpha : E \times D \rightarrow C$ at $e \in \|E\|$ coincides with the slice of the cone $\alpha : *D \rightarrow \{E, C\}$ at $e$, both being given by the component of the cylinder $\kappa \alpha : E \rightarrow \{*, D, C\}$ at $e$.

Note. The following is a special case of Definition 4.5.14.

4.9.23 Definition. Given a pair of cone and wedge

$\begin{align*}
\star \rightarrow & \quad E \\
\nu \uparrow & \quad \mu \quad \mu \quad \nu \\
\left(\overline{c}\right) & \rightarrow C - \rightarrow \left(\overline{c}\right) \rightarrow C
\end{align*}$

or componentwise by

$\begin{align*}
\nu \circ \mu & = [E \times D \circ \nu] \circ \mu \quad \text{op.} \\
\mu \circ \nu & = \mu \circ \nu [E \times D \circ \nu]
\end{align*}$

defined by

$\left[\nu \circ \mu\right]_{(e, d)} = \nu_e \circ \mu_{(e, d)} \quad \text{op.} \\
\left[\mu \circ \nu\right]_{(e, d)} = \mu_{(e, d)} \circ \nu_e$

for $(e, d) \in \|E \times D\|$.

4.9.24 Proposition. The assignment

$\nu \mapsto \nu \circ \mu \quad \text{op.} \quad \nu \mapsto \mu \circ \nu$

yields a right [op. left] module morphism

$\begin{align*}
\star \rightarrow C \quad \text{op.} \quad \star \rightarrow C
\end{align*}$

Proof. For a $C$-arrow $f : c' \rightarrow c$, we need to show that the associative law

$f \circ \left[\nu \circ \mu\right] = [f \circ \nu] \circ \mu$

holds. But this is immediate from the definition of the composition $\nu \circ \mu$ (cf. Proposition 4.5.15). \qed

4.10 Cones and wedges in Set

Just like we introduced the notation $\Psi : \mathcal{M} \rightarrow \mathcal{N} : * \rightarrow E$ for a natural transformation $\Psi : \mathcal{M} \rightarrow \mathcal{N} : E \rightarrow \text{Set}$, we will introduce special notations for cones and wedges in the category $\text{Set}$—our ambient category—to avoid the explicit specification of it.

4.10.1 Definition. A cone $\xi$, between a set $S$ and a left module $M : * \rightarrow E$ (i.e. a functor $M : E \rightarrow \text{Set}$) is denoted by

$\xi : S \rightarrow M : * \rightarrow E \quad \text{op.} \quad \xi : M \rightarrow S : * \rightarrow E$

rather than

$\xi : S \rightarrow M : *E \rightarrow \text{Set} \quad \text{op.} \quad \xi : M \rightarrow S : E \rightarrow \text{Set}$

and the component of $\xi$, at $e \in \|E\|$ is written as

$\langle \xi \rangle e : S \rightarrow \langle M \rangle e \quad \text{op.} \quad \langle \xi \rangle e : \langle M \rangle e \rightarrow S$

; the module of cones $*E \rightarrow \text{Set}$ is denoted by

$\langle *E \rangle : \text{Set} \rightarrow \{[E] \}$ \quad \text{op.} \quad \langle E \rangle : \{E\} \rightarrow \text{Set}$
rather than
\[ \langle \ast E, \text{Set} \rangle : \text{Set} \to [E, \text{Set}] \] op. \[ \langle E^\ast, \text{Set} \rangle : [E, \text{Set}] \to \text{Set} \]

- A cone \( \xi \) between a set \( S \) and a right module \( \mathcal{M} : E \to \ast \) (i.e. a functor \( \mathcal{M} : E^\ast \to \text{Set} \)) is denoted by
  \[ \xi : S \to \mathcal{M} : \ast \to \ast \] op. \[ \xi : \mathcal{M} \to S : \ast \to \ast \]
rather than
  \[ \xi : S \to \mathcal{M} : \ast E \to \ast \] op. \[ \xi : \mathcal{M} \to S : \ast E \to \ast \]

, and the component of \( \xi \) at \( e \in \|E\| \) is written as
  \[ e(\xi) : S \to e(\mathcal{M}) \] op. \[ e(\xi) : e(\mathcal{M}) \to S \]

; the module of cones \( \ast E \to \text{Set} \) is denoted by
  \[ \langle \ast E \rangle : \text{Set} \to [E : \mathcal{M}] \] op. \[ \langle E^\ast \rangle : [E : \mathcal{M}] \to \text{Set} \]
rather than
  \[ \langle \ast E^\ast \rangle : \text{Set} \to [E^\ast : \mathcal{M}] \] op. \[ \langle E^\ast \rangle : [E^\ast : \mathcal{M}] \to \text{Set} \]

4.10.2 Remark. By Remark 4.9.4(2), the modules of cones in Definition 4.10.1 are the representable [op. corepresentable] modules of the diagonal functors in Example 1.1.31(10); that is,
- the module
  \[ \langle \ast E \rangle : \text{Set} \to [E : \mathcal{M}] \] op. \[ \langle E^\ast \rangle : [E : \mathcal{M}] \to \text{Set} \]
is given by the composition
  \[ \text{Set} \xrightarrow{[\mathcal{M}]} [E : \mathcal{M}] \xrightarrow{\langle E \rangle} [E : \mathcal{M}] \] op. \[ [E : \mathcal{M}] \xrightarrow{\langle E \rangle} [E : \mathcal{M}] \xrightarrow{[\mathcal{M}]} \text{Set} \]

; a cone
  \[ \xi : S \to \mathcal{M} : \ast \to \ast E \] op. \[ \xi : \mathcal{M} \to S : \ast \to \ast E \]
is the same thing as a left module morphism
  \[ \xi : \Delta_E S \to \mathcal{M} : \ast \to E \] op. \[ \xi : \mathcal{M} \to \Delta_E S : \ast \to E. \]

- the module
  \[ \langle \ast E \rangle : \text{Set} \to [E : \mathcal{M}] \] op. \[ \langle E^\ast \rangle : [E : \mathcal{M}] \to \text{Set} \]
is given by the composition
  \[ \text{Set} \xrightarrow{[\mathcal{M}]} [E : \mathcal{M}] \xrightarrow{\langle E \rangle} [E : \mathcal{M}] \] op. \[ [E : \mathcal{M}] \xrightarrow{\langle E \rangle} [E : \mathcal{M}] \xrightarrow{[\mathcal{M}]} \text{Set} \]

; a cone
  \[ \xi : S \to \mathcal{M} : \ast E \to \ast \] op. \[ \xi : \mathcal{M} \to S : \ast E \to \ast \]
is the same thing as a right module morphism
  \[ \xi : \Delta_E S \to \mathcal{M} : E \to \ast \] op. \[ \xi : \mathcal{M} \to \Delta_E S : E \to \ast. \]

4.10.3 Definition.
- A wedge \( \xi \) between a right module \( \mathcal{L} : X \to \ast \) (i.e. a functor \( \mathcal{L} : X^\ast \to \text{Set} \)) and a module \( \mathcal{M} : X \to E \) (i.e. a functor \( \mathcal{M} : X^\ast \times E \to \text{Set} \)) is denoted by
  \[ \xi : \mathcal{L} \to \mathcal{M} : X \to \ast E \] op. \[ \xi : \mathcal{M} \to \mathcal{L} : X \to E^\ast \]
rather than
  \[ \xi : \mathcal{L} \to \mathcal{M} : X^\ast \times \ast E \to \text{Set} \] op. \[ \xi : \mathcal{M} \to \mathcal{L} : X^\ast \times E^\ast \to \text{Set} \]

, and the component of \( \xi \) at \( (x, e) \in \|X \times E\| \) is written as
  \[ x(\xi) : x(\mathcal{L}) \to x(\mathcal{M})e \] op. \[ x(\xi) : x(\mathcal{M})e \to x(\mathcal{L}) \]

; the module of wedges \( X^\ast \times \ast E \to \text{Set} \) is denoted by
  \[ \langle X : \ast E \rangle : [X : \mathcal{M}] \to [X : E] \] op. \[ \langle X : E^\ast \rangle : [X : E] \to [X : \mathcal{M}] \]
4.10.5 Definition.

\[(X^- \times \*E, \text{Set}) : \{X^-, \text{Set}\} \rightarrow [X^- \times E, \text{Set}] \quad \text{op.} \quad (X^- \times \*E, \text{Set}) : \{X^- \times E, \text{Set}\} \rightarrow [X^-, \text{Set}].\]

- A wedge \(\xi\), between a left module \(\mathcal{L} : X \rightarrow A\) (i.e. a functor \(\mathcal{L} : A \rightarrow \text{Set}\)) and a module \(\mathcal{M} : E \rightarrow A\) (i.e. a functor \(\mathcal{M} : E \times A \rightarrow \text{Set}\)) is denoted by

\[\xi : \mathcal{L} \leadsto \mathcal{M} : \*E \rightarrow A \quad \text{op.} \quad \xi : \mathcal{M} \leadsto \mathcal{L} : \*E \rightarrow A\]

rather than

\[\xi : \mathcal{L} \leadsto \mathcal{M} : \*E \times A \rightarrow \text{Set} \quad \text{op.} \quad \xi : \mathcal{M} \leadsto \mathcal{L} : E \times \*A \rightarrow \text{Set}\]

, and the component of \(\xi\), at \((e, a) \in \|E \times A\|\) is written as

\[e(\xi) a : (\mathcal{L}) a \rightarrow e(\mathcal{M}) a \quad \text{op.} \quad e(\xi) a : e(\mathcal{M}) a \rightarrow (\mathcal{L}) a\]

; the module of wedges \(\*E \times A \rightarrow \text{Set}\) is denoted by

\[(\*E : A) : [\*A] \rightarrow [E : A] \quad \text{op.} \quad (\*E : A) : [E : A] \rightarrow [\*A]\]

rather than

\[(\*E \times A, \text{Set}) : [A, \text{Set}] \rightarrow [E \times A, \text{Set}] \quad \text{op.} \quad (E^- \times A, \text{Set}) : [E^- \times A, \text{Set}] \rightarrow [A, \text{Set}].\]

4.10.4 Remark. By Remark 4.9.18(2), the modules of wedges in Definition 4.10.3 are the representable [op. corepresentable] modules of the diagonal functors in Example 1.1.31(8)); that is,

- the module

\[(X : \*E) : [X : ] \rightarrow [X : E] \quad \text{op.} \quad (X : \*E) : [X : E] \rightarrow [X : ]\]

is given by the composition

\[[X : ] \xrightarrow{[X : E]} [X : E] \xrightarrow{[E : X]} [X : ] \quad \text{op.} \quad [X : E] \xrightarrow{([E : X])} [X : E] \xrightarrow{([E : X])} [X : ]\]

; a wedge

\[\xi : \mathcal{L} \leadsto \mathcal{M} : X \rightarrow \*E \quad \text{op.} \quad \xi : \mathcal{M} \leadsto \mathcal{L} : X \rightarrow E^+\]

is the same thing as a module morphism

\[\xi : (\mathcal{L}) [\!*E] \rightarrow \mathcal{M} : X \rightarrow E \quad \text{op.} \quad \xi : \mathcal{M} \rightarrow (\mathcal{L}) [\!*E] : X \rightarrow E.\]

- the module

\[(\*E : A) : [\*A] \rightarrow [E : A] \quad \text{op.} \quad (\*E : A) : [E : A] \rightarrow [\*A]\]

is given by the composition

\[[\*A] \xrightarrow{[\*E : A]} [E : A] \xrightarrow{[E : A]} [\*A] \quad \text{op.} \quad [E : A] \xrightarrow{([E : A])} [E : A] \xrightarrow{([E : A])} [\*A]\]

; a wedge

\[\xi : \mathcal{L} \leadsto \mathcal{M} : \*E \rightarrow A \quad \text{op.} \quad \xi : \mathcal{M} \leadsto \mathcal{L} : E^+ \rightarrow A\]

is the same thing as a module morphism

\[\xi : [\!*E] (\mathcal{L}) \rightarrow \mathcal{M} : E \rightarrow A \quad \text{op.} \quad \xi : \mathcal{M} \rightarrow [\!*E] (\mathcal{L}) : E \rightarrow A.\]

Note. The following is an instance of the exponential transpose of a wedge in 4.9.19.

4.10.5 Definition.

- Given a right module \(\mathcal{L} : X \rightarrow \*\) and a module \(\mathcal{M} : X \rightarrow E\), the right exponential transpose of a wedge

\[\xi : \mathcal{L} \leadsto \mathcal{M} : X \rightarrow \*E \quad \text{op.} \quad \xi : \mathcal{M} \leadsto \mathcal{L} : X \rightarrow E^+\]

is the cone

\[\begin{array}{c}
\begin{array}{ccc}
\ast & \xrightarrow{\mathcal{L}} & E \\
\mathcal{M} & \xrightarrow{\mathcal{M}} & \mathcal{L}
\end{array} \\
\begin{array}{c}
\xi \uparrow \downarrow \mathcal{L} \\
\xi \uparrow \downarrow \mathcal{M}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{ccc}
X & \xrightarrow{\mathcal{L}} & X \\
\mathcal{M} & \xrightarrow{\mathcal{M}} & \mathcal{L}
\end{array} \\
\begin{array}{c}
\xi \uparrow \downarrow \mathcal{L} \\
\xi \uparrow \downarrow \mathcal{M}
\end{array}
\end{array}\]
whose component at \( e \in |E| \) is the right module morphism
\[
(\xi) e : L \to (M) e : X \to \ast \quad \text{op.} \quad (\xi) e : (M) e \to L : X \to \ast
\]
— the right slice of \( \xi \), at \( e \)—defined by
\[
x((\xi) e) = x(\xi) e
\]
for each \( x \in |X| \).

\* Given a left module \( L : \ast \to A \) and a module \( M : E \to A \), the left exponential transpose of a wedge
\[
\xi : L \to (M) : \ast \to A \quad \text{op.} \quad \xi : M \to L : E^* \to A
\]
is the cone
\[
\begin{array}{c}
\xymatrix{
\ast & E^- \ar[l]_{\xi} \ar[r]^{\xi_L} & (A) \ar[r]^\sim & [A] \\
L \ar[u]^L \ar[r]^L & M \ar[u]^M \ar[r]^L & L \\

[: A] \ar[r]^\sim & [A] \\
\}
\end{array}
\]
whose component at \( e \in |E| \) is the left module morphism
\[
e(\xi) e : L \to (M) e : \ast \to A \quad \text{op.} \quad e(\xi) e : (M) e \to L : E^* \to A
\]
— the left slice of \( \xi \), at \( e \)—defined by
\[
(e(\xi)) a = e(\xi) a
\]
for each \( a \in |A| \).

4.10.6 Remark. The right exponential transposition of wedges \( X \to E^* \) [op. \( X \to E^* \)] form the iso cell
\[
\begin{array}{c}
\xymatrix{
[X] : & [X : E] \ar[l]_{(X \to E^*)} \ar[r]^{\sim} & [X : E] \\
\}
\end{array}
\]
and the left exponential transposition of wedges \( E^* \times A \to \text{Set} \) [op. \( E^* \times A \to \text{Set} \)] form the iso cell
\[
\begin{array}{c}
\xymatrix{
[: A] & [E : A] \ar[l]_{(E \to A)} \ar[r]^{\sim} & [E : A] \\
\}
\end{array}
\]
; in fact, the cell \( (X : E) \xrightarrow{\sim} (E^* \times (A, X)) \) [op. \( (E^* \times (A, X)) \xrightarrow{\sim} (E^* \times (A, X)) \)] is obtained from the hom-cell of the functor \( [X : E] \xrightarrow{\sim} [E, [X :]] \) by the pasting composition
\[
\begin{array}{c}
\xymatrix{
[X] : & [X : E] \ar[l]_{(X \to E)} \ar[r]^{\sim} & [X : E] \\
\}
\end{array}
\]
with the commutative diagram of diagonal functors in Remark 2.1.2(5), and the cell \( (E^* : A) \xrightarrow{\sim} (E^* \times (A, A)) \) [op. \( (E^* \times (A, A)) \xrightarrow{\sim} (E^* \times (A, A)) \)] is obtained from the hom-cell of the functor \( [E : A] \xrightarrow{\sim} [E, [A]] \) by the pasting composition
\[
\begin{array}{c}
\xymatrix{
[: A] & [E : A] \ar[l]_{(E \to A)} \ar[r]^{\sim} & [E : A] \\
\}
\end{array}
\]
(c.f. Remark 4.9.20). In Section 5.4, the cell \( (E^* : A) \xrightarrow{\sim} (E^* \times (A, A)) \) [op. \( (E^* \times (A, A)) \xrightarrow{\sim} (E^* \times (A, A)) \)]
appears in the opposite form

$$\text{[E : A]^{-}} \xrightarrow{\text{(E')}^{-}} \text{[A]^{-}} \quad \text{op.} \quad \text{[A]^{-}} \xrightarrow{\text{(E')}^{-}} \text{[E : A]^{-}}$$

$$\text{[E, [A]^{-}]_{(E')}^{-}} \xrightarrow{\text{[E, [A]^{-}]}} \text{[A]^{-}}$$

$$\text{[A]^{-}} \xrightarrow{\text{[(E')-1]}^{-}} \text{[E, [A]^{-}]_{(E')}^{-}}$$

(note that \((\text{E')}^{-}, (\text{A})^{-}) \cong (\text{E}^{-*}, (\text{A})^{-}) \) [op. \((\text{E}^{-*}, (\text{A})^{-}) \cong (\text{E'}, (\text{A})^{-})\)] by Remark 4.6(4)).

### 4.11 Orbits

We conclude the chapter with the definition of orbits of an endomodule and of a one-sided module. The notion may be regarded as the dual of that of frames; the set of orbits of a module is seen as a categorical generalization of the notion of the disjoint union of an indexed family of sets. We will see in Section 8.6 that the set of orbits of a small one-sided module \(\mathcal{M}\) gives a colimit of \(\mathcal{M}\) in the category \(\text{Set}\), and see in Section 13.4 that the set of orbits of a small endomodule \(\mathcal{M}\) gives a coend of \(\mathcal{M}\) in \(\text{Set}\).

**4.11.1 Definition.** Two endoarrows \(m : e \sim e\) and \(m' : e' \sim e'\) of an endomodule \(\mathcal{M} : E \to E\) are said to be connected, written \(m \equiv m'\), if there is a finite sequence of \(E\)-arrows \(h_1, \ldots, h_n\) connecting \(e\) and \(e'\) (see Preliminary 0.0.10) and diagonal \(\mathcal{M}\)-arrows \(m_1, \ldots, m_n\) making the diagram

\[
\begin{array}{c}
e \xrightarrow{h_1} e_1 \xrightarrow{h_2} e_2 \ldots \xrightarrow{h_n} e' \\
m \xrightarrow{m_1} e_1 \xrightarrow{m_2} e_2 \ldots \xrightarrow{m_n} e'
\end{array}
\]

commute (the direction of each \(h_i\) is arbitrary, and each \(m_i\) goes in the direction opposite to \(h_i\)).

**4.11.2 Remark.** If two endoarrows \(m : e \sim e\) and \(m' : e' \sim e'\) are connected in \(\mathcal{M}\), they are connected in the comma category \([\mathcal{M}]\) in the sense of Preliminary 0.0.10. (The converse is not the case in general.)

**4.11.3 Proposition.** All pairs \((m, m')\) of connected endoarrows form an equivalence relation on the set \(\coprod_{e \in \text{E}} [\mathcal{M}, e]\) of all endoarrows of \(\mathcal{M}\). \(\square\)

**4.11.4 Remark.** The equivalence relation \(m \equiv m'\) (\(m\) and \(m'\) are connected) is generated by the pairs \((h \circ m, m \circ h)\) for every \(E\)-arrow \(h : e \to e'\) and every \(\mathcal{M}\)-arrow \(m : e' \sim e\) as in the commutative diagram

\[
\begin{array}{c}
e \xrightarrow{h} e' \\
h \circ m \xrightarrow{\cdot} m \circ h
\end{array}
\]

**4.11.5 Proposition.** Any module morphism \(\psi : \mathcal{M} \to \mathcal{N} : E \to E\) preserves connectedness; that is, if two endoarrow \(m : e \sim e\) and \(m' : e' \sim e'\) of \(\mathcal{M}\) are connected, then their images under \(\psi\) is connected in \(\mathcal{N}\).

**Proof.** Any module morphism \(\psi\), by its naturality, preserves the commutativity of the diagram in Definition 4.11.1. \(\square\)

**4.11.6 Definition.** For an endomodule \(\mathcal{M} : E \to E\), the equivalence classes of the equivalence relation in Proposition 4.11.3 are called orbits of \(\mathcal{M}\). The orbit including an endoarrow \(m : e \sim e\) is written as \(m^{\circ}\), and the set of orbits of \(\mathcal{M}\) is denoted by \(\coprod E \mathcal{M}\); that is,

\[
\coprod E \mathcal{M} := \left(\coprod_{e \in \text{E}} [\mathcal{M}, e]\right) / \approx.
\]
4.11.7 Remark. 
(1) If $\mathcal{M}$ is small, so is the set $\coprod_{E} \mathcal{M}$.
(2) If $E$ is discrete, no two distinct endoarrows of $\mathcal{M}$ are connected and hence $\coprod_{E} \mathcal{M} \cong \coprod_{e \in [E]} e(\mathcal{M}) e$.

Note. Proposition 4.11.5 justifies the following definition.

4.11.8 Definition. Given a module morphism $\psi : \mathcal{M} \to N : E \to E$, the function

$$\coprod_{E} \psi : \coprod_{E} \mathcal{M} \to \coprod_{E} N$$

is defined so that

$$m^* \cdot \coprod_{E} \psi = (m \cdot \psi)^*$$

for any endoarrow $m$ in $\mathcal{M}$; that is, so that the diagram

$$\begin{array}{ccc}
\coprod_{e \in [E]} e(\mathcal{M}) e & \xrightarrow{[\cdot]^*} & \coprod_{E} \mathcal{M} \\
\coprod_{e \in [E]} e(\psi)e & \downarrow \coprod_{E} \psi & \coprod_{E} N \\
\coprod_{e \in [E]} e(N) e & \xrightarrow{[\cdot]^*} & \coprod_{E} N
\end{array}$$

commutes.

4.11.9 Proposition. The assignment $\psi \mapsto \coprod_{E} \psi$ is functorial and defines a functor

$$\coprod_{E} : [E : E] \to \text{Set}$$

, i.e. a (locally small) left module

$$\coprod_{E} : * \to [E : E]$$

for $E$ small.

4.11.10 Definition. Given a left [op. right] module $\mathcal{M}$ over a category $E$, two $\mathcal{M}$-arrows $m$ and $m'$ are said to be connected, written $m \equiv m'$, if there is a finite sequence of $E$-arrows $h_1, \ldots, h_n$ connecting $e$ and $e'$ (see Preliminary 0.0.10) and $\mathcal{M}$-arrows $m_1, \ldots, m_n$ making the diagram

$$\begin{array}{c}
m \xrightarrow{h_1} m_1 \xrightarrow{h_2} \cdots \xrightarrow{h_n} m' \equiv m' \xrightarrow{e'} \end{array}$$

to commute (the direction of each $h_i$ is arbitrary).

4.11.11 Remark. Two $\mathcal{M}$-arrows $m$ and $m'$ are connected in $\mathcal{M}$ if and only if they are connected in the comma category $[\mathcal{M}]$ in the sense of Preliminary 0.0.10.

4.11.12 Proposition. For a left [op. right] module $\mathcal{M}$ over a category $E$, all pairs $(m, m')$ of connected $\mathcal{M}$-arrows form an equivalence relation on the set $\coprod_{e \in [E]} e(\mathcal{M}) e$ [op. $\coprod_{e \in [E]} e(\mathcal{M})$] of all $\mathcal{M}$-arrows.

4.11.13 Remark. For a left [op. right] module $\mathcal{M}$ over a category $E$, the equivalence relation $m \equiv m'$ (and $m'$ are connected) is generated by the pairs $(m, m \circ h)$ [op. $(m, m \circ h)$] for every $E$-arrow $h : e \to e'$ [op. $h : e' \to e$] and every $\mathcal{M}$-arrow $m : \ast \to e$ [op. $m : e \to \ast$] as in the commutative diagram.

$$\begin{array}{c}
m \xrightarrow{h} m \circ h \end{array}$$

4.11.14 Proposition. Any left [op. right] module morphism $\psi : \mathcal{M} \to N$ preserves connectedness; that is, if two $\mathcal{M}$-arrows $m$ and $m'$ are connected in $\mathcal{M}$, then their images under $\psi$ is connected in $N$. 
Proof. Any module morphism \( \psi \), by its naturality, preserves the commutativity of the diagram in Definition 4.11.10.

4.11.15 Definition. For a left [op. right] module \( \mathcal{M} \) over a category \( \mathbf{E} \), the equivalence classes of the equivalence relation in Proposition 4.11.12 are called orbits of \( \mathcal{M} \). The orbit including an \( \mathcal{M} \)-arrow \( m \) is written as \( m^\circ \), and the set of orbits of \( \mathcal{M} \) is denoted by \( \prod_{\mathbf{E}} \mathcal{M} \) [op. \( \prod_{\mathbf{E}} \mathcal{M} \)]; that is,

\[
\prod_{\mathbf{E}} \mathcal{M} := \left\{ \{ \mathbf{e} \mid \mathcal{E} \} \mathcal{M} \mathbf{e} \right\} / \sim \quad \text{op.} \quad \prod_{\mathbf{E}} \mathcal{M} := \left\{ \{ \mathbf{e} \mid \mathcal{E} \} \mathcal{E} \mathbf{e} \right\} / \sim.
\]

4.11.16 Remark. (1) Recall from Remark 1.1.4(3) that a right module over a category \( \mathbf{E} \) is the same thing as a left module over the opposite category \( \mathbf{E}^{\text{op}} \), and note that the set of orbits of a right module \( \mathcal{M} : \mathbf{E} \to \star \) is the same thing as the set of orbits of a left module \( \mathcal{M} : \star \to \mathbf{E}^{\text{op}} \); \( \prod_{\mathbf{E}} \mathcal{M} \) denotes the set of orbits of a right module \( \mathcal{M} : \mathbf{E} \to \star \), i.e. the set of orbits of a left module \( \mathcal{M} : \star \to \mathbf{E}^{\text{op}} \).

(2) If \( \mathbf{E} \) is discrete, no two distinct arrows of a left module \( \mathcal{M} : \star \to \mathbf{E} \) are connected and hence \( \prod_{\mathbf{E}} \mathcal{M} \cong \prod_{\mathcal{E}} \mathcal{E} \mathbf{e} \). The set of orbits of a left module is thus seen as a generalization of the disjoint union of an indexed family of sets.

Note. The diagonal functor \( \left[ \mathbf{1}_{\mathbf{E}} : \mathbf{E} \right] : \left[ \mathbf{E} : \mathbf{E} \right] \to \left[ \mathbf{E} : \mathbf{E} \right] \) (see Example 1.1.31(8)) embeds the category of left module \( \mathbf{E} \to \mathbf{E} \) into the category of endomodules \( \mathbf{E} \to \mathbf{E} \). A left module \( \mathbf{E} \to \mathbf{E} \) may thus be seen as a special instance of an endomodule \( \mathbf{E} \to \mathbf{E} \), and

4.11.17 Proposition. The orbits of a left module \( \mathcal{M} : \star \to \mathbf{E} \) [op. right module \( \mathcal{M} : \mathbf{E} \to \star \)] is the same thing as the orbits of the endomodule \( \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M} : \mathbf{E} \to \mathbf{E} \) [op. \( \mathbf{E} \mathcal{M} \)]

\[
\prod_{\mathbf{E}} \mathcal{M} = \prod_{\mathbf{E}} \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M} \quad \text{op.} \quad \prod_{\mathbf{E}} \mathcal{M} = \prod_{\mathbf{E}} \mathbf{E} \mathcal{M} \mathbf{e}.
\]

Proof. First note that \( \mathcal{M} \mathbf{e} = \mathbf{e} \left( \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M} \mathbf{e} \right) \) for each \( \mathbf{e} \in \mathbf{E} \): an arrow \( m : \star \to \mathbf{e} \) of a left module \( \mathcal{M} : \star \to \mathbf{E} \) is the same thing as an endoarrow \( \mathbf{e} \to \mathbf{e} \) of the endomodule \( \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M} : \mathbf{E} \to \mathbf{E} \). Now since \( \mathbf{h} \circ m = m \) for any \( \mathbf{E} \)-arrow \( \mathbf{h} : \mathbf{e} \to \mathbf{e}^\prime \) and any \( \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M} \)-arrow \( \mathbf{e} \to \mathbf{e} \), the pairs \( (h \circ m, m \circ h) \) in Remark 4.11.4 coincides with the pairs \( (m, m \circ h) \) in Remark 4.11.13, and generate the same equivalence relation on \( \prod_{\mathcal{E}} \mathcal{E} \mathbf{e} = \prod_{\mathcal{E}} \mathbf{e} \mathcal{E} \mathbf{e} \mathcal{E} \mathbf{e} \).

Note. Proposition 4.11.14 justifies the following definition.

4.11.18 Definition. Given a left [op. right] module morphism \( \psi : \mathcal{M} \to \mathcal{N} \) over a category \( \mathbf{E} \), the function

\[
\prod_{\mathbf{E}} \mathcal{M} \to \prod_{\mathbf{E}} \mathcal{N}
\]

\[
\prod_{\mathbf{E}} \mathcal{M} \to \prod_{\mathbf{E}} \mathcal{N}
\]

is defined so that

\[
\mathbf{m}^\circ : \prod_{\mathbf{E}} \mathcal{M} = \left( \mathbf{m} : \mathcal{M} \right)^\circ \quad \text{op.} \quad \mathbf{m}^\circ : \prod_{\mathbf{E}} \mathcal{M} = \left( \mathbf{m} : \mathcal{M} \right)^\circ
\]

for any \( \mathcal{M} \)-arrow \( \mathbf{m} \); that is, so that the diagram

\[
\begin{array}{ccc}
\prod_{\mathbf{E}} \mathcal{M} & \xrightarrow{\left[ \mathbf{e} \mathcal{M} \right]} & \prod_{\mathbf{E}} \mathcal{M} \\
\prod_{\mathbf{E}} \mathcal{N} & \xrightarrow{\left[ \mathbf{e} \mathcal{N} \right]} & \prod_{\mathbf{E}} \mathcal{N}
\end{array}
\]

commutes;

4.11.19 Proposition. The diagram

\[
\begin{array}{ccc}
\prod_{\mathbf{E}} \mathcal{M} & \xrightarrow{\left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M}} & \prod_{\mathbf{E}} \mathcal{M} \\
\prod_{\mathbf{E}} \mathcal{N} & \xrightarrow{\left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{N}} & \prod_{\mathbf{E}} \mathcal{N}
\end{array}
\]

(see Proposition 4.11.17 for the identities on the top and bottom) commutes; that is,

\[
\prod_{\mathbf{E}} \mathcal{M} = \prod_{\mathbf{E}} \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{M}
\]

\[
\prod_{\mathbf{E}} \mathcal{N} = \prod_{\mathbf{E}} \left[ \mathbf{1}_{\mathbf{E}} \right] \mathcal{N}
\]
Proof. This follows from the commutativity of
\[
\begin{align*}
\Pi_{\text{op}} M &\cong \Pi_{\text{op}} E (\Pi_{\text{op}} M) e \\
\Pi_{\text{op}} (\Pi_{\text{op}} E \Pi_{\text{op}} M) e &\cong \Pi_{\text{op}} (\Pi_{\text{op}} E \Pi_{\text{op}} M) e
\end{align*}
\]
and the commutativity of the diagrams in Definition 4.11.18 and Definition 4.11.8 (replace \( M \) by \( \Pi_{\text{op}} M \)).

\( \square \)

4.11.20 Proposition. The assignment \( \psi \mapsto \Pi_{\text{op}} E \psi \) \([\text{op.} \ \psi \mapsto \Pi_{\text{op}} E \psi]\) is functorial and defines a functor
\[
\Pi_{\text{op}} E : [\cdot : E] \to \text{Set}
\]
, i.e., a (locally small) left module
\[
\Pi_{\text{op}} E : * \to [\cdot : E]
\]
for \( E \) small; in fact, the functor is obtained from the functor \( \Pi_{\text{op}} E \) in Proposition 4.11.9 by the composition
\[
[\cdot : E] \xrightarrow{[\Pi_{\text{op}} E]} [E : E] \xrightarrow{\Pi_{\text{op}} E} \text{Set}
\]
\([\text{op.} \ \ [E : E] \xrightarrow{\Pi_{\text{op}} E} \text{Set} \])

Proof. The second assertion is immediate from Proposition 4.11.19. The first assertion follows from the second.

\( \square \)

4.11.21 Theorem. Given a cone \( \alpha : L \to c : E \to C \) and an object \( x \in \| C \| \), suppose that two \( C \)arrow \( m : x \to L \cdot e \) and \( m' : x \to L \cdot e' \) are connected in the composite left module \( x (C) L : * \to E \). Then the diagram
\[
\begin{array}{c}
x \xrightarrow{m} L \cdot e \xrightarrow{\alpha_e} c \\
\end{array}
\]
commutes.

Proof. The interior of the diagram is divided into a finite number of commutative diagrams as in
\[
\begin{array}{c}
c \xrightarrow{c} c \xrightarrow{c} c \xrightarrow{c} c \xrightarrow{c} c \\
\end{array}
\]
5 Yoneda Morphisms

5.1 Yoneda modules

For a category $X$, the right Yoneda module $\langle X, \cdot \rangle : X \to [X:]$ is defined so that it incorporates all arrows of all right modules over $X$, and for a pair of categories $X$ and $A$, the right generalized Yoneda module $\langle X, \cdot A \rangle : [A, X] \to [X : A]$ is defined so that it incorporates all right cylinders along all modules $X \to A$. In Section 5.2 and Section 5.3, we will see that these cosmic modules are in fact represented by the right Yoneda functor $[X, \cdot] : X \to [X:]$ and the right generalized Yoneda functor $[X, \cdot A] : [A, X] \to [X : A]$ respectively (hence the name “Yoneda module”).

5.1.1 Definition.

- The right Yoneda module for a category $X$ is the module

$$\langle X, \cdot \rangle : X \to [X:]$$

given by the evaluation

$$(x, M) \mapsto x(M) : X^\cdot [X:] \to \text{Set}$$

; that is,

$$(x)_\langle X, \cdot \rangle (M) := x(M)$$

for $x \in X$ and $M \in [X:]$.

- The left Yoneda module for a category $A$ is the module

$$\langle \cdot, A \rangle : [: A]^\cdot \to A$$

given by the evaluation

$$(M, a) \mapsto (M) a : [: A] \times A \to \text{Set}$$

; that is,

$$(M) (\cdot, A) (a) := (M) a$$

for $a \in A$ and $M \in [: A]$.

5.1.2 Remark.

1. The module $X^\cdot$ [op. $\cdot, A$] is called the Yoneda module just because it is represented [op. corepresented] by the Yoneda functor $X^\cdot$ [op. $\cdot, A$] (see Theorem 5.2.15).

2. For a right module $M : X \to \ast$ and an object $x \in \|X\|$, the set

$$(x) (\langle X, \cdot \rangle) (M) = (x) (M)$$

consists of all $M$-arrows $x \rightsquigarrow \ast$, and for a left module $M : \ast \to A$ and an object $a \in \|A\|$, the set

$$(M) (\cdot, A) (a) = (M) (a)$$

consists of all $M$-arrows $\ast \to a$.

3. For any $X$-arrow $g : x' \rightarrow x$ and any $(X, \cdot)$-arrow $m : x \rightarrow M$ (i.e. $M$-arrow $m : x \rightarrow \ast$), the composition $g \circ m$ in $X^\cdot$ is given by that in $M$, and for any $A$-arrow $f : a \rightarrow a'$ and any $(\cdot, A)$-arrow $m : M \rightarrow a$ (i.e. $M$-arrow $m : \ast \rightarrow a$), the composition $m \circ f$ in $\cdot, A$ is given by that in $M$.

4. For any $(X, \cdot)$-arrow $m : x \rightarrow M$ (i.e. $M$-arrow $m : x \rightarrow \ast$) and any right module morphism $\psi : M \rightarrow M'$ : $X \rightarrow \ast$, the composition $m \circ \psi$ in $X^\cdot$ is given by the evaluation $m \circ \psi$, and for any $(\cdot, A)$-arrow $m : M \rightarrow a$ (i.e. $M$-arrow $m : \ast \rightarrow a$) and any left module morphism $\psi : M \rightarrow M'$ : $\ast \rightarrow A$, the composition $\psi \circ m$ (where $\psi$ denotes the opposite of $\psi : M \rightarrow M'$) in $\cdot, A$ is given by the evaluation $m \circ \psi$.
5.1.3 Proposition.  
- The right exponential transpose of the right Yoneda module \( (X^\circ) : X \to [X:] \) is the identity \([X:] \to [X:]\):
  \[
  [X:] \xrightarrow{(X^\circ)\circ = 1} [X:]
  \]
  ; hence the right slice of \(X^\circ\) at a right module \(M : X \to \ast \) is \(M\) itself:
  \[(X^\circ)(M) = M.\]

- The left exponential transpose of the left Yoneda module \((\ast, A) : [A]^{-} \to A\) is the identity \([A] \to [A]:\):
  \[
  [A] \xrightarrow{\ast (\ast, A) = 1} [A]
  \]
  ; hence the left slice of \(\ast, A\) at a left module \(M : \ast \to A\) is \(M\) itself:
  \[(M)(\ast, A) = M.\]

Proof. Immediate from the definition. \qed

5.1.4 Proposition. Given a module (resp. module morphism) \(M : X \to A\),
- the identity
  \[
  M = (X^\circ)[M^\circ]
  \]
  holds; that is, \(M\) is recovered from its right exponential transpose by the composition
  \[
  X \xrightarrow{X^\circ} [X:] \xrightarrow{M^\circ} A
  \]
  ; hence the right action of the right Yoneda module \(X^\circ\) on the functor category \([A,[X:]]\) yields the inverse of the right exponential transposition \([X:A] \xrightarrow{\ast} [A,[X:]]\); that is,
  \[
  [X:A] \xleftarrow{\ast(X^\circ)A \ast} [A,[X:]].
  \]

- the identity
  \[
  M = [\ast M](\ast, A)
  \]
  holds; that is, \(M\) is recovered from its left exponential transpose by the composition
  \[
  X \xrightarrow{\ast M} [:A]^{-} \xrightarrow{\ast A} A
  \]
  ; hence the left action of the left Yoneda module \(\ast, A\) on the functor category \([X,[:A]^{-}]\) yields the inverse of the left exponential transposition \([X:A]^{-} \xrightarrow{\ast} [:A]^{-}\); that is,
  \[
  [X:A]^{-} \xleftarrow{\ast(X, \ast)(\ast, A)} [:A]^{-}.
  \]

Proof. By the bijectivity of exponential transposition, it suffices to show that
  \[
  M^\circ = ((X^\circ)[M^\circ])^\circ
  \]
  . But by Proposition 2.1.6 and Proposition 5.1.3,
  \[
  ((X^\circ)[M^\circ])^\circ = ((X^\circ)^\circ \circ \circ [M^\circ] = M^\circ.
  \]

\qed

5.1.5 Remark. For a module \(M : X \to A\), the identity in Proposition 5.1.4 is expressed by the fully faithful cell
  \[
  X \xrightarrow{M} A \quad \text{op.} \quad X \xrightarrow{\ast M} A
  \]
  , which identifies each \(M\)-arrow \(m : x \to a\) with
  - the \((X^\circ)\)-arrow \(m : x \to (M)a\), i.e. the \((M)a\)-arrow \(m : x \to \ast\).
  - the \((\ast, A)\)-arrow \(m : x(M) \to a\), i.e. the \(x(M)\)-arrow \(m : \ast \to a.\)
Note. The compositions in Remark 4.3.6(3) and Remark 4.3.12(2) yield the module consisting of all right cylinders along all modules \( \mathbf{X} \to \mathbf{A} \); the functorialities in Proposition 4.1.7 and Remark 1.1.27 allow the following definition.

5.1.6 Definition. Given categories \( \mathbf{X} \) and \( \mathbf{A} \),
\begin{itemize}
\item the right generalized Yoneda module
\[ \langle \mathbf{X} \ast \mathbf{A} \rangle : [\mathbf{A}, \mathbf{X}] \to [\mathbf{X} : \mathbf{A}] \]
for \([\mathbf{A}, \mathbf{X}]\) is defined by
\[ (G) \langle \mathbf{X} \ast \mathbf{A} \rangle (M) = \prod_A G(M) \]
for \( G \in [\mathbf{A}, \mathbf{X}] \) and \( M \in [\mathbf{X} : \mathbf{A}] \).
\item the left generalized Yoneda module
\[ \langle \mathbf{X} \ast \mathbf{A} \rangle : [\mathbf{X} : \mathbf{A}]^\ast \to [\mathbf{X} : \mathbf{A}] \]
for \([\mathbf{X} : \mathbf{A}]\) is defined by
\[ (M) \langle \mathbf{X} \ast \mathbf{A} \rangle (F) = \prod_X (M) F \]
for \( M \in [\mathbf{X} : \mathbf{A}] \) and \( F \in [\mathbf{X} : \mathbf{A}] \).
\end{itemize}

5.1.7 Remark.
\begin{itemize}
\item[(1)] The module \( \mathbf{X} \ast \mathbf{A} \) \( \iff \mathbf{X} \ast \mathbf{A} \) is called the generalized Yoneda module just because it is represented \( \iff \) corepresented \( \iff \) by the generalized Yoneda functor \( \mathbf{X} \ast \mathbf{A} \) \( \iff \mathbf{X} \ast \mathbf{A} \) (see Theorem 5.3.19).
\item[(2)] For a functor \( G : \mathbf{A} \to \mathbf{X} \) and a module \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \), the set
\[ (G) \langle \mathbf{X} \ast \mathbf{A} \rangle (\mathcal{M}) = \prod_A G(\mathcal{M}) \]
consists of all frames of the composite endomodule \( G(\mathcal{M}) : \mathbf{A} \to \mathbf{A} \), i.e. all right cylinders \( G \sim \mathcal{M} \), and for a functor \( F : \mathbf{X} \to \mathbf{A} \) and a module \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \), the set
\[ (M) \langle \mathbf{X} \ast \mathbf{A} \rangle (F) = \prod_X (M) F \]
consists of all frames of the composite endomodule \( (\mathcal{M}) F : \mathbf{X} \to \mathbf{X} \), i.e. all left cylinders \( \mathcal{M} \sim F \).
\item[(3)] The composition in the module \( \langle \mathbf{X} \ast \mathbf{A} \rangle \) \( \iff \langle \mathbf{X} \ast \mathbf{A} \rangle \) is that given in Remark 4.3.6(3) and Remark 4.3.12(2); indeed, by definition, the compositions
\begin{align*}
\begin{array}{ccc}
\tau \circ \alpha & 
\overset{\alpha}{\Rightarrow} & \tau \circ \sigma \\
\downarrow \quad \tau & 
\overset{\alpha \circ \sigma}{\Rightarrow} & \downarrow \\
\mathcal{M} & 
\overset{\alpha \circ \sigma}{\Rightarrow} & \mathcal{M}'
\end{array}
\end{align*}
and
\begin{align*}
\begin{array}{ccc}
\tau \circ \alpha & 
\overset{\alpha \circ \sigma}{\Leftrightarrow} & \tau \circ \sigma \\
\downarrow \quad \alpha \circ \sigma & 
\overset{\psi \circ \delta \circ \alpha}{\Leftrightarrow} & \downarrow \\
\mathcal{M} & 
\overset{\psi \circ \delta \circ \alpha}{\Leftrightarrow} & \mathcal{M}'
\end{array}
\end{align*}
are in \( \langle \mathbf{X} \ast \mathbf{A} \rangle \) \( \iff \langle \mathbf{X} \ast \mathbf{A} \rangle \) is given by
\[ \tau \circ \alpha = \alpha \circ \tau (\mathcal{M}) \quad \text{and} \quad \alpha \circ \sigma = \alpha \circ \sigma (\mathcal{M}) \]
and
\[ \alpha \circ \sigma = \alpha \circ \sigma (\mathcal{M}) \quad \text{and} \quad \alpha \circ \sigma (\mathcal{M}) \]
\item[(4)] The identification in Remark 4.3.4(2) yields canonical isomorphisms
\[ \langle \mathbf{X} \ast \mathbf{A} \rangle \cong \langle \mathbf{X} \ast \mathbf{A} \ast \rangle \quad \text{and} \quad \langle \ast \mathbf{A} \rangle \cong \langle \ast \mathbf{A} \rangle \]
\end{itemize}

5.1.8 Proposition. Given a category \( \mathbf{E} \) and a module \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \), the identity
\[ \begin{array}{ccc}
\mathbf{E}, \mathbf{X} & \overset{\mathbf{E}, \mathbf{X}}{\Rightarrow} & \mathbf{E}, \mathbf{A} \\
\downarrow 1 & \quad \downarrow 1 & \quad \downarrow 1 \\
\mathbf{E}, \mathbf{X} & \overset{\mathbf{E}, \mathbf{X}}{\Rightarrow} & \mathbf{E}, \mathbf{A}
\end{array} \]
\[ \begin{array}{ccc}
\mathbf{E}, \mathbf{X} & \overset{\mathbf{E}, \mathbf{X}}{\Rightarrow} & \mathbf{E}, \mathbf{A} \\
\downarrow 1 & \quad \downarrow 1 & \quad \downarrow 1 \\
\mathbf{E}, \mathbf{X} & \overset{\mathbf{E}, \mathbf{X}}{\Rightarrow} & \mathbf{E}, \mathbf{A}
\end{array} \]
Yoneda modules are thus special instances of generalized Yoneda modules.
5.1. Yoneda modules

This is the identity stated in Remark 4.3.2(6): the fully faithful cell

\[
G(\mathbf{E}, \mathbf{M}) F = \prod_{\mathbf{E}} G(\mathbf{M}) F
\]

\[
= \prod_{\mathbf{E}} G(\langle \mathbf{M} \rangle F)
\]

\[
= G(\langle \mathbf{X} \rangle \mathbf{E}) (\langle \mathbf{M} \rangle F)
\]

\[
= G(\langle \mathbf{X} \rangle \mathbf{E})([\mathbf{M} \mathbf{E}] : F)
\]

\[
= G(\langle \mathbf{X} \rangle \mathbf{E})[\mathbf{M} \mathbf{E}] F.
\]

\[
\square
\]

5.1.9 Remark.

(1) This is the identity stated in Remark 4.3.2(6): the fully faithful cell \(\langle \mathbf{E}, \mathbf{M} \rangle \to \langle \mathbf{X} \rangle \mathbf{E} \mathbf{A} \) in Proposition 5.1.8 identifies each two-sided cylinder

\[
\xymatrix{ \mathbf{X} \ar[r]_{\alpha} & \mathbf{E} \ar@/^/[l]^{\mathbf{A}} \ar@/_/[r]_{F} & \mathbf{A} \ar@/^/[l]^{G} }\]

with the right [op. left] cylinder

\[
\xymatrix{ \mathbf{X} \ar[r]_{\alpha} & \mathbf{E} \ar@/^/[l]^{\mathbf{A}} \ar@/_/[r]_{F} & \mathbf{A} \ar@/^/[l]^{G(\mathbf{M}) F} \ar@/_/[r]_{\mathbf{A}} . \ar@/^/[l]^{\mathbf{E}} \ar@/_/[r]_{\alpha} & \mathbf{A} \ar@/^/[l]^{\mathbf{E}} \ar@/_/[r]_{\alpha} & }\]

(2) If we replace \(\mathbf{E}\) with the terminal category, then we obtain the cells in Remark 5.1.5.

5.1.10 Proposition. Given categories \(\mathbf{X}\) and \(\mathbf{A}\), the identity

\[
\begin{array}{c}
\xymatrix{ \mathbf{[A, X]} \ar[r]^{X \times \mathbf{A}} & \mathbf{[X : A]} & \mathbf{op.} & \xymatrix{ \mathbf{[X : A]} \ar[r]^{X \times \mathbf{A}} & \mathbf{[X, A]} & \mathbf{op.} & } \\
\mathbf{[A, X]} \ar[r]_{\langle X, \mathbf{X} \rangle} & \mathbf{[A, [X :]]} & \mathbf{op.} & \xymatrix{ \mathbf{[X, [A :]]} \ar[r]^{X \times \mathbf{A}} & \mathbf{[X, A]} & \mathbf{op.} & } \\
\end{array}
\]

holds.

Proof. Replacing \(\mathbf{M}\) in Proposition 5.1.8 with the right and left Yoneda modules, we have the identity

\[
\begin{array}{c}
\xymatrix{ \mathbf{[A, X]} \ar[r]^{\langle X, \mathbf{A} \rangle} & \mathbf{[A, [X :]]} & \mathbf{op.} & \xymatrix{ \mathbf{[X, [A :]]} \ar[r]^{\langle X, \mathbf{A} \rangle} & \mathbf{[X, A]} & \mathbf{op.} & } \\
\mathbf{[A, X]} \ar[r]_{\langle X, \mathbf{A} \rangle} & \mathbf{[A, [X :]]} & \mathbf{op.} & \xymatrix{ \mathbf{[X, [A :]]} \ar[r]^{\langle X, \mathbf{A} \rangle} & \mathbf{[X, A]} & \mathbf{op.} & } \\
\end{array}
\]

\]; since \(\langle X, \mathbf{A} \rangle \mathbf{A} = \mathbf{A}\) by Proposition 5.1.4, the assertion follows by taking the inverse. \(\square\)

5.1.11 Remark. The iso cell \(\langle X, \mathbf{A} \rangle \to \langle \mathbf{A}, \langle X, \mathbf{A} \rangle \rangle \mathbf{op.} (\langle X, \mathbf{A} \rangle \to \langle \mathbf{A}, \langle X, \mathbf{A} \rangle \rangle)\) identifies each right [op. left] cylinder

\[
\xymatrix{ \mathbf{X} \ar[r]_{\alpha} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{F} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{\alpha} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{\alpha} & }\]

with the two-sided cylinder

\[
\xymatrix{ \mathbf{X} \ar[r]_{\alpha} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{F} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{\alpha} & \mathbf{A} \ar@/^/[l]^{\mathbf{M}} \ar@/_/[r]_{\alpha} & }\]

Note. Recall from Remark 1.1.14(3) that a right module \([\mathbf{A, X}] \times [\mathbf{X, A}] \to \mathbf{A}\) is the same thing as a two-sided module \([\mathbf{A, X}] \to [\mathbf{X, A}]\).
5.1.12 Proposition. Given categories $X$ and $A$,

- the composition
  \[
  [A, X] \xrightarrow{X \cdot A} [X : A] \xrightarrow{X \cdot A} [A, A]^-
  \]
  of the right generalized Yoneda module for $[A, X]$ and the left generalized Yoneda functor for $[X, A]$ yields the same module as the composition
  \[
  [A, X] \times [X, A] \xrightarrow{[-, -]} [A, A] \xrightarrow{[A, A]} [A, A] \xrightarrow{1_A} *
  \]
  giving the identity
  \[
  (G) (X \cdot A)(F(A)) = (G \circ F)(A, A)(1_A)
  \]
  for any pair of functors $G : A \to X$ and $F : X \to A$.

- the composition
  \[
  [A, X]^\sim \xrightarrow{X \cdot A} [X : A]^\sim \xrightarrow{X \cdot A} [A, A]^\sim
  \]
  of the left generalized Yoneda module for $[X, A]$ and the right generalized Yoneda functor for $[A, X]$ yields the same module as the composition
  \[
  * \xrightarrow{1_X} [X, X] \xrightarrow{[-, -]} [X, X] \xrightarrow{[-, -]} [A, X] \times [X, A]
  \]
  giving the identity
  \[
  (X) G(X \cdot A)(F) = (1_X)(X, X)(G \circ F)
  \]
  for any pair of functors $G : A \to X$ and $F : X \to A$.

Proof. For any $G \in [A, X]$ and any $F \in [X, A]$, we have
\[
(G)(X \cdot A)(F(A)) = \prod_A (F(A)) = \prod_A [G \circ F](A)[1_A] = (G \circ F)(A, A)(1_A).
\]

5.1.13 Remark. These are the identities in Remark 4.3.4(3):

- a right cylinder $X \xrightarrow{\epsilon} A$ (an element of $(G)(X \cdot A)(F(A))$) is the same thing as a natural transformation $\epsilon : G \circ F \to 1_A$ (an element of $(G \circ F)(A, A)(1_A)$).

- a left cylinder $X \xrightarrow{\eta} A$ (an element of $(X) G(X \cdot A)(F)$) is the same thing as a natural transformation $\eta : 1_X \to G \circ F$ (an element of $(1_X)(X, X)(G \circ F)$).

5.2 Yoneda morphisms

The right Yoneda morphism for a module $\mathcal{M} : X \to A$ is a cell $\mathcal{M} \to \langle X \rangle$ that sends each $\mathcal{M}$-arrow $m : x \rightsquigarrow a$ to the right module morphism $X \downarrow m : (X) x \to (\mathcal{M}) a : X \to *$ from the representable right module of $x \in |X|$ to the right slice of $\mathcal{M}$ at $a \in |A|$. Theorem 5.2.10 shows that the Yoneda morphism is fully faithful, generalizing both the Yoneda lemma and the Yoneda embedding. If applied to the Yoneda module $\langle X \cdot A \rangle : X \to [A :]$ of $\mathcal{M}$, the Yoneda morphism yields a representation of it by the Yoneda functor $[X, \cdot A] : X \to [A :]$ (the Yoneda representation). Section 6.2 manifests the importance of Theorem 5.2.10: the Yoneda morphism establishes a link between the notion of a universal arrow and that of an isomorphism.

Note. The exponential transpose of the collage envelope (see Definition 3.1.18) of a module yields the following (the definition should be compared with Definition 2.3.1).

5.2.1 Definition. The right [op. left] Yoneda functor for a module $\mathcal{M} : X \to A$ is the functor
\[
\langle \smallint (\mathcal{M} \downarrow) \rangle : |\mathcal{M}| \to [X :] \quad \text{op.} \quad \langle \smallint (\mathcal{M} \uparrow) \rangle : |\mathcal{M}| \to [: A]^\sim
\]
given by the right [op. left] exponential transpose of the right [op. left] collage envelope
\[ \langle X \upharpoonright M \rangle : X \to [M] \quad \text{op.} \quad \langle M \upharpoonright A \rangle : [M] \to A \]
of \( M \).

5.2.2 Remark.
- By Remark 3.1.19(2), the right slices of \( \langle X \upharpoonright M \rangle \) at \( x \in \|X\| \) and \( a \in \|A\| \) are given by
  \[ \langle X \upharpoonright M \rangle x = (X \upharpoonright M) (M_0 \cdot x) = \langle (X \upharpoonright M) M_0 \rangle x = \langle X \rangle x \]
and
  \[ \langle X \upharpoonright M \rangle a = (X \upharpoonright M) (M_1 : a) = \langle (X \upharpoonright M) M_1 \rangle a = \langle M \rangle a \]
; hence the right Yoneda functor \( \langle X \upharpoonright M \rangle \) sends each \( M \)-arrow \( m : x \to a \) to the right module morphism
  \[ \langle X \upharpoonright M \rangle m : \langle X \rangle x \to \langle M \rangle a : X \to A \]
which maps each \( X \)-arrow \( h : x' \to x \) to the \( M \)-arrow \( h \circ m : x' \to a \) as indicated in

\[
\begin{array}{c}
\xymatrix{ x \ar@{~}[r]^m & a \\
\overset{h\circ m}{\Longrightarrow} & \ar@{~}[u]^{h \downarrow} \\
x' \ar@{~}[r]_{m = (X \upharpoonright M) m} & \langle X \upharpoonright M \rangle x \ar@{~}[u]^{\langle X \upharpoonright M \rangle h} \}
\end{array}
\]
(cf. Remark 2.1.2(1)). When \( M \) is understood, \( \langle X \upharpoonright M \rangle m \) is also written as
\[ X \upharpoonright m : \langle X \rangle x \to \langle M \rangle a : X \to A \]
and called the right module morphism generated by \( X \) direct along \( m \).
- By Remark 3.1.19(2), the left slices of \( \langle M \upharpoonright A \rangle \) at \( x \in \|X\| \) and \( a \in \|A\| \) are given by
  \[ x \langle M \upharpoonright A \rangle = (x : M_0) \langle M \upharpoonright A \rangle = x (M_0 \langle M \upharpoonright A \rangle) = x \langle M \rangle \]
and
  \[ a \langle M \upharpoonright A \rangle = (a : M_1) \langle M \upharpoonright A \rangle = a (M_1 \langle M \upharpoonright A \rangle) = a \langle A \rangle \]
; hence the left Yoneda functor \( \langle M \upharpoonright A \rangle \) sends each \( M \)-arrow \( m : x \to a \) to the left module morphism
  \[ m \langle M \upharpoonright A \rangle : a \langle A \rangle \to x \langle M \rangle : * \to A \]
which maps each \( A \)-arrow \( h : a \to a' \) to the \( M \)-arrow \( m \circ h : x \to a' \) as indicated in

\[
\begin{array}{c}
\xymatrix{ x \ar@{~}[r]^{m} & a \\
\ar@{~}[u]_{m = \langle M \upharpoonright A \rangle m} & \langle X \rangle x \ar@{~}[u]^{\langle X \rangle m} \\
\ar@{~}[r]_{h \downarrow} & \langle M \rangle x \ar@{~}[u]_{\langle M \rangle h} \}
\end{array}
\]
(cf. Remark 2.1.2(1)). When \( M \) is understood, \( m \langle M \upharpoonright A \rangle \) is also written as
\[ m \upharpoonright A : a \langle A \rangle \to x \langle M \rangle : * \to A \]
and called the left module morphism generated by \( A \) inverse along \( m \).

5.2.3 Definition. The right [op. left] Yoneda morphism for a module \( M : X \to A \) is the cell

\[
\begin{array}{c}
X \ar@{:>}[r]^{M} & A \\
\langle X \upharpoonright M \rangle \ar@{<:}[u]_{\langle X \upharpoonright M \rangle} & \langle M \rangle \ar@{<:}[u]_{M} \\
\langle X \rangle \ar@{<:}[u]_{\langle X \rangle} & \langle A \rangle \ar@{<:}[u]_{\langle A \rangle} \end{array}
\]

sending each \( M \)-arrow \( m : x \to a \) to the right [op. left] module morphism
\[ X \upharpoonright m = \langle X \upharpoonright M \rangle m : \langle X \rangle x \to \langle M \rangle a : X \to A \quad \text{op.} \quad m \upharpoonright A = m \langle M \upharpoonright A \rangle : a \langle A \rangle \to x \langle M \rangle : * \to A. \]
5.2.4 Remark.

(1) The cell $(X\uparrow \mathcal{M}) \xrightarrow{\text{op. }} \langle \mathcal{M} \downarrow \mathcal{A} \rangle$ is formally given by the adjunct (see Theorem 3.1.16) of the right [op. left] Yoneda functor for $\mathcal{M}$, i.e. by the composition

\[
\begin{align*}
\xymatrix{X \ar[r]^{\mathcal{M}} & \mathcal{A} \\
\mathcal{M}_0 \ar[r]^1 & \mathcal{M}_1 \ar@{<-}[l]_{\text{op. left}}}
\end{align*}
\]

\[
\begin{align*}
\langle \mathcal{M} \downarrow \mathcal{A} \rangle \ar[r]_{\text{op. left}} & \langle \mathcal{M} \rangle \\
\mathcal{M}_0 \ar[r]^1 & \mathcal{M}_1 \ar@{<-}[l]_{\text{op. left}}
\end{align*}
\]

of the unit cell of $\mathcal{M}$ and the hom-cell of the right [op. left] Yoneda functor for $\mathcal{M}$.

(2) • The Yoneda morphism for a right module $\mathcal{M} : X \to \ast$ is the right conic cell

\[
\begin{align*}
X \ar[r]^{\mathcal{M}} & \ast \\
x \ar[r] & \langle X \downarrow \mathcal{M} \rangle \\
[\mathcal{M}] \ar@{<-}[l]_{\text{op. left}} & [X] \ar@{<-}[l]_{\text{op. left}}
\end{align*}
\]

defined as a special case of Definition 5.2.3 where $\mathcal{A}$ is the terminal category under the identification $[X] \cong [X : \ast]$. This conic cell sends each $\mathcal{M}$-arrow $m : x \sim \ast$ to the right module morphism

\[
X \uparrow m : \langle X \rangle x \to \mathcal{M} : X \to \ast
\]

which maps each $X$-arrow $h : x' \to x$ to the $\mathcal{M}$-arrow $h \circ m : x' \sim \ast$ as indicated in

\[
\begin{array}{c}
x \ar[r]^m & \ast \\
h \ar[u]^x & h \ar[u]^m \circ \langle X \downarrow [X] \rangle
\end{array}
\]

Conversely, given an arrow $m : x \sim a$ of a two-sided module $\mathcal{M} : X \to \mathcal{A}$, the right module morphism

\[
X \uparrow m : \langle X \rangle x \to \langle \mathcal{M} \rangle a : X \to \ast
\]

coincides with that generated by $X$ direct along the arrow $m : x \sim \ast$ of the right module $\langle \mathcal{M} \rangle a : X \to \ast$.

• The Yoneda morphism for a left module $\mathcal{M} : \ast \to \mathcal{A}$ is the left conic cell

\[
\begin{align*}
\ast \ar[r]^{\mathcal{M}} & \mathcal{A} \\
\mathcal{M} \ar[r]^1 & \langle \mathcal{M} \rangle \ar@{<-}[l]_{\text{op. left}} \langle \mathcal{A} \rangle \\
\langle \mathcal{A} \rangle \ar[r]^1 & \langle \mathcal{A} \rangle \ar@{<-}[l]_{\text{op. left}}
\end{align*}
\]

defined as a special case of Definition 5.2.3 where $\mathcal{X}$ is the terminal category under the identification $[\mathcal{A}] \cong [\ast : \mathcal{A}]$. This conic cell sends each $\mathcal{M}$-arrow $m : x \sim a$ to the left module morphism

\[
m \uparrow \mathcal{A} : a \langle \mathcal{A} \rangle \to \mathcal{M} : \ast \to \mathcal{A}
\]

which maps each $\mathcal{A}$-arrow $h : a \to a'$ to the $\mathcal{M}$-arrow $m \circ h : \ast \sim a'$ as indicated in

\[
\begin{array}{c}
\ast \ar[r]^m & a \\
h \ar[u]^m \circ \langle \mathcal{M} \rangle h & a' \ar[u]^h \circ \langle \mathcal{M} \rangle a
\end{array}
\]

Conversely, given an arrow $m : x \sim a$ of a two-sided module $\mathcal{M} : X \to \mathcal{A}$, the left module morphism

\[
m \uparrow \mathcal{A} : a \langle \mathcal{A} \rangle \to x \langle \mathcal{M} \rangle : \ast \to \mathcal{A}
\]

coincides with that generated by $\mathcal{A}$ inverse along the arrow $m : \ast \sim a$ of the left module $x \langle \mathcal{M} \rangle : \ast \to \mathcal{A}$.

Note. Recall from Definition 2.1.8 that a cell is sliced into pieces of right [op. left] conic cells.

5.2.5 Proposition. Let $\mathcal{M} : X \to \mathcal{A}$ be a module.
5.2. Yoneda morphisms

The right slice

\[
\begin{array}{c}
X \rightarrow \langle M \rangle a \\
X \times \langle M \rangle \rightarrow \langle (M) a \rangle \\
[X:] \rightarrow \langle X \rangle[a]
\end{array}
\]

at \(a \in \|A\|\) of the right Yoneda morphism for \(M\) is given by the Yoneda morphism

\[
\begin{array}{c}
X \rightarrow \langle M \rangle a \\
X \times \langle (M) a \rangle \rightarrow \langle (M) a \rangle \\
[X:] \rightarrow \langle X \rangle[a]
\end{array}
\]

for the right module \(\langle M \rangle a\), the right slice of \(M\) at \(a\).

The left slice

\[
\begin{array}{c}
\ast \rightarrow \langle x(M) \rangle a \\
\langle x(M) \rangle \rightarrow \langle (x(M)) a \rangle \\
[\langle A \rangle] \rightarrow \langle A \rangle[a]
\end{array}
\]

at \(x \in \|X\|\) of the left Yoneda morphism for \(M\) is given by the Yoneda morphism

\[
\begin{array}{c}
\ast \rightarrow \langle x(M) \rangle a \\
\langle x(M) \rangle \rightarrow \langle (x(M)) a \rangle \\
[\langle A \rangle] \rightarrow \langle A \rangle[a]
\end{array}
\]

for the left module \(x\langle M \rangle\), the left slice of \(M\) at \(x\).

Proof. Immediate from Remark 5.2.4(2).

5.2.6 Proposition.

Given a category \(X\), the right Yoneda morphism

\[
\begin{array}{c}
X \rightarrow \langle X \rangle a \\
X \times \langle X \rangle \rightarrow \langle X \rangle a \\
[X:] \rightarrow \langle X \rangle[a]
\end{array}
\]

for the hom-module of \(X\) is the same thing as the hom-cell

\[
\begin{array}{c}
X \rightarrow \langle X \rangle a \\
X \times \langle X \rangle \rightarrow \langle X \rangle a \\
[X:] \rightarrow \langle X \rangle[a]
\end{array}
\]

of the right Yoneda functor for \(X\); that is, for any \(X\)-arrow \(f : x \rightarrow a\), \(Xf\) and \(\langle X \rangle f\) yield the same right module morphism

\[Xf = \langle X \rangle f : \langle X \rangle x \rightarrow \langle X \rangle a : X \rightarrow \ast\,.
\]

Given a category \(A\), the left Yoneda morphism

\[
\begin{array}{c}
A \rightarrow \langle A \rangle a \\
\langle A \rangle \rightarrow \langle (A) a \rangle \\
[\langle A \rangle] \rightarrow \langle A \rangle[a]
\end{array}
\]

for the hom-module of \(A\) is the same thing as the hom-cell

\[
\begin{array}{c}
A \rightarrow \langle A \rangle a \\
\langle A \rangle \rightarrow \langle (A) a \rangle \\
[\langle A \rangle] \rightarrow \langle A \rangle[a]
\end{array}
\]
of the left Yoneda functor for $A$; that is, for any $A$-arrow $f : x \to a$, $f \uparrow A$ and $f(A)$ yield the same left module morphism

\[ f \uparrow A = f(A) : a(A) \to x(A) : * \to A. \]

Proof. Both $X \uparrow f$ and $(X)f$ map each $X$-arrow $h : x' \to x$ to the $X$-arrow $h \circ f : x' \to a$ (see Remark 2.3.2). 

\[ \square \]

5.2.7 Example. 
(1) Consider a module and functors as in

\[ E \xrightarrow{G} X \xrightarrow{M} A \xleftarrow{F} D \]

- The right Yoneda morphism for the composite module $G(M) F : E \to D$ sends each $G(M) F$-arrow $m : e \to e'$ to the right module morphism

\[ E \uparrow m : (E)e \to (G(M) F)d : E \to * \]

which maps each $E$-arrow $h : e' \to e$ to the $G(M) F$-arrow $h \circ m : e' \to e$ as indicated in

\[ e \xrightarrow{h} e : G \xrightarrow{m} F : d \]

(cf. Example 1.1.31(1)); the commutativity of the triangle above for every $h \in (E)e$ translates into the commutativity of

\[ (E)e \xrightarrow{(G)e} (G(X) G)e = G(X) (G : e) \]

\[ (G(M) F)d \xrightarrow{G(M) (F : d)} G(M) (F : d) \]

- The left Yoneda morphism for the composite module $G(M) F : E \to D$ sends each $G(M) F$-arrow $m : e \to d$ to the left module morphism

\[ m \uparrow D : d(D) \to e(G(M) F) : * \to D \]

which maps each $D$-arrow $h : d \to d'$ to the $G(M) F$-arrow $m \circ h : e \to d'$ as indicated in

\[ e : G \xrightarrow{m} F : d \xrightarrow{d} d' \]

(cf. Example 1.1.31(1)); the commutativity of the triangle above for every $h \in d(D)$ translates into the commutativity of

\[ d(D) \xrightarrow{d(F)} d(A) F = (d : F) (A) F \]

\[ m \uparrow D : d(D) \xrightarrow{(m \uparrow D) : F} (e : G) (M) F \]

(2) As a special case of (1) above, consider a composable pair of a functor and a right [op. left] module as in

\[ E \xrightarrow{G} X \xrightarrow{M} * \xrightarrow{op.} * \xrightarrow{M} A \xleftarrow{F} E \]

- The Yoneda morphism for the composite right module $G(M) : E \to *$ sends each $G(M)$-arrow $m : e \to *$ (i.e. $M$-arrow $m : e : G \to *$) to the right module morphism

\[ E \uparrow m : (E)e \to G(M) : E \to * \]

which maps each $E$-arrow $h : e' \to e$ to the $G(M)$-arrow $h \circ m : e' \to *$ as indicated in

\[ e \xrightarrow{h} e : G \xrightarrow{m} * \]

\[ h \xrightarrow{h : G} h : (M) \]

\[ e' \xrightarrow{e'} e : G \xrightarrow{h : (M)} \]
As a special case of (1) above, for any functor $m : * \to e$ (i.e. $\mathcal{M}$-arrow $m : * \to F \cdot e$) to the left module morphism

$$m \upharpoonright E : e \langle E \rangle \to \langle \mathcal{M} \rangle F : * \to E$$

which maps each $E$-arrow $h : e \to e'$ to the $(\mathcal{M})$-arrow $m \circ h : * \to e'$ as indicated in

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{m} & \mathcal{M}
\end{array}
\end{array}$$

; the commutativity of the triangle above for every $h \in (\mathcal{E})$ translates into the commutativity of

$$\begin{array}{c}
\begin{array}{ccc}
\langle \mathcal{E} \rangle e & \xrightarrow{(G)e} & (G(\mathcal{X}) \mathcal{G}) e \\
\mathcal{E} \mathcal{M} & \xrightarrow{G \mathcal{M}} & \mathcal{G}(\mathcal{X}) \mathcal{G}(\mathcal{M})
\end{array}
\end{array}$$

(3) As a special case of (1) above, for any functor $K : \mathcal{D} \to \mathcal{E}$, consider its representable and corepresentable modules (see Definition 2.3.7).

- The right Yoneda morphism for the representable module $K(\mathcal{E}) : \mathcal{D} \to \mathcal{E}$ sends each $K(\mathcal{E})$-arrow $f : d \to e$ (i.e. $\mathcal{E}$-arrow $f : d \cdot K \to e$) to the right module morphism

$$D \upharpoonright f : (\mathcal{D}) d \to K(\mathcal{E}) e : D \to *$$

which maps each $\mathcal{D}$-arrow $h : d' \to d$ to the $K(\mathcal{E})$-arrow $h \circ f : d' \to e$ as indicated in

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{h} & \mathcal{D}
\end{array}
\end{array}$$

; the commutativity of the triangle above for every $h \in (\mathcal{D}) d$ translates into the commutativity of

$$\begin{array}{c}
\begin{array}{ccc}
\langle \mathcal{D} \rangle d & \xrightarrow{(K)d} & \langle K(\mathcal{E}) \mathcal{K} \rangle d \\
\mathcal{K} \mathcal{E} \mathcal{D} & \xrightarrow{K(\mathcal{E}) \mathcal{D}} & \mathcal{K}(\mathcal{E}) \mathcal{K}(\mathcal{D})
\end{array}
\end{array}$$

- The left Yoneda morphism for the corepresentable module $\langle E \rangle K : \mathcal{E} \to \mathcal{D}$ sends each $\langle E \rangle K$-arrow $f : e \to d$ (i.e. $\mathcal{D}$-arrow $f : e \to K \cdot d$) to the left module morphism

$$f \upharpoonright D : d \langle \mathcal{D} \rangle \to e \langle E \rangle K : * \to \mathcal{D}$$

which maps each $\mathcal{D}$-arrow $h : d \to d'$ to the $\langle E \rangle K$-arrow $f \circ h : e \to d'$ as indicated in

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D} & \xleftarrow{h} & \mathcal{D}
\end{array}
\end{array}$$

; the commutativity of the triangle above for every $h \in d \langle \mathcal{D} \rangle$ translates into the commutativity of

$$\begin{array}{c}
\begin{array}{ccc}
da \langle \mathcal{D} \rangle d & \xrightarrow{d(K)} & d(\langle E \rangle K) \\
\mathcal{E} \mathcal{D} & \xrightarrow{\mathcal{E} \mathcal{K}} & \mathcal{E}(\mathcal{D} \mathcal{K})
\end{array}
\end{array}$$

Note. We saw in Remark 5.2.4(2) that the Yoneda morphism transforms an arrow $m : x \to *$ of a
right module $\mathcal{M} : X \to \ast$ to the right module morphism $X \uparrow m : (X) x \to \mathcal{M} : X \to \ast$. The following crucial lemma in category theory asserts the bijectivity of this transformation.

### 5.2.8 Theorem. (Yoneda Lemma : Part one).

- The Yoneda morphism for a right module $\mathcal{M} : X \to \ast$ is fully faithful. Specifically, for each object $x \in |X|$, the assignment $m \mapsto X \uparrow m \equiv (X) x \mapsto (\mathcal{M}) x$ yields a bijection

$$x(\mathcal{M}) \cong ((X) x) (X : (\mathcal{M}))$$

from the set of right module morphisms $X \downarrow\to \mathcal{M}$, whose inverse sends each right module morphism $\theta : (X) x \to \mathcal{M}$ to the $\mathcal{M}$-arrow $1_x : \theta : x \to \ast$—the image of the identity $x \to x$ under the function $X(\theta) : x(X) x \to x(\mathcal{M})$.

- The Yoneda morphism for a left module $\mathcal{M} : \ast \to A$ is fully faithful. Specifically, for each object $a \in |A|$, the assignment $m \mapsto m \uparrow A \equiv (M) a \equiv \theta$ yields a bijection

$$(M) a \cong (a(A)) (A : (M) a)$$

from the set of left module morphisms $A \downarrow\to \mathcal{M}$, whose inverse sends each left module morphism $\theta : a(A) \to \mathcal{M}$ to the $\mathcal{M}$-arrow $\theta : \ast \to a$—the image of the identity $a \to a$ under the function $A(\theta) : a(A) a \to (M) a$.

**Proof.** Let $m : x \to \ast$ be an $\mathcal{M}$-arrow and $\theta : (X) x \to \mathcal{M}$ be a right module morphism. We need to show that $m = 1_x \cdot (X \downarrow m)$ and $\theta = X \uparrow (1_x \cdot \theta)$.

Replacing $h : x' \to x$ with $1_x : x \to x$ in the triangle in Remark 5.2.4(2), we have

$$x \xrightarrow{1_x} \ast \xleftarrow{1_x \cdot (X \downarrow m)} x, \quad i.e. \quad m = 1_x \cdot (X \downarrow m).$$

For any object $x' \in |X|$ and any arrow $h : x' \to x$, the commutative triangle

$$x \xrightarrow{1_x} \ast \xleftarrow{1_x \cdot \theta} x'$$

yields a commutative triangle

by the naturality of $\theta$; comparing this triangle with that in Remark 5.2.4(2), we have

$$\theta = X \uparrow (1_x \cdot \theta).$$

\[\blacksquare\]

### 5.2.9 Remark. The bijection $x(\mathcal{M}) \cong ((X) x) (X : (\mathcal{M}))$ is natural in $x$ because it is the component at $x$ of the Yoneda morphism. We will see in Theorem 5.2.19 that the bijection is also natural in $\mathcal{M}$.

### 5.2.10 Theorem. Let $\mathcal{M} : X \to A$ be a module.

- The right Yoneda morphism

$$X \xrightarrow{\mathcal{M}} A, \quad x \mapsto (X : (\mathcal{M})) \xrightarrow{\mathcal{M}} [X : \ast]$$

for $\mathcal{M}$ is fully faithful. Specifically, for each pair of objects $x \in |X|$ and $a \in |A|$, the assignment $m \mapsto X \uparrow m \equiv (M) a$ yields a bijection

$$x(\mathcal{M}) a \cong ((X) x) (X : ((M) a))$$
Recalling Proposition 5.2.6, we see that the assertion is a special case of Theorem 5.2.10.

Proof. Since a cell is fully faithful iff so is each right slice (see Proposition 2.1.10), by Proposition 5.2.5, the assertion is reduced to Theorem 5.2.8.

5.2.11 Remark. As we have just seen, Theorem 5.2.10 follows from the Yoneda lemma (Theorem 5.2.8). Conversely, the Yoneda lemma is a special case of Theorem 5.2.10 where $\mathcal{A}$ [op. $\mathcal{X}$] is the terminal category. We will see below that Theorem 5.2.10 is also a generalization of the Yoneda embedding.

5.2.12 Corollary. (Yoneda Embedding).

- For any category $\mathcal{X}$, the right Yoneda functor $[\mathcal{X} \dashv]: \mathcal{X} \to [\mathcal{X} :]$ is fully faithful. Specifically, for each pair of objects $x, a \in [\mathcal{X}]$, the assignment $f \mapsto ([\mathcal{X}] f)$ yields a bijection

\[
\left(\mathcal{X} a \cong \{a(\mathcal{A})\} : \mathcal{A}\right)(\mathcal{X} a)
\]

from the set of $\mathcal{X}$-arrows $x \to a$ to the set of right module morphisms $(\mathcal{X}) x \to (\mathcal{X}) a : \mathcal{X} \to \mathcal{X}$.

- For any category $\mathcal{A}$, the left Yoneda functor $[\dashv \mathcal{A}] : \mathcal{A} \to [\mathcal{A} :]$ is fully faithful. Specifically, for each pair of objects $x, a \in [\mathcal{A}]$, the assignment $f \mapsto f(\mathcal{A})$ yields a bijection

\[
\left(\mathcal{X} a \cong \{a(\mathcal{A})\} : \mathcal{A}\right)(\mathcal{X} a)
\]

from the set of $\mathcal{A}$-arrows $x \to a$ to the set of left module morphisms $a(\mathcal{A}) \to x(\mathcal{A}) : \mathcal{A} \to \mathcal{A}$.

Proof. Recalling Proposition 5.2.6, we see that the assertion is a special case of Theorem 5.2.10 where $\mathcal{M}$ is given by the hom-module of $\mathcal{X}$.

5.2.13 Corollary. Let $K : \mathcal{D} \to \mathcal{E}$ be a functor.

- The assignment $f \mapsto \mathcal{D} f$ yields a bijection

\[
(\mathcal{D} : \mathcal{E}) (e) \cong \{(\mathcal{D} d) (\mathcal{E} e)\}
\]

, natural in $d \in [\mathcal{D}]$ and $e \in [\mathcal{E}]$, from the set of $\mathcal{D}$-arrows $d : \mathcal{K} \to e$ to the set of right module morphisms $\mathcal{D} d \to K(E) e : \mathcal{D} \to \mathcal{E}$, whose inverse sends each right module morphism $\theta : \mathcal{D} d \to K(E) e$ to the $\mathcal{E}$-arrow $1_a : \theta d : \mathcal{K} \to e$ — the image of the identity $d \to d$ under the function $d(\theta) : d(\mathcal{D} d) \to (d : \mathcal{K}) (\mathcal{E} e)$. 

Note. If $\mathcal{M}$ in Theorem 5.2.10 is given by a representable module as in Example 5.2.7(3), the theorem reads as follows.
5.2. Yoneda morphisms

- The assignment \( f \mapsto f|_D \) yields a bijection
  \[
  (e)(E)(K;d) \cong (d(D))(e(E)K)
  \]
  natural in \( d \in \|D\| \) and \( e \in \|E\| \), from the set of \( E\)-arrows \( e \to K;d \) to the set of left module morphisms \( d(D) \to e(E)K \), whose inverse sends each left module morphism \( \theta : d(D) \to e(E)K \) to the \( E\)-arrow \( \theta \cdot 1_d : e \to K;d \) — the image of the identity \( d \to d \) under the function \( \theta \).

Proof. By replacing \( M : X \to A \) in Theorem 5.2.10 with the representable module \( K(E) : D \to E \), we have the bijection
  \[
  (d'K)(E)(e) \cong d(K(E))e \cong (d(D)d)(K(E)e)
  \]
  natural in \( d \in \|D\| \) and \( e \in \|E\| \). \( \square \)

5.2.14 Remark. Corollary 5.2.13 is a special case of Theorem 5.2.10 where \( M \) is representable, and the Yoneda embedding (Corollary 5.2.12) is a special case of Theorem 5.2.10 where \( M \) is given by the hom-module of a category. The Yoneda embedding may also be seen as a special case of Corollary 5.2.13 where \( F \) is an identity functor.

Note. The following shows that the right Yoneda module \( X.\ast \) introduced in Definition 5.1.1 is represented by the right Yoneda functor \( X.\ast ; \) the representation in fact incorporates Yoneda morphisms \( (X : M)\ast \) (see Remark 5.2.4(2)) for all right modules \( M : X \to \ast \).

5.2.15 Theorem.

- Given a category \( X \), the right Yoneda morphism
  \[
  X - \xrightarrow{X.\ast} \hat{[X :]}
  \]
  \[
  X.\ast(\hat{(X:\ast)})\ast = \ast
  \]
  \[
  [X :] - \xrightarrow{X.\ast} \hat{[X :]}
  \]
  for the right Yoneda module \( X.\ast \) yields a representation
  \[
  (X.\ast) \cong \hat{[X.\ast]}(\hat{[X :]}) : X \to \hat{[X :]}
  \]
  of the right Yoneda module \( X.\ast \) by the right Yoneda functor \( X.\ast \); the right slice of the representation cell at a right module \( M : X \to \ast \) is exactly the Yoneda morphism for \( M \); that is, the cell sends each \( X.\ast \)-arrow \( m : x \to M \) (i.e. \( M\)-arrow \( m : x \to \ast \)) to the right module morphism \( X \upharpoonright m : (X)x \to M \).

- Given a category \( A \), the left Yoneda morphism
  \[
  [: A] - \xrightarrow{\ast A} \hat{[A :]}
  \]
  \[
  \times(\times A) = 1 \quad \times((\times A)A) = \times A
  \]
  \[
  [: A] - \xrightarrow{\ast A} \hat{[A :]}
  \]
  for the left Yoneda module \( \ast A \) yields a corepresentation
  \[
  (\ast A) \cong \hat{[A :]}(\times A) : [A] \to \hat{[A :]}
  \]
  of the left Yoneda module \( \ast A \) by the left Yoneda functor \( \ast A \); the left slice of the corepresentation cell at a left module \( M : \ast \to A \) is exactly the Yoneda morphism for \( M \); that is, the cell sends each \( \ast A \)-arrow \( m : M \to a \) (i.e. \( M\)-arrow \( m : \ast \to a \)) to the left module morphism \( m \upharpoonright A : a(A) \to M \).

Proof. See Proposition 5.1.3 for the identity \( (X.\ast)\ast = 1 \). The first assertion now follows from the fully faithfulness of the Yoneda morphism (Theorem 5.2.10). By Proposition 5.2.5 and Proposition 5.1.3, the right slice of \( (X \upharpoonright (X.\ast))(\ast M) \) at \( M \) is given by
  \[
  (X \upharpoonright (X.\ast))(\ast M) = (X \upharpoonright ((X.\ast)(M)))(\ast M) = (X \upharpoonright M)(\ast M)
  \]
  i.e. by the Yoneda morphism for \( M \). \( \square \)
5.2.16 Definition. The representation \([\text{op. corepresentation}]\) in Theorem 5.2.15 is called the Yoneda representation \([\text{op. corepresentation}]\) and denoted by

\[
\begin{align*}
X & \rightarrow X^* \rightarrow [X : \cdot] & \text{op.} & \quad [A] \rightarrow [A]^{\ast} \rightarrow A \\
\text{X} & \quad \text{X}^* & \rightarrow & \quad \cdot & \rightarrow & \quad [X : \cdot] & \rightarrow & \quad [A] & \rightarrow & \quad [A]^{\ast} \rightarrow & \quad A
\end{align*}
\]

5.2.17 Remark. Each component of the Yoneda representation \([\text{op. corepresentation}]\) gives the bijection

\[
(x) (X \rightarrow (M)) : (x) (X \rightarrow (M)) \cong ((x) (X) (X)) (M)
\]

in Theorem 5.2.8.

5.2.18 Proposition. \([\text{Yoneda Lemma : Part two}]\). The bijection in Theorem 5.2.8 is natural in \([\text{op. a}]\) and \([M]\).

\[
\begin{align*}
\text{For any } X\text{-arrow } g : x' \rightarrow x, \text{ any } M\text{-arrow } m : x \rightarrow *, \text{ and any right module morphism } \psi : M \rightarrow M' : X \rightarrow *; & \\
X^\rightarrow (g \circ m) = (X) g \circ (X \rightarrow m) \quad \text{and} \quad X^\rightarrow (m \circ \psi) = (X \rightarrow m) \circ \psi.
\end{align*}
\]

\[
\begin{align*}
\text{For any } A\text{-arrow } f : a \rightarrow a', \text{ any } M\text{-arrow } m : * \rightarrow a, \text{ and any left module morphism } \psi : M \rightarrow M' : * \rightarrow A; & \\
(m \circ f) \to A = (m \to A) \circ f (A) \quad \text{and} \quad (m \circ \psi) \to A = \psi \circ (m \to A).
\end{align*}
\]

Proof. Recalling (Remark 5.1.2(3), (4)) the composition in the Yoneda module, we see that this just expresses the naturality of the Yoneda representation \(X^\rightarrow\).

5.2.19 Theorem. \([\text{Yoneda Lemma : Part two}]\). The bijection in Theorem 5.2.8 is natural in \([\text{op. a}]\) and \([M]\).

\[
\begin{align*}
\text{Given a composable pair of a functor and a right module as in } & \\
E \xrightarrow{G} X \xrightarrow{\cdot} * \\
, \text{consider a } G(X)\text{-arrow } g : e \rightarrow x \text{ (i.e. } X\text{-arrow } g : e : G \rightarrow x \text{) and an } M\text{-arrow } m : x \rightarrow *. \text{ Then for the composite } G(M)\text{-arrow } g \circ m : e \rightarrow * \text{ (i.e. } M\text{-arrow } g \circ m : e : G \rightarrow *), \text{ we have } & \\
E^\rightarrow (g \circ m) = (E \rightarrow g) \circ G(X \rightarrow m) & \\
; \text{ that is, the right module morphism } E^\rightarrow (g \circ m) : (E) e \rightarrow G(M) \text{ is given by the composition } & \\
\langle E \rangle e \xrightarrow{E \rightarrow g} \langle G(X) \rangle x = G(\langle X \rangle x) \xrightarrow{G(X \rightarrow m)} G(M) & \\
\end{align*}
\]

\[
\begin{align*}
\text{Given a composable pair of a functor and a left module as in } & \\
* \xrightarrow{\cdot} A \xrightarrow{F} E \\
, \text{consider an } (A)\text{-arrow } f : a \rightarrow e \text{ (i.e. } A\text{-arrow } f : a : F \rightarrow e \text{) and an } M\text{-arrow } m : * \rightarrow a. \text{ Then for the composite } (M)\text{-arrow } m \circ f : * \rightarrow e \text{ (i.e. } M\text{-arrow } m \circ f : * : F \rightarrow e), \text{ we have } & \\
(m \circ f) \to E = (f \to E) \circ (m \to A) F & \\
; \text{ that is, the right module morphism } (m \circ f) \to E : e (E) \rightarrow (M)F \text{ is given by the composition } & \\
e (E) \xrightarrow{f \to E} a (\langle A \rangle F) = (a \langle A \rangle) F \xrightarrow{(m \to A)F} (M)F.
\end{align*}
\]
Similarly, it sends a right cylinder \( \mathbf{E} \downarrow (g \circ m) \).

Consider a cylinder \( E \downarrow \alpha \)

\[
\begin{align*}
E & \downarrow (g \circ m) = \langle G \rangle e \circ G \langle X \downarrow (g \circ m) \rangle \\
& = \langle G \rangle e \circ G \langle (X)g \circ (X \downarrow m) \rangle \\
& = \langle G \rangle e \circ G \langle X \rangle g \circ G \langle X \downarrow m \rangle \\
& = \langle E \rangle g \circ G \langle X \downarrow m \rangle \quad (*^3)
\end{align*}
\]

\( (*^1 \) by Example 5.2.7(2), \( *^2 \) by Proposition 5.2.18, \( *^3 \) by Example 5.2.7(3)).

\[\square\]

### 5.3 Yoneda morphisms for cylinders

In this section, we generalize the notions and results in Section 5.2 to cylinders. The right Yoneda morphism sends a cylinder \( \langle X \rangle G \rightarrow \langle M \rangle F : X \rightarrow E \).

Similarly, it sends a right cylinder \( \langle X \rangle G \rightarrow \langle M \rangle F : X \rightarrow A \).

The generalized Yoneda lemma states that this correspondence is bijective: the morphisms from the corepresentable module \( \langle X \rangle G : X \rightarrow A \) to an arbitrary module \( M : X \rightarrow A \) correspond one-to-one with the right cylinders from \( G \) to \( M \). Using the lemma, we establish a variety of bijective correspondences between frames and cells in Section 5.5. Later in Section 6.3 and Section 6.5, the Yoneda morphism for one-sided cylinders and that for two-sided cylinders are used respectively to define a unit of a two-sided module and to characterize a pointwise lift. Theorem 5.3.24, which is a special case of the generalized Yoneda lemma where \( M : X \rightarrow A \) is a representable module, plays an important role in Section 7.3, allowing the definition of the unit and counit of an adjunction.

**Note.** The action of the collage envelope (see Definition 3.1.18) of a module yields the following (the definition should be compared with Definition 2.3.10).

**5.3.1 Definition.** Given a category \( \mathbf{E} \) and a module \( M : X \rightarrow A \),

- the right action

\[
\langle X \rangle M \triangleright E : \mathbf{E}([E, [M]]) \rightarrow \mathbf{E} \quad
\]

of the right collage envelope of \( M \) on the functor category \([\mathbf{E}, [M]]\) is called the right generalized Yoneda functor for \( (\mathbf{E}, M) \).

- the left action

\[
\mathbf{E} \ll \langle M \rangle A : \mathbf{E}([E, [M]]) \rightarrow \mathbf{E} \quad
\]

of the left collage envelope of \( M \) on the functor category \([\mathbf{E}, [M]]\) is called the left generalized Yoneda functor for \( (\mathbf{E}, M) \).

**5.3.2 Remark.** Consider a cylinder \( \begin{xy}
 0;0;<500,0>;<0,500>::\xymatrix{ G \ar@{-}^{\alpha} & \mathbf{E} \ar@{-}^{F} & A \\
 0;0;<500,0>;<0,500>::X \ar@{-}^{M} & A \\
 0;0;<500,0>;<0,500>::M \ar@{-}^{E} \ar@{-}^{F} & A \\
 \end{xy} \) (see Remark 4.3.2(2)).

- By Remark 3.1.19(2), the module \( \langle X \rangle M \) acts on \( M_0 \circ G \) and \( M_1 \circ F \) and yields

\[
\langle X \rangle M \circ [M_0 \circ G] = \langle (X \rangle M \rangle M_0 \rangle G = \langle X \rangle G
\]

and

\[
\langle X \rangle M \circ [M_1 \circ F] = \langle (X \rangle M \rangle M_1 \rangle F = \langle M \rangle F
\]

; hence \( \langle X \rangle M \) acts on \( \alpha \) and gives the module morphism

\[
\langle X \rangle M \alpha : \langle X \rangle G \rightarrow \langle M \rangle F : X \rightarrow \mathbf{E}
\]
which maps each \((X)\) \(G\)-arrow \(h : x \to e\) to the \((M)\) \(F\)-arrow \(h \circ \alpha : x \to e\) as indicated in
\[
\begin{array}{c}
e : G \\ h \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
5.3.5 Example.

(1) • Given a right cylinder \( X \xrightarrow{\alpha} A \), i.e. a two-sided cylinder \( X \xrightarrow{\alpha} A \) \( \xleftarrow{\beta} \), the category \( X \) acts on \( \alpha \) and generates a module morphism

\[
X \uparrow \alpha = (X \uparrow \mathcal{M}) \alpha : (X) G \rightarrow \mathcal{M} : X \rightarrow A
\]

direct along \( \alpha \), mapping each \((X) G\)-arrow \( h : x \rightarrow a \) to the \( \mathcal{M}\)-arrow \( h \circ \alpha : x \rightarrow a \) as indicated in

\[
\xymatrix{ a : G \ar[r]^\alpha & a \ar[d]_h \cr & h : (X \uparrow \alpha) \ar[l]_\alpha \ar[u]^h }
\]

and the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha (\mathcal{M} \uparrow A) : (A) \rightarrow G (\mathcal{M}) : A \rightarrow A
\]

inverse along \( \alpha \), mapping each \( A\)-arrow \( h : a \rightarrow b \) to the \( G(\mathcal{M})\)-arrow \( \alpha_a \circ h : a \rightarrow b \) as indicated in

\[
\xymatrix{ a : G \ar[r]^\alpha & a \ar[d]_h \cr & h : (\alpha \uparrow A) \ar[l]_\alpha \ar[u]^h }
\]

• Given a left cylinder \( X \xleftarrow{\alpha} A \), i.e. a two-sided cylinder \( X \xleftarrow{\alpha} A \) \( \xrightarrow{\beta} \), the category \( A \) acts on \( \alpha \) and generates a module morphism

\[
\alpha \uparrow A = \alpha (\mathcal{M} \uparrow A) : F (A) \rightarrow \mathcal{M} : X \rightarrow A
\]

inverse along \( \alpha \), mapping each \( F(A)\)-arrow \( h : x \rightarrow a \) to the \( \mathcal{M}\)-arrow \( \alpha_x \circ h : x \rightarrow a \) as indicated in

\[
\xymatrix{ x : F \ar[r]^{\alpha_x} & x \ar[d]_h \cr & h : (\alpha \uparrow A) \ar[l]_\alpha \ar[u]^h }
\]

and the category \( X \) acts on \( \alpha \) and generates a module morphism

\[
X \uparrow \alpha = (X \uparrow \mathcal{M}) \alpha : (X) F \rightarrow \mathcal{M} : X \rightarrow X
\]

direct along \( \alpha \), mapping each \( X\)-arrow \( h : y \rightarrow x \) to the \( \mathcal{M}\)-arrow \( h \circ \alpha : y \rightarrow x \) as indicated in

\[
\xymatrix{ x : F \ar[r]^{\alpha_x} & x \ar[d]_h \cr & h : (X \uparrow \alpha) \ar[l]_\alpha \ar[u]^h }
\]

(2) Consider a pair of functors \( X \xrightarrow{G} A \).

• Given a natural transformation \( \epsilon : G \circ F \rightarrow 1_A : A \rightarrow A \), i.e. a right cylinder \( X \xrightarrow{G} A \) \( \xleftarrow{\epsilon} \) (see Remark 4.3.4(3)), the category \( X \) acts on \( \epsilon \) and generates a module morphism

\[
X \uparrow \epsilon = (X \uparrow (F(A))) \epsilon : (X) G \rightarrow F (A) : X \rightarrow A
\]

direct along \( \epsilon \), mapping each \((X) G\)-arrow \( h : x \rightarrow a \) to the \( F(A)\)-arrow \( h \circ \epsilon : x \rightarrow a \) as indicated in

\[
\xymatrix{ G \ar@{-->}[r]^\epsilon & A \ar[d]^h \cr & h : F \ar[l]_{\epsilon} \ar[r]_{h : (X \uparrow \epsilon)} & A }
\]
, and the category $\mathbf{A}$ acts on $\epsilon$ and generates a module morphism

$$\epsilon \downarrow \mathbf{A} = \epsilon \downarrow \langle F(\mathbf{A}) \uparrow \mathbf{A} \rangle : \langle \mathbf{A} \rangle \to G \langle F(\mathbf{A}) \rangle : \mathbf{A} \to \mathbf{A}$$

inverse along $\epsilon$, mapping each $\mathbf{A}$-arrow $h : a \to b$ to the $G \langle F(\mathbf{A}) \rangle$-arrow $\epsilon_a \circ h : a \to b$ as indicated in

$$\begin{array}{c}
\begin{array}{c}
 a \xrightarrow{\epsilon_a} a \\
\downarrow h
\end{array} \\
\begin{array}{c}
 \epsilon \downarrow \mathbf{A} \\
\downarrow h
\end{array}
\end{array}$$

; the action $\epsilon \downarrow \mathbf{A}$ of $\mathbf{A}$ on a right cylinder $\underline{\mathbf{X}} \xrightarrow{\mathbf{G}} \mathbf{A}$ is the same thing as the action $\epsilon \downarrow \langle \mathbf{A} \rangle$ (see Remark 2.3.11(1)) of $\mathbf{A}$ on a natural transformation $\epsilon : \mathbf{G} \circ \mathbf{F} \to 1_\mathbf{A} : \mathbf{A} \to \mathbf{A}$: the diagram

$$\begin{array}{c}
\begin{array}{c}
\epsilon \downarrow \mathbf{A} \\
\downarrow \epsilon(\mathbf{A})
\end{array} \\
\begin{array}{c}
 G \langle F(\mathbf{A}) \rangle = [G \circ \mathbf{F}] \langle \mathbf{A} \rangle
\end{array}
\end{array}$$

commutes.

• Given a natural transformation $\eta : 1_\mathbf{X} \to \mathbf{G} \circ \mathbf{F} : \mathbf{X} \to \mathbf{X}$, i.e. a left cylinder $\underline{\mathbf{X}} \xleftarrow{\mathbf{F} \eta} \mathbf{A}$ (see Remark 4.3.4(3)), the category $\mathbf{A}$ acts on $\eta$ and generates a module morphism

$$\eta \downarrow \mathbf{A} = \eta \downarrow \langle (\mathbf{X}) \mathbf{G} \uparrow \mathbf{A} \rangle : F(\mathbf{A}) \to \langle (\mathbf{X}) \mathbf{G} \rangle : \mathbf{X} \to \mathbf{A}$$

inverse along $\eta$, mapping each $F(\mathbf{A})$-arrow $h : x \to a$ to the $\langle (\mathbf{X}) \mathbf{G} \rangle$-arrow $\eta_a \circ h : x \to a$ as indicated in

$$\begin{array}{c}
\begin{array}{c}
 x \xrightarrow{\eta_x} G \circ \mathbf{F} : x \\
\downarrow h
\end{array} \\
\begin{array}{c}
 \eta \downarrow \mathbf{A} \\
\downarrow \epsilon(\mathbf{A})
\end{array}
\end{array}$$

, and the category $\mathbf{X}$ acts on $\eta$ and generates a module morphism

$$\underline{\mathbf{X}} \uparrow \eta = \langle \mathbf{X} \uparrow \mathbf{M} \rangle \eta : \langle \mathbf{X} \rangle \to \langle (\mathbf{X}) \mathbf{G} \rangle \mathbf{F} : \mathbf{X} \to \mathbf{X}$$

direct along $\eta$, mapping each $\mathbf{X}$-arrow $h : y \to x$ to the $\langle (\mathbf{X}) \mathbf{G} \rangle \mathbf{F}$-arrow $\eta_x \circ h : y \to x$ as indicated in

$$\begin{array}{c}
\begin{array}{c}
 x \xrightarrow{\eta_x} G \circ \mathbf{F} : x \\
\downarrow h \\
 y
\end{array} \\
\begin{array}{c}
 h : \langle \mathbf{X} \uparrow \eta \rangle
\end{array}
\end{array}$$

; the action $\underline{\mathbf{X}} \uparrow \eta$ of $\mathbf{X}$ on a left cylinder $\underline{\mathbf{X}} \xleftarrow{\mathbf{F} \eta} \mathbf{A}$ is the same thing as the action $\langle \mathbf{X} \rangle \eta$ (see Remark 2.3.11(1)) of $\mathbf{X}$ on a natural transformation $\eta : 1_\mathbf{X} \to \mathbf{G} \circ \mathbf{F} : \mathbf{X} \to \mathbf{X}$: the diagram

$$\begin{array}{c}
\begin{array}{c}
 \langle \mathbf{X} \rangle \uparrow \eta \\
\downarrow \langle \mathbf{X} \rangle \eta
\end{array} \\
\begin{array}{c}
 \langle \mathbf{X} \rangle \ [G \circ \mathbf{F}] = \langle (\mathbf{X}) \mathbf{G} \rangle \mathbf{F}
\end{array}
\end{array}$$

commutes.

(3) • Given a right $K$-weighted cylinder $\underline{\mathbf{D}} \xrightarrow{K} \mathbf{E}$, i.e. a right cylinder $\underline{\mathbf{D}} \xrightarrow{K \alpha} \mathbf{E}$ (see Remark 4.5.2(1)), the category $\mathbf{D}$ acts on $\alpha$ and generates a module morphism

$$\underline{\mathbf{D}} \uparrow \alpha = \langle \underline{\mathbf{D}} \uparrow \langle \mathbf{G}(\mathbf{M}) \mathbf{F} \rangle \rangle \alpha : \langle \mathbf{D} \rangle \mathbf{K} \to \mathbf{G}(\mathbf{M}) \mathbf{F} : \mathbf{D} \to \mathbf{E}$$
Given a left $\mathcal{K}$-weighted cylinder $\mathcal{E} \xrightarrow{\mathcal{K}} \mathcal{D}$, i.e. a left cylinder $\mathcal{E} \xrightarrow{\mathcal{K} \mathcal{D}} \mathcal{D}$ (see Remark 4.5.2(1)), the category $\mathcal{D}$ acts on $\alpha$ and generates a module morphism

$$\alpha \mathcal{D} = \alpha (\langle \mathcal{G} \mathcal{M} \rangle \mathcal{F} \mathcal{D}) : \mathcal{K} \mathcal{D} \to \mathcal{G} \mathcal{M} \mathcal{F} \mathcal{E} \to \mathcal{D}$$

, i.e. a cell

$$\begin{array}{c}
\mathcal{E} \xrightarrow{\mathcal{K}} \mathcal{D} \\
\mathcal{G} \xrightarrow{\alpha} \mathcal{D} \\
\mathcal{X} \xrightarrow{\mathcal{M}} \mathcal{A}
\end{array}$$

inverse along $\alpha$, mapping each $\mathcal{K}(\mathcal{D})$-arrow $\mathcal{h} : \mathcal{e} \sim \mathcal{d}$ to the $\mathcal{M}$-arrow $\mathcal{h} : \mathcal{e} \sim \mathcal{F} \mathcal{e}$ as indicated in

$$\begin{array}{c}
\mathcal{e} \mathcal{K} \\
\mathcal{h} \mathcal{G} \\
\mathcal{d} \mathcal{F} \mathcal{h}
\end{array}$$

(4) Given a cone $\alpha : \mathcal{X} \sim \mathcal{F} : \mathcal{E} \sim \mathcal{M}$, i.e. a cylinder $\mathcal{X} \xrightarrow{\mathcal{F} \mathcal{E}} \mathcal{M}$, the category $\mathcal{X}$ acts on $\alpha$ and generates a wedge

$$\mathcal{X} \uparrow \alpha = \langle \mathcal{X} \uparrow \mathcal{M} \rangle \alpha : \langle \mathcal{X} \rangle \mathcal{X} \sim \langle \mathcal{M} \rangle \mathcal{F} : \mathcal{X} \sim \mathcal{E}$$

direct along $\alpha$, mapping each $\mathcal{X}$-arrow $\mathcal{h} : \mathcal{x} \sim \mathcal{x}'$ to the $\langle \mathcal{M} \rangle$-arrow $\mathcal{h} : \mathcal{x} \sim \mathcal{F} \mathcal{x}$ as indicated in

$$\begin{array}{c}
\mathcal{x} \mathcal{F} \mathcal{x} \\
\mathcal{h} \mathcal{h} \mathcal{x}'
\end{array}$$

for each $\mathcal{e} \in \mathcal{E}$.

Given a cone $\alpha : \mathcal{G} \sim \mathcal{A} : \mathcal{E} \sim \mathcal{M}$, i.e. a cylinder $\mathcal{E} \xrightarrow{\mathcal{G} \mathcal{A}} \mathcal{M}$, the category $\mathcal{A}$ acts on $\alpha$ and generates a wedge

$$\alpha \mathcal{A} = \alpha \langle \mathcal{M} \mathcal{A} \rangle \alpha : \langle \mathcal{M} \rangle \mathcal{A} \sim \mathcal{G} \mathcal{M} \mathcal{E} \sim \mathcal{A}$$

inverse along $\alpha$, mapping each $\mathcal{A}$-arrow $\mathcal{a} \sim \mathcal{a}'$ to the $\langle \mathcal{M} \rangle$-arrow $\mathcal{a} \sim \mathcal{F} \mathcal{a}$ as indicated in

$$\begin{array}{c}
\mathcal{e} \mathcal{G} \mathcal{a} \\
\mathcal{h} \mathcal{h} \mathcal{a}'
\end{array}$$

for each $\mathcal{e} \in \mathcal{E}$.
5.3.6 Proposition. For any right \([\text{op. left}]\) cylinder

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow G & & \downarrow M \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\downarrow F & & \downarrow M \\
\end{array}
\]

, the triangle

\[
\begin{array}{ccc}
G(X) G & \xrightarrow{(G)} & \langle A \rangle \\
\downarrow G(X\alpha) & & \downarrow \alpha|A \\
G(M) & \xrightarrow{\alpha|A} & \langle A \rangle \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
\langle X \rangle & \xrightarrow{(F)} & F\langle A \rangle F \\
\downarrow (X\alpha) & & \downarrow (\alpha|A)F \\
\langle M \rangle F & \xrightarrow{(\alpha|A)F} & \langle M \rangle F \\
\end{array}
\]

commutes.

Proof. We need to show that the identity

\[ h \circ G : \langle X \alpha \rangle = \langle \alpha|A \rangle \circ h \]

holds for any \(A\)-arrow \(h : a \to b\). But since the triangles

\[
\begin{array}{ccc}
a \circ G & \xrightarrow{\alpha|a} & a \\
\downarrow h \circ G & & \downarrow h \\
b \circ G & \xrightarrow{\alpha|b} & b \\
\end{array}
\]

commute (see Example 5.3.5(1)), \(h \circ G : \langle X \alpha \rangle\) and \(\langle \alpha|A \rangle \circ h\) are identical, being the diagonal of the naturality square

\[
\begin{array}{ccc}
a \circ G & \xrightarrow{\alpha|a} & a \\
\downarrow h \circ G & & \downarrow h \\
b \circ G & \xrightarrow{\alpha|b} & b \\
\end{array}
\]

Note. The following is a special case of Proposition 5.3.6 where \(M\) is given by a representable \([\text{op. corepresentable}]\) module.

5.3.7 Proposition. Consider a pair of functors \(X \xrightarrow{G F} A\). Given a natural transformation

\[
\epsilon : G \circ F \to 1_A : A \to A \quad \text{op.} \quad \eta : 1_X \to G \circ F : X \to X
\]

, the diagram

\[
\begin{array}{ccc}
G(X) G & \xrightarrow{(G)} & \langle A \rangle \\
\downarrow G(X\alpha) & & \downarrow \epsilon|A \\
G(F\langle A \rangle) & \xrightarrow{G\epsilon|A} & [G \circ F]\langle A \rangle \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
\langle X \rangle & \xrightarrow{(F)} & F\langle A \rangle F \\
\downarrow (X\eta) & & \downarrow (\eta|A)F \\
\langle X \rangle F\langle A \rangle & \xrightarrow{(\eta|A)F} & \langle X \rangle F\langle A \rangle F \\
\end{array}
\]

commutes.

Proof. In Proposition 5.3.6, replace \(M\) with the representable module \(F\langle A \rangle\) and replace \(\alpha\) with \(\epsilon\), and we have the commutative triangle depicted in the upper left-hand side of the diagram. The commutativity of the lower right triangle was seen in Example 5.3.5(2).

Note. A two-sided cylinder \(X \xrightarrow{G F} A\) is also depicted as a right cylinder \(X \xrightarrow{\alpha}_{M} \text{op. left} E\) \(\xrightarrow{\alpha}_{G(M)} E\) \(\text{op. left}\) cylinder \(E \xrightarrow{\alpha}_{G(M)} A\) \(\text{op. right}\). Proposition 5.3.8 says that \(X\) \([\text{op. } A]\) generates the same module morphism direct \([\text{op. inverse}]\) along \(\alpha\) irrespective of the way \(\alpha\) is depicted.

5.3.8 Proposition. Consider a cylinder as in Note above. Then
the module morphism
\[ \alpha \mapsto (X \uparrow M) \alpha : (X) G \to (M) F : X \to E \]

coincides with
\[ X \uparrow \alpha = (X \uparrow (M) F) \alpha : (X) G \to (M) F : X \to E. \]

the module morphism
\[ \alpha \uparrow A = \alpha (G \uparrow A) : F(A) \to G(M) : E \to A \]

coincides with
\[ \alpha \uparrow A = \alpha (G(M) \uparrow A) : F(A) \to G(M) : E \to A. \]

Proof. Both map each \( (X) G \)-arrow \( h : x \sim e \) to the \( (M) F \)-arrow \( h \circ \alpha_e : x \sim e \) (see Example 5.3.5(1)). □

Note. Given a pair of categories \( X \) and \( A \), we saw in Remark 4.3.8(3) that the hom-module of the functor category \([A, X]\) is the same thing as the module \((A, X)\), and will see below that the hom-cell of the right generalized Yoneda functor for \([A, X]\) is the same thing as the right generalized Yoneda morphism for \((A, X)\).

5.3.9 Proposition. Given a pair of categories \( X \) and \( A \), the right \([ \text{op. left} \] generalized Yoneda morphism

\[
\begin{align*}
\text{op.} & \quad (A, X) \quad \text{op.} & \quad (X, A) \\
& \quad (A, X) \quad \text{op.} & \quad (X, A) \\
& \quad (X, A) \quad \text{op.} & \quad (A, X)
\end{align*}
\]

for \((\text{op. } (A, X))\) \([ \text{op. } (X, A)]\) is the same thing as the hom-cell

\[
\begin{align*}
\text{op.} & \quad (A, X) \quad \text{op.} & \quad (X, A) \\
& \quad (A, X) \quad \text{op.} & \quad (X, A) \\
& \quad (X, A) \quad \text{op.} & \quad (A, X)
\end{align*}
\]

of the right \([ \text{op. left} \] generalized Yoneda functor for \([A, X]\) \([ \text{op. } (X, A)]\); that is, for any natural transformation \( \tau : G \to F : A \to X \) \([ \text{op. } \tau : G \to F : X \to A]\),

\[ X \uparrow \tau = (X) \tau \quad \text{op.} \quad \tau \uparrow A = \tau(A). \]

Proof. Both \( X \uparrow \tau \) and \((X) \tau\) map each \( (X) G \)-arrow \( h : x \sim a \) to the \( (X) F \)-arrow \( h \circ \tau_a : x \sim a \) (see Remark 2.3.11(1)). □

Note. Recall from Definition 4.3.17 that any cell \( \psi : M \to N \) yields the postcomposition cell \((E, \psi) : (E, M) \to (E, N)\) for a given category \( E \). We will see below that the generalized Yoneda morphism \((X \uparrow X) \tau) : (E, M) \to (E, X)\) in Definition 5.3.3 is in fact given (upon the exponential transposition \((X : E) \to (E, X)\)) in Definition 2.1.1) as the postcomposition cell \((E, (X \uparrow M) \tau) : (E, M) \to (E, (X : E))\) the Yoneda morphism \((X \uparrow M) \tau) : M \to (X : E)\) yields.

5.3.10 Theorem. Given a category \( E \) and a module \( X : E \to A \), the triangle

\[
\begin{align*}
& \quad \text{op.} & \quad \text{op.} \\
& \quad \text{op.} & \quad \text{op.} \\
& \quad \text{op.} & \quad \text{op.}
\end{align*}
\]

commutes; that is, the composition

\[
\begin{align*}
& \quad \text{op.} & \quad \text{op.} \\
& \quad \text{op.} & \quad \text{op.} \\
& \quad \text{op.} & \quad \text{op.}
\end{align*}
\]
of the right [op. left] generalized Yoneda morphism for \((E,M)\) and the right [op. left] exponential transposition yields the postcomposition cell

\[
\begin{array}{c}
\{E,X]\rightarrow [E,A] \quad \text{op.} \quad \{E,X]\rightarrow [E,A] \\
\end{array}
\]

—postcomposition with the right [op. left] Yoneda morphism for \(M\).

**Proof.** By Proposition 2.2.3, the diagram

\[
\begin{array}{c}
\langle X\downarrow M\rangle \xrightarrow{(E,\{X\downarrow M\})} \{E,\{X\downarrow M\}\} \\
\end{array}
\]

commutes. Now consider their hom-cells and the postcomposition cell in Example 4.3.23, shown in the commutative diagram

\[
\begin{array}{c}
\langle E,M\rangle \xrightarrow{\langle E,\{X\downarrow M\}\rangle} \{E,\{X\downarrow M\}\} \\
\end{array}
\]

\((\langle E,\{X\downarrow M\}\rangle) = (E,\langle\{X\downarrow M\}\rangle)\) by Remark 4.3.16(2), and the diagram commutes by the functoriality of the hom-cell assignment. Since the composition of \(\langle E,1_{M}\rangle\) and \(\langle\{X\downarrow M\}\rangle\) gives \(\langle\{X\downarrow M\}\rangle \xrightarrow{E} \) by Remark 5.3.4(1), and the composition of \(\langle E,1_{M}\rangle\) and \(\langle E,\langle\{X\downarrow M\}\rangle\rangle\) gives \(\langle E,\{X\downarrow M\}\rangle\) by Proposition 4.3.21 and Remark 5.2.4(1), we have the desired commutative triangle.

\(\square\)

5.3.11 Remark. Theorem 5.3.10 says that, given a cylinder

\[
\begin{array}{c}
\begin{array}{c}
\text{op.} \quad \langle X\downarrow M\rangle \xrightarrow{(E,\{X\downarrow M\})} \{E,\{X\downarrow M\}\} \\
\end{array}
\end{array}
\]

holds; that is, the right [op. left] exponential transpose of the module morphism

\[
\begin{array}{c}
\langle X\downarrow M\rangle \alpha : \langle X\rangle G \rightarrow \langle M\rangle F \rightarrow X \quad \text{op.} \quad \alpha \langle M\downarrow A\rangle : F\langle A\rangle \rightarrow G\langle M\rangle : E \rightarrow A \\
\end{array}
\]

is the natural transformation given by the composition

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

of \(\alpha\) and the right [op. left] Yoneda morphism for \(M\).

**Note.** The following is a pointwise description of Remark 5.3.11.

5.3.12 Corollary. For any cylinder

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

and any object \(e \in \|E\|\),

- the identity

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

holds; that is, the right slice

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

at \(e\) of the module morphism generated by \(X\) direct along \(\alpha\) is given by the right module

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]
Proof. By Theorem 5.3.10, the cell \(X\) direct along the component of \(\alpha\) at \(e\).

- the identity

\[ e(\alpha \downarrow A) = \alpha_e \downarrow A \]

holds; that is, the left slice

\[ e(\alpha \downarrow A) : e(F(A)) \to e(G(M)) : \ast \to A \]

at \(e\) of the module morphism generated by \(A\) inverse along \(\alpha\) is given by the left module

\[ \alpha_e \downarrow A : (e \cdot F)(A) \to (e \cdot G)(M) : \ast \to A \]

generated by \(A\) inverse along the component of \(\alpha\) at \(e\).

5.3.13 Theorem. Let \(E\) be a category and \(M : X \to A\) be a module.

- The right generalized Yoneda morphism

\[
\begin{align*}
\{E, X\}^{(E, M)} & \circ - \circ \{E, A\} \\
X \cdot E & \downarrow (X \cdot M) \cdot E \downarrow M, E \\
\{X : E\} & \circ \{E : X\}
\end{align*}
\]

for \((E, M)\) is fully faithful. Specifically, for each pair of functors \(G : E \to X\) and \(F : E \to A\), the assignment \(\alpha \mapsto X \downarrow \alpha\) yields a bijection

\[(G)(\{E, M\})(F) \cong \{(X) G \cdot (X) E\}(\{M\) F)\]

from the set of cylinders \(G : E \to M\) to the set of module morphisms \((X) G \to \{M\) F\) : \(X \to E\), whose inverse sends each module morphism \(\theta : \{X\) G \to \{M\) F\) to the cylinder \(X \downarrow \theta : G \to F : E \to M\) defined by

\[
[X \downarrow \theta]_e = 1_{(e \cdot G)} \cdot \theta
\]

for \(e \in \|E\|\), where \(1_{(e \cdot G)} : \theta\) is the image of the identity \(X\)-arrow \(e : G \to G\cdot e\) (i.e. \(X\) \(-\)-arrow \(e : G \to e\)) under the function

\[(e \cdot G)(\{X\}) (G \cdot e) \cdot e = (e \cdot G)(\{X\} G \cdot e) (e \cdot G)(\{M\) F\) \cdot e = (e \cdot G)(\{M\) F\) \cdot e .

- The left generalized Yoneda morphism

\[
\begin{align*}
\{E, X\}^{(E, M)} & \circ - \circ \{E, A\} \\
E \cdot X & \downarrow (E \cdot M) \cdot A \downarrow E, A \\
\{E : A\} & \circ \{A : E\}
\end{align*}
\]

for \((E, M)\) is fully faithful. Specifically, for each pair of functors \(G : E \to X\) and \(F : E \to A\), the assignment \(\alpha \mapsto \alpha \downarrow A\) yields a bijection

\[(G)(\{E, M\})(F) \cong (F(A)) (E : A)(G(M))\]

from the set of cylinders \(G : F : E \to M\) to the set of module morphisms \(F(A) \to G(M) : E \to A\), whose inverse sends each module morphism \(\theta : F(A) \to G(M)\) to the cylinder \(\theta \downarrow A : G \to F : E \to M\) defined by

\[
[\theta \downarrow A]_e = \theta \cdot 1_{(F \cdot e)}
\]

for \(e \in \|E\|\), where \(\theta \cdot 1_{(F \cdot e)}\) is the image of the identity \(A\)-arrow \(e : F \to F \cdot e\) (i.e. \(F(A)\) \(-\)-arrow \(e \to F \cdot e\)) under the function

\[(e \cdot F)(A)(F \cdot e) = e(F(A))(F \cdot e) \cdot e(\theta(F \cdot e)) \cdot e(G(M))(F \cdot e) = (e \cdot G)(M)(F \cdot e) .
\]

Proof. By Theorem 5.3.10, the cell \((X \cdot M) \cdot E\) is fully faithful iff so is the cell \((E, (X \cdot M) \cdot)\).
But since the cell $(X \downarrow \mathcal{M})^\cdot$ is fully faithful (see Theorem 5.2.10), so is $(E, (X \downarrow \mathcal{M})^\cdot)$ by Proposition 4.3.19. For the second assertion, it suffices to show that a cylinder $\alpha : G \Rightarrow F : E \Rightarrow \mathcal{M}$ is recovered from the module morphism $(X \uparrow \alpha)$ by $\alpha_e = 1_{(e : G)} : (e : G)(X \uparrow \alpha)e$. But by Theorem 5.2.10 and Corollary 5.3.12, we have

$$\alpha_e = 1_{(e : G)} : (e : G)(X \uparrow \alpha)e = 1_{(e : G)} : (e : G)(X \uparrow \alpha)e.$$  

□

5.3.14 Remark. Theorem 5.2.10 is regarded as a special case of Theorem 5.3.13 where $E$ is the terminal category.

5.3.15 Corollary. (Generalized Yoneda embedding). Let $X$ and $A$ be categories.
- The right generalized Yoneda functor $[X \times A] : [A, X] \rightarrow [X : A]$ is fully faithful. Specifically, for each pair of functors $G, F : A \rightarrow X$, the assignment $\tau \mapsto (X \tau)$ yields a bijection

$$(G)(A, X)(F) \cong ((X)G)((X)A)((X)F)$$

from the set of natural transformations $G \Rightarrow F : A \rightarrow X$ to the set of module morphisms $(X)G \Rightarrow (X)F : X \Rightarrow A$, whose inverse sends each module morphism $\theta : (X)G \Rightarrow (X)F$ to the natural transformation $[0] : G \Rightarrow F$ defined by

$$[0]_a = 1_{(a : G)} : \theta$$

for $a \in \|A\|$, where $1_{(a : G)} : \theta$ is the image of the identity $X$-arrow $a : G \Rightarrow a$ (i.e. $(X)$-arrow $a : G \Rightarrow a$) under the function

$$(a : G)(X)(G \cdot a) = (a : G)((X)G)a \xrightarrow{(a : (G)(\theta)a)} (a : G)((X)F)a = (a : G)((X)(F \cdot a)).$$

- The left generalized Yoneda functor $[X \times A][A, X] \Rightarrow [X : A]$ is fully faithful. Specifically, for each pair of functors $G, F : X \Rightarrow A$, the assignment $\tau \mapsto \tau(A)$ yields a bijection

$$(G)(X, A)(F) \cong (F(A))(A : X)(A : G)$$

from the set of natural transformations $G \Rightarrow F : X \Rightarrow A$ to the set of module morphisms $F(A) \Rightarrow G(A) : X \Rightarrow A$, whose inverse sends each module morphism $\theta : F(A) \Rightarrow G(A)$ to the natural transformation $[0] : G \Rightarrow F$ defined by

$$[0]_x = \theta \cdot 1_{(F \cdot x)}$$

for $x \in \|X\|$, where $1_{(F \cdot x)}$ is the image of the identity $X$-arrow $x : F \Rightarrow F \cdot x$ (i.e. $F(A)$-arrow $x \Rightarrow F \cdot x$) under the function

$$(x : F)(A)(F \cdot x) = x(A)(F \cdot x) \xrightarrow{x(\theta)(F \cdot x)} x(A)(F \cdot x) = (x : G)(A)(F \cdot x).$$

Proof. Since the hom-cell of the right generalized Yoneda functor $X \times A$ is the same thing as the right generalized Yoneda morphism for $(A, (X))$ (see Proposition 5.3.9), this is a special case of Theorem 5.3.13 where $\mathcal{M}$ is given by the hom-module of a category. □

5.3.16 Remark. Corollary 5.2.12 (Yoneda embedding) is regarded as a special case of Corollary 5.3.15 where $A$ [op. $X$] is the terminal category.

Note. We saw in Example 5.3.5(1) that the generalized Yoneda morphism transforms a right cylinder $\alpha : G \Rightarrow \mathcal{M}$ into the module morphism $X \uparrow \alpha : (X)G \Rightarrow \mathcal{M}$. The following corollary of Theorem 5.3.13 states the bijectivity of this transformation. Recall from Remark 5.1.7(2) that $(G)(X \times A)(\mathcal{M})$ denotes the set of right cylinders $G \Rightarrow \mathcal{M}$.

5.3.17 Corollary. (Generalized Yoneda Lemma : Part one).
- Given a module $M : X \Rightarrow A$ and a functor $G : A \Rightarrow X$, the assignment $\alpha \mapsto X \uparrow \alpha$ yields a bijection

$$(G)(X \times A)(\mathcal{M}) \cong ((X)G)(X : A)(\mathcal{M}).$$
Proof. The bijective correspondence follows from Theorem 5.3.13 by identifying right cylinders $G \rightsquigarrow M$ to the set of module morphisms $(X)G \rightarrow M$, whose inverse sends each module morphism $\theta : (X)G \rightarrow M$ to the right cylinder $X|\theta : G \rightsquigarrow M$ defined by

$$[X|\theta]_a = 1_{(a;G)}\cdot \theta$$

for $a \in \|A\|$, where $1_{(a;G)}\cdot \theta$ is the image of the identity $X$-arrow $a : G \rightarrow G$ (i.e. $(X)G$-arrow $a : G \rightsquigarrow a$) under the function

$$(a : G)(X)(G : a) = (a : G)((X)G) a \xrightarrow{(a;G)(\theta)a} (a : G)(M)a.$$  

- Given a module $M : X \rightarrow A$ and a functor $F : X \rightarrow A$, the assignment $\alpha \mapsto \alpha|A$ yields a bijection $$(M)(X : A)(F) \cong (F(A))(X : A)(M)$$

from the set of left cylinders $M \rightsquigarrow F$ to the set of module morphisms $F(A) \rightarrow M$, whose inverse sends each module morphism $\theta : F(A) \rightarrow M$ to the left cylinder $\theta|A : M \rightsquigarrow F$ defined by

$$[\theta|A]_x = \theta \cdot 1_{(F,x)}$$

for $x \in \|X\|$, where $\theta \cdot 1_{(F,x)}$ is the image of the identity $A$-arrow $x : F \rightarrow F$ (i.e. $F(A)$-arrow $x \rightsquigarrow F : x$) under the function

$$(x : F)(A)(F : x) = x(F(A))(F : x) \xrightarrow{x(\theta)(F:x)} x(M)(F : x).$$

5.3.18 Remark. Theorem 5.2.8 is a special case of Corollary 5.3.17 where $A$ is the terminal category. Indeed, if $G : A \rightarrow X$ [op. $F : X \rightarrow A$] in Corollary 5.3.17 is replaced by $x : \ast \rightarrow X$ [op. $a : \ast \rightarrow A$], we have

$$(x)(X : \ast)(M) \cong ((X)x)(X : \ast)(M) \quad \text{op.} \quad (M)(\ast : A)(a) \cong (a(A))(\ast : A)(M)$$

, which is identified with the bijection in Theorem 5.2.8 by the isomorphisms in Remark 5.1.7(4) and Remark 1.1.14(4).

Note. The following shows that the right generalized Yoneda module $X : \ast A$ introduced in Definition 5.1.6 is represented by the right generalized Yoneda functor $X : \ast A$.

5.3.19 Theorem. Let $X$ and $A$ be categories.

- The right generalized Yoneda morphism for $(A, (X : \ast))$ composed with the iso cell in Proposition 5.1.10 as shown in

$$\begin{array}{ccc}
[A, X] & \xrightarrow{\sim} & [X : A] \\
\downarrow \sim & & \downarrow \sim \\
[X : A] & \xrightarrow{(X : \sim A)} & [A, [X :]] \\
\downarrow (X : \ast A) & & \downarrow (X : \ast A) = \sim \\
[A, X] & \xrightarrow{\sim} & [X : A]
\end{array}$$

yields a representation

$$(X : \ast A) \cong [X : \ast A](X : A) : [A, X] \rightarrow [X : A]$$

of the right generalized Yoneda module $X : \ast A$ by the right generalized Yoneda functor $X : \ast A$; the component of the representation cell at a functor $G : A \rightarrow X$ and a module $M : X \rightarrow A$ is exactly the bijection in the generalized Yoneda lemma; that is, the cell sends each $(X : \ast A)$-arrow $\alpha : G \rightarrow M$ (i.e. right cylinder $\alpha : G \rightsquigarrow M$) to the right module morphism $X|\alpha : (X)G \rightarrow M$.

- The left generalized Yoneda morphism for $(X, (\ast A))$ composed with the iso cell in Proposi-
Let \( \sigma : G \to G : A \to X \) be a natural transformation and \( \psi : M \to M' : X \to A \) be a module morphism. Then for any left cylinder \( \alpha : G \to M \),

\[
X \downarrow [\sigma \circ \alpha] = \langle X \rangle \sigma \circ \langle X \rangle \alpha \quad \text{and} \quad X \downarrow [\alpha \circ \psi] = \langle X \rangle \alpha \circ \psi
\]

, and for any module morphism \( \theta : \langle X \rangle \to M : X \to A \),

\[
X \downarrow [\langle X \rangle \sigma \circ \theta] = \sigma \circ [X \downarrow \theta] \quad \text{and} \quad X \downarrow [\theta \circ \psi] = [X \downarrow \theta] \circ \psi.
\]

Let \( \tau : F \to F' : X \to A \) be a natural transformation and \( \psi : M \to M' : X \to A \) be a module morphism. Then for any left cylinder \( \alpha : M \to F \),

\[
[\alpha \circ \tau] \downarrow A = \langle \alpha \downarrow A \rangle \circ \tau (A) \quad \text{and} \quad [\psi \circ \alpha] \downarrow A = \psi \circ \langle \alpha \downarrow A \rangle
\]

, and for any module morphism \( \theta : F \downarrow A : X \to A \),

\[
(\theta \circ \tau) \downarrow A = [\theta \downarrow A] \circ \tau \quad \text{and} \quad (\psi \circ \theta) \downarrow A = \psi \circ [\theta \downarrow A].
\]
Proof. Recalling (Remark 5.1.7(3)) the composition in the generalized Yoneda module, we see that this just expresses the naturality of the generalized Yoneda representation $X \uparrow A$ and its inverse. □

5.3.23 Theorem. (Generalized Yoneda Lemma : Part two). The bijection in Corollary 5.3.17 is natural in $G$ [op. $F$] and $M$.

Proof. Immediate from Remark 5.3.21(1). □

Note. We saw in Example 5.3.5(2) that the generalized Yoneda morphism transforms a natural transformation $\epsilon : G \circ F \rightarrow 1_A$ into the module morphisms $X \uparrow \epsilon : (X) G \rightarrow F(A)$. The following corollary of Theorem 5.3.13 states the bijectivity of this transformation.

5.3.24 Theorem. Given a pair of functors $X \xrightarrow{G} F \xrightarrow{A}$, 

- the assignment $\epsilon \mapsto X \uparrow \epsilon$ yields a bijection 
  $$(G \circ F)(A, A)(1_A) = (G)(X \rightharpoonup A)(F(A)) \cong ((X) G)(X : A)(F(A))$$
  from the set of natural transformations $G \circ F \rightarrow 1_A : A \rightarrow A$ to the set of module morphisms $(X) G \rightarrow F(A) : X \rightarrow A$, whose inverse sends each module morphism $\theta : (X) G \rightarrow F(A)$ to the natural transformation $X \downarrow \theta : G \circ F \rightarrow 1_A$ defined by
  $$[X \downarrow \theta]_a = 1_{(a : G)} \cdot \theta$$
  for $a \in \| A \|$, where $1_{(a : G)} \cdot \theta$ is the image of the identity $X$-arrow $a : G \rightarrow a$ (i.e. $(X) G$-arrow $a : G \sim a$) under the function 
  $$(a : G)(X : G)(a) \xrightarrow{a : G((a : G) \cdot \theta)} (a : G)(F(A)) a = (a : G : F)(A) a$$
  moreover, the bijection is natural in $G$ and $F$.

- the assignment $\eta \mapsto \eta \uparrow A$ yields a bijection 
  $$(1_X)(X, X) (G \circ F) = ((X) G)(X \rightharpoonup A)(F) \cong (F(A))(X : A)(X) G$$
  from the set of natural transformations $1_X \rightarrow G \circ F : X \rightarrow X$ to the set of module morphisms $F(A) \rightarrow (X) G : X \rightarrow A$, whose inverse sends each module morphism $\theta : F(A) \rightarrow (X) G$ to the natural transformation $\theta \downarrow A : 1_X \rightarrow G \circ F$ defined by 
  $$[\theta \downarrow A]_x = \theta \cdot 1_{(F : X)}$$
  for $x \in \| X \|$, where $\theta \cdot 1_{(F : X)}$ is the image of the identity $A$-arrow $x : F \rightarrow x$ (i.e. $F(A)$-arrow $x \sim F : x$) under the function 
  $$(x : F)(A)(F : x) \xrightarrow{x : F(\theta : x)} x((X) G)(F : x) = x(X)(G : F : x)$$
  moreover, the bijection is natural in $G$ and $F$.

Proof. We saw the identity $(G \circ F)(A, A)(1_A) = (G)(X \rightharpoonup A)(F(A))$ in Proposition 5.1.12. Now replacing $M$ with $F(A)$ in Corollary 5.3.17, we have a bijection 
  $$(G)(X \rightharpoonup A)(F(A)) \cong ((X) G)(X : A)(F(A))$$
  which is natural in $G$ and $F$ by Theorem 5.3.23. □

5.3.25 Remark. Just like the generalized Yoneda lemma is a componentwise description of the generalized Yoneda representation (see (Remark 5.3.21(1))), Theorem 5.3.24 is a componentwise description of the following module isomorphism.

- The module isomorphism 
  $$(X \uparrow A)[X \rightharpoonup A] : (X \rightharpoonup A)[X \rightharpoonup A] \rightarrow [X \rightharpoonup A](X : A)[X \rightharpoonup A] : [A, X] \rightarrow [X, A]$$
is defined by the composition

\[
[A, X] \xrightarrow{\beta} X \mapsto \frac{X \cdot A}{X \otimes A} \xrightarrow{\sim} [X : A] \xrightarrow{\sim} [X, A]^{-}\]

of the generalized Yoneda representation and the left generalized Yoneda functor; the component of \((X \vdash A)[X \otimes A]\) at \((G, F)\) gives the bijection

\[
(G)(X \otimes A)(F(A)) \cong ((X) G)(X : A)(F(A))
\]

- The module isomorphism

\[
[X \otimes A](X \dashv A) : [X \otimes A](X \otimes A) \to [X \otimes A](X \otimes A)[X, X]^{-} \to [X, A]
\]

is defined by the composition

\[
[A, X]^{-} \xrightarrow{\sim} \frac{X \cdot A}{X \otimes A} \xrightarrow{\sim} \frac{X \cdot A}{X \otimes A} \xrightarrow{\sim} [X, A]
\]

of the generalized Yoneda corepresentation and the right generalized Yoneda functor; the component of \([X \otimes A](X \dashv A)\) at \((G, F)\) gives the bijection

\[
((X) G)(X \otimes A)(F) \cong (F(A))(X : A)((X) G)
\]

5.3.26 Theorem. Given a pair of natural transformations as in

\[
\begin{array}{c}
\begin{array}{ccc}
G & \xrightarrow{\beta} & F \\
\downarrow & & \downarrow \\
D & \xrightarrow{\beta} & E
\end{array} \\
\begin{array}{ccc}
\downarrow & & \downarrow \\
P & \xrightarrow{\gamma} & Q
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{ccc}
G & \xleftarrow{\beta} & F \\
\downarrow & & \downarrow \\
E & \xleftarrow{\beta} & D
\end{array} \\
\begin{array}{ccc}
\downarrow & & \downarrow \\
P & \xleftarrow{\gamma} & Q
\end{array}
\end{array}
\]

, their pasting composite

\[
[\beta \circ \alpha] \circ \beta F ] \circ \beta \alpha : G \circ \alpha P \to Q \circ \beta F \circ \beta \alpha
\]

is given by the composition

\[
\begin{array}{c}
\begin{array}{ccc}
G & \xrightarrow{\beta} & F \\
\downarrow & & \downarrow \\
D & \xrightarrow{\beta} & E
\end{array} \\
\begin{array}{ccc}
\downarrow & & \downarrow \\
P & \xrightarrow{\gamma} & Q
\end{array}
\end{array}
\]

of the cylinder \(\beta\) (see Remark 4.3.2(3)) and the cell \(D \vdash \alpha \ [\alpha \vdash D\] (see Example 5.3.5(3)).

Proof. For any \(x \in \|X\|\),

\[
[[\beta \circ \alpha] \circ \beta F ] \circ \beta \alpha \circ \beta \alpha \circ \beta \alpha = (\beta \circ \alpha \circ \beta F ) \circ \beta \alpha \circ \beta \alpha = (\beta \circ \alpha \circ \beta F ) \circ \beta \alpha \circ \beta \alpha
\]

and

\[
[\beta \circ (D \vdash \alpha)] _{x} = \beta x : (D \vdash \alpha)
\]

We thus need to show that \(\beta x : (D \vdash \alpha) = (\beta x : P) \circ \alpha : (x : F)\). But by replacing \(h : d \to K : e\) with \(\beta x : x : G \to K : F : x\) in the commutative diagram of Example 5.3.5(3), we have

\[
\begin{array}{ccc}
x : F : K & x : F : K : P & Q : F : x \\
\uparrow \beta x & \uparrow \beta x : P & \downarrow \beta x : (D \vdash \alpha)
\end{array}
\]

as required. \(\square\)
5.4.1 Definition. Let \( \mathcal{M} \to \mathcal{E} \to \mathcal{A} \) be a module.

- The right generalized Yoneda morphism for \( \langle \ast \mathcal{E} \mathcal{M} \rangle \) [op. \( \langle \mathcal{E} \mathcal{A} \rangle \)] is the cell

\[
\begin{array}{cc}
\ast \mathcal{E} & \ast \mathcal{M} \\
\mathcal{E} & \mathcal{A} \\
\mathcal{M} & \mathcal{E} \\
\cdot & \cdot \\
\langle \mathcal{E} \mathcal{M} \rangle & \langle \mathcal{E} \mathcal{A} \rangle \\
\end{array}
\]

sending each cone

\[ \alpha : \mathcal{M} \to \ast \mathcal{E} \to \langle \mathcal{M} \rangle \to \langle \mathcal{E} \rangle \to \langle \mathcal{A} \rangle \]

to the wedge

\[ \mathcal{M} \to \langle \mathcal{M} \rangle \to \langle \mathcal{E} \rangle \to \langle \mathcal{A} \rangle \]

- The left generalized Yoneda morphism for \( \langle \ast \mathcal{E} \mathcal{M} \rangle \) [op. \( \langle \mathcal{E} \mathcal{A} \rangle \)] is the cell

\[
\begin{array}{cc}
\ast \mathcal{E} & \ast \mathcal{M} \\
\mathcal{E} & \mathcal{A} \\
\mathcal{M} & \mathcal{E} \\
\cdot & \cdot \\
\langle \mathcal{E} \mathcal{M} \rangle & \langle \mathcal{E} \mathcal{A} \rangle \\
\end{array}
\]

sending each cone

\[ \alpha : \mathcal{M} \to \langle \mathcal{E} \rangle \to \langle \mathcal{A} \rangle \to \langle \mathcal{M} \rangle \to \langle \mathcal{E} \rangle \]

to the wedge

\[ \mathcal{E} \to \langle \mathcal{E} \rangle \to \langle \mathcal{A} \rangle \to \langle \mathcal{M} \rangle \to \langle \mathcal{E} \rangle \]
sending each cone
\[ \alpha : \text{x} \rightsquigarrow F : \ast \text{E} \rightsquigarrow \text{M} \quad \text{op.} \quad \alpha : G \rightsquigarrow a : \text{E}^\ast \rightsquigarrow \text{M} \]
to the wedge
\[ \alpha \upharpoonright \text{A} = \alpha \langle \text{M} \upharpoonright \text{A} \rangle : F \langle \text{A} \rangle \rightsquigarrow \text{x} \langle \text{M} \rangle : \ast \text{E} \rightarrow \text{A}. \quad \text{op.} \quad \alpha \upharpoonright \text{A} = \alpha \langle \text{M} \upharpoonright \text{A} \rangle : a \langle \text{A} \rangle \rightsquigarrow G \langle \text{M} \rangle : \ast \text{E} \rightarrow \text{A}. \]

5.4.2 Remark. By virtue of Theorem 4.6.22 and Remark 4.10.4, the cell \( \langle \text{X} \upharpoonright \text{M} \rangle \searrow \ast \text{E} \) [op. \( \langle \text{X} \upharpoonright \text{M} \rangle \searrow \ast \text{E} \)] is formally given by the pasting composition
\[
\begin{array}{c}
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\end{array}
\]
of the right [op. left] generalized Yoneda morphism for \( \langle \text{E}, \text{M} \rangle \) and the commutative diagram in Proposition 2.3.12 and Proposition 2.2.4, and the cell \( \ast \text{E} \searrow \text{X} \rangle \text{M} \langle \text{A} \rangle \) [op. \( \text{E}^\ast \searrow \text{X} \rangle \text{M} \langle \text{A} \rangle \)] is formally given by the pasting composition
\[
\begin{array}{c}
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} & \text{op.} & \text{E} \xrightarrow{\langle \text{E}, \text{X} \rangle} \text{[E, X]} \xrightarrow{\langle \text{E}, \text{A} \rangle} \text{[E, A]} \\
\end{array}
\]

5.4.3 Theorem. The generalized Yoneda morphism, right or left, for \( \langle \ast \text{E}, \text{M} \rangle \) [op. \( \langle \text{E}^\ast, \text{M} \rangle \)] is fully faithful.

Proof. By Remark 5.4.2 and Proposition 1.2.35, this follows from the fully faithfulness of the generalized Yoneda morphism for \( \langle \text{E}, \text{M} \rangle \).

Note. Recall from Definition 4.6.17 that any cell \( \psi : \text{M} \rightarrow \text{N} \) yields the postcomposition cell \( \langle \ast \text{E}, \psi \rangle : \langle \ast \text{E}, \text{M} \rangle \rightarrow \langle \ast \text{E}, \text{N} \rangle \) for a given category \( \text{E} \). We will see below that the generalized Yoneda morphism \( \langle \langle \text{X} \rangle \upharpoonright \text{M} \rangle \searrow \ast \text{E} \rangle : \langle \ast \text{E}, \text{M} \rangle \rightarrow \langle \text{X} \rangle \upharpoonright \ast \text{E} \rangle \) in Definition 5.4.1 is in fact given (upon the exponential transposition \( \langle \text{X} \rangle \rightarrow \ast \text{E} \rangle \rightarrow \langle \ast \text{E}, \text{X} \rangle \rangle \) in Remark 10.6) as the postcomposition cell \( \langle \ast \text{E}, \langle \text{X} \rangle \rangle : \langle \ast \text{E}, \text{M} \rangle \rightarrow \langle \ast \text{E}, \text{X} \rangle \rangle \) the Yoneda morphism \( \langle \langle \text{X} \rangle \upharpoonright \text{M} \rangle \rangle : \text{M} \rightarrow \langle \text{X} \rangle \rangle \).

5.4.4 Theorem. Given a category \( \text{E} \) and a module \( \text{M} : \text{X} \rightarrow \text{A} \),

\[ \text{X} \] the triangle
\[ \langle \text{X} \rangle \upharpoonright \text{E} \rangle \rightarrow \langle \ast \text{E}, \text{X} \rangle \rangle \] of the right generalized Yoneda morphism for \( \langle \ast \text{E}, \text{M} \rangle \) [op. \( \langle \text{E}^\ast, \text{M} \rangle \)] and the right exponential transposition of wedges \( \text{X} \rightarrow \ast \text{E} \) [op. \( \text{X} \rightarrow \text{E}^\ast \)] yields the postcomposition cell
\[ \text{X} \] the triangle
\[ \langle \text{X} \rangle \upharpoonright \text{E} \rangle \rightarrow \langle \ast \text{E}, \text{X} \rangle \rangle \] —postcomposition with the right Yoneda morphism for \( \text{M} \).
Proof. Consider the cells and commutative diagrams as in

\[
\xymatrix{
\ast E \wedge (\mathcal{M}|A) \ar[r]^{\ast E \wedge \langle \mathcal{M}|A \rangle} \ar[dr]_{\mathcal{E} \wedge \mathcal{A}} & \langle \ast E, \mathcal{M} \rangle \ar[d]_{\mathcal{E} \wedge \mathcal{A}} \\
(E \ast : A)^\heartsuit \ar[r]_{\heartsuit} & \langle \ast E, (\cdot : A)^\heartsuit \rangle
}
\]

(\ast E \ast : M) → \langle \ast E, (\cdot : M)^\heartsuit \rangle

op. \quad \xymatrix{
\mathcal{E} \wedge \mathcal{A} \ar[r]^{\mathcal{E} \wedge \mathcal{A}} \ar[dr]_{\mathcal{E} \wedge \mathcal{A}} & \langle \mathcal{E}, \mathcal{M} \rangle \ar[d]_{\mathcal{E} \wedge \mathcal{A}} \\
\langle \ast E, (\cdot : A)^\heartsuit \rangle \ar[r]_{\heartsuit} & \langle \ast E, (\cdot : A)^\heartsuit \rangle
}

commutes; that is, the composition

\[
\xymatrix{
X \ar[r]_{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \ Waves}
the identity
\[ \langle \langle X \uparrow M \rangle \, \alpha \rangle \circ \langle \langle X \uparrow M \rangle \, \alpha \rangle = \langle \langle X \uparrow M \rangle \, \alpha \rangle \circ \alpha \]
holds; that is, the right exponential transpose of the wedge
\[ \langle X \uparrow M \rangle \, \alpha : \langle X \rangle x \twoheadrightarrow \langle M \rangle F : X \to *E \quad \text{op.} \quad \langle X \uparrow M \rangle \, \alpha : \langle X \rangle G \twoheadrightarrow \langle M \rangle a : X \to E^* \]
is the cone given by the composition
\[ \begin{array}{ccc}
* & \xrightarrow{1} & E \\
\xymatrix{X & \ar[l]_{\alpha} \ar[r]_{F} & \langle X \rangle \ar[r] & \langle X \rangle \ar[r] & A}
\end{array} \]
\[ \begin{array}{ccc}
E & \xrightarrow{1} & * \\
\xymatrix{X & \ar[l]_{\alpha} \ar[r]_{G} & \langle X \rangle \ar[r] & \langle X \rangle \ar[r] & A}
\end{array} \]
of \( \alpha \) and the right Yoneda morphism for \( M \).

- for any cone
\[ \alpha : x \twoheadrightarrow F : *E \twoheadrightarrow M \quad \text{op.} \quad \alpha : G \twoheadrightarrow a : E^* \twoheadrightarrow M \]
the identity
\[ \nabla \langle \alpha \cdot (M \uparrow A) \rangle = \alpha \circ \nabla \langle \langle M \uparrow A \rangle \rangle \]
holds; that is, the left exponential transpose of the wedge
\[ \alpha \cdot (M \uparrow A) : F(A) \twoheadrightarrow \langle X \rangle (M) : *E \twoheadrightarrow A \quad \text{op.} \quad \alpha \cdot (M \uparrow A) : a(A) \twoheadrightarrow G(\langle M \rangle) : *E \twoheadrightarrow A \]
is the cone given by the composition
\[ \begin{array}{ccc}
* & \xrightarrow{1} & E \\
\xymatrix{X & \ar[l]_{\alpha} \ar[r]_{F} & \langle X \rangle \ar[r] & \langle X \rangle \ar[r] & A}
\end{array} \]
\[ \begin{array}{ccc}
E & \xrightarrow{1} & * \\
\xymatrix{X & \ar[l]_{\alpha} \ar[r]_{G} & \langle X \rangle \ar[r] & \langle X \rangle \ar[r] & A}
\end{array} \]
of \( \alpha \) and the left Yoneda morphism for \( M \).

**Note.** The following definition is a special case of Definition 5.4.1: if \( M \) in Definition 5.4.1 is given by the hom-module of a category \( C \), we have a cell \( (*E, C) \rightarrow (C : *E) \) from the module of cones \( *E \rightarrow C \) (see Definition 4.9.3) to the module of the wedges \( C \rightarrow *E \) (see Definition 4.10.3).

### 5.4.6 Definition.

Let \( E \) and \( C \) be categories.

- The right generalized Yoneda morphism for \( \langle *E, C \rangle \) [op. \( \langle E^*, C \rangle \)] is the cell
\[ \begin{array}{ccc}
C & \xrightarrow{\alpha} & [E, C] \\
\xymatrix{C \ar[r]^{\cdot (E \cdot C)} & (C : *E) \ar[r]_{\cdot E \cdot C} & \langle C \rangle \ar[r]_{\cdot C \cdot E} & \langle C \rangle}
\end{array} \]
\[ \begin{array}{ccc}
[C] & \xrightarrow{\cdot (E \cdot C)} & [E : C] \\
\xymatrix{[C] \ar[r]_{\cdot (E \cdot C)} & [E : C] \ar[r]_{\cdot E \cdot C} & \langle E : C \rangle \ar[r]_{\cdot E \cdot C} & \langle E : C \rangle}
\end{array} \]
sending each cone
\[ \alpha : c \twoheadrightarrow L : *E \rightarrow C \quad \text{op.} \quad \alpha : L \twoheadrightarrow c : E^* \rightarrow C \]
to the wedge
\[ \langle C \rangle \alpha : (C) c \twoheadrightarrow (C) L : C \rightarrow *E \quad \text{op.} \quad \langle C \rangle \alpha : (C) L \twoheadrightarrow (C) C : C \rightarrow E^* \]

- The left generalized Yoneda morphism for \( \langle *E, C \rangle \) [op. \( \langle E^*, C \rangle \)] is the cell
\[ \begin{array}{ccc}
C & \xrightarrow{\alpha} & [E, C] \\
\xymatrix{C \ar[r]^{\cdot (E \cdot C)} & (C : *E) \ar[r]_{\cdot E \cdot C} & \langle C \rangle \ar[r]_{\cdot C \cdot E} & \langle C \rangle}
\end{array} \]
\[ \begin{array}{ccc}
[C] & \xrightarrow{\cdot (E \cdot C)} & [E : C] \\
\xymatrix{[C] \ar[r]_{\cdot (E \cdot C)} & [E : C] \ar[r]_{\cdot E \cdot C} & \langle E : C \rangle \ar[r]_{\cdot E \cdot C} & \langle E : C \rangle}
\end{array} \]
sending each cone
\[ \alpha : c \twoheadrightarrow L : *E \rightarrow C \quad \text{op.} \quad \alpha : L \twoheadrightarrow c : E^* \rightarrow C \]
to the wedge
\[ \alpha(C) : \text{L}(C) \simeq \text{c}(C) : E^* \to C \quad \text{op.} \quad \alpha(C) : c(C) \simeq \text{L}(C) : *E \to C. \]

5.4.7 Remark.
(1) By Remark 4.9.4(2) and Remark 4.10.4, the cell \( \langle C \rhd *E \rangle \) is obtained from the hom-cell of the right generalized Yoneda functor for \([E, C]\) by the pasting composition

\[
\begin{array}{c}
\begin{array}{c}
E \xrightarrow{\text{E} \rhd \text{C} \xrightarrow{\text{E} \rhd \text{C}}} \text{E} \xrightarrow{\text{E} \rhd \text{C}} \text{E} \\
\end{array}
\end{array}
\]

with the commutative diagram in Proposition 2.3.12, and the cell \( \langle *E \backslash \text{C} \rangle \) is obtained from the hom-cell of the left generalized Yoneda functor for \([E, C]\) by the pasting composition

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{E} \xrightarrow{\text{E} \rhd \text{C}} \text{E} \xrightarrow{\text{E} \rhd \text{C}} \text{E} \\
\end{array}
\end{array}
\end{array}
\]

(2) Comparing the compositions above and those in Remark 5.4.2, and noting Proposition 5.3.9, we see that the right and left generalized Yoneda morphisms for \( \langle *E, C \rangle \) are just special instances of the right and left generalized Yoneda morphism for \( \langle *E, M \rangle \) where \( M \) is given by the hom-module of \( C \); that is,

\[ \langle C \rhd *E \rangle = \langle C \uparrow \langle C \rangle \rangle \rhd *E \quad \text{op.} \quad \langle C \rhd *E \rangle = \langle C \uparrow \langle C \rangle \rangle \rhd *E \]

and

\[ \langle *E \backslash \text{C} \rangle = *E \backslash \langle \langle C \rangle \rangle \quad \text{op.} \quad \langle *E \backslash \text{C} \rangle = *E \backslash \langle \langle C \rangle \rangle \]

Note. The following is a special case of Theorem 5.4.3 where \( M \) is given by the hom-module of a category.

5.4.8 Theorem. The generalized Yoneda morphism, right or left, for \( \langle *E, C \rangle \) is fully faithful.

Proof. Since the cell \( \langle C \rhd *E \rangle \) is obtained from the cell \( \langle C \rhd E \rangle \) (the hom-cell of the right generalized Yoneda functor for \([E, C]\)) by the pasting composition in Remark 5.4.7(1), the fully faithfulness of \( \langle C \rhd *E \rangle \) follows from that of \( \langle C \rhd E \rangle \) (see Corollary 5.3.15) by applying Proposition 1.2.35. \( \square \)

Note. The following is a special case of Theorem 5.4.4 where \( M \) is given by the hom-module of a category: the generalized Yoneda morphism \( \langle C \rhd *E \rangle : \langle *E, C \rangle \to \langle C : *E \rangle \) in Definition 5.4.1 is given as the postcomposition cell \( \langle *E, C \rangle \) \( \to \langle *E, (C :) \rangle \) the Yoneda functor \( \langle C \rhd : \rangle \) \( : \to \langle C :) \) yields (recall from Definition 4.9.5 that any functor \( H : C \to B \) yields the postcomposition cell \( \langle *E, H \rangle : \langle *E, C \rangle \to \langle *E, B \rangle \) for a given category \( E \).

5.4.9 Theorem. Given categories \( E \) and \( C \),

\[ \text{the triangle} \]

\[ C \rhd *E \xrightarrow{\langle *E, C \rangle} \langle *E, (C :) \rangle \quad \text{op.} \quad C \rhd E^* \xrightarrow{\langle E^*, C \rangle} \langle E^*, (C :) \rangle \]

\[ \langle C : *E \rangle \xrightarrow{\varphi} \langle *E, (C :) \rangle \quad \text{op.} \quad \langle C : E^* \rangle \xrightarrow{\varphi} \langle E^*, (C :) \rangle \]

\( \varphi \)
commutes; that is, the composition

\[
\begin{array}{ccc}
C \rightarrow [E, C] & \rightarrow [E, E, C] & \rightarrow [E, C] \\
\downarrow & \downarrow & \downarrow \\
[C] & \rightarrow [C, E] & \rightarrow [C, E] \\
\uparrow & \uparrow & \uparrow \\
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C]
\end{array}
\]

\[
\begin{array}{ccc}
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

of the right generalized Yoneda morphism for \((\ast E, C)\) \([\text{op.} \ (E, C)]\) and the right exponential transposition of wedges \(C \rightarrow \ast C \ [\text{op.} \ C \rightarrow E] \) yields the postcomposition cell

\[
\begin{array}{ccc}
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C] \\
\downarrow & \downarrow & \downarrow \\
[C] & \rightarrow [C, E] & \rightarrow [C, E] \\
\uparrow & \uparrow & \uparrow \\
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C]
\end{array}
\]

\[
\begin{array}{ccc}
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

—postcomposition with the right Yoneda functor for \(C\).

The triangle

\[
\begin{array}{ccc}
\ast E \times C & \rightarrow & \ast E \times (C) \\
\downarrow & \downarrow & \downarrow \\
\ast E \times C & \rightarrow & \ast E \times (C)
\end{array}
\]

commutes; that is, the composition

\[
\begin{array}{ccc}
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C] \\
\downarrow & \downarrow & \downarrow \\
[C] & \rightarrow [C, E] & \rightarrow [C, E] \\
\uparrow & \uparrow & \uparrow \\
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C]
\end{array}
\]

\[
\begin{array}{ccc}
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

of the left generalized Yoneda morphism for \((\ast E, C)\) \([\text{op.} \ (E, C)]\) and the left exponential transposition of wedges \(E \rightarrow C \ [\text{op.} \ E \rightarrow C]\) yields the postcomposition cell

\[
\begin{array}{ccc}
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C] \\
\downarrow & \downarrow & \downarrow \\
[C] & \rightarrow [C, E] & \rightarrow [C, E] \\
\uparrow & \uparrow & \uparrow \\
C \rightarrow [E, C] & \rightarrow [E, C] & \rightarrow [E, C]
\end{array}
\]

\[
\begin{array}{ccc}
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\text{op.} & & \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

—postcomposition with the left Yoneda functor for \(C\).

Proof. Since \((C \triangleright \ast E) = (C \uparrow (C)) \triangleright \ast E\) (see Remark 5.4.7(2)) and \((C \triangleright) = (C \uparrow (C)) \triangleright\) (see Proposition 5.2.6), the desired commutative diagram is obtained from that in Theorem 5.4.4 by replacing \(M\) with the hom-module of \(C\).

\begin{flushright}
\Box
\end{flushright}

5.4.10 Remark. Theorem 5.4.9 says that

- for any cone

\[
\alpha : c \rightarrow L : \ast E \rightarrow C \quad \text{op.} \quad \alpha : L \rightarrow c : E \rightarrow C
\]

the identity

\[
\{(C) \triangleright \alpha\} = [C \triangleright] \delta \alpha
\]

holds; that is, the right exponential transpose of the wedge

\[
\{C\} \alpha : \langle C \rangle c \rightarrow \langle C \rangle L : C \rightarrow \ast E \quad \text{op.} \quad \{C\} \alpha : \langle C \rangle L \rightarrow \langle C \rangle c : C \rightarrow E
\]
is the cone given by the composition
\[
\begin{array}{ccc}
\ast & \rightarrow & E \\
\vert & \alpha & \vert \\
C & \rightarrow (\L C) & \rightarrow C \\
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E & \rightarrow & \ast \\
\vert & \alpha & \vert \\
C & \rightarrow (\L C) & \rightarrow C \\
\end{array}
\]
of \(\alpha\) and the right Yoneda functor for \(C\).

* for any cone
\[
\alpha : c \rightsquigarrow L : \ast E \rightarrow C \quad \text{op.} \quad \alpha : L \rightsquigarrow c : E \ast \rightarrow C
\]
the identity
\[
\preceq \{\alpha(C)\} = \alpha \circ [\preceq C]
\]
holds; that is, the left exponential transpose of the wedge
\[
\alpha(C) : L(C) \rightarrow c(C) : E \ast \rightarrow C \quad \text{op.} \quad \alpha(C) : c(C) \rightsquigarrow L(C) : \ast E \rightarrow C
\]
is the cone given by the composition
\[
\begin{array}{ccc}
\ast & \rightarrow & E \\
\vert & \alpha & \vert \\
C & \rightarrow (\L C) & \rightarrow C \\
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
E & \rightarrow & \ast \\
\vert & \alpha & \vert \\
C & \rightarrow (\L C) & \rightarrow C \\
\end{array}
\]
of \(\alpha\) and the left Yoneda functor for \(C\).

## 5.5 Correspondences between frames and cells

In this section, we establish a variety of bijective correspondences between frames and cells using the generalized Yoneda lemma presented in Section 5.3. For a functor \(K : E \rightarrow D\) and a module \(M : X \rightarrow A\), Theorem 5.5.1 establishes an isomorphism \((K \cdot M) \cong \langle \langle D \rangle K, M \rangle\) between the module of weighted cylinders (see Definition 4.5.3) from \(K\) to \(M\) and the module of cells (see Definition 1.2.8) from the representable module of \(K\) to \(M\). This isomorphism allows us to transform a cylindrical extension into a cellular extension in Chapter 12. Theorem 5.5.1 turns out to be very powerful and versatile. All results in this section (and the generalized Yoneda lemma itself) are derived from this single theorem as special instances of it.

Note. We saw in Example 5.3.5(3) that the generalized Yoneda morphism transforms a weighted cylinder into a cell. The following corollary of Theorem 5.3.13 states the bijectivity of this transformation.

### 5.5.1 Theorem. Given a functor \(K : E \rightarrow D\) and a module \(M : X \rightarrow A\),

* the assignment

\[
\begin{array}{ccc}
D & \xrightarrow{K} & E \\
\vert & \alpha & \vert \\
X & \xrightarrow{-M} & A \\
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
D & \xrightarrow{(D)K} & E \\
\vert & \alpha & \vert \\
X & \xrightarrow{-M} & A \\
\end{array}
\]

yields a module isomorphism
\[
\Psi_{\alpha, M}^{K} : (K \cdot M) \cong \langle \langle D \rangle K, M \rangle : [D, X] \rightarrow [E, A]
\]
, natural in \(K\) and \(M\), from the module of right \(K\)-weighted cylinders along \(M\) to the module of
cells \( (D)K \to M \), whose inverse

\[
\begin{align*}
D \xrightarrow{K} E & \quad \leftrightarrow \quad D \xrightarrow{(D)K} E \\
G \downarrow \, D \theta \downarrow F & \quad \text{and} \quad G \downarrow \theta \downarrow F
\end{align*}
\]

sends each cell \( \theta \) to the right \( K \)-weighted cylinder \( D \downarrow \theta \) defined by

\[
[D \downarrow \theta]_e = 1_{(e; K)} : \theta
\]

for \( e \in \parallel E \parallel \), where \( 1_{(e; K)} : \theta \) is the image of the identity \( D \)-arrow \( e : K \to K \cdot e \), i.e. \( (D)K \)-arrow \( e : K \rightrightarrows e \) under the cell \( \theta \); moreover, \( \Psi^K_M \) is natural in \( M \) with \( M \) varying in \( \text{MOD} \); that is, the square

\[
\begin{align*}
\langle K \cdot \mathcal{M} \rangle & \xrightarrow{\psi^K_M} \langle (D)K, \mathcal{M} \rangle \\
\langle (K \cdot \mathcal{M}) \rangle & \quad \downarrow \langle (D)K, \mathcal{M} \rangle
\end{align*}
\]

commutes for every cell \( \psi : M \to N \).

- the assignment

\[
\begin{align*}
E \xrightarrow{K} D & \quad \leftrightarrow \quad E \xrightarrow{(D)K} D \\
G \downarrow \theta \downarrow D & \quad \text{and} \quad G \downarrow \theta \downarrow D
\end{align*}
\]

yields a module isomorphism

\[
\Psi^K_M : \langle K \cdot \mathcal{M} \rangle \cong \langle (D)K, \mathcal{M} \rangle : [E, X] \to [D, A]
\]

, natural in \( K \) and \( M \), from the module of left \( K \)-weighted cylinders along \( M \) to the module of cells \( K(D) \to M \), whose inverse

\[
\begin{align*}
E \xrightarrow{K} D & \quad \leftrightarrow \quad E \xrightarrow{(D)K} D \\
G \downarrow \theta \downarrow D & \quad \text{and} \quad G \downarrow \theta \downarrow D
\end{align*}
\]

sends each cell \( \theta \) to the left \( K \)-weighted cylinder \( \theta \downarrow D \) defined by

\[
[\theta \downarrow D]_e = \theta : 1_{(e; K)}
\]

for \( e \in \parallel E \parallel \), where \( \theta : 1_{(e; K)} \) is the image of the identity \( D \)-arrow \( e : K \to K \cdot e \), i.e. \( K(D) \)-arrow \( e \rightrightarrows K \cdot e \), under the cell \( \theta \); moreover, \( \Psi^K_M \) is natural in \( M \) with \( M \) varying in \( \text{MOD} \); that is, the square

\[
\begin{align*}
\langle K \cdot \mathcal{M} \rangle & \xrightarrow{\psi^K_M} \langle (D)K, \mathcal{M} \rangle \\
\langle (K \cdot \mathcal{M}) \rangle & \quad \downarrow \langle (D)K, \mathcal{M} \rangle
\end{align*}
\]

commutes for every cell \( \psi : M \to N \).

\textbf{Proof.} In Remark 5.3.21(1), replace \( \mathcal{M} : X \to A \) with \( G(\mathcal{M}) F : D \to E \) and replace \( G : A \to X \) with \( K : E \to D \). Then we have a bijection

\[
(K) (D \downarrow E) (G(\mathcal{M}) F) : (K) (D \cdot \mathcal{E}) (G(\mathcal{M}) F) \cong ((D)K) (D : E) (G(\mathcal{M}) F)
\]

, natural in \( K, G, F, \) and \( \mathcal{M} \). But

\[
(K) (D \cdot \mathcal{E}) (G(\mathcal{M}) F) = \prod_E K(G(\mathcal{M}) F) = (K \circ G) (E, M)(F) = (G)(K \cdot \mathcal{M})(F)
\]

and

\[
((D)K) (D : E) (G(\mathcal{M}) F) = (G)(((D)K, \mathcal{M}))(F)
\]
Theorem 5.3.24 is also viewed as a special case of Theorem 5.5.1 by depicting natural transformations.

\[ \Psi^{K\iota}_{\mathcal{M}} : (K \iota, \mathcal{M}) \cong (\langle D \rangle K, \mathcal{M}) \]

is given by

\[ (G) (\Psi^{K\iota}_{\mathcal{M}})(F) = (K) (\langle D \rangle E) (G(\mathcal{M}) F) \]

The second assertion holds since for any cell \[ \xymatrix{ X \ar@{|-}[r]_{\iota} & A } \]
the square

\[ \xymatrix{ (G) (K \iota, \mathcal{M}) (F) \ar[r]^{(G)(\Psi^{K\iota}_{\mathcal{M}})(F)} \ar[d]_{(G)(K \iota, \mathcal{M})(\phi)} & (G) (\langle D \rangle K, \mathcal{M}) (F) \\ (G \circ \phi)(K \iota, \mathcal{M}) (Q \delta F) & (G \circ \phi)(\langle D \rangle K, \mathcal{M}) (Q \delta F) } \]

(i.e.

\[ \xymatrix{ (K) (D \circ E) (G(\mathcal{M}) F) \ar[r]^{(K)(D \circ E)(G(\mathcal{M}) F)} \ar[d]_{(K)(D \circ E)(G(\mathcal{M}) F)(\phi)} & (\langle D \rangle K) (D : E) (G(\mathcal{M}) F) \\ (K)(D \circ E)(G(\mathcal{M}) F)(\phi) } \]

) commutes by the naturality of \( D \circ E \).

\[ \square \]

5.5.2 Remark.

(1) Although derived as a special case of Corollary 5.3.17 (generalized Yoneda lemma), Theorem 5.5.1 is, as we will see below, more versatile; first of all, Corollary 5.3.17 itself becomes a special case of Theorem 5.5.1 where \( G \) and \( F \) are identities.

(2) Theorem 5.3.13 is viewed as a special case of Theorem 5.5.1 by depicting a cylinder

\[ \xymatrix{ G \ar@{|-}[r]_{\alpha} & E \ar@{|-}[r]_{F} & X \ar@{|-}[r]_{\iota} & A } \]

as \[ \xymatrix{ X \ar@{|-}[r]_{\iota} & A } \], \[ \xymatrix{ E \ar@{|-}[r]_{F} & A } \]

\[ \xymatrix{ \alpha \ar@{|-}[r]_{\iota} & A } \]

The category \( X \) acts on \( \alpha \) and generates a cell \( \xymatrix{ X \ar@{|-}[r]_{\iota} & A } \) direct along \( \alpha \), and the assignment \( \alpha \mapsto X \iota \alpha \) yields a bijection from the set of cylinders \( G \to F : E \to \mathcal{M} \) to the set of cells \( 1_{X} : F : (X)G \to \mathcal{M} \), i.e. the set of module morphisms \( (X)G \to (\mathcal{M})F : X \to E \).

\[ \bullet \]

The category \( A \) acts on \( \alpha \) and generates a cell \( \xymatrix{ E \ar@{|-}[r]_{F} & A } \) inverse along \( \alpha \), and the assignment \( \alpha \mapsto \alpha \iota A \) yields a bijection from the set of cylinders \( G \to F : E \to \mathcal{M} \) to the set of cells \( G \to 1_{A} : F(A) \to \mathcal{M} \), i.e. the set of module morphisms \( F(A) \to G(\mathcal{M}) : E \to A \).

(3) Theorem 5.3.24 is also viewed as a special case of Theorem 5.5.1 by depicting natural transformations \( \epsilon : G \delta F \to 1_{A} : A \to A \) [op. \( \eta : 1_{X} \to G \delta F : X \to X \)] as \[ \xymatrix{ X \ar@{|-}[r]_{\iota} & A } \], \[ \xymatrix{ E \ar@{|-}[r]_{F} & A } \]

\[ \xymatrix{ \alpha \ar@{|-}[r]_{\iota} & A } \]

The category \( X \) acts on \( \epsilon \) and generates a cell \( \xymatrix{ X \ar@{|-}[r]_{\iota} & A } \) direct along \( \epsilon \), and the assignment \( \epsilon \mapsto X \iota \epsilon \) yields a bijection from the set of natural transformations \( G \delta F \to 1_{A} : A \to A \) to the set of cells \( F \to 1_{A} : (X)G \to (A) \), i.e. the set of module morphisms \( (X)G \to F(A) : X \to A \).
The category $\mathbf{A}$ acts on $\eta$ and generates a cell $\mathbf{X} \xrightarrow{F(\alpha)} \mathbf{A}$ inverse along $\eta$, and the assignment $\eta \mapsto \eta | A$ yields a bijection from the set of natural transformations $1_{\mathbf{X}} \rightarrow G \circ F : \mathbf{X} \rightarrow \mathbf{X}$ to the set of cells $1_{\mathbf{X}} \rightarrow G : (\mathbf{A}) \rightarrow (\mathbf{X})$, i.e. the set of module morphisms $F(\mathbf{A}) \rightarrow (\mathbf{X}) G : \mathbf{X} \rightarrow \mathbf{A}$.

Note. The following is a special case of Theorem 5.5.1 where $K : E \rightarrow D$ is given by the identity functor $E \rightarrow E$.

5.5.3 Corollary. Given a category $E$ and a module $\mathcal{M} : \mathbf{X} \rightarrow \mathbf{A}$, there is a canonical module isomorphism

$$\Psi^E_{\mathcal{M}} : \langle (E), \mathcal{M} \rangle : [E, \mathbf{X}] \rightarrow [E, \mathbf{A}]$$

from the module of cylinders $E \rightarrow \mathcal{M}$ to the modules of cells $(E) \rightarrow \mathcal{M}$; the bijection

$$\begin{array}{ccc}
G & \xrightarrow{\alpha} & E \\
\downarrow F & \downarrow & \downarrow E \xrightarrow{(\alpha)} E \\
\mathbf{X} & \xrightarrow{\mathcal{M}} \xrightarrow{\gamma} & \mathbf{A}
\end{array}$$

sends each cylinder $\alpha$ to the cell $(\alpha)$ which maps each $E$-arrow $h : e \rightarrow e'$ to the $\mathcal{M}$-arrow

$$(h : G) \circ \alpha e' = \alpha e \circ (F \circ h)$$

as indicated in

$\begin{array}{ccc}
e & \xleftarrow{\cdot} & \mathbf{e} \\
\uparrow & \uparrow & \uparrow \\
h : \mathbf{G} & \xleftarrow{\cdot} & \mathbf{F} : \mathbf{e} \\
e' & \xleftarrow{\cdot} & \mathbf{e'}
\end{array}$

, and the inverse

$$\begin{array}{ccc}
E & \xleftarrow{\mathcal{M}} \xleftarrow{\mathcal{M}} & \mathbf{A} \\
\downarrow F & \downarrow \gamma & \downarrow E \xleftarrow{\mathcal{M}} E \\
\mathbf{X} & \xleftarrow{\mathcal{M}} \xleftarrow{\mathcal{M}} & \mathbf{A}
\end{array}$$

sends each cell $\theta$ to the cylinder $[\theta]$ defined by $[\theta]_e = \mathbf{e} \cdot \theta$

for $e \in \mathcal{E}$, where $1_e : \theta$ is the image of the identity $e \rightarrow e$ under the function

$e(\mathcal{E}) e \xrightarrow{e(\theta)} e(\mathcal{G}(\mathcal{M}) F) e = (e : \mathcal{G} (\mathcal{M}) (F \cdot e))$.

; moreover, $\Psi^E_{\mathcal{M}}$ is natural in $\mathcal{M}$ with $\mathcal{M}$ varying in MOD; that is, the square

$$\begin{array}{ccc}
\langle (E), \mathcal{M} \rangle & \xrightarrow{\phi^E_{\mathcal{M}}} & \langle (E), \mathcal{M} \rangle \\
(\mathcal{E}, \psi) \downarrow & \downarrow & \downarrow (\mathcal{E}, \psi) \\
\langle (E), \mathcal{N} \rangle & \xrightarrow{\phi^E_{\mathcal{N}}} & \langle (E), \mathcal{N} \rangle
\end{array}$$

commutes for every cell $\psi : \mathcal{M} \rightarrow \mathcal{N}$.

Proof. Since the hom-module $(E)$ is represented and corepresented by the identity functor $1_E : E \rightarrow E$, and since

$$(E, \mathcal{M}) = (1_E \cdot \mathcal{M}) = (1_E \triangleright \mathcal{M}) \quad \text{and} \quad (E, \psi) = (1_E \cdot \psi) = (1_E \triangleright \psi)$$

, respectively, by Remark 4.5.4(3) and Remark 4.5.8(2), the assertion follows as a special case of Theorem 5.5.1 where $K$ is given by $1_E$, with $\Psi^E_{\mathcal{M}}$ defined by

$\Psi^E_{\mathcal{M}} = \Psi^{1_E \mathcal{M}}_{\mathcal{M}} = \Psi^{1_E \mathcal{M}}_{\mathcal{M}}$;

the cell $(\alpha)$ is given by $E \triangleright \alpha = \alpha \triangleright E$ and the cylinder $[\theta]$ is given by $E \triangleright \theta = \theta \triangleright E$. \qed
5.5.4 Remark. If the module $\mathcal{M}$ in Corollary 5.5.3 is replaced by the hom-module of a category $\mathbf{C}$, we have a bijection between the set of natural transformations $\alpha : G \to F : E \to \mathbf{C}$ and the set of cells in $E^\Delta$. A morphism between two paralleling functors $G, F : E \to \mathbf{C}$ is thus defined either by $G \downarrow \alpha \downarrow F$ or $\alpha : E \downarrow \Delta$. The relation between cells and frames can be compared with that between linear maps and matrices.

Note. The following is a special case of Theorem 5.5.1 where $K : E \to D$ is given by the unique functor $\mathbf{C} \downarrow \alpha \downarrow E$ from the module of cones $G \downarrow \alpha \downarrow E$ to the terminal category. Recall from Example 1.1.31(11) that $\ast \Delta_E$ denotes the corepresentable module of the functor $\mathbf{1}_E : E \to \ast$.

5.5.5 Corollary. Given a category $\mathbf{E}$ and a module $\mathcal{M} : X \to \mathbf{A}$,
- there is a canonical module isomorphism
  $$\Psi^E_M : (\ast \Delta_E, \mathcal{M}) \cong (\ast \ast \Delta_E, \mathcal{M}) : X \to [E, A]$$

  from the module of cones $\ast \mathbf{E} \to \mathcal{M}$ to the module of conic cells $\ast \Delta_E \to \mathcal{M}$; the bijection

  \[
  \begin{array}{ccc}
  \ast & \mathbf{E} & \ast \\
  \downarrow & \Delta_E & \downarrow \\
  X \mathcal{M} & A & X \mathcal{M} = A \\
  \end{array}
  \]

  sends each cone $\alpha$ to the conic cell $(\alpha)$ which maps each $(\ast \Delta_E)$-arrow $\ast : \ast \to e$ to the component of $\alpha$ at $e$, and the inverse

  \[
  \begin{array}{ccc}
  \ast & \mathbf{E} & \ast \\
  \downarrow & \Delta_E & \downarrow \\
  X \mathcal{M} & A & X \mathcal{M} = A \\
  \end{array}
  \]

  sends each conic cell $\theta$ to the cone $[\theta]$ defined by

  $[\theta]_e = \ast \theta$

  for $e \in \mathbf{E}$, where $\ast \theta$ is the image of the $(\ast \Delta_E)$-arrow $\ast : \ast \to e$ under the cell $\theta$; moreover, $\Psi^E_M$ is natural in $\mathcal{M}$ with $\mathcal{M}$ varying in $\mathbf{MOD}$; that is, the square

  \[
  \begin{array}{ccc}
  (\ast \mathbf{E}, \mathcal{M}) & \Psi^E_M & (\ast \Delta_E, \mathcal{M}) \\
  \downarrow & \downarrow & \downarrow \\
  (\ast \mathbf{E}, \psi) & (\ast \Delta_E, \psi) & (\ast \mathbf{E}, \mathcal{N}) \\
  \end{array}
  \]

  commutes for every cell $\psi : \mathcal{M} \to \mathcal{N}$.
- there is a canonical module isomorphism

  $$\Psi^E_M : (E^* \mathcal{M}) \cong (\Delta_E^*, \mathcal{M}) : [E, X] \to \mathbf{A}$$

  from the module of cones $E^* \to \mathcal{M}$ to the module of conic cells $\Delta_E^* \to \mathcal{M}$; the bijection

  \[
  \begin{array}{ccc}
  E & \mathbf{E} & E \\
  \downarrow & \Delta_E^* & \downarrow \\
  X \mathcal{M} & A & X \mathcal{M} = A \\
  \end{array}
  \]

  sends each cone $\alpha$ to the conic cell $(\alpha)$ which maps each $(\Delta_E^*)$-arrow $\ast : e \to \ast$ to the component of $\alpha$ at $e$, and the inverse

  \[
  \begin{array}{ccc}
  E & \mathbf{E} & E \\
  \downarrow & \Delta_E^* & \downarrow \\
  X \mathcal{M} & A & X \mathcal{M} = A \\
  \end{array}
  \]
5.5. Correspondences between frames and cells

sends each conic cell \( \theta \) to the cone \([\theta]\) defined by
\[
[\theta]_e = \theta \circ \ast
\]
for \( e \in \vert E \vert \), where \( \ast : \theta \) is the image of the \((\Delta_E \ast)\)-arrow \( \ast : e \to \ast \) under the cell \( \theta \); moreover, \( \Psi^E_{\mathcal{M}} \) is natural in \( \mathcal{M} \) with \( \mathcal{M} \) varying in \( \text{MOD} \); that is, the square
\[
\begin{array}{ccc}
(E \ast, \mathcal{M}) & \xrightarrow{\psi \ast \mathcal{M}} & (\Delta_E \ast, \mathcal{M}) \\
\downarrow \phi & & \downarrow \Delta_E \ast \\
(E \ast, \mathcal{N}) & \xrightarrow{\psi \ast \mathcal{N}} & (\Delta_E \ast, \mathcal{N})
\end{array}
\]
commutes for every cell \( \psi : \mathcal{M} \to \mathcal{N} \).

**Proof.** Since \( \ast \Delta_E \) is the corepresentable module of the functor \( !_E : E \to \ast \), and since
\[
(\ast E, \mathcal{M}) = (\ast \mathcal{M}) \quad \text{and} \quad (\ast E, \psi) = (\ast \psi)
\]
, respectively, by Theorem 4.6.22 and Corollary 4.6.23, the assertion follows as a special case of Theorem 5.5.1 where \( K \) is given by \( !_E \), with \( \Psi^E_{\mathcal{M}} \) defined by
\[
\psi \ast \mathcal{M} = \psi \ast \mathcal{N}
\]
; the conic cell \((\alpha)\) is given by \( \ast \uparrow \alpha \) and the cone \([\theta]\) is given by \( \ast \uparrow \theta \).

\[\Box\]

5.5.6 Theorem. Given an endomodule \( \mathcal{M} : E \to E \), there is a canonical bijection
\[
\Phi^E_{\mathcal{M}} : \Pi_{E\mathcal{M}} \cong (\langle E \rangle)(E : E)(\mathcal{M})
\]
, from the set of frames of \( \mathcal{M} \) to the set of module morphisms \( \langle E \rangle : \mathcal{M} : E \to E \), sending each frame \( \alpha \) of \( \mathcal{M} \) to the module morphism \( \langle \alpha \rangle : \langle E \rangle \to \mathcal{M} \) which maps each \( E \)-arrow \( h : e \to e' \) to the \( \mathcal{M} \)-arrow
\[
h \circ \alpha \circ e = \alpha_{e'} \circ h
\]
as indicated in
\[
\begin{array}{ccc}
e & \xrightarrow{\alpha} & e' \\
\downarrow h & & \downarrow h' \\
e' & \xleftarrow{\alpha'} & e'
\end{array}
\]
, with the inverse sending each module morphism \( \theta : \langle E \rangle \to \mathcal{M} \) to the frame \([\theta]\) of \( \mathcal{M} \) defined by
\[
[\theta]_e = 1_e : \theta
\]
for \( e \in \vert E \vert \), where \( 1_e : \theta \) is the image of the identity \( e \to e \) under the function
\[
e(\theta) e : e(\langle E \rangle) e \to e(\mathcal{M}) e
\]
; moreover, the bijection is natural in \( \mathcal{M} \).

**Proof.** This follows from Corollary 5.5.3 by setting \( X = A = E \) and \( G = F = 1_E \) and noting that
\[
(1_E)(\langle E \rangle, \mathcal{M})(1_E) = \Pi_{E\mathcal{M}} \quad \text{and} \quad (1_E)(\langle E \rangle, \mathcal{M})(1_E) = (\langle E \rangle)(E : E)(\mathcal{M})
\]
, respectively, by Definition 4.3.7 and Definition 1.2.8.

\[\Box\]

**Note.** In Theorem 5.5.7 and Theorem 5.5.8, the hom-module of a category \( \mathcal{E} \) is regarded as a left module \( \langle \mathcal{E} \rangle : \ast \to \mathcal{E}^{-} \times \mathcal{E} \) [op. right module \( \langle \mathcal{E} \rangle : \mathcal{E} \times \mathcal{E}^{-} \to \ast \)].

5.5.7 Theorem.

\( \triangleright \) Given a left module \( \mathcal{M} : \ast \to \mathcal{E}^{-} \times \mathcal{E} \), there is a canonical bijection
\[
\Phi^E_{\mathcal{M}} : \Pi_{E\mathcal{M}} \cong (\langle E \rangle)(\mathcal{E}^{-} \times \mathcal{E})(\mathcal{M})
\]
, from the set of cylindrical frames of \( \mathcal{M} \) to the set of module morphisms \( \langle E \rangle : \mathcal{M} : \ast \to \mathcal{E}^{-} \times \mathcal{E} \), sending each cylindrical frame \( \alpha \) of \( \mathcal{M} \) to the module morphism \( \langle \alpha \rangle : \langle E \rangle \to \mathcal{M} \) which maps each \( E \)-arrow \( h : e \to e' \) to the \( \mathcal{M} \)-arrow
\[
\alpha_{e'} \circ (h, e') = \alpha_{e} \circ (e, h)
\]
as indicated in

\[
\begin{array}{ccc}
\ast & \overset{\alpha_e}{\longrightarrow} & (e, e') \\
\{ (e, h) \} & \overset{h : (\alpha)}{\longrightarrow} & (e, e') \\
(e', e') & \overset{\alpha_{e'}}{\longrightarrow} & (e', e')
\end{array}
\]

, with the inverse sending each module morphism \( \theta : (E) \rightarrow M \) to the cylindrical frame \([\theta] \) of \( M \) defined by

\[
[\theta]_e = \theta : 1_e
\]

for \( e \in \|E\| \), where \( 1_e : 1_e \) is the image of the identity \( e \rightarrow e \) under the function

\[
(\theta)(e, e) : (E) (e, e) \rightarrow (M) (e, e)
\]

; moreover, the bijection is natural in \( M \).

\* Given a right module \( M : E \times E^* \rightarrow * \), there is a canonical bijection

\[
\Phi_{M}^{E} : \prod_{E} M \cong ((E)) (E \times E^* : ) (M)
\]

, from the set of cylindrical frames of \( M \) to the set of module morphisms \( (E) \rightarrow M : E \times E^* \rightarrow * \), sending each cylindrical frame \( \alpha \) of \( M \) to the module morphism \( (\alpha) : (E) \rightarrow M \) which maps each \( E \)-arrow \( h : e \rightarrow e' \) to the \( M \)-arrow

\[
(h, e') \circ \alpha_{e'} = (e, h) \circ \alpha_e
\]

as indicated in

\[
\begin{array}{ccc}
(e, e') & \overset{(h, e')}{\longrightarrow} & (e', e') \\
\{ (e, h) \} & \overset{h : (\alpha)}{\longrightarrow} & \{ (e', h) \}
\end{array}
\]

, with the inverse sending each module morphism \( \theta : (E) \rightarrow M \) to the cylindrical frame \([\theta] \) of \( M \) defined by

\[
[\theta]_e = 1_e : \theta
\]

for \( e \in \|E\| \), where \( 1_e : \theta \) is the image of the identity \( e \rightarrow e \) under the function

\[
(e, e)(\theta) : (e, e) (E) \rightarrow (e, e) (M)
\]

; moreover, the bijection is natural in \( M \).

Proof. This is a restatement of Theorem 5.5.6 with \( M : E \rightarrow E \) regarded as a left module \( (E) : * \rightarrow E^* \times E \).

\[\square\]

5.5.8 Theorem. Given a category \( E \) and a module \( M : X \rightarrow A \),
\* there is a canonical module isomorphism

\[
\Psi_{M}^{E} : (E, M) \cong ((E), M) : X \rightarrow \left[ E^* \times E, A \right]
\]

from the module of extraordinary cylinders \( E \rightarrow M \) to the module of cells \( (E) \rightarrow M \); the bijection

\[
\begin{array}{ccc}
\ast & \overset{E^* \times E}{\longrightarrow} & \ast \\
\{ (\alpha) \} & \overset{F}{\longrightarrow} & \{ (\alpha) \}
\end{array}
\]

\[
\begin{array}{ccc}
X & \overset{M}{\longrightarrow} & A \\
\{ (\alpha) \} & \overset{F}{\longrightarrow} & \{ (\alpha) \}
\end{array}
\]

sends each cylinder \( \alpha \) to the cell \( (\alpha) \) which maps each \( E \)-arrow \( h : e \rightarrow e' \) to the \( M \)-arrow

\[
\alpha_{e'} \circ F(h, e') = \alpha_e \circ F(e, h)
\]

as indicated in

\[
\begin{array}{ccc}
X & \overset{\alpha_e}{\longrightarrow} & F(e, e) \\
\{ (\alpha) \} & \overset{h}{\longrightarrow} & \{ F(e, h) \}
\end{array}
\]

\[
\begin{array}{ccc}
F(e', e') & \overset{F(h, e')}{\longrightarrow} & F(e, e')
\end{array}
\]
and the inverse

\[ \begin{array}{c c c}
\star & E^e \times E & \leftrightarrow & E^e \times E \\
\uparrow & [\theta] & \uparrow & \frac{\theta}{F} \\
X - \frac{\mathcal{M}}{A} & \cdot & X - \frac{\mathcal{M}}{A}
\end{array} \]

sends each cell \( \theta \) to the cylinder \([\theta]\) defined by

\[ [\theta]_e = \theta:1_e \]

for \( e \in \|E\| \), where \( \theta:1_e \) is the image of the identity \( e \to e \) under the function

\[ (E)(e,e) \xrightarrow{(e,e)(\theta)} (x(M)F)(e,e) = x(M)(F(e,e)) \]

; moreover, \( \Psi^E_{\mathcal{M}} \) is natural in \( \mathcal{M} \) with \( \mathcal{M} \) varying in \( \text{MOD} \); that is, the square

\[ \begin{array}{c c c}
\langle E,\mathcal{M} \rangle & \xrightarrow{\Psi^E_{\mathcal{M}}} & \langle (E),\mathcal{M} \rangle \\
\langle (E,\psi) \rangle & \downarrow & \downarrow \langle (E,\psi) \rangle \\
\langle E,\mathcal{N} \rangle & \xrightarrow{\Psi^E_{\mathcal{N}}} & \langle (E),\mathcal{N} \rangle
\end{array} \]

commutes for every cell \( \psi : \mathcal{M} \to \mathcal{N} \).

there is a canonical module isomorphism

\[ \Psi^E_{\mathcal{M}} : \langle E^e,\mathcal{M} \rangle \cong \langle (E^e),\mathcal{M} \rangle : [E^e \times E, X] \to A \]

from the module of extraordinary cylinders \( G \to a : E^e \to \mathcal{M} \to \mathcal{M} \); the bijection

\[ \begin{array}{c c c}
E^e \times E & \star & \leftrightarrow & E^e \times E \\
\uparrow \alpha & \uparrow a & \uparrow \frac{\alpha}{A} & \uparrow a \\
X - \frac{\mathcal{M}}{A} & \cdot & X - \frac{\mathcal{M}}{A}
\end{array} \]

sends each cylinder \( \alpha \) to the cell \( \langle \alpha \rangle \) which maps each \( E^e \)-arrow \( h : e' \to e \) to the \( \mathcal{M} \)-arrow

\[ G(e',h) \circ \alpha_e' = G(h,e) \circ \alpha_e \]

as indicated in

\[ G(e',e) \xrightarrow{G(e',h)} G(e',e') \]

\[ \xrightarrow{G(h,e)} \xrightarrow{h: \langle \alpha \rangle} \xrightarrow{\alpha_e} \xrightarrow{a} a \]

, and the inverse

\[ \begin{array}{c c c}
E^e \times E & \star & \leftrightarrow & E^e \times E \\
\uparrow & [\theta] & \uparrow & \frac{\theta}{F} \\
X - \frac{\mathcal{M}}{A} & \cdot & X - \frac{\mathcal{M}}{A}
\end{array} \]

sends each cell \( \theta \) to the cylinder \([\theta]\) defined by

\[ [\theta]_e = 1_e : \theta \]

for \( e \in \|E\| \), where \( \theta:1_e \) is the image of the identity \( e \to e \) under the function

\[ (e,e)(\langle E^e \rangle) \xrightarrow{(e,e)(\theta)} (e,e)(G\{M\}a) = (G(e,e))(\langle M \rangle a) \]

; moreover, \( \Psi^E_{\mathcal{M}} \) is natural in \( \mathcal{M} \) with \( \mathcal{M} \) varying in \( \text{MOD} \); that is, the square

\[ \begin{array}{c c c}
\langle E^e,\mathcal{M} \rangle & \xrightarrow{\Psi^E_{\mathcal{M}}} & \langle (E^e),\mathcal{M} \rangle \\
\langle (E^e,\psi) \rangle & \downarrow & \downarrow \langle (E^e,\psi) \rangle \\
\langle E^e,\mathcal{N} \rangle & \xrightarrow{\Psi^E_{\mathcal{N}}} & \langle (E^e),\mathcal{N} \rangle
\end{array} \]

commutes for every cell \( \psi : \mathcal{M} \to \mathcal{N} \).
Proof. Replacing \( M \) with \( x(\langle M \rangle)F \) in Theorem 5.5.7, we have a bijection
\[
\Phi_{x(\langle M \rangle)F}^E: \Pi_E x(\langle M \rangle)F \cong ((E)) (E^{-} \times E) (x \langle M \rangle)F
\]
, natural in \( x \) and \( F \). But
\[
\Pi_E x(\langle M \rangle)F = (x) \langle E, M \rangle (F)
\]
and
\[
((E)) (E^{-} \times E) (x \langle M \rangle)F = (x) \langle \langle E \rangle, M \rangle (F)
\]
by the definitions. Hence the isomorphism
\[
\Psi^E_{\langle M \rangle} : \langle E, M \rangle \cong \langle \langle E \rangle, M \rangle
\]
is given by
\[
(x) \langle \Psi^E_{\langle M \rangle} \rangle (F) = \Phi^E_{x(\langle M \rangle)F}.
\]

The last assertion holds since for any cell \( \begin{array}{c}
X \xrightarrow{M} A \\
P \xrightarrow{\psi} Q \\
Y \xrightarrow{\varphi} B
\end{array} \),
the square
\[
\begin{array}{c}
\xrightarrow{(x \cdot P) \langle E, N \rangle (Q \varphi F)}
\end{array}
\]
(i.e. \( \Pi_E x(\langle M \rangle)F \xrightarrow{\phi^E_{x(\langle M \rangle)F}} ((E)) (E^{-} \times E) (x \langle M \rangle)F \))
commutes by the naturality of \( \Phi^E_{\langle M \rangle} \). \qed
6 Universals

6.1 Units of one-sided modules

The notion of universality is ultimately formulated by a unit\(^1\) of a one-sided module. A unit of a right module \(M : X \rightarrow \ast\) is an \(M\)-arrow \(u : r \rightarrow \ast\) satisfying the following universal mapping property: to every \(M\)-arrow \(m : x \rightarrow \ast\) there is a unique \(X\)-arrow \(m/u : x \rightarrow r\) (the adjunct of \(m\) along \(u\)) making the diagram

\[
\begin{array}{c}
\ast \\
\downarrow m \\
x
\end{array}
\begin{array}{c}
\ast \\
\downarrow m/u \\
r
\end{array}
\]

commute. A unit of a left module is defined dually. An arrow of a one-sided module is a unit precisely when the Yoneda morphism sends it to a module isomorphism, and we define a unit in Definition 6.1.1 using this characterization. By virtue of the Yoneda lemma, units and representations of a one-sided module correspond one-to-one. Regarding a one-sided module cell, we define the notion of preservation, reflection, and creation of units.

A unit of a one-sided module is essentially the same thing as a universal arrow of a two-sided module to be introduced in Section 6.2; a unit may be alternatively called a universal arrow of a one-sided module.

Note. As mentioned above, we use the mechanism described in Remark 5.2.4(2) in the following.

6.1.1 Definition.

\(^\wedge\) An arrow \(u : r \rightarrow \ast\) of a right module \(M : X \rightarrow \ast\) is called a unit if the right module morphism \(X/u : (X)r \rightarrow M : X \rightarrow \ast\) is iso.

\(^\wedge\) An arrow \(u : \ast \rightarrow r\) of a left module \(M : \ast \rightarrow A\) is called a unit if the left module morphism \(u/A : r(A) \rightarrow M : \ast \rightarrow A\) is iso.

6.1.2 Remark.

(1) A right [op. left] module has a unit if and only if it is representable. In fact, by the Yoneda lemma (Theorem 5.2.8), representations and units correspond one-to-one:

\(^\wedge\) if \(u : r \rightarrow \ast\) is a unit of a right module \(M : X \rightarrow \ast\), then the object \(r\) represents \(M\) by the isomorphism \(X/u : (X)r \rightarrow M\); conversely, if an object \(r \in \lVert X \rVert\) and a right module isomorphism \(\phi : (X)r \rightarrow M\) form a representation of \(M\), then the \(M\)-arrow \(1_r : \phi : r \rightarrow \ast\) is a unit of \(M\).

\(^\wedge\) if \(u : \ast \rightarrow r\) is a unit of a left module \(M : \ast \rightarrow A\), then the object \(r\) represents \(M\) by the isomorphism \(u/A : r(A) \rightarrow M\); conversely, if an object \(r \in \lVert A \rVert\) and a left module isomorphism \(\phi : r(A) \rightarrow M\) form a representation of \(M\), then the \(M\)-arrow \(1_r : \phi : \ast \rightarrow r\) is a unit of \(M\).

(2) \(^\wedge\) Let \(u : r \rightarrow \ast\) be a unit of a right module \(M : X \rightarrow \ast\). Given an \(M\)-arrow \(m : x \rightarrow \ast\), its inverse image under \(X/u\) is called the adjunct of \(m\) along \(u\) and written \(m/u\); that is,

\[
m/u := m : (X/u)^{-1}
\]

; this is the unique \(X\)-arrow \(x \rightarrow r\) making the triangle

\[
\begin{array}{c}
\ast \\
\downarrow m \\
x
\end{array}
\begin{array}{c}
\ast \\
\downarrow m/u \\
r
\end{array}
\]

commute. An \(M\)-arrow \(u : r \rightarrow \ast\) is a unit if and only if to every \(M\)-arrow \(m : x \rightarrow \ast\) there is a unique \(X\)-arrow \(m/u : x \rightarrow r\) as above.

\(^\wedge\) Let \(u : \ast \rightarrow r\) be a unit of a left module \(M : \ast \rightarrow A\). Given an \(M\)-arrow \(m : \ast \rightarrow a\), its inverse

\(^1\)A unit is called a universal element in the literature.
image under \( u \uparrow A \) is called the adjunct of \( m \) along \( u \) and written \( u \downarrow m \); that is,
\[
u \downarrow m := (u \uparrow A)^{-1} : m\]
; this is the unique \( A \)-arrow \( r \to a \) making the triangle
\[
\begin{tikzcd}
\ast & r \\
& m \\
a & u \downarrow m
\end{tikzcd}
\]
commute. An \( M \)-arrow \( u : \ast \to r \) is a unit if and only if to every \( M \)-arrow \( m : \ast \to a \) there is a unique \( A \)-arrow \( u \downarrow m : r \to a \) as above.

### 6.1.3 Proposition.
- An arrow of a right module \( M \) is a unit if and only if it is a terminal object of the comma category \([M]\).
- An arrow of a left module \( M \) is a unit if and only if it is an initial object of the comma category \([M]\).

**Proof.** Immediate from Remark 6.1.2(2). \(\square\)

### 6.1.4 Proposition.
- A right module \( M \) is representable if and only if the comma category \([M]\) has a terminal object.
- A left module \( M \) is representable if and only if the comma category \([M]\) has an initial object.

**Proof.** Since \( M \) is representable iff it has a unit (see Remark 6.1.2(1)), this is immediate from Proposition 6.1.3. \(\square\)

### 6.1.5 Theorem.
- Suppose that a right module \( M : X \to \ast \) has a unit \( u : r \to \ast \). Then an \( X \)-arrow \( f : s \to r \) is iso if and only if the composite \( f \circ u : s \to \ast \) is a unit of \( M \); to put it the other way round, an \( M \)-arrow \( v : s \to \ast \) is a unit if and only if its adjunct \( v/u : s \to r \) along \( u \) is an iso \( X \)-arrow.
- Suppose that a left module \( M : \ast \to A \) has a unit \( u : \ast \to r \). Then an \( A \)-arrow \( f : r \to s \) is iso if and only if the composite \( u \circ f : \ast \to s \) is a unit of \( M \); to put it the other way round, an \( M \)-arrow \( v : \ast \to s \) is a unit if and only if its adjunct \( u/v : r \to s \) along \( u \) is an iso \( A \)-arrow.

**Proof.** By the naturality of the Yoneda morphism
\[
X \uparrow (f \circ u) = (X) f \circ X \uparrow u.
\]
Since \( u \) is a unit, \( X \uparrow u \) is iso. Hence \( X \uparrow (f \circ u) \) is iso iff \( (X) f \) is iso. But \( X \uparrow (f \circ u) \) is iso iff \( f \circ u \) is a unit, and, since the Yoneda functor is fully faithful, \( (X) f \) is iso iff \( f \) is iso. \(\square\)

**Note.** By Theorem 6.1.5, if \( u : r \to \ast \) and \( v : s \to \ast \) are two units of a right module, then the adjunct \( v/u : s \to r \) of \( v \) along \( u \) and the adjunct \( u/v : r \to s \) of \( u \) along \( v \) are isomorphisms. In fact, they are the inverse of each other as will be shown below.

### 6.1.6 Theorem.
- If \( u : r \to \ast \) and \( v : s \to \ast \) are two units of a right module, then the adjunct \( v/u : s \to r \) of \( v \) along \( u \) and the adjunct \( u/v : r \to s \) of \( u \) along \( v \) (shown in
\[
\begin{tikzcd}
S & \ast \\
& r \\
v/u \downarrow & u/v \downarrow
\end{tikzcd}
\]
) are the inverse of each other.
If \( u: * \to r \) and \( v: * \to s \) are two units of a left module, then the adjunct \( u \downarrow v: r \to s \) of \( v \) along \( u \) and the adjunct \( v \downarrow u: s \to r \) of \( u \) along \( v \) (shown in

\[
\begin{array}{c}
u \\
\downarrow \quad v \downarrow u \\
\uparrow \\
u \downarrow v
\end{array}
\]

) are the inverse of each other.

Proof. Since

\[
u/\downarrow v \downarrow u = u/\downarrow v \downarrow v = u
\]

and \( u \) is a unit, \( u/\downarrow v \downarrow v = 1_r \) by the uniqueness of the factorization. Symmetrically, \( v/\downarrow u \downarrow v = 1_s \).

1. Prove the following

6.1.7 Corollary. If a right \( \text{op. left} \) module is representable, then a representing object is unique up to isomorphism.

Proof. Since a representing object has a unit associated with it (see Remark 6.1.2(1)), this is immediate from Theorem 6.1.6 (or Theorem 6.1.5).

6.1.8 Theorem.

- Consider a composable pair of a functor and a right module as in

\[
E \xrightarrow{G} X \xrightarrow{M} *
\]

with \( G \) fully faithful. If an \( M \)-arrow \( u: r \Rightarrow * \) is a unit (of \( M \)), then the \( G(M) \)-arrow \( u: r \Rightarrow * \) is a unit (of the composite right module \( G(M) \)).

- Consider a composable pair of a functor and a left module as in

\[
* \xrightarrow{M} A \xleftarrow{F} E
\]

with \( F \) fully faithful. If an \( M \)-arrow \( u: * \Rightarrow r \) is a unit (of \( M \)), then the \( (M) F \)-arrow \( u: * \Rightarrow r \) is a unit (of the composite left module \( (M) F \)).

Proof. By Example 5.2.7(2), the diagram

\[
\begin{array}{ccc}
\langle E \rangle r & \xrightarrow{(G) r} & G \langle X \rangle G r \\
\downarrow \cong & & \downarrow \cong \\
G \langle M \rangle & \xrightarrow{(G \langle X \downarrow u \rangle)} & G \langle M \rangle
\end{array}
\]

commutes. Hence to show that \( u: r \Rightarrow * \) is a unit of \( G \langle M \rangle \) (i.e. \( E \uparrow u \) is iso), it suffices to show that \( (G) r \) and \( G \langle X \uparrow u \rangle \) are iso. But \( (G) r \) is iso because \( G \) is fully faithful, and \( X \uparrow u \) is iso because \( u: r \Rightarrow G \Rightarrow * \) is a unit, and so is \( G \langle X \uparrow u \rangle \) by Proposition 1.1.33.

6.1.9 Definition.

- A right module cell \( X \xrightarrow{M} * \) is said to

\[
\begin{array}{c}
P \downarrow \\
\psi \downarrow \\
Y \xrightarrow{\psi} * \xrightarrow{M} *
\end{array}
\]

(1) preserve units if \( \psi \) sends each unit \( u: r \Rightarrow * \) of \( M \) to a unit \( u: \psi: r: P \Rightarrow * \) of \( N \);
(2) reflect units if an \( M \)-arrow \( u: r \Rightarrow * \) is a unit whenever the \( N \)-arrow \( u: \psi: r: P \Rightarrow * \) is a unit;
(3) create units if for every unit \( v: s \Rightarrow * \) of \( N \) there is exactly one \( M \)-arrow \( u: r \Rightarrow * \) with \( u: \psi = v \), and if this \( u \) is a unit.

- A left module cell \( * \xleftarrow{N} A \) is said to

\[
\begin{array}{c}
\psi \downarrow \\
\downarrow \\
* \xrightarrow{\psi} * \xleftarrow{N} B
\end{array}
\]

(1) preserve units if \( \psi \) sends each unit \( u: * \Rightarrow r \) of \( M \) to a unit \( u: \psi: * \Rightarrow Q: r \) of \( N \);
(2) reflect units if an $\mathcal{M}$-arrow $u: * \rightarrow r$ is a unit whenever the $\mathcal{N}$-arrow $u: \psi : * \rightarrow Q: r$ is a unit;
(3) create units if for every unit $v: * \rightarrow s$ of $\mathcal{N}$ there is exactly one $\mathcal{M}$-arrow $u: * \rightarrow r$ with $u: \psi = v$, and if this $u$ is a unit.

6.1.10 Proposition. If a right [op. left] module cell creates units, then it reflects units.

Proof. Obvious by the definitions.

6.1.11 Proposition. If a right [op. left] module cell is iso, then it preserves, reflects, and creates units.

Proof. Immediate on noting Remark 1.2.22.

6.1.12 Proposition. Consider a right [op. left] module cell $\psi$ as in Definition 6.1.9.

- If $P$ and $\psi$ are fully faithful, then $\psi$ reflects units.
- If $Q$ and $\psi$ are fully faithful, then $\psi$ reflects units.

Proof. Let $u: r \rightarrow *$ be an $\mathcal{M}$-arrow, and assume that the $\mathcal{N}$-arrow $u: \psi : r : P \rightarrow *$ is a unit. Since $\psi$ is fully faithful, $\psi : M \rightarrow P(\mathcal{N})$ is a right module isomorphism. Hence an $\mathcal{M}$-arrow $u: r \rightarrow *$ is a unit iff the $P(\mathcal{N})$-arrow $u: \psi : r : \rightarrow *$ is a unit. But this is the case by Theorem 6.1.8 because $P$ is fully faithful and the $\mathcal{N}$-arrow $u: \psi : r : P \rightarrow *$ is a unit.

6.1.13 Proposition. If a right [op. left] module cell preserves a unit, then it preserves every unit; that is,

- if a right module cell $\begin{array}{c} X \\ P \\ Y \\ \end{array} \xrightarrow{\psi} \begin{array}{c} \mathcal{M} \\ \mathcal{N} \\ \end{array} \xrightarrow{\psi} \begin{array}{c} \mathcal{N} \\ \mathcal{N} \\ \end{array}$ sends a unit $u: r \rightarrow *$ of $\mathcal{M}$ to a unit $u: \psi : r : P \rightarrow *$ of $\mathcal{N}$, then $\psi$ sends any other unit $v: s \rightarrow *$ of $\mathcal{M}$ to a unit $v: \psi : s : P \rightarrow *$ of $\mathcal{N}$ as well.

- if a left module cell $\begin{array}{c} s \\ Q \\ s \\ \end{array} \xrightarrow{\psi} \begin{array}{c} \mathcal{N} \\ \mathcal{N} \\ \end{array} \xrightarrow{\psi} \begin{array}{c} \mathcal{N} \\ \mathcal{N} \\ \end{array}$ sends a unit $u: * \rightarrow r$ of $\mathcal{M}$ to a unit $u: \psi : * \rightarrow Q: r$ of $\mathcal{N}$, then $\psi$ sends any other unit $v: * \rightarrow s$ of $\mathcal{M}$ to a unit $v: \psi : * \rightarrow Q: s$ of $\mathcal{N}$ as well.

Proof. If $u$ is a unit of $\mathcal{M}$, another unit $v$ of $\mathcal{M}$ is written as $v = v/u \circ u$, with $v/u$ an isomorphism by Theorem 6.1.5. $\psi$ thus sends $v$ to the $\mathcal{N}$-arrow $v: \psi = (v/u : P) \circ (u : \psi)$ (see Proposition 1.2.3), with $(v/u : P)$ an isomorphism (because any functor preserves isomorphisms). Hence if $u: \psi$ is a unit of $\mathcal{N}$, so is $v: \psi$ again by Theorem 6.1.5.

6.1.14 Proposition. Consider a cell $\psi$ as in Definition 6.1.9. If $\mathcal{N}$ has a unit and $\psi$ creates units, then $\mathcal{M}$ has a unit as well and $\psi$ preserves units.

Proof. Clearly $\mathcal{M}$ has a unit. Since $\psi$ preserves the units it has created, it preserves every unit by Proposition 6.1.13.

6.2 Universal arrows

Inverse and direct universal arrows of a two-sided module are defined in Definition 6.2.1 using the Yoneda morphism: an arrow of a two-sided module is inverse [op. direct] universal precisely when the right [op. left] Yoneda morphism sends it to a module isomorphism. Regarding a two-sided module cell, we define the notion of preservation, reflection, and creation of universal arrows. Universal arrows formulate the notion of universalality; in the sequel, various universal constructions such as limits, ends, lifts, and extensions are defined by universal arrows of some module. The theory developed in this section in an abstract framework can thus be applied to any concrete universal construction.
As noted in Remark 6.2.2(2), the notion of a universal arrow of a two-sided module is reduced to that of a unit of a one-sided module; many of results in this section are thus in fact restatements of those obtained in Section 6.1. Nevertheless, it is beneficial to study the notion in the wider context of a two-sided module.

**Note.** As mentioned above, we use the mechanism described in Definition 5.2.3 in the following.

### 6.2.1 Definition

Let $\mathcal{M} : X \to A$ be a module.

- An $\mathcal{M}$-arrow $u : r \sim a$ is inverse universal if the right module morphism $X \downarrow u : (X) \rightharpoonup (\mathcal{M})a : X \to *$ is iso.
- An $\mathcal{M}$-arrow $u : x \sim r$ is direct universal if the left module morphism $u \uparrow A : r(A) \to x(\mathcal{M}) : * \to A$ is iso.

### 6.2.2 Remark

1. An $\mathcal{M}$-arrow $u : x \sim r$ is direct universal if and only if the $\mathcal{M}^\to$-arrow $u : r \sim x$ is inverse universal.
2. By Remark 5.2.4(2),
   - an $\mathcal{M}$-arrow $u : r \sim a$ is inverse universal if and only if it is a unit of the right module $\langle \mathcal{M} \rangle a : X \to *$, and a unit $u : r \sim *$ of a right module $\mathcal{M} : X \to *$ is the same thing as an inverse universal arrow of $\mathcal{M}$ regarded as a two-sided module from $X$ to the terminal category.
   - an $\mathcal{M}$-arrow $u : x \sim r$ is direct universal if and only if it is a unit of the left module $x(\mathcal{M}) : * \to A$, and a unit $u : * \sim r$ of a left module $\mathcal{M} : * \to A$ is the same thing as a direct universal arrow of $\mathcal{M}$ regarded as a two-sided module from the terminal category to $A$.
3. Remark 6.1.2(2) is repeated below in terms of universal arrows.
   - Let $u : r \sim a$ be an inverse universal $\mathcal{M}$-arrow. Given an $\mathcal{M}$-arrow $m : x \sim a$, its inverse image under $X \downarrow u$ is called the adjunct of $m$ along $u$ and written $m / u$; that is,
     $$ m / u := m : (X \downarrow u)^{-1} $$
     ; this is the unique $X$-arrow $x \to r$ making the triangle
     \[
     \begin{array}{ccc}
     r & \sim & a \\
     m / u \downarrow & \downarrow & m \\
     x & \sim & u / m \\
     \end{array}
     \]
     commute. An $\mathcal{M}$-arrow $u : r \sim a$ is inverse universal if and only if to every $\mathcal{M}$-arrow $m : x \sim a$ there is a unique $X$-arrow $m / u : x \to r$ as above.
   - Let $u : x \sim r$ be a direct universal $\mathcal{M}$-arrow. Given an $\mathcal{M}$-arrow $m : x \sim a$, its inverse image under $u \uparrow A$ is called the adjunct of $m$ along $u$ and written $u \setminus m$; that is,
     $$ u \setminus m := m : (u \uparrow A)^{-1} $$
     ; this is the unique $A$-arrow $r \to a$ making the triangle
     \[
     \begin{array}{ccc}
     x & \sim & r \\
     u \setminus m \downarrow & \downarrow & u / m \\
     a & \sim & u \setminus m \\
     \end{array}
     \]
     commute. An $\mathcal{M}$-arrow $u : x \sim r$ is direct universal if and only if to every $\mathcal{M}$-arrow $m : x \sim a$ there is a unique $A$-arrow $u \setminus m : r \to a$ as above.
4. An $\mathcal{M}$-arrow $u : r \to s$ is called two-way universal if it is both inverse and direct universal.

### 6.2.3 Example

For any functor $K : \mathcal{D} \to \mathcal{E}$, consider its representable module $K(\mathcal{E}) : \mathcal{D} \to \mathcal{E}$ and corepresentable modules $(\mathcal{E})K : \mathcal{E} \to \mathcal{D}$ (see Definition 2.3.7).

- A $(\mathcal{E})K$-arrow $u : r \to e$ (i.e. $\mathcal{E}$-arrow $u : r : K \to e$) is inverse universal if the right module morphism $\mathcal{D} \downarrow u : (\mathcal{D})r \to K(\mathcal{E})e : D \to *$ (see Example 5.2.7(3)) is iso; that is, if to every $K(\mathcal{E})$-arrow $f : d \to e$ (i.e. $\mathcal{E}$-arrow $f : d : K \to e$), there is a unique $\mathcal{D}$-arrow $f / u : d \to r$, the adjunct of $f$ along $u$, such that the triangle
   \[
   \begin{array}{ccc}
   r & \sim & e \\
   f / u \downarrow & \downarrow & (f / u) : K \\
   d & \sim & d : K \\
   \end{array}
   \]
   commutes.
6.2. Universal arrows

commutes. An $E$-arrow $u : r \to e$ is said to be universal from $K$ to $e$ if the $K(E)$-arrow $u : r \sim e$ is inverse universal.

- An $(E)K$-arrow $u : e \sim r$ (i.e. $E$-arrow $u : e \to K \cdot r$) is direct universal if the left module morphism $u \uparrow D : r(D) \to e(E)K : * \to D$ (see Example 5.2.7(3)) is iso; that is, if to every $(E)K$-arrow $f : e \sim d$ (i.e. $E$-arrow $f : e \to K \cdot d$), there is a unique $D$-arrow $u\uparrow f : r \to d$, the adjunct of $f$ along $u$, such that the triangle

\[
\begin{array}{c}
e \\
\downarrow f \\
K \cdot d \\
\downarrow u\uparrow f \\
d
\end{array}
\]

commutes. An $E$-arrow $u : e \to K \cdot r$ is said to be universal from $e$ to $K$ if the $(E)K$-arrow $u : e \sim r$ is direct universal.

6.2.4 Remark. The module $K(E) : D \to E$ [op. $(E)K : E \to D$] yields

\[
\begin{array}{c}
\bullet & u & e \\
\downarrow f \\
r
\end{array}
\] \hspace{1cm}
\begin{array}{c}
e & u & r \\
\downarrow f \\
d
\end{array}
\]

by abstracting away the details from the commutative diagrams above, presenting a simpler and more conceptual view of universal arrows.

6.2.5 Proposition. For any arrow $f : r \to s$ of a category $C$, the following conditions are equivalent:

1. $C$-arrow $f : r \to s$ is iso;
2. $(C)\text{-arrow } f : r \sim s$ is inverse universal;
3. $(C)\text{-arrow } f : r \sim s$ is direct universal;
4. $(C)\text{-arrow } f : r \sim s$ is two-way universal,

where $(C)$ is the hom-module of $C$.

Proof. By definition, a $(C)\text{-arrow } f : r \sim s$ is inverse universal iff the right module morphism $C \uparrow f : (C) r \to (C) s$ is iso. By Proposition 5.2.6, $C \uparrow f = (C)f$, and by the fully faithfulness of the Yoneda functor, $(C)f$ is iso iff $f$ is iso. The conditions (1) and (2) are thus equivalent. The equivalence of the conditions (1) and (3) is proved dually, and the equivalence of the conditions (1) and (4) follows.

Note. The following is a restatement of Theorem 6.1.5 in terms of universal arrows.

6.2.6 Theorem. Let $M : X \to A$ be a module.

- Let $u : r \sim a$ be an inverse universal $M$-arrow. Then an $X$-arrow $f : s \to r$ is iso if and only if the composite $f \circ u : s \sim a$ is an inverse universal $M$-arrow; to put it the other way round, an $M$-arrow $v : s \sim a$ is inverse universal if and only if its adjunct $v\uparrow u$ along $u$ is an iso $X$-arrow.

- Let $u : x \sim r$ be a direct universal $M$-arrow. Then an $A$-arrow $f : r \to s$ is iso if and only if the composite $u \circ f : x \sim s$ is a direct universal $M$-arrow; to put it the other way round, an $M$-arrow $v : x \sim s$ is direct universal if and only if its adjunct $u\uparrow v$ along $u$ is an iso $A$-arrow.

Proof. Since an inverse universal $M$-arrow $u : r \sim a$ (resp. $v : s \sim a$) is the same thing as a unit of the right module $(M) a : X \to *$ (see Remark 6.2.2(2)), the assertion follows from Theorem 6.1.5.

Note. The following is a restatement of Theorem 6.1.6 in terms of universal arrows.

6.2.7 Theorem.

- If $u : r \sim a$ and $v : s \sim a$ are two inverse universal arrows of a module, then the adjunct $v\uparrow u : s \to r$ of $v$ along $u$ and the adjunct $u\uparrow v : r \to s$ of $u$ along $v$ (shown in

\[
\begin{array}{c}
s \\
\downarrow v\uparrow u \\
r
\end{array}
\] \hspace{1cm}
\begin{array}{c}
a \\
\downarrow u\uparrow v \\
\end{array}
\]

is an inverse universal $M$-arrow.

6.2. Universal arrows

6.2.10 Theorem. 

Let \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) be a module.

- Consider a composable pair of a functor and a module \( \mathcal{E} \xrightarrow{G} \mathbf{X} \xrightarrow{\mathcal{M}} \mathbf{A} \) with \( G \) fully faithful. If an \( \mathcal{M} \)-arrow \( u : r \to a \) is inverse universal, so is the \( G\{\mathcal{M}\} \)-arrow \( u : r \to a \).

- Consider a composable pair of a functor and a module \( \mathbf{X} \xrightarrow{\mathcal{M}} \mathbf{A} \xleftarrow{F} \mathcal{E} \) with \( F \) fully faithful. If an \( \mathcal{M} \)-arrow \( u : x \to r \) is direct universal, so is the \( \{\mathcal{M}\} \)\( \mathcal{F} \)-arrow \( u : x \to r \).

Proof. Since an inverse universal \( \mathcal{M} \)-arrow \( u : r \to a \) (resp. \( v : s \to a \)) is the same thing as a unit of the right module \( \langle \mathcal{M} \rangle a : \mathbf{X} \to * \) (see Remark 6.2.2(2)), the assertion follows from Theorem 6.1.6.

\[ \text{Proof.} \] Immediate from Theorem 6.2.7 (or Theorem 6.2.6).

Note. The following is a restatement of Corollary 6.1.7 in terms of universal arrows.

6.2.8 Corollary. Let \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) be a module.

- For any object \( a \in \| \mathbf{A} \| \), inverse universal \( \mathcal{M} \)-arrows to \( a \), if exist, are unique up to isomorphism; that is, if \( u : r \to a \) and \( v : s \to a \) are two inverse universal \( \mathcal{M} \)-arrows, then \( r \) and \( s \) are isomorphic.

- For any object \( x \in \| \mathbf{X} \| \), direct universal \( \mathcal{M} \)-arrows from \( x \), if exist, are unique up to isomorphism; that is, if \( u : x \to r \) and \( v : x \to s \) are two direct universal \( \mathcal{M} \)-arrows, then \( r \) and \( s \) are isomorphic.

\[ \text{Proof.} \] Immediate from Theorem 6.2.7 (or Theorem 6.2.6).

Note. The following is a restatement of Theorem 6.1.8 in terms of universal arrows.

6.2.9 Theorem.

- Given a composable pair of a functor and a right module as in \( \mathcal{E} \xrightarrow{G} \mathbf{X} \xrightarrow{\mathcal{M}} * \) , if an \( \mathbf{X} \)-arrow \( u : r \to s \) is universal from \( \mathbf{G} \) to \( s \) and an \( \mathcal{M} \)-arrow \( v : s \to * \) is a unit of \( \mathcal{M} \), then the \( G\{\mathcal{M}\} \)-arrow \( u \circ v : r \to * \) (i.e. \( \mathcal{M} \)-arrow \( u \circ v : r \to \mathbf{G} \to * \) ) is a unit of the composite right module \( G\{\mathcal{M}\} \).

- Given a composable pair of a functor and a left module as in \( * \xleftarrow{\mathcal{M}} \mathbf{A} \xleftarrow{F} \mathcal{E} \) , if an \( \mathbf{A} \)-arrow \( u : s \to \mathbf{F} \cdot r \) is universal from \( s \) to \( \mathbf{F} \) and an \( \mathcal{M} \)-arrow \( v : * \to s \) is a unit of \( \mathcal{M} \), then the \( \{\mathcal{M}\} \)\( \mathbf{F} \)-arrow \( v \circ u : * \to r \) (i.e. \( \mathcal{M} \)-arrow \( v \circ u : * \to \mathcal{F} \cdot r \) ) is a unit of the composite left module \( \{\mathcal{M}\} \) \( \mathbf{F} \).
Proof. By Theorem 5.2.20, \( E \uparrow (u \circ v) = (E \uparrow u) \circ G(X \uparrow v) \). Hence if \( E \uparrow u \) and \( X \uparrow v \) are isomorphisms, so is \( E \uparrow (u \circ v) \).

\[ \tag{6.2.11} \]

**Theorem.** Let \( K : D \to E \) be a fully faithful functor.

\( \bullet \) An \( E \)-arrow \( u : r \cdot K \to e \) is universal from \( K \) to \( e \) (i.e. \( K(E) \)-arrow \( u : r \to e \) is inverse universal) if and only if the right module morphism \( K(E) u : K(E)(K \cdot r) \to K(E) e : D \to * \) is iso.

\( \bullet \) An \( E \)-arrow \( u : e \to K \cdot r \) is universal from \( e \) to \( K \) (i.e. \( (E)K \)-arrow \( u : e \to r \) is direct universal) if and only if the left module morphism \( u(E)K : (r \cdot K)(E)K \to e(E)K : * \to D \) is iso.

Proof. As we saw in Example 5.2.7(3), the diagram

\[
\begin{array}{ccc}
(D) r & \rightarrow & (K(E)(K \cdot r) \\
\downarrow & & \downarrow \text{iso} \\
K(E) e & \rightarrow & K(E) e
\end{array}
\]

commutes. Since \( K \) is fully faithful, \( (K) r \) is iso. Hence \( D \uparrow u \) is iso iff \( K(E) u \) is iso. But \( D \uparrow u \) is iso iff \( u \) is inverse universal by the definition of inverse universality.

\[ \tag{6.2.12} \]

**Corollary.** Let \( K : D \to E \) be a fully faithful functor.

\( \bullet \) If an \( E \)-arrow \( u : r \cdot K \to e \) is iso, then it is universal from \( K \) to \( e \), i.e. the \( K(E) \)-arrow \( u : r \to e \) is inverse universal.

\( \bullet \) If an \( E \)-arrow \( u : e \to K \cdot r \) is iso, then it is universal from \( e \) to \( K \), i.e. the \( (E)K \)-arrow \( u : e \to r \) is direct universal.

Proof. If \( u \) is iso, so is \( K(E) u \). The assertion thus follows from Theorem 6.2.11.

\[ \tag{6.2.13} \]

**Remark.** Noting Proposition 6.2.5, we see that Corollary 6.2.12 is a special case of Theorem 6.2.9 where \( M \) is given by the hom-module of a category.

\[ \tag{6.2.14} \]

**Theorem.** Let \( K : D \to E \) be a functor and \( e \) be an object of \( E \).

\( \bullet \) If there is an iso \( E \)-arrow \( u : r \cdot K \to e \) universal from \( K \) to \( e \), then every \( E \)-arrow \( v : s \cdot K \to e \) universal from \( K \) to \( e \) is iso.

\( \bullet \) If there is an iso \( E \)-arrow \( u : e \to K \cdot r \) universal from \( e \) to \( K \), then every \( E \)-arrow \( v : e \to K \cdot s \) universal from \( e \) to \( K \) is iso.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
r & \rightarrow & e \\
\downarrow \text{iso} & & \downarrow \text{iso} \\
s & \rightarrow & s
\end{array}
\]

\( \text{v/u} \)

; we need to show that if \( u \) is iso, so is \( v \). For this, it suffices to show that \( (v/u) : K \) is iso. But this holds because \( v/u \) is iso by Theorem 6.2.6 and any functor preserves isomorphisms.

\[ \tag{6.2.15} \]

**Theorem.** Let \( M : X \to A \) be a module.

\( \bullet \) Let \( f : a \to b \) be an \( A \)-arrow such that the right module morphism \( (M) f : (M) a \to (M) b : X \to * \) is iso. Then an \( M \)-arrow \( u : r \to a \) is inverse universal if and only if the composite \( M \)-arrow \( u \circ f : r \to b \) is.

\( \bullet \) Let \( f : y \to x \) be an \( X \)-arrow such that the left module morphism \( f(M) : x(M) \to y(M) : * \to A \) is iso. Then an \( M \)-arrow \( u : x \to r \) is direct universal if and only if the composite \( M \)-arrow \( f \circ u : y \to r \) is.

Proof. By the naturality of the Yoneda morphism (see Definition 5.2.3), we have

\[
X \uparrow (u \circ f) = X \uparrow u \circ (M) f
\]

; since \( (M) f \) is iso by assumption, \( X \uparrow (u \circ f) \) is iso (i.e. \( u \circ f \) is inverse universal) iff \( X \uparrow u \) is iso (i.e. \( u \) is inverse universal).
6.2.16 Remark. The pullback lemma follows from Theorem 6.2.15. Consider a commutative diagram

\[
\begin{array}{ccc}
x & \xrightarrow{a} & a' \\
\downarrow{h} & & \downarrow{h'} \\
c & \xleftarrow{k} & b' \\
\end{array}
\]

with the right-hand square a pullback; the pullback lemma says that the left-hand square is a pullback if and only if the outer rectangle is a pullback. To see this, first note (Definition 8.1.9) that a pullback in a category \( C \) is the same thing as an inverse universal arrow of the module \( \langle \ast E, C \rangle : C \rightarrow [E, C] \) with \( E \) be a category which looks like \( 0 \rightarrow 1 \leftarrow 2 \). Now transform the commutative diagram into

\[
\begin{array}{ccc}
x & \xrightarrow{a} & a' \\
\downarrow{h} & & \downarrow{h'} \\
b & \xleftarrow{g} & b' \\
\end{array}
\]

, and observe that the triple \( \tau := (f, g, c) \) forms a natural transformation from the cospan \( G := (a \xrightarrow{h} b \xleftarrow{k} c) \) to the cospan \( F := (a' \xrightarrow{h'} b' \xleftarrow{k'g} c) \), and that the composition \( \circ \tau \) maps the left-hand square in the pullback lemma to the outer rectangle of it. This observation allows us to treat the pullback lemma as an instance of Theorem 6.2.15: it suffices to prove that the assignment \( \alpha \mapsto \alpha \circ \tau \) is bijective. To see this bijectivity, let \( \alpha : x \rightarrow F \) be a cone, i.e. a commutative square

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha_0} & a' \\
\downarrow{\alpha_2} & & \downarrow{h'} \\
c & \xleftarrow{k} & b' \\
\end{array}
\]

; then the unique arrow \( \alpha'_0 : x \rightarrow a \) making the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha_0} & a' \\
\downarrow{\alpha_2} & & \downarrow{h'} \\
c & \xleftarrow{k} & b' \\
\end{array}
\]

commute (the right-hand square is a pullback by assumption) forms with \( \alpha_2 \) the unique cone \( \alpha' : x \rightarrow G \) such that \( \alpha = \alpha' \circ \tau \).

6.2.17 Corollary. Let \( M : X \rightarrow A \) be a module.

- If \( f : a \rightarrow b \) is an isomorphism in \( A \), then an \( M \)-arrow \( u : r \rightarrow a \) is inverse universal if and only if the composite \( M \)-arrow \( u \circ f : r \rightarrow b \) is.
- If \( f : y \rightarrow x \) is an isomorphism in \( X \), then an \( M \)-arrow \( u : x \rightarrow r \) is direct universal if and only if the composite \( M \)-arrow \( f \circ u : y \rightarrow r \) is.

Proof. Since (like any other functor) the functor \( [M \cdot] : A \rightarrow [X :] \) preserves isomorphisms, if \( f : a \rightarrow b \) is an isomorphism, so is the right module morphism \( \langle M \rangle f : \langle M \rangle a \rightarrow \langle M \rangle b \). The assertion thus follows from Theorem 6.2.15. \( \square \)

6.2.18 Theorem. Let \( M : X \rightarrow A \) be a module and consider a commutative square

\[
\begin{array}{ccc}
x & \xrightarrow{u} & a \\
g & & \downarrow{f} \\
y & \xrightarrow{v} & b \\
\end{array}
\]

consisting of \( M \)-arrows \( u \) and \( v \), an iso \( X \)-arrow \( g \), and an iso \( A \)-arrow \( f \). In this square, if \( u \) is inverse \( \text{[op. direct]} \) universal, so is \( v \).

Proof. Suppose that \( u \) is inverse universal. By Corollary 6.2.17, \( u \circ f \) is inverse universal. Hence, by Theorem 6.2.6, \( v = g^{-1} \circ u \circ f \) is inverse universal. \( \square \)
6.2.19 Theorem. 
- Given a composable pair of a module and a functor as in 
  \[ X \xrightarrow{M} A \xleftarrow{F E} \]
  an \( M \)-arrow \( u : r \rightsquigarrow F : e \) is inverse universal if and only if the \( (M)F \)-arrow \( u : r \rightsquigarrow e \) is.
- Given a composable pair of a module and a functor as in 
  \[ E \xrightarrow{G} X \xleftarrow{M} A \]
  an \( M \)-arrow \( u : e \rightsquigarrow r \) is direct universal if and only if the \( G(M) \)-arrow \( u : e \rightsquigarrow r \) is.

Proof. By Corollary 6.2.12, the proof follows from Theorem 6.2.19 where \( M \) is given by the hom-module of a category.

6.2.20 Corollary. Let \( K : D \rightarrow E \) be a functor.
- An \( E \)-arrow \( u : r \rightarrow K : d \) is iso if and only if the \( (E)K \)-arrow \( u : r \rightarrow d \) is inverse universal.
- An \( E \)-arrow \( u : d : K \rightarrow r \) is iso if and only if the \( K(E) \)-arrow \( u : d \rightarrow r \) is direct universal.

Proof. By Proposition 6.2.5, the proof follows from Theorem 6.2.19.

6.2.21 Theorem. For a pair of fully faithful functors \( X \xrightarrow{C} C \xleftarrow{F} A \), consider the composite module \( G(C)F : X \rightarrow A \) (see Example 1.1.31(12)). If a \( C \)-arrow \( u : r \rightarrow G \rightarrow F : s \) is iso, then the \( G(C)F \)-arrow \( u : r \rightarrow s \) is two-way universal.

Proof. By Corollary 6.2.12, the \( G(C) \)-arrow \( u : r \rightarrow F : s \) (resp. \( (C)F \)-arrow \( u : r : G \rightarrow s \)) is inverse (resp. direct) universal; hence, by Theorem 6.2.19, the \( G(C)F \)-arrow \( u : r \rightarrow s \) is inverse (resp. direct) universal.

6.2.22 Definition. A cell \( X \xrightarrow{M} A \) is said to 
\[
\begin{align*}
p \parallel & \psi \quad q \\
Y \xrightarrow{\psi} & B
\end{align*}
\]
(1) preserve
- inverse universal arrows if \( \psi \) sends each inverse universal \( M \)-arrow \( u : r \rightarrow a \) to an inverse universal \( N \)-arrow \( u : r : P \rightarrow Q : a \).
- direct universal arrows if \( \psi \) sends each direct universal \( M \)-arrow \( u : x \rightarrow r \) to a direct universal \( N \)-arrow \( u : x : P \rightarrow Q : r \).

(2) reflect
- inverse universal arrows if an \( M \)-arrow \( u : r \rightarrow a \) is inverse universal whenever the \( N \)-arrow \( u : r : P \rightarrow Q : a \) is inverse universal.
- direct universal arrows if an \( M \)-arrow \( u : x \rightarrow r \) is direct universal whenever the \( N \)-arrow \( u : x : P \rightarrow Q : r \) is direct universal.

(3) create
- inverse universal arrows if for every object \( a \in \parallel A \) and for every inverse universal \( N \)-arrow \( v : s \rightarrow Q : a \) there is exactly one \( M \)-arrow \( u : r \rightarrow a \) with \( u : \psi = v \), and if this \( u \) is inverse universal.
- direct universal arrows if for every object \( x \in \parallel X \) and for every direct universal \( N \)-arrow \( v : x : P \rightarrow s \) there is exactly one \( M \)-arrow \( u : x \rightarrow r \) with \( u : \psi = v \), and if this \( u \) is direct universal.

6.2.23 Remark. By Remark 6.2.2(2),
- \( \psi \) preserves (resp. reflects, creates) inverse universal arrows if and only if each right module cell \( (\psi)a \) (see Definition 2.1.8) preserves (resp. reflects, creates) units in the sense of Definition 6.1.9.
- Ψ preserves (resp. reflects, creates) direct universal arrows if and only if each left module cell x(Ψ) (see Definition 2.1.8) preserves (resp. reflects, creates) units in the sense of Definition 6.1.9.

Note. The following is a restatement of Proposition 6.1.10 in terms of universal arrows.

### 6.2.24 Proposition
If a cell creates inverse [op. direct] universal arrows, then it reflects inverse [op. direct] universal arrows.

**Proof.** Obvious by the definitions.

Note. The following is a restatement of Proposition 6.1.11 in terms of universal arrows.

### 6.2.25 Proposition
Consider a cell Ψ as in Definition 6.2.22.

- If Ψ is fully faithful and P is iso, then Ψ preserves, reflects, and creates inverse universal arrows.
- If Ψ is fully faithful and Q is iso, then Ψ preserves, reflects, and creates direct universal arrows.

**Proof.** If Ψ is fully faithful and P is iso, then each right module cell ⟨Ψ⟩a is iso by Proposition 2.1.9. The assertion thus follows from Proposition 6.1.11 by noting Remark 6.2.23.

Note. The following is a restatement of Proposition 6.1.12 in terms of universal arrows.

### 6.2.26 Proposition
Consider a cell Ψ as in Definition 6.2.22.

- If P and Ψ are fully faithful, then Ψ reflects inverse universal arrows.
- If Q and Ψ are fully faithful, then Ψ reflects direct universal arrows.

**Proof.** If Ψ is fully faithful, so is each right module cell ⟨Ψ⟩a by Proposition 2.1.10. The assertion thus follows from Proposition 6.1.12 by noting Remark 6.2.23.

### 6.2.27 Proposition
Given a cell and a commutative square of functors as in

\[
\begin{array}{ccc}
X \ar[r]^M \ar[d]_P & A \ar[d]^F \ar[r]^D & D \ar[d]^Q \ar[r]^K & \ar[l]_\psi X \ar[r]^\cdot \ar[d]_\psi \ar[r]^\cdot \ar[l]_\cdot & A \ar[d]_Q \\
Y \ar[r]^-N \ar[d]_P & B \ar[r]^-G \ar[d]^F & E \ar[d]^E \ar[r]^-G & \ar[l]_\psi Y \ar[r]^\cdot \ar[d]_\cdot \ar[r]^\cdot \ar[l]_\cdot & B \\
\end{array}
\]

, if the cell Ψ preserves (resp. reflects, creates) inverse [op. direct] universal arrows, so does the composite

\[
\begin{array}{ccc}
X \ar[r]^{\langle M \rangle F} \ar[d]_P & D \ar[d]^K \ar[r]^{\cdot F} & \ar[l]_{\psi F} X \ar[r]^\cdot \ar[d]_\cdot \ar[r]^\cdot \ar[l]_\cdot & A \ar[d]_Q \\
Y \ar[r]^{\langle N \rangle G} & E \ar[r]^{G} & \ar[l]_{\cdot G(N)} D \ar[r]^{\cdot Q} & B \\
\end{array}
\]

**Proof.** Suppose that Ψ preserves (resp. reflects) inverse universal arrows. By Theorem 6.2.19, we see that the cell ⟨Ψ⟩F also preserves (resp. reflects) inverse universal arrows. Now suppose that Ψ creates inverse universal arrows. To see that ⟨Ψ⟩F also creates inverse universal arrows, let d ∈ [D], and suppose that an ⟨N⟩G-arrow v : s ↠ K·d is inverse universal. Then, by Theorem 6.2.19, N-arrow v : s ↠ G·K·d = Q·F·d is inverse universal. Hence there is exactly one M-arrow u : r ↠ F·d whose image under Ψ is the N-arrow v : s ↠ Q·F·d; that is, there is exactly one ⟨M⟩F-arrow u : r ↠ F·d whose image under ⟨Ψ⟩F is the ⟨N⟩G-arrow v : s ↠ K·d. Since the M-arrow u : r ↠ F·d is inverse universal, again by Theorem 6.2.19, so is the ⟨M⟩F-arrow u : r ↠ F·d.

### 6.2.28 Proposition
If two cells Ψ : M → N and φ : M → N are isomorphic (see Definition 1.3.3), then

- Ψ preserves (resp. reflects) inverse universal arrows if and only if φ preserves (resp. reflects) inverse universal arrows.
- Ψ preserves (resp. reflects) direct universal arrows if and only if φ preserves (resp. reflects) direct universal arrows.
6.3.2 Remark. The right Yoneda morphism preserves and reflects inverse universal.

Proof. Noting Proposition 1.3.4, this is immediate from Theorem 6.2.18.

6.2.29 Theorem. A right (op. left) Yoneda morphism preserves and reflects inverse [op. direct] universal.

Proof. The right Yoneda morphism for a module $\mathcal{M} : X \to A$ (see Definition 5.2.3) sends each $\mathcal{M}$-arrow $u : r \to a$ to the right module $X \uparrow u : (X) r \to (\mathcal{M}) a : X \to X$. But by definition, $u : r \to a$ is inverse universal iff $X \uparrow u$ is iso, and by Proposition 6.2.5, $X \uparrow u$ is iso iff $X \uparrow u$ is inverse universal.

6.3 Conjugation

This section introduces the notion of conjugation along a pair of universal arrows (see Definition 6.3.1). We borrowed the term “conjugate” from [ML98], which in fact uses the term in the context of a pair of adjunctions. Here we present the notion in a more abstract context of universal arrows, and apply it to adjunctions later in Section 7.6. Conjugation also plays an important role in Section 6.4.

6.3.1 Definition. Let $\mathcal{M} : X \to A$ be a module.

- Given a pair of inverse universal $\mathcal{M}$-arrows $u : r \to a$ and $v : s \to b$, the conjugate of an $A$-arrow $f : a \to b$ inverse along $(u,v)$ is the $X$-arrow given by the adjunct of the composite $u \circ f$ along $v$, i.e. the unique $X$-arrow $g : r \to s$ making the square

$$
\begin{array}{c}
  r \sim u\sim a \\
  s \sim v \sim b \\
\end{array}
\begin{array}{c}
  g \\
  f \\
\end{array}
$$

commute. This commutative square, or the assignment $f \mapsto g$, is called an inverse conjugation by $(u,v)$.

- Given a pair of direct universal $\mathcal{M}$-arrows $u : x \to r$ and $v : y \to s$, the conjugate of an $X$-arrow $g : x \to y$ direct along $(u,v)$ is the $A$-arrow given by the adjunct of the composite $g \circ v$ along $u$, i.e. the unique $A$-arrow $f : r \to s$ making the square

$$
\begin{array}{c}
  x \sim u\sim r \\
  y \sim v \sim s \\
\end{array}
\begin{array}{c}
  g \\
  f \\
\end{array}
$$

commute. This commutative square, or the assignment $g \mapsto f$, is called a direct conjugation by $(u,v)$.

6.3.2 Remark. Let $\mathcal{M} : X \to A$ be a module.

- The category $\mathcal{U}i[\mathcal{M}]$ of inverse universal arrows of $\mathcal{M}$ is given by the full subcategory of the comma category $[\mathcal{M}]$ consisting of all inverse universal arrows; given inverse universal $\mathcal{M}$-arrows $u : r \sim a$ and $v : s \sim b$, an arrow $u \to v$ in $\mathcal{U}i[\mathcal{M}]$ is a pair of an $X$-arrow $g : r \to s$ and an $A$-arrow $f : a \to b$ forming an inverse conjugation by $(u,v)$. The restriction of the comma fibration $\mathcal{M}_1' : [\mathcal{M}] \to A$ to $\mathcal{U}i(\mathcal{M})$ yields a fully faithful forgetful functor $\mathcal{M}_1' : \mathcal{U}i[\mathcal{M}] \to A$ sending each inverse conjugation

$$
\begin{array}{c}
  r \sim u\sim a \\
  s \sim v \sim b \\
\end{array}
\begin{array}{c}
  g \\
  f \\
\end{array}
$$

to the $A$-arrow $f : a \to b$.

- The category $\mathcal{U}d[\mathcal{M}]$ of direct universal arrows of $\mathcal{M}$ is given by the full subcategory of the comma category $[\mathcal{M}]$ consisting of all direct universal arrows; given direct universal $\mathcal{M}$-arrows $u : x \sim r$ and $v : y \sim s$, an arrow $u \to v$ in $\mathcal{U}d[\mathcal{M}]$ is a pair of an $X$-arrow $g : x \to y$ and an $A$-arrow $f : r \to s$ forming a direct conjugation by $(u,v)$. The restriction of the comma fibration
\( \mathcal{M}_0 : [\mathcal{M}] \to \mathbf{X} \) to \( \text{Ud}[\mathcal{M}] \) yields a fully faithful forgetful functor \( \mathcal{M}_0^1 : \text{Ud}[\mathcal{M}] \to \mathbf{X} \) sending each direct conjugation

\[
\begin{array}{c}
x \xrightarrow{u} r \\
\downarrow^g \downarrow \downarrow^f \\
y \xrightarrow{v} s
\end{array}
\]

to the \( \mathbf{X} \)-arrow \( g : x \to y \).

**6.3.3 Proposition.** Let \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) be a module.

1. Conjugation is functorial in the following sense:
   a) for any inverse [op. direct] universal \( \mathcal{M} \)-arrow \( u : r \leadsto s \), the pair of identities

   \[
   \begin{array}{c}
r \xrightarrow{u} s \\
\downarrow^1 \\
r \xrightarrow{u} s
\end{array}
\]

   forms an inverse [op. direct] conjugation.

   b) if, in the diagram

   \[
   \begin{array}{c}
r \xrightarrow{u} s \\
\downarrow^g \downarrow \downarrow^f \\
r' \xrightarrow{u'} s' \\
\downarrow^{g'} \downarrow \downarrow^{f'} \\
r'' \xrightarrow{u''} s''
\end{array}
\]

   , each of the two inner squares is an inverse [op. direct] conjugation, so is the outer rectangle.

2. Conjugation preserve isomorphisms; that is, given an inverse [op. direct] conjugation

   \[
   \begin{array}{c}
r \xrightarrow{u} a \\
\downarrow^g \downarrow \downarrow^f \\
s \xrightarrow{v} b
\end{array}
\]

   , if \( f \) [op. \( g \)] is an isomorphism, so is \( g \) [op. \( f \)].

3. Conjugation is universal in the following sense: given an inverse [op. direct] conjugation

   \[
   \begin{array}{c}
r \xrightarrow{u} a \\
\downarrow^g \downarrow \downarrow^f \\
s \xrightarrow{v} b
\end{array}
\]

   and any commutative square

   \[
   \begin{array}{c}
x \xrightarrow{m} a \\
\downarrow^h \downarrow \downarrow^f \\
y \xrightarrow{n} b
\end{array} \quad \text{op.} \quad \begin{array}{c}
x \xrightarrow{m} a \\
\downarrow^g \downarrow \downarrow^h \\
y \xrightarrow{n} b
\end{array}
\]

   , the adjunct of \( m \) along \( u \) and the adjunct of \( n \) along \( v \) give a unique pair of \( \mathbf{X} \)-arrows [op. \( \mathbf{A} \)-arrows] making the diagram

   \[
   \begin{array}{c}
r \xrightarrow{u} a \\
\downarrow^g \downarrow \downarrow^m \\
s \xrightarrow{v} b
\end{array} \quad \text{op.} \quad \begin{array}{c}
r \xrightarrow{u} a \\
\downarrow^g \downarrow \downarrow^m \\
s \xrightarrow{v} b
\end{array}
\]

   commute.

**Proof.**

1. Evident.
2. Any functorial operation preserves isomorphisms.
3. The uniqueness of such a pair follows from the uniqueness of the adjunct. The proof is thus
6.4 Units of two-sided modules

A counit of a two-sided module \( \mathcal{M} \) is a right cylinder along \( \mathcal{M} \) consisting of a family of inverse universal arrows of \( \mathcal{M} \). A right cylinder along a two-sided module is a counit precisely when the Yoneda morphism sends it to a module isomorphism, and we define a counit in Definition 6.4.1 using this characterization. By virtue of the generalized Yoneda lemma, counits and corepresentations of a two-sided module correspond one-to-one. In Section 7.3, we will see that the counit of an adjunction is a special instance of a counit of a two-sided module. A unit of a two-sided module \( \mathcal{M} \) is defined dually by a pointwise universal left cylinder along \( \mathcal{M} \).

A unit of a one-sided module is seen as a special instance of a unit of a two-sided module.

**6.4.1 Definition.** Let \( M : X \to A \) be a module.

- A right cylinder \( X \xrightarrow{M} X \xrightarrow{R} M \) or the pair \((R, \mu)\) is called a counit of \( \mathcal{M} \) if the module morphism \( X \mu : X \to M \) is iso.

- A left cylinder \( X \xrightarrow{R} X \xrightarrow{M} A \) or the pair \((R, \mu)\) is called a unit of \( \mathcal{M} \) if the module morphism \( \mu : R(A) \to M : X \to A \) is iso.

**6.4.2 Remark.** By the generalized Yoneda lemma (Corollary 5.3.17), representations and units correspond one-to-one:

- if a right cylinder \( \mu : R \to \mathcal{M} \) is a counit of a module \( \mathcal{M} : X \to A \), then the functor \( R \) corepresents \( \mathcal{M} \) by the isomorphism \( X \mu : X \to M \); conversely, if a functor \( R : A \to X \) and a module isomorphism \( \phi : R \to \mathcal{M} \) form a corepresentation of \( \mathcal{M} \), then the right cylinder \( X \mu \phi : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \).

- if a left cylinder \( \mu : \mathcal{M} \to R \) is a unit of a module \( \mathcal{M} : X \to A \), then the functor \( R \) represents \( \mathcal{M} \) by the isomorphism \( \mu \phi : R(A) \to \mathcal{M} \); conversely, if a functor \( R : X \to A \) and a module isomorphism \( \phi : R(A) \to \mathcal{M} \) form a representation of \( \mathcal{M} \), then the left cylinder \( \phi \mu : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \).

**6.4.3 Proposition.** Let \( M : X \to A \) be a module.

- A right cylinder \( \mu : R \to \mathcal{M} \) is a counit of \( \mathcal{M} \) if and only if each component \( \mu_a : a \cdot R \to a \) is an inverse universal \( \mathcal{M} \)-arrow.

- A left cylinder \( \mu : \mathcal{M} \to R \) is a unit of \( \mathcal{M} \) if and only if each component \( \mu_x : x \to R \cdot x \) is a direct universal \( \mathcal{M} \)-arrow.

**Proof.** By Proposition 2.1.3, \( X \mu \) is iso iff its each slice \( (X \mu) \) is iso. Since \( (X \mu) a = X \mu a \) (see Corollary 5.3.12), \( (X \mu) a \) is iso iff \( \mu_a \) is inverse universal. \( \square \)
6.4.4 Remark.
(1) Proposition 6.4.3 gives an alternative definition of units of a module.
(2) By Proposition 6.4.3 and Remark 6.2.2(2), under the identification in Remark 4.3.4(2),
   - a unit of a right module \( M : X \to * \) is the same thing as a counit of \( M \) regarded as the
two-sided module from \( X \) to the terminal category.
   - a unit of a left module \( M : * \to A \) is the same thing as a unit of \( M \) regarded as the two-sided
module from the terminal category to \( A \).

6.4.5 Proposition. Let \( M : X \to A \) be a module.
- A counit \( (R, \mu) \) of \( M \) gives an inverse universal arrow \( \mu : R \to M \) of the right generalized Yoneda module
  \( X \cdot A \), i.e. a unit \( \mu : R \to * \) (in the sense of Definition 6.1.1) of the right module
  \( (X \cdot A)(M) \).
- A unit \( (R, \mu) \) of \( M \) gives a direct universal arrow \( \mu : M \to R \) of the left generalized Yoneda module
  \( X^* \cdot A \), i.e. a unit \( \mu : * \to R \) (in the sense of Definition 6.1.1) of the left module \( (M)(X^* \cdot A) \).

Proof. For a right cylinder \( \alpha : G \to M \), we need to show that there is a unique natural transformation
\( \alpha/\mu : G \to R : A \to X \) making the triangle

\[
\begin{array}{ccc}
R & \xrightarrow{\mu} & M \\
\alpha/\mu \downarrow & & \downarrow \alpha \\
G & & A
\end{array}
\]

commute. For an \( A \)-arrow \( f : a \to b \), consider the naturality squares

\[
\begin{array}{ccc}
a : R & \xrightarrow{\mu_a} & a \\
\alpha_a/\mu_a \downarrow & & \downarrow \alpha_a \\
G & & b
\end{array}
\quad\quad\quad
\begin{array}{ccc}
a : G & \xrightarrow{\alpha_a} & a \\
\alpha_a \downarrow & & \downarrow \alpha_a \\
b : G & & b
\end{array}
\]

given by \( \mu \) and \( \alpha \). Since \( \mu_a \) and \( \mu_b \) are inverse universal by Proposition 6.4.3, the square on the left
forms an inverse conjugation. Hence, by Proposition 6.3.3(3), the adjunct of \( \alpha_a \) along \( \mu_a \) and the
adjunct of \( \alpha_b \) along \( \mu_b \) yield a unique pair of \( X \)-arrows making the diagram

\[
\begin{array}{ccc}
a : R & \xrightarrow{\mu_a} & a \\
\alpha_a/\mu_a \downarrow & & \downarrow \alpha_a \\
G & & b
\end{array}
\quad\quad\quad
\begin{array}{ccc}
a : G & \xrightarrow{\alpha_a} & a \\
\alpha_a \downarrow & & \downarrow \alpha_a \\
b : G & & b
\end{array}
\]

commute. The family of \( X \)-arrows \( \alpha_a/\mu_a \), one for each \( a \in \|A\| \), thus forms a unique natural transformation
\( \alpha/\mu : G \to R \) such that \( \alpha/\mu \circ \mu = \alpha \). \qed

6.4.6 Remark. The converse does not hold. See Example 6.4.7.

6.4.7 Example. Consider a module \( M : 2 \to 2 \) from the discrete category \( 2 = \{0, 1\} \) to the interval
category \( 2 \) which looks like

\[
\begin{array}{c}
0 \xrightarrow{\Delta 0} 0 \\
\uparrow \\
1 \xrightarrow{\Delta 1} 1
\end{array}
\]

; \( M \) admits only one right cylinder

\[
2 \xrightarrow{\mu} M \to 2
\]

along it. This right cylinder \( \mu \) is an inverse universal arrow of the right generalized Yoneda module
\( 2 \cdot 2 \), being the sole \( (2 \cdot 2) \)-arrow to \( M \); however, \( \mu \) is not a unit of \( M \) since \( \mu_1 : 0 \to 1 \) is not an
inverse universal \( M \)-arrow.
6.4.8 Theorem. Let $\mathcal{M} : X \to A$ be a module.

- If $(R, \mu)$ and $(S, \nu)$ are two counits of $\mathcal{M}$, then $R$ and $S$ are isomorphic.
- If $(R, \mu)$ and $(S, \nu)$ are two units of $\mathcal{M}$, then $R$ and $S$ are isomorphic.

Proof. By Proposition 6.4.5, $(R, \mu)$ and $(S, \nu)$ give inverse universal $(X, A)$-arrows $\mu : R \sim \mathcal{M}$ and $\nu : S \sim \mathcal{M}$. Hence $R$ and $S$ are isomorphic by Corollary 6.2.8.

6.4.9 Corollary. A representing functor of a module, if exists, is unique up to isomorphism.

Proof. By Remark 6.4.2, this is just a restatement of Theorem 6.4.8.

6.4.10 Theorem. Let $\mathcal{M} : X \to A$ be a module.

- If there is a family of inverse universal $\mathcal{M}$-arrows $\mu_a : r_a \sim a$, one for each object $a \in |A|$, then there is a unique functor $R : A \to X$ with $R(a) = r_a$ such that $\mu := (\mu_a)_{a \in |A|}$ forms a right cylinder $R \sim \mathcal{M}$, and moreover $\mu$ is a counit of $\mathcal{M}$.
- If there is a family of direct universal $\mathcal{M}$-arrows $\mu_x : x \sim r_x$, one for each object $x \in |X|$, then there is a unique functor $R : X \to A$ with $R(x) = r_x$ such that $\mu := (\mu_x)_{x \in |X|}$ forms a left cylinder $R \sim \mathcal{M}$, and moreover $\mu$ is a unit of $\mathcal{M}$.

Proof. The arrow function of $R$ is given by the inverse conjugation

$$\begin{align*}
r_a &\sim_{\mu_a}^R a \\
r_b &\sim_{\mu_b}^R b
\end{align*}$$

(see Definition 6.3.1) for each $A$-arrow $f : a \to b$. $R$ is functorial by Proposition 6.3.3(1) and the uniqueness $R$ follows from the uniqueness of a conjugate. Since each $\mu_a$ is inverse universal, $\mu$ forms a counit of $\mathcal{M}$ by Proposition 6.4.3.

Note. The axiom of choice is used in the proof of the following.

6.4.11 Corollary. Given a module $\mathcal{M} : X \to A$,

- the following conditions are equivalent:
  1. $\mathcal{M}$ is corepresentable;
  2. $\mathcal{M}$ has a counit;
  3. for every object $a \in |A|$, the right module $(\mathcal{M})a : X \to \ast$ is representable;
  4. for every object $a \in |A|$, the right module $(\mathcal{M})a : X \to \ast$ has a unit; that is, for every $a \in |A|$, there is an inverse universal $\mathcal{M}$-arrow to $a$.
- the following conditions are equivalent:
  1. $\mathcal{M}$ is representable;
  2. $\mathcal{M}$ has a unit;
  3. for every object $x \in |X|$, the left module $x(\mathcal{M}) : \ast \to A$ is representable;
  4. for every object $x \in |X|$, the left module $x(\mathcal{M}) : \ast \to A$ has a unit; that is, for every $x \in |X|$, there is a direct universal $\mathcal{M}$-arrow from $x$.

Proof. (1)$\iff$(2) By Remark 6.4.2.
(3)$\iff$(4) By Remark 6.1.2(1).
(2)$\Rightarrow$(4) Immediate from Proposition 6.4.3.
(4)$\Rightarrow$(2) A family of inverse universal $\mathcal{M}$-arrows $\mu_a : r_a \sim a$, one chosen for each $a \in |A|$, yields a counit of $\mathcal{M}$ by Theorem 6.4.10.

6.4.12 Theorem. Consider a cell $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{\mu} A \\
\parallel \psi \parallel \downarrow
\end{array}
\end{array}
\end{array}$

$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Y \xrightarrow{\nu} B
\end{array}
\end{array}
\end{array}$
6.5. Lifts

If \( N \) has a counit and \( \psi \) creates inverse universal arrows, then \( M \) has a counit as well and \( \psi \) preserves inverse universal arrows.

If \( N \) has a unit and \( \psi \) creates direct universal arrows, then \( M \) has a unit as well and \( \psi \) preserves direct universal arrows.

Proof. By the equivalence of (2) and (4) in Corollary 6.4.11, and by Remark 6.2.23, this is reduced to Proposition 6.1.14.

6.4.13 Theorem. Let \( M : X \to A \) be a module.

- For a counit \( X \overset{\mu}{\to} M \quad R \quad \) of \( M \), the following conditions are equivalent:
  1. the functor \( R \) is fully faithful;
  2. each component of \( \mu \) is direct universal (hence two-way universal).

- For a unit \( X \overset{\mu}{\to} M \quad R \quad \) of \( M \), the following conditions are equivalent:
  1. the functor \( R \) is fully faithful;
  2. each component of \( \mu \) is inverse universal (hence two-way universal).

Proof. Let \( a \) and \( b \) be objects of \( A \). The commutativity of the naturality square

\[
\begin{array}{ccc}
a & \overset{\mu_a}{\to} & a \\
f \downarrow & & \downarrow f \\
b & \overset{\mu_b}{\to} & b
\end{array}
\]

for every \( A \)-arrow \( f : a \to b \), translates into the commutativity of the triangle

\[
\begin{array}{ccc}
a \cdot f \cdot R & \to & a \cdot (R \cdot b) \\
\mu_a \cdot f & \to & h \cdot h \cdot \mu_a \cdot f
\end{array}
\]

Since \( \mu_b \) is inverse universal, the assignment \( h \mapsto h \cdot \mu_b \) is bijective. Hence the assignment \( f \mapsto f \cdot R \) is bijective (i.e. \( R \) is fully faithful) iff the assignment \( f \mapsto \mu_a \cdot f \) is bijective (i.e. \( \mu_a \) is direct universal).

6.5 Lifts

A lift is defined in Definition 6.5.1 as a universal two-sided cylinder; specifically, a lift of a functor \( E \to A \) along a module \( M : X \to A \) is defined by a universal arrow of the module \( \langle E, M \rangle \) introduced in Section 4.3. A pointwise (or strong) lift is also defined in Definition 6.5.3: a two-sided cylinder is a pointwise lift if each component is universal. The propositions that follow the definition show that a two-sided cylinder is a pointwise lift precisely when the Yoneda morphism sends it to a module isomorphism, and show that the notion of a pointwise lift subsumes each other as a special case. Later in Section 12.4, we will see that the notion of pointwise lift subsumes that of pointwise extension and vice versa. All natural examples of lifts are pointwise.

Note. In the following, we consider a cylinder \( E \rightsquigarrow M \) as an arrow of the module \( \langle E, M \rangle \) defined in Definition 4.3.7.

6.5.1 Definition. A cylinder \( E \overset{\mu}{\to} M \quad R \quad \) is called inverse universal if it is an inverse universal \( \langle E, M \rangle \)-arrow.

Given a functor \( F : E \to A \), an inverse universal cylinder \( \mu : R \rightsquigarrow F : E \rightsquigarrow M \) or the pair \( (R, \mu) \), or the functor \( R \) itself, is called a lift of \( F \) along \( M \).
• A cylinder \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) is called direct universal if it is a direct universal \( (\mathbf{E}, \mathbf{M}) \)-arrow.

Given a functor \( \mathbf{G} : \mathbf{E} \to \mathbf{X} \), a direct universal cylinder \( \mu : \mathbf{G} \Rightarrow \mathbf{R} : \mathbf{E} \Rightarrow \mathbf{M} \) or the pair \( (\mathbf{R}, \mu) \), or the functor \( \mathbf{R} \) itself, is called a colift of \( \mathbf{G} \) along \( \mathbf{M} \).

6.5.2 Remark.
(1) A cylinder \( \mu : \mathbf{R} \Rightarrow \mathbf{F} : \mathbf{E} \Rightarrow \mathbf{M} \) is inverse universal if and only if to every cylinder \( \alpha : \mathbf{G} \Rightarrow \mathbf{F} : \mathbf{E} \Rightarrow \mathbf{M} \) there is a unique natural transformation \( \alpha / \mu : \mathbf{G} \Rightarrow \mathbf{R} \) (the adjunct of \( \alpha \) along \( \mu \)) such that \( \alpha = \alpha / \mu \circ \mu \). Dually, a cylinder \( \mu : \mathbf{G} \Rightarrow \mathbf{R} : \mathbf{E} \Rightarrow \mathbf{M} \) is direct universal if and only if to every cylinder \( \alpha : \mathbf{G} \Rightarrow \mathbf{F} : \mathbf{E} \Rightarrow \mathbf{M} \), there is a unique natural transformation \( \mu \backslash \alpha : \mathbf{R} \Rightarrow \mathbf{F} \) such that \( \alpha = \mu \circ \mu \alpha \).

(2) A cylinder \( \mu : \mathbf{G} \Rightarrow \mathbf{F} : \mathbf{E} \Rightarrow \mathbf{M} \) is called two-way universal if it is both inverse and direct universal.

(3) Lifts are unique up to isomorphism by Corollary 6.2.8.

6.5.3 Definition.
• A cylinder \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) is called pointwise inverse universal if each component \( \mu_e : \mathbf{e} : \mathbf{R} \Rightarrow \mathbf{F} : \mathbf{e} \) \( \xymatrix{ \mathbf{R} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) is an inverse universal \( \mathbf{M} \)-arrow; in this case, the cylinder \( \mu \) or the pair \( (\mathbf{R}, \mu) \), or the functor \( \mathbf{R} \) itself, is called a pointwise lift of \( \mathbf{G} \) along \( \mathbf{M} \).

• A cylinder \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) is called pointwise direct universal if each component \( \mu_e : \mathbf{e} : \mathbf{G} \Rightarrow \mathbf{R} : \mathbf{e} \) \( \xymatrix{ \mathbf{G} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) is a direct universal \( \mathbf{M} \)-arrow; in this case, the cylinder \( \mu \) or the pair \( (\mathbf{R}, \mu) \), or the functor \( \mathbf{R} \) itself, is called a pointwise colift of \( \mathbf{G} \) along \( \mathbf{M} \).

6.5.4 Remark.
(1) A cylinder \( \mu : \mathbf{G} \Rightarrow \mathbf{F} : \mathbf{E} \Rightarrow \mathbf{M} \) is called two-way pointwise universal if each component \( \mu_e : \mathbf{e} : \mathbf{R} \Rightarrow \mathbf{F} : \mathbf{e} \) is a two-way universal \( \mathbf{M} \)-arrow.

(2) We will see in Proposition 6.5.10 that a pointwise lift is a lift in the sense of Definition 6.5.1.

Note. The following characterizes a pointwise lift using the generalized Yoneda morphism (see Definition 5.3.3).

6.5.5 Proposition. For a cylinder \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \) \( \xymatrix{ \mathbf{E} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} } \), the following conditions are equivalent:

1. \( \mu \) is pointwise inverse [op. direct] universal;
2. the module morphism

\[ (\mathbf{X} \uparrow \mathbf{M}) \mu : (\mathbf{X} \uparrow \mathbf{M}) \mathbf{F} : \mathbf{X} \Rightarrow \mathbf{E} \]

\[ \mu (\mathbf{M} \uparrow \mathbf{A}) : \mathbf{R}(\mathbf{A}) \Rightarrow \mathbf{G}(\mathbf{M}) : \mathbf{E} \Rightarrow \mathbf{A} \]

is iso;
3. the composition

\[ \xymatrix{ \mathbf{X} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} \ar[r] & \mathbf{X} \\ \mathbf{X} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} \ar[r] & \mathbf{X} } \]

\[ \mathbf{X} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} \ar[r] & \mathbf{X} \]

\[ \mathbf{X} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} \ar[r] & \mathbf{X} \]

\[ \mathbf{X} \ar[r]^\mu & \mathbf{M} \ar[r] & \mathbf{A} \ar[r] & \mathbf{X} \]

of \( \mu \) and the right [op. left] Yoneda morphism for \( \mathbf{M} \) yields a natural isomorphism.

Proof. (1)\(\Rightarrow\) (3) The component of the natural transformation \( (\mathbf{X} \uparrow \mathbf{M}) \Rightarrow \) \( \mu \) at \( \mathbf{e} \in \mathbf{E} \) is given by the right module morphism \( \mathbf{X} \uparrow \mu_e = (\mathbf{X} \uparrow \mathbf{M}) \mu_e \). Since a natural transformation is iso iff each
Proof. By definition, a right cylinder $\mu$ is iso iff $X \downarrow \mu$ is iso, i.e., $\mu$ is inverse universal, for each $e \in \mathcal{E}$.

(2) $\iff$ (3) By Proposition 2.1.3, $(X \uparrow M) \mu$ is iso iff $(\langle X \uparrow M \rangle \mu)$ is iso. But

$\langle (X \uparrow M) \mu \rangle \tau = \langle (X \uparrow M) \tau \rangle \delta \mu$

by Remark 5.3.11.

6.5.6 Remark. Proposition 6.5.5 gives alternative definitions of the pointwise universality of a cylinder.

6.5.7 Proposition.

- A counit $\xrightarrow{\mu} \xrightarrow{\mathcal{M}} A$ of a module $\mathcal{M}$ is the same thing as a pointwise lift

  $\xrightarrow{\mu} \xrightarrow{\mathcal{M}} A \xrightarrow{1} A$  

  of the identity $A \to A$ along $\mathcal{M}$.

- A unit $\xrightarrow{\mu} \xrightarrow{\mathcal{M}} A$ of a module $\mathcal{M}$ is the same thing as a pointwise colift

  $\xrightarrow{\mu} \xrightarrow{\mathcal{M}} A \xleftarrow{1} A$  

  of the identity $X \to X$ along $\mathcal{M}$.

Proof. By definition, $(R, \mu)$ is a unit of $\mathcal{M}$ iff the module morphism $(X \uparrow \mathcal{M}) \mu : (X) R \to \mathcal{M}$ is iso. But by Proposition 6.5.5, this is the case iff $(R, \mu)$ is a pointwise lift of the identity $A \to A$ along $\mathcal{M}$. □

6.5.8 Remark. Proposition 6.4.3 now follows from Proposition 6.5.7.

6.5.9 Proposition.

- A pointwise lift $\xrightarrow{\mu} \xrightarrow{(M)F} E$ of $F$ along $\mathcal{M}$ is the same thing as a counit $\xrightarrow{\mu} \xrightarrow{(M)F} E$ of the composite module $\langle (M)F \rangle$.

- A pointwise colift $\xrightarrow{\mu} \xrightarrow{G(M)} A$ of $G$ along $\mathcal{M}$ is the same thing as a unit $\xrightarrow{\mu} \xrightarrow{G(M)} A$ of the composite module $G \langle M \rangle$.

Proof. By definition, a right cylinder $\xrightarrow{\mu} \xrightarrow{(M)F} E$ is a unit iff the module morphism $(X \uparrow \langle (M)F \rangle) \mu : (X) R \to \langle (M)F \rangle$ is iso, and by Proposition 6.5.5, a two-sided cylinder $\xrightarrow{\mu} \xrightarrow{(M)F} E$ is a pointwise lift iff the module morphism $(X \uparrow \mathcal{M}) \mu : (X) R \to \langle (M)F \rangle$ is iso. But these module isomorphisms coincide by Proposition 5.3.8. □

6.5.10 Proposition. A pointwise inverse [op. direct] universal cylinder in Definition 6.5.3 is inverse [op. direct] universal in the sense of Definition 6.5.1.

Proof. Let $\mu : R \Rightarrow F : E \Rightarrow \mathcal{M}$ be a pointwise inverse universal cylinder. By Proposition 6.5.9, $(R, \mu)$ is a unit of $(\mathcal{M})F$. Hence, by Proposition 6.4.5, $\mu : R \Rightarrow \langle (M)F \rangle$ is an inverse universal $(X, \mathcal{A})$-arrow, and by the identity in Proposition 5.1.8, this is the same thing as an inverse universal $(E, \mathcal{M})$-arrow $\mu : R \Rightarrow F$. □

6.5.11 Proposition. Let $E$ be a category and $M : X \to A$ be a module.
If a cylinder \( \mu : R \rightarrow F : E \rightarrow \mathcal{M} \) is inverse universal (resp. pointwise inverse universal), then a natural transformation \( \tau : S \rightarrow R : E \rightarrow X \) is iso if and only if the cylinder \( \tau \circ \mu : S \rightarrow F : E \rightarrow \mathcal{M} \) is inverse universal (resp. pointwise inverse universal).

If a cylinder \( \mu : G \rightarrow R : E \rightarrow \mathcal{M} \) is direct universal (resp. pointwise direct universal), then a natural transformation \( \tau : R \rightarrow S : E \rightarrow A \) is iso if and only if the cylinder \( \mu \circ \tau : G \rightarrow S : E \rightarrow \mathcal{M} \) is direct universal (resp. pointwise direct universal).

Proof. Suppose that \( \mu \) is inverse universal, i.e. an inverse universal \( \langle E, \mathcal{M} \rangle \)-arrow. Then the assertion is just an instance of Theorem 6.2.6 where \( \mathcal{M} \) is given by \( \langle E, \mathcal{M} \rangle \). Now suppose that \( \mu \) is pointwise inverse universal. Since a natural transformation is iso iff each component is an isomorphism, it suffices to show that, given \( e \in \|E\| \), \( [\tau \circ \mu]_e \) is an inverse universal \( \mathcal{M} \)-arrow iff \( \tau_e \) is an iso \( X \)-arrow. But since \( [\tau \circ \mu]_e = \tau_e \circ \mu_e \), this follows again from Theorem 6.2.6.

6.5.12 Proposition. Let \( E \) be a category and \( \mathcal{M} : X \rightarrow A \) be a module.

- If a functor \( F : E \rightarrow A \) has a pointwise lift along \( \mathcal{M} \), then every lift of \( F \) along \( \mathcal{M} \) is pointwise.
- If a functor \( G : E \rightarrow X \) has a pointwise colift along \( \mathcal{M} \), then every colift of \( G \) along \( \mathcal{M} \) is pointwise.

Proof. Let \( \mu : R \rightarrow F : E \rightarrow \mathcal{M} \) and \( \nu : S \rightarrow F : E \rightarrow \mathcal{M} \) be two lifts of \( F \) along \( \mathcal{M} \). Then, by Theorem 6.2.7, there are natural isomorphisms \( \nu/\mu : S \rightarrow R \) and \( \mu/\nu : R \rightarrow S \) inverse to each other making the diagram commute. Hence, by Proposition 6.5.11, if one of \( \mu \) and \( \nu \) is pointwise inverse universal, so is the other.

6.5.13 Theorem. For a natural transformation \( \tau : G \rightarrow F : E \rightarrow C \), i.e. a cylinder \( \tau : G \rightarrow F : E \rightarrow \langle C \rangle \), the following conditions are equivalent:

1. \( \tau \) is a natural isomorphism in \( C \);
2. \( \tau \) is an inverse universal cylinder along the hom-module \( \langle C \rangle \);
3. \( \tau \) is a direct universal cylinder along the hom-module \( \langle C \rangle \);
4. \( \tau \) is a two-way universal cylinder along the hom-module \( \langle C \rangle \);
5. \( \tau \) is a pointwise inverse universal cylinder along the hom-module \( \langle C \rangle \);
6. \( \tau \) is a pointwise direct universal cylinder along the hom-module \( \langle C \rangle \);
7. \( \tau \) is a pointwise two-way universal cylinder along the hom-module \( \langle C \rangle \).

Proof. Since a natural transformation \( \tau : G \rightarrow F : E \rightarrow C \) is the same thing as an arrow \( \tau : G \rightarrow F \) in the category \( [E, C] \), and since a cylinder \( \tau : G \rightarrow F : E \rightarrow \langle C \rangle \) is the same thing as an arrow \( \tau : G \rightarrow F \) in the module \( \langle E, \langle C \rangle \rangle = \langle E, C \rangle \) (see Remark 4.3.8(3)), the equivalence of conditions (1), (2), (3), (4), and (5) follows by applying Proposition 6.2.5 to the category \( [E, C] \). Since a natural transformation is iso iff each component is an isomorphism, the equivalence of conditions (1), (5), (6), and (7) follows again from Proposition 6.2.5.

Note. By Proposition 6.5.9, the following is a special case of Theorem 6.4.10 (and vice versa by Proposition 6.5.7).

6.5.14 Theorem. Let \( E \) be a category and \( \mathcal{M} : X \rightarrow A \) be a module.

- Given a functor \( F : E \rightarrow A \), if there is a family of inverse universal \( \mathcal{M} \)-arrows \( \mu_e : r_e \rightarrow F : e \), one for each object \( e \in \|E\| \), then there is a unique functor \( R : E \rightarrow X \) with \( e \cdot R = r_e \) such that \( \mu := (\mu_e)_{e \in \|E\|} \) forms a cylinder \( X \rightarrow \mathcal{M} \rightarrow A \), and \( \mu \) is pointwise inverse universal.
Given a functor $G : E \to X$, if there is a family of direct universal $M$-arrows $\mu_e : e \cdot G \to r_e$, one for each object $e \in \|E\|$, then there is a unique functor $R : E \to A$ with $e \cdot R = r_e$ such that $\mu := (\mu_e)_{e \in \|E\|}$ forms a cylinder $\xrightarrow{G} \xrightarrow{\mu} \xrightarrow{R} \xrightarrow{\mu} \xrightarrow{A}$, and $\mu$ is pointwise direct universal.

**Proof.** Since an $M$-arrow $\mu_e : r_e \sim F \cdot e$ is inverse universal iff so is the $\langle M \rangle F$-arrow $\mu_e : r_e \sim e$ (see Theorem 6.2.19), by Proposition 6.5.9, the assertion is reduced to an instance of Theorem 6.4.10 where $M$ is given by the composite module $\langle M \rangle F$.

**Note.** The composition in Definition 4.3.26 preserves pointwise lifts:

**6.5.15 Theorem.** If $\mu$ is pointwise inverse [op. direct] universal cylinder and $K$ is a functor as in

$$
\begin{array}{ccc}
D & \xrightarrow{\mu} & F \\
\downarrow{K} & & \downarrow{F} \\
X & \xrightarrow{\mu} & A
\end{array}
$$

then their composite

$$
\begin{array}{ccc}
K \circ G & \xrightarrow{\mu} & F \circ K \\
\downarrow{K \circ \mu} & & \downarrow{F \circ K} \\
X & \xrightarrow{\mu} & A
\end{array}
$$

is pointwise inverse [op. direct] universal as well.

**Proof.** Obvious since $[K \circ \mu]_d = \mu[K \cdot d]$ for each $d \in \|D\|$.

**Note.** The precomposition cell $(K, M)$ in Definition 4.3.28 thus preserves pointwise universality:

**6.5.16 Corollary.** Given a functor $K : D \to E$ and a module $M : X \to A$, the precomposition cell $(K, M) : (E, M) \to (D, M)$ preserves pointwise universality; that is, $(K, M)$ sends each pointwise inverse [op. direct] universal cylinder $\mu : E \to M$ to a pointwise inverse [op. direct] universal cylinder $K \circ \mu : D \to M$.

**Proof.** Immediate from Theorem 6.5.15.

**Note.** Recall from Remark 4.3.35(2) that any cylinder $\alpha$ yields the postcomposition cylinder $[D, \alpha]$.

**6.5.17 Corollary.** If a cylinder $\mu$ is pointwise inverse [op. direct] universal, so is the postcomposition cylinder $[D, \mu]$ for any category $D$, and moreover each component of $[D, \mu]$ is a pointwise inverse [op. direct] universal cylinder.

**Proof.** Since the component of $[D, \mu]$ at a functor $K$ is given by the cylinder $K \circ \mu$, the assertion follows from Theorem 6.5.15 by noting that (see Proposition 6.5.10) a pointwise universal cylinder is universal.

**Note.** Recall from Remark 4.3.27(3) that the composition of a right cylinder and a functor yields a two sided cylinder; we see below that the composition of a counit and a functor yields a pointwise universal cylinder.

**6.5.18 Theorem.** If a module $M : X \to A$ has a counit [op. unit] $(R, \mu)$, then the composition of $\mu$ and a functor $K$ as in

$$
\begin{array}{ccc}
X & \xrightarrow{\mu} & A \\
\downarrow{\mu} & & \downarrow{\mu} \\
\mu & & \mu
\end{array}
$$

op.

$$
\begin{array}{ccc}
E & \xrightarrow{K} & X \\
\downarrow{R} & & \downarrow{R} \\
A & & A
\end{array}
$$
yields a pointwise inverse [op. direct] universal cylinder

\[
\begin{array}{c}
\xymatrix{K \circ \mu \ar[r] & K \ar[r] & M \ar[r] & \mathcal{E} \ar[r] & X \\
\mathcal{M} \ar[r] & A}
\end{array}
\quad \text{op.}
\begin{array}{c}
\xymatrix{K \circ \mu \ar[r] & K \ar[r] & M \ar[r] & \mathcal{E} \ar[r] & X \\
\mathcal{M} \ar[r] & A}
\end{array}
\]

**Proof.** Obvious since \([K \circ \mu]_e = \mu(K_e)\) for each \(e \in |\mathcal{E}|\).

\(\square\)

6.5.19 Remark. By Proposition 6.5.7, Theorem 6.5.18 is a special case of Theorem 6.5.15 (and vice versa by Proposition 6.5.9).

**Note.** Recall from Remark 4.3.35(3) that any right cylinder \(\alpha\) yields the postcomposition right cylinder \([\mathcal{E}, \alpha]\).

6.5.20 Corollary. If a right [op. left] cylinder \(\mu\) is a counit [op. unit], so is the postcomposition right [op. left] cylinder \([\mathcal{E}, \mu]\) for any category \(\mathcal{E}\), and moreover each component of \([\mathcal{E}, \mu]\) is a pointwise inverse [op. direct] universal cylinder.

**Proof.** Since the component of \([\mathcal{E}, \mu]\) at a functor \(K\) is given by the cylinder \(K \circ \mu\), the assertion follows from Theorem 6.5.18 by noting that (see Proposition 6.5.10) a pointwise universal cylinder is universal.

\(\square\)

6.5.21 Remark. Corollary 6.5.20 is a special case of Corollary 6.5.17 (cf. Remark 6.5.19).

**Note.** Recall from Remark 4.3.16(3) that the composition of a right cylinder and a cell yields a two sided cylinder; in the following, we consider the composition of a counit and a cell.

6.5.22 Theorem. Given a cell \(X \xrightarrow{\mathcal{M}} A\), \(P \xrightarrow{\psi} Q \xleftarrow{N}\)

\(\bullet\) suppose that \(\mathcal{M}\) has a counit \(X \xrightarrow{\mu} A\), then its composite \(P \xleftarrow{\mu \circ \psi} Q \xrightarrow{\psi} N\) with \(\psi\) is pointwise inverse universal if and only if \(\psi\) preserves inverse universal arrows.

\(\bullet\) suppose that \(\mathcal{M}\) has a unit \(X \xleftarrow{\mu} A\), then its composite \(P \xrightarrow{\mu \circ \psi} Q \xleftarrow{\psi} N\) with \(\psi\) is pointwise direct universal if and only if \(\psi\) preserves direct universal arrows.

**Proof.** By Remark 4.3.16(3), \([\mu \circ \psi]_a = \mu_a \circ \psi\) for each \(a \in |A|\). The “if” part is thus obvious on noting Proposition 6.4.3. To prove the “only if” part, by Remark 6.2.23, it suffices to show that each right module cell \((\psi)_a\) preserves units. But this is immediate on noting Proposition 6.1.13.

\(\square\)

6.5.23 Corollary. Given a cell \(X \xrightarrow{\mathcal{M}} A\), \(P \xrightarrow{\psi} Q \xleftarrow{N}\)

\(\bullet\) if \(\mathcal{M}\) and \(\mathcal{N}\) has counits \(X \xrightarrow{\mu} A\) and \(Y \xrightarrow{\nu} B\), then there is the canonical natural transformation

\[
\begin{array}{c}
\xymatrix{X \ar[r]^-R & A \\
\ar[r]^-S & B}
\end{array}
\]

; this natural transformation is an isomorphism if and only if the cell \(\psi\) preserves inverse universal arrows.
• if \( M \) and \( N \) has units \( X \xrightarrow{R} A \) and \( Y \xrightarrow{S} B \), then there is the canonical natural transformation

\[
\begin{array}{ccc}
X & \xrightarrow{R} & A \\
\downarrow \quad \downarrow \psi & & \downarrow \downarrow \\
Y & \xrightarrow{S} & B
\end{array}
\]

; this natural transformation is an isomorphism if and only if the cell \( \psi \) preserves direct universal arrows.

Proof. By Theorem 6.5.18, the composition of \( Y \xrightarrow{S} B \) and \( B \xleftarrow{Q} A \) yields the pointwise inverse universal cylinder \( \begin{array}{cc} A & \xleftarrow{\mu \circ \psi} Q \\
\downarrow & \downarrow \downarrow \\
Y & \xrightarrow{\mu \circ \psi} B \end{array} \) \( Q \circ \mu \), while the composition of \( X \xrightarrow{R} A \) and \( X \xleftarrow{\mu} A \) yields \( \begin{array}{cc} A & \xleftarrow{\mu \circ \psi} Q \\
\downarrow & \downarrow \downarrow \\
Y & \xrightarrow{\mu \circ \psi} B \end{array} \) \( Q \circ \mu \), giving a naturality square

\[
\begin{array}{ccc}
d \colon G \xrightarrow{d} F \xrightarrow{\mu} \ x \xrightarrow{\psi} dh & \xrightarrow{F \cdot d} & F \cdot e \\
h \colon G \xrightarrow{h} & \xrightarrow{F \cdot h} & \ x \xrightarrow{\psi} e
\end{array}
\]

with \( h \colon G \) and \( F \cdot h \) iso (because any functor preserves isomorphisms). Now, since \( \mu_d \) is universal by assumption, so is \( \mu_e \) by Theorem 6.2.18.

6.5.25 Remark. Since a natural isomorphism is the same thing as a pointwise universal cylinder along the hom-module of a category (see Theorem 6.5.13), the lemma in Preliminary 0.0.11 is a special case of Theorem 6.5.24.

6.5.26 Theorem. Let \( G \xrightarrow{E} F \) be a cylinder and \( D \) be an isomorphism-dense subcategory of \( E \). Then \( \mu \) is pointwise inverse [op. direct] universal if and only if its restriction to \( D \) (see Remark 4.3.27(1)) is; that is, if and only if the component \( \mu_d \) is inverse [op. direct] universal for each \( d \in [D] \).

Proof. The forward implication is immediate from Theorem 6.5.15. Assume now that \( \mu \) is pointwise universal on \( D \). We need to show that the component \( \mu_e \) is universal for each \( e \in [E] \). Since \( D \) is isomorphism-dense in \( E \), there is an object \( d \in [D] \) and an iso \( E \)-arrow \( h \colon d \to e \), giving a naturality square

\[
\begin{array}{ccc}
d \colon G \xrightarrow{d} F \xrightarrow{\mu} \ x \xrightarrow{\psi} dh & \xrightarrow{F \cdot d} & F \cdot e \\
h \colon G \xrightarrow{h} & \xrightarrow{F \cdot h} & \ x \xrightarrow{\psi} e
\end{array}
\]

with \( h \colon G \) and \( F \cdot h \) iso (because any functor preserves isomorphisms). Now, since \( \mu_d \) is universal by assumption, so is \( \mu_e \) by Theorem 6.2.18.

6.5.27 Remark. Since a natural isomorphism is the same thing as a pointwise universal cylinder along the hom-module of a category (see Theorem 6.5.13), the lemma in Preliminary 0.0.11 is a special case of Theorem 6.5.24.
Consider a pair of direct universal cylinders $\xymatrix{ X \ar[rr]_{\mu} \ar[rr]^{M} & & A \ar[rr]_{\mu'} \ar[rr]^{M} & & A }$. Then for any natural transformation $\sigma : G \rightarrow G'$, there exists a unique natural transformation $\tau : R \rightarrow R'$ making the square

\[
\xymatrix{ G \ar[r]^\mu \ar[d]_{\sigma} & R \ar[d]^{\tau} \\
G' \ar[r]_{\mu'} & R' }
\]
commute in the module $(E, M) : [E, X] \rightarrow [E, A]$.

**Proof.** The unique $\tau$ is given by the conjugate (see Definition 6.3.1) of $\sigma$ inverse along $(\mu, \mu')$. □

**6.5.27 Remark.** By Remark 4.3.6(1), the square

\[
\xymatrix{ R \ar[r]^\mu \ar[d]_{\tau} & F \ar[d]^{\sigma} \\
R' \ar[r]_{\mu'} & F' }
\]

in Theorem 6.5.26 commutes if and only if the square

\[
\xymatrix{ e : R \ar[r]^\mu \ar[d]_{\tau_e} & F \ar[d]^{\sigma_e} \\
e' : R' \ar[r]_{\mu'} & F' }
\]

commutes in the module $M : X \rightarrow A$ for every object $e \in E$. Hence if $\mu$ and $\mu'$ are pointwise inverse [op. direct] universal, each $\tau_e$ is given by the conjugate of $\sigma_e$ along $(\mu_e, \mu'_e)$; Theorem 6.5.26 says that if $\sigma_e$ is natural in $e$, so will be $\tau_e$.

### 6.6 Kan lifts

A lift along a representable module has a special name—a Kan lift. The notion of pointwise lift and all results in Section 6.5 also apply to Kan lifts. In Section 7.3, we will see that an adjoint functor is a special instance of a pointwise Kan lift.

**Note.** Remark 4.3.2(3) allows the following definition.

**6.6.1 Definition.** Given a pair of functors $D \xrightarrow{K} E \xleftarrow{L} C$,

- a natural transformation

\[
\xymatrix{ D \ar[rr]_{\mu} \ar[rr]^{K} & & E \ar[rr]_{\mu'} \ar[rr]^{K} & & E }
\]

itself, is called a right Kan lift (resp. pointwise right Kan lift) of $L$ along $K$ if the cylinder is inverse universal (resp. pointwise inverse universal).

- a natural transformation

\[
\xymatrix{ E \ar[rr]_{\mu} \ar[rr]^{L} & & D \ar[rr]_{\mu'} \ar[rr]^{L} & & D }
\]

is called a left Kan lift (resp. pointwise left Kan lift) of $L$ along $K$ if the cylinder

is direct universal (resp. pointwise direct universal).
6.6.2 Remark.

(1) A Kan lift is thus a special instance of a lift defined in Definition 6.5.1 where \(\mathcal{M}\) is representable, and similarly a pointwise Kan lift is a special instance of a pointwise lift defined in Definition 6.5.3.

(2) A natural transformation \(\mu : R \circ K \to L\) forms a right Kan lift if and only if to every natural transformation \(\alpha : G \circ K \to L\) there is a unique natural transformation \(\alpha / \mu : G \to R\) (the adjunct of \(\alpha\) along \(\mu\)) such that \(\alpha = (\alpha / \mu \circ K) \circ \mu\) \([\alpha / \mu \circ K] \circ \mu\) is the pasting composite of \(\alpha / \mu\) and \(\mu\). Dually, a natural transformation \(\mu : L \to K \circ R\) forms a left Kan lift if and only if to every natural transformation \(\alpha : L \to K \circ F\) there is a unique natural transformation \(\mu \circ \alpha : R \to F\) such that \(\alpha = \mu \circ [\mu \circ \alpha \circ K]\).

(3) By Proposition 6.5.9,
- a pointwise right Kan lift of \(L\) along \(K\) is the same thing as a counit of the composite module \(K(C) \circ L : D \to C\).
- a pointwise left Kan lift of \(L\) along \(K\) is the same thing as a unit of the composite module \(L(C) \circ K : C \to D\).

(4) Using the terms introduced in Example 6.2.3, a pointwise Kan lift is described as follows:
- a natural transformation \(\mu : R \circ K \to L\) forms a pointwise right Kan lift of \(L\) along \(K\) if and only if each component \(\mu_c : c \circ R : K \to L \circ c\) is isomorphic from \(K\) to \(L \circ c\).
- a natural transformation \(\mu : L \to K \circ R\) forms a pointwise left Kan lift of \(L\) along \(K\) if and only if each component \(\mu_c : c \circ L \to K \circ R \circ c\) is isomorphic from \(c \circ L\) to \(K\).

(5) By Proposition 6.5.10, a pointwise Kan lift is a Kan lift. See Example 6.6.5(1) for a (very artificial) example of a non-pointwise Kan lift.

6.6.3 Theorem. Assume that a natural transformation

\[
\begin{array}{ccc}
D & \xrightarrow{\mu} & E \\
\downarrow \scriptstyle{R} & & \downarrow \scriptstyle{L} \\
K & \xleftarrow{\scriptstyle{\alpha}} & C
\end{array}
\]

is a natural isomorphism. Under this assumption, if \(K\) is fully faithful, then \(\mu\) is a pointwise right \([\text{op. left}]\) Kan lift of \(L\) along \(K\). The converse holds if we assume in addition that \(R\) is essentially surjective.

Proof. For each \(c \in \parallel C\), we have a commutative diagram

\[
\begin{array}{ccc}
(D)(R \circ c) & \xrightarrow{(K \circ (R \circ c))} (K(E) \circ K \circ (R \circ c)) & K(E)(K \circ R \circ c) \\
\downarrow D \circ \mu c & & \downarrow K(E) \circ \mu c \\
K(E)(L \circ c) & \xrightarrow{(K(E)(L \circ c))} K(E)(L \circ c)
\end{array}
\]

by replacing \(f : d \Rightarrow e\) in Example 5.2.7(3) with \(\mu_c : R \circ c \Rightarrow L \circ c\). Since \(\mu_c\) is an iso \(E\)-arrow by assumption, \(K(E) \circ \mu_c\) is iso. Hence \(D \uparrow \mu_c\) is iso iff \((K)(R \circ c)\) is iso. Suppose now that \(K\) is fully faithful. Then \((K)(R \circ c)\) and hence \(D \uparrow \mu_c\) is iso for every \(c \in \parallel C\); \(\mu\) is thus pointwise inverse universal. Suppose conversely that \(\mu\) is pointwise inverse universal. Then \(D \uparrow \mu_c\) and hence \((K)(R \circ c)\) is iso for every \(c \in \parallel C\); \(K\) is thus fully faithful by Proposition 2.1.5 under the condition that \(R\) is essentially surjective.

\(\square\)

6.6.4 Remark. If \(K\) is not fully faithful in Theorem 6.6.3, then a natural isomorphism \(\mu\), even an identity, need not form a Kan lift. See Example 6.6.5(2) for an example.

6.6.5 Example.

(1) Example 6.4.7 is repeated below in terms of Kan lifts. Let 2 and 2 be as in Example 6.4.7 and let \(K : 2 \to 2\) be the inclusion functor. Then the representable module of \(K\) looks like

\[
\begin{array}{c}
0 \xrightarrow{\sim} 0 \\
\downarrow \\
1 \xrightarrow{\sim} 1
\end{array}
\]
Consider functors as in

\[
\begin{array}{ccc}
\Delta_0 & \stackrel{\mu}{\longrightarrow} & 1 \\
2 & \mathrel{\longrightarrow} & 2
\end{array}
\]

along \(K\). The constant functor \(\Delta_0\) and \(\mu\) form a right Kan lift of the identity \(2 \to 2\) along \(K\); however, the lift is not pointwise since \(\mu_1 : 0 \to 1\) is not universal from \(K\) to \(1\).

(2) Consider functors as in

\[
\begin{array}{ccc}
1 & \mathrel{\longrightarrow} & 1 \\
E & \mathrel{\longleftarrow} & 1
\end{array}
\]

, where \(1\) is the terminal category and \(E\) is any category. Given \(e \in \|E\|\), the functor \(e : 1 \to E\), and the identity natural transformation

\[
\begin{array}{ccc}
E & \mathrel{\longleftarrow} & 1 \\
e & \mathrel{\longleftarrow} & 1
\end{array}
\]

form a right Kan lift of the identity \(1 \to 1\) along the unique functor \(E \to 1\) only when \(e\) is a terminal object of \(E\).

**Note.** Since the hom-module \((C)\) of a category \(C\) is the same thing as the representable module of the identity functor \(1_C : C \to C\), Theorem 6.5.13 is restated as follows.

**6.6.6 Theorem.** For a natural transformation \(\tau : G \to F : E \to C\), the following conditions are equivalent:

1. \(\tau\) is a natural isomorphism in \(C\);
2. \(\tau\) is a right Kan lift of \(F\) along the identity functor \(1_C\);
3. \(\tau\) is a left Kan lift of \(G\) along the identity functor \(1_C\);
4. \(\tau\) is a pointwise right Kan lift of \(F\) along the identity functor \(1_C\);
5. \(\tau\) is a pointwise left Kan lift of \(G\) along the identity functor \(1_C\).

**Note.** The following is a special case of Theorem 6.6.14 where \(M\) is representable.

**6.6.7 Theorem.** Given a pair of functors \(D \stackrel{K}{\longrightarrow} E \stackrel{L}{\longleftarrow} C\),

- if there is a family of \(E\)-arrows \(\mu_e : R_e : L \to C\), one for each \(e \in \|C\|\), universal from \(K\) to \(L\), \(c\), then there is a unique functor \(R : C \to D\) with \(c : R = R_e\) such that \(\mu = (\mu_e)_{e \in \|C\|}\) forms a natural transformation \(D \mathrel{\longleftarrow} C \mathrel{\longleftarrow} E\), and \(\mu\) is a pointwise right Kan lift of \(L\) along \(K\).

- if there is a family of \(E\)-arrows \(\mu_e : c : L \to K\), one for each \(e \in \|C\|\), universal from \(c : L\) to \(K\), then there is a unique functor \(R : C \to D\) with \(c : R = R_e\) such that \(\mu = (\mu_e)_{e \in \|C\|}\) forms a natural transformation \(E \mathrel{\longleftarrow} L \mathrel{\longleftarrow} D\), and \(\mu\) is a pointwise left Kan lift of \(L\) along \(K\).

**6.6.8 Corollary.**

- Given a functor \(F : E \to C\), if there is a family of iso \(C\)-arrows \(\tau_e : r_e \to F \cdot e\), one for each \(e \in \|E\|\), then there is a unique functor \(G : E \to C\) with \(e \cdot G = r_e\) such that \(\tau = (\tau_e)_{e \in \|E\|}\) forms a natural isomorphism \(\tau : G \to F : E \to C\).

- Given a functor \(G : E \to C\), if there is a family of iso \(C\)-arrows \(\tau_e : c : G \to r_e\), one for each \(e \in \|E\|\), then there is a unique functor \(F : E \to C\) with \(e \cdot F = r_e\) such that \(\tau = (\tau_e)_{e \in \|E\|}\) forms a natural isomorphism \(\tau : G \to F : E \to C\).

**Proof.** Noting Theorem 6.6.6, we see that this is a special case of Theorem 6.6.7 where \(K\) is the identity. \(\Box\)
7  Adjunction and Equivalence

7.1  Symmetric cells

This section introduces a different type of module cell called symmetric. A symmetric cell $A \xleftrightarrow{Q} B$ between modules $\mathcal{M}$ and $\mathcal{N}$ is defined by a pair of functors $P$ and $Q$ going opposite directions and a module morphism $\psi: \langle \mathcal{M} \rangle Q \to P\langle \mathcal{N} \rangle: X \to B$. In many ways, symmetric cells are no different from normal cells:

- modules and symmetric cells assemble into a category;
- symmetric cells between two modules form a module between two functor categories;
- a symmetric cell yields a postcomposition symmetric cell with a module or with a category.

7.1.1 Definition. Given a pair of modules $\mathcal{M}: X \to A$ and $\mathcal{N}: Y \to B$, and given a pair of functors $P: X \to Y$ and $Q: B \to A$,

- a right symmetric cell $\psi: Q \to P: \mathcal{M} \to \mathcal{N}$, written diagrammatically as

  \[
  \begin{array}{cccccc}
  X & \xrightarrow{M} & A & \xleftarrow{N} & Y \\
  P & \xleftarrow{\psi} & Q & \xrightarrow{\psi} & Q \\
  Y & \xrightarrow{\psi} & B & \xleftarrow{\psi} & A \\
  \
  \end{array}
  \]

  , is defined by a module morphism $\psi: \langle \mathcal{M} \rangle Q \to P\langle \mathcal{N} \rangle: X \to B$.

- a left symmetric cell $\psi: Q \to P: \mathcal{M} \to \mathcal{N}$, written diagrammatically as

  \[
  \begin{array}{cccccc}
  X & \xrightarrow{M} & A & \xleftarrow{N} & Y \\
  P & \xleftarrow{\psi} & Q & \xrightarrow{\psi} & Q \\
  Y & \xleftarrow{\psi} & B & \xrightarrow{\psi} & A \\
  \
  \end{array}
  \]

  , is defined by a module morphism $\psi: P\langle \mathcal{N} \rangle \to \langle \mathcal{M} \rangle Q: X \to B$.

7.1.2 Remark.

1. For a pair of objects $x \in \|X\|$ and $b \in \|B\|$,

   - the component of a right symmetric cell

     \[
     \begin{array}{ccc}
     A & \xleftarrow{Q} & B \\
     X & \xrightarrow{\rho} & Y \\
     \end{array}
     \]

     at $(x, b)$ is the function

     \[
     x\langle \mathcal{M} \rangle (Q\cdot b) = x\langle \mathcal{M} \rangle Qb \xrightarrow{x\langle \psi \rangle b} x\langle P\langle \mathcal{N} \rangle \rangle b = (x\cdot P)\langle \mathcal{N} \rangle b
     \]

     , which sends each $\mathcal{M}$-arrow $m: x \to Q\cdot b$ to the $\mathcal{N}$-arrow $m\cdot \psi: x\cdot P \to b$.

   - the component of a left symmetric cell

     \[
     \begin{array}{ccc}
     A & \xrightarrow{Q} & B \\
     X & \xleftarrow{\rho} & Y \\
     \end{array}
     \]

     at $(x, b)$ is the function

     \[
     x\langle \mathcal{M} \rangle (Q\cdot b) = x\langle \mathcal{M} \rangle Qb \xleftarrow{x\langle \psi \rangle b} x\langle P\langle \mathcal{N} \rangle \rangle b = (x\cdot P)\langle \mathcal{N} \rangle b
     \]

     , which sends each $\mathcal{N}$-arrow $n: x\cdot P \to b$ to the $\mathcal{M}$-arrow $\psi\cdot n: x \to Q\cdot b$.
(2) The identity module morphism $M \to M$ yields the identity right and left symmetric cells:

<table>
<thead>
<tr>
<th>A</th>
<th>M</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>X</td>
</tr>
</tbody>
</table>

(3) Cells introduced in Section 1.2 are referred to as ordinary cells to distinguish them from symmetric cells introduced here.

7.1.3 Definition. Given a pair of right [op. left] symmetric cells as in

$$
\begin{array}{ccc}
A & \xrightarrow{Q} & A' & \xrightarrow{Q'} & A'' \\
\downarrow{\psi} & \downarrow{\psi'} & \downarrow{\psi''} & \downarrow{\psi'''} & \downarrow{\psi''''} \\
X & \xrightarrow{p} & X' & \xrightarrow{p'} & X''
\end{array}

$$

their composite $\psi \circ \psi' = \psi' \circ \psi$ is the right [op. left] symmetric cell

$$
\begin{array}{ccc}
A & \xrightarrow{Q \circ Q'} & A'' \\
\downarrow{\psi \circ \psi'} & \downarrow{\psi'' \circ \psi'')} & \downarrow{\psi'''} & \downarrow{\psi''''} \\
X & \xrightarrow{p \circ p'} & X''
\end{array}

$$

defined by the module morphism

$$
\psi \circ \psi' : \langle M \rangle [Q \circ Q'] \to [P \circ P'] \langle M'' \rangle \quad \text{op.} \quad \psi \circ \psi' : [P \circ P'] \langle M'' \rangle \to \langle M \rangle [Q \circ Q']
$$
given by the composition

$$
\langle M \rangle [Q \circ Q'] = \langle \langle M \rangle Q \circ \psi' \rangle \circ \langle P \circ (M') \rangle Q' = P \langle \langle M' \rangle Q' \circ \psi' \rangle \circ \langle P \circ (M'') \rangle = [P \circ P'] \langle M'' \rangle \quad \text{op.}
$$

$$
\langle M \rangle [Q \circ Q'] = \langle \langle M \rangle Q \circ \psi' \rangle \circ \langle P \circ (M') \rangle Q' = P \langle \langle M' \rangle Q' \circ \psi' \rangle \circ \langle P \circ (M'') \rangle = [P \circ P'] \langle M'' \rangle
$$

7.1.4 Proposition. Modules and right [op. left] symmetric cells among them form a category with the composition given in Definition 7.1.3 and the identity symmetric cells in Remark 7.1.2(2).

Proof. The only non-trivial part is the verification of the associativity of the composition. Given right symmetric cells

$$
\begin{array}{ccc}
A & \xrightarrow{Q} & A' & \xrightarrow{Q'} & A'' & \xrightarrow{Q''} & A''' \\
\downarrow{\psi} & \downarrow{\psi'} & \downarrow{\psi''} & \downarrow{\psi'''} & \downarrow{\psi''''} & \downarrow{\psi''''' & \downarrow{\psi''''''} \\
X & \xrightarrow{p} & X' & \xrightarrow{p'} & X'' & \xrightarrow{p''} & X'''
\end{array}

$$

, the two cell compositions $\langle \langle \psi Q \circ P \psi' \rangle \circ Q' \rangle \circ P \langle \psi'' \rangle$ and $\langle \langle \psi Q' \circ P \psi'' \rangle \circ Q'' \rangle \circ P \langle \psi'''' \rangle$ are defined by the module morphisms $\langle \langle \psi Q' \circ P \psi'' \circ P \psi' \rangle \circ Q'' \rangle \circ P \langle \psi'''' \rangle$ respectively. But by the functoriality and associativity of the composition, we have

$$
\langle \langle \psi Q' \circ P \psi'' \circ P \psi' \rangle \circ Q'' \rangle \circ P \langle \psi''''' \rangle = \langle \langle \psi Q' \circ P \psi'' \rangle \circ Q'' \circ P \langle \psi''''' \rangle \rangle \circ P \langle \psi'''' \rangle
$$

\[= \langle \langle \psi Q' \circ P \psi'' \rangle \circ Q'' \circ P \langle \psi''''' \rangle \rangle \circ P \langle \psi''''' \rangle \rangle
\]

\boxed{}

7.1.5 Definition. Given a right [op. left] symmetric cell $\psi$ and natural transformations $\tau$ and $\sigma$ as in

$$
\begin{array}{ccc}
A & \xrightarrow{Q} & B \\
\downarrow{\psi} & \downarrow{\psi'} & \downarrow{\psi''} \\
X & \xrightarrow{\sigma} & Y
\end{array}

$$

op.

$$
\begin{array}{ccc}
A & \xrightarrow{Q} & B \\
\downarrow{\psi} & \downarrow{\psi'} & \downarrow{\psi''} \\
X & \xrightarrow{\tau} & Y
\end{array}

$$
The composition in the module allows the following definition. The composition in Definition 7.1.5 yields the module of symmetric cells

\[ \tau \circ \psi \circ \sigma : (\mathcal{M}) Q' \rightarrow P'('N) \]

given by the composition

\[ (\mathcal{M}) Q' \xrightarrow{\rho(N)} (\mathcal{M}) Q \xrightarrow{\psi} P('N) \xrightarrow{\sigma(N)} P'('N). \]

Note. The composition in Definition 7.1.5 yields the module of symmetric cells \( \mathcal{J} \rightarrow \mathcal{M} \); the covariant and contravariant functors in Definition 2.2.1 allow the following definition.

### 7.1.6 Definition

Given a pair of modules \( \mathcal{M} : X \rightarrow A \) and \( \mathcal{N} : Y \rightarrow B \),

- the module

\[ (\mathcal{M} \downarrow \mathcal{N}) : [B, A] \rightarrow [X, Y]^{-} \]

of right symmetric cells \( \mathcal{M} \rightarrow \mathcal{N} \) is defined by the composition

\[ [B, A] \xrightarrow{M \cdot B} [X : B] \xrightarrow{(X B)} [X : B] \xrightarrow{X \cdot N} [X, Y]^{-} \]

where \( M \cdot B \) is the right action of \( M \) on the functor category \([B, A]\) and \( X \cdot N \) is the left action of \( N \) on the functor category \([X, Y]\).

- the module

\[ (\mathcal{M} \downarrow \mathcal{N}) : [B, A]^{-} \rightarrow [X, Y] \]

of left symmetric cells \( \mathcal{M} \rightarrow \mathcal{N} \) is defined by the composition

\[ [B, A]^{-} \xrightarrow{M \cdot B} [X : B]^{-} \xrightarrow{(X B)} [X : B]^{-} \xrightarrow{X \cdot N} [X, Y] \]

where \( M \cdot B \) is the right action of \( M \) on the functor category \([B, A]\) and \( X \cdot N \) is the left action of \( N \) on the functor category \([X, Y]\).

### 7.1.7 Remark

1. For each pair of functors \( Q : B \rightarrow A \) and \( P : X \rightarrow Y \), the set

\[ (Q) (\mathcal{M} \downarrow \mathcal{N}) (P) = (\mathcal{M}) Q \cdot (X : B) \cdot (P \cdot 'N) \]

consists of all module morphisms \( \mathcal{M} Q \rightarrow P \cdot 'N : X \rightarrow B \), i.e. all right symmetric cells \( Q \rightarrow P : \mathcal{M} \rightarrow \mathcal{N} \), and the set

\[ (Q) (\mathcal{M} \downarrow \mathcal{N}) (P) = (\mathcal{M}) Q \cdot (X : B)^{-} \cdot (P \cdot 'N) \]

consists of all module morphisms \( P \cdot 'N \rightarrow (\mathcal{M}) Q : X \rightarrow B \), i.e. all left symmetric cells \( Q \rightarrow P : \mathcal{M} \rightarrow \mathcal{N} \).

2. The composition in the module \( (\mathcal{M} \downarrow \mathcal{N}) \) [op. \( (\mathcal{M} \downarrow \mathcal{N}) \)] is that defined in Definition 7.1.5; indeed, by definition, the composition

\[ Q \xrightarrow{\psi} P \]

\[ \tau \circ \psi \circ \sigma \]

in \( (\mathcal{M} \downarrow \mathcal{N}) \) [op. \( (\mathcal{M} \downarrow \mathcal{N}) \)] is given by

\[ \tau \circ \psi \circ \sigma = \psi : (\mathcal{M}) \tau (X : B) (\sigma (\mathcal{N})) = (\mathcal{M}) \tau \circ \psi \circ \sigma (\mathcal{N}) \]

op.

\[ \tau \circ \psi \circ \sigma = \psi : (\mathcal{M}) \tau (X : B)^{-} (\sigma (\mathcal{N})) = (\mathcal{M}) \tau \circ \psi \circ \sigma (\mathcal{N}). \]
(3) For any modules \( \mathcal{M} \) and \( \mathcal{N} \),
\[
\langle \mathcal{M} \upharpoonright \mathcal{N} \rangle^* \cong \langle \mathcal{N} \upharpoonright \mathcal{M}^* \rangle.
\]

\textbf{Note.} The postcomposition module morphism in Definition 1.2.14 and the identity in Proposition 1.2.10 allow the following definition (cf. Definition 1.2.25).

\subsection{7.1.8 Definition.}
Let \( \mathcal{J} : E \to D \) be a module. Given a right \( \text{op. left} \) symmetric cell
\[
\begin{array}{c}
A \xrightarrow{Q} B \\
\downarrow_{\mathcal{M}_1} \downarrow_{\mathcal{N}} \\
X \xrightarrow{\psi} Y
\end{array}
\quad \text{op.}
\begin{array}{c}
A \xrightarrow{Q} B \\
\downarrow_{\mathcal{M}_1} \downarrow_{\mathcal{N}} \\
X \xrightarrow{\psi} Y
\end{array}
\]
the postcomposition right \( \text{op. left} \) symmetric cell
\[
\begin{array}{c}
[D, A] \xrightarrow{[D, Q]} [D, B] \\
\downarrow_{(\mathcal{J}, \mathcal{M})} \downarrow_{(\mathcal{J}, \mathcal{N})} \\
[E, X] \xrightarrow{[E, P]} [E, Y]
\end{array}
\quad \text{op.}
\begin{array}{c}
[D, A] \xrightarrow{[D, Q]} [D, B] \\
\downarrow_{(\mathcal{J}, \mathcal{M})} \downarrow_{(\mathcal{J}, \mathcal{N})} \\
[E, X] \xrightarrow{[E, P]} [E, Y]
\end{array}
\]
is defined by the postcomposition module morphism
\[
\langle \mathcal{J}, \mathcal{M} \rangle [D, Q] = \langle \mathcal{J}, \mathcal{M} \rangle Q \xrightarrow{(\mathcal{J}, \psi)} \langle \mathcal{J}, \mathcal{P} \rangle \langle \mathcal{N} \rangle = \langle \mathcal{E}, \mathcal{P} \rangle \langle \mathcal{J}, \mathcal{N} \rangle
\]
—postcomposition with \( \psi : \langle \mathcal{M} \rangle Q \to \mathcal{P} \langle \mathcal{N} \rangle \) [op. \( \psi : \mathcal{P} \langle \mathcal{N} \rangle \to \langle \mathcal{M} \rangle Q \)].

\subsection{7.1.9 Remark.}
Given a pair of functors \( G : E \to X \) and \( S : D \to B \),
\begin{itemize}
\item the right symmetric cell \( (\mathcal{J}, \psi) \) sends each cell \( \theta : G \sim Q \circ S : \mathcal{J} \to \mathcal{M} \) to the cell \( \theta \circ \psi : G \circ P \sim \mathcal{S} : \mathcal{J} \to \mathcal{N} \) defined by the module morphism \( \theta \circ \psi : \mathcal{J} \xrightarrow{\theta \circ \psi} \mathcal{G} \circ \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathcal{S} \) given by the composition
\[
\mathcal{J} \xrightarrow{\theta} \mathcal{G}(\langle \mathcal{M} \rangle \langle Q \circ S \rangle) \mathcal{S} \xrightarrow{\mathcal{G}(\psi) \mathcal{S}} \mathcal{G}(\langle \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathcal{S}) = \mathcal{G}(\mathcal{P} \circ \mathcal{Q} \circ \mathcal{S})
\]
\item the left symmetric cell \( (\mathcal{J}, \psi) \) sends each cell \( \theta : G \circ P \sim \mathcal{S} : \mathcal{J} \to \mathcal{N} \) to the cell \( \theta \circ \psi : G \circ Q \circ S : \mathcal{J} \to \mathcal{M} \) defined by the module morphism \( \theta \circ \psi : \mathcal{J} \xrightarrow{\theta \circ \psi} \mathcal{G} \circ \mathcal{M} \rangle \langle Q \circ \mathcal{S} \rangle \) given by the composition
\[
\mathcal{G}(\langle \mathcal{M} \rangle \langle Q \circ \mathcal{S} \rangle) \mathcal{S} \xrightarrow{\mathcal{G}(\psi) \mathcal{S}} \mathcal{G}(\langle \mathcal{P} \rangle \langle \mathcal{N} \rangle \mathcal{S}) = \mathcal{G}(\mathcal{P} \circ \mathcal{Q} \circ \mathcal{S})
\]
\end{itemize}
\textbf{Note.} The postcomposition module morphism in Definition 4.3.13 and the identity in Proposition 4.3.9 allow the following definition (cf. Definition 4.3.17).

\subsection{7.1.10 Definition.}
Let \( E \) be a category. Given a right \( \text{op. left} \) symmetric cell
\[
\begin{array}{c}
A \xrightarrow{Q} B \\
\downarrow_{\mathcal{M}_1} \downarrow_{\mathcal{N}} \\
X \xrightarrow{\psi} Y
\end{array}
\quad \text{op.}
\begin{array}{c}
A \xrightarrow{Q} B \\
\downarrow_{\mathcal{M}_1} \downarrow_{\mathcal{N}} \\
X \xrightarrow{\psi} Y
\end{array}
\]
the postcomposition right \( \text{op. left} \) symmetric cell
\[
\begin{array}{c}
[E, A] \xrightarrow{[E, Q]} [E, B] \\
\downarrow_{(\mathcal{E}, \mathcal{M})} \downarrow_{(\mathcal{E}, \mathcal{N})} \\
[E, X] \xrightarrow{[E, P]} [E, Y]
\end{array}
\quad \text{op.}
\begin{array}{c}
[E, A] \xrightarrow{[E, Q]} [E, B] \\
\downarrow_{(\mathcal{E}, \mathcal{M})} \downarrow_{(\mathcal{E}, \mathcal{N})} \\
[E, X] \xrightarrow{[E, P]} [E, Y]
\end{array}
\]
is defined by the postcomposition module morphism
\[
\langle \mathcal{E}, \mathcal{M} \rangle [E, Q] = \langle \mathcal{E}, \mathcal{M} \rangle Q \xrightarrow{(E, \psi)} \langle \mathcal{E}, \mathcal{P} \rangle \langle \mathcal{N} \rangle = \langle \mathcal{E}, \mathcal{P} \rangle \langle \mathcal{E}, \mathcal{N} \rangle
\]
—postcomposition with \( \psi : \langle \mathcal{M} \rangle Q \to \mathcal{P} \langle \mathcal{N} \rangle \) [op. \( \psi : \mathcal{P} \langle \mathcal{N} \rangle \to \langle \mathcal{M} \rangle Q \)].
7.1.11 Remark. Given a pair of functors $G : E \to X$ and $S : E \to B$, 

- the right symmetric cell $(E, \psi)$ sends each cylinder $\alpha : G \to Q \delta S : E \to M$ to the cylinder $\alpha \circ \psi : G \circ \delta P \to S : E \to N$ defined by 
  \[ \alpha \circ \psi = \alpha \circ G(\psi)S \]

— the composition of a frame $\alpha$ of the endomodule $G(M)[Q \delta S] : E \to E$ and the module morphism 
  \[ G(M)[Q \delta S] = G((M)Q)S \xrightarrow{G(\psi)S} G(P(N))S = [G \circ \delta P](N)S. \]

- the left symmetric cell $(E, \psi)$ sends each cylinder $\alpha : G \circ \delta P \to S : E \to N$ to the cylinder $\alpha \circ \psi : G \to Q \delta S : E \to M$ defined by 
  \[ \alpha \circ \psi = \alpha \circ G(\psi)S \]

— the composition of a frame $\alpha$ of the endomodule $[G \circ \delta P](N)S : E \to E$ and the module morphism 
  \[ G(M)[Q \delta S] = G((M)Q)S \xleftarrow{G(\psi)S} G(P(N))S = [G \circ \delta P](N)S. \]

Note. The postcomposition module morphism in Definition 4.6.13 and the identity in Proposition 4.6.9 allow the following definition (cf. Definition 4.6.17).

7.1.12 Definition. Let $E$ be a category. Given a right [op. left] symmetric cell 

\[
\begin{array}{ccc}
A & \xrightarrow{Q} & B \\
M & \xrightarrow{\psi} & N \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]

, the postcomposition right [op. left] symmetric cell 

\[
\begin{array}{ccc}
[E, A] & \xrightarrow{[E, Q]} & [E, B] \\
(\ast E, M) & \xrightarrow{\psi} & (\ast E, N) \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]

is defined by the postcomposition module morphism 

\[ (\ast E, M)Q = (\ast E, (M)Q) \xrightarrow{(\ast E, \psi)} (\ast E, P(N)) = P(\ast E, N) \]

op.

\[ (E_+, M)Q = (E_+, (M)Q) \xleftarrow{(E_+, \psi)} (E_+, P(N)) = [E, P](E_+, N) \]

— postcomposition with $\psi : (M)Q \to P(N)$ [op. $\psi : P(N) \to (M)Q$].

7.1.13 Remark. 

- Given an object $x \in \|X\|$ and a functor $S : E \to B$, the right symmetric cell $(\ast E, \psi)$ sends each cone $\alpha : x \to Q \delta S : \ast \to M$ to the cone $\alpha \circ \psi : x \circ \delta P \to S : \ast \to N$ defined by 
  \[ \alpha \circ \psi = \alpha \circ x(\psi)S \]

— the composition of a frame $\alpha$ of the left module $x(M)[Q \delta S] : \ast \to E$ and the module morphism 
  \[ x(M)[Q \delta S] = x((M)Q)S \xrightarrow{x(\psi)S} x(P(N))S = (x \circ \delta P)(N)S. \]

- Given an object $b \in \|B\|$ and a functor $T : E \to X$, the left symmetric cell $(E_+, \psi)$ sends each cone $\alpha : T \circ \delta P \to b : E_+ \to N$ to the cone $\alpha \circ \psi : T \circ Q \delta b : E_+ \to M$ defined by 
  \[ \alpha \circ \psi = \alpha \circ T(\psi)b \]

— the composition of a frame $\alpha$ of the right module $[T \circ \delta P](N)b : E \to \ast$ and the module morphism 
  \[ T(M)(Q \cdot b) = T((M)Q)b \xrightarrow{T(\psi)b} T(P(N))b = [T \circ \delta P](N)b. \]
7.2 Adjunctions between modules

A symmetric cell \( \text{A} \xleftarrow{\phi} \text{B} \) is called adjunctive, or an adjunction, if the module morphism
\[
\begin{array}{c}
\text{M}^\phi \xrightarrow{\phi} \text{N}^\phi \\
\text{X} \xrightarrow{P} \text{Y}
\end{array}
\]
\( \phi : (\text{M}) \text{Q} \to \text{P} (\text{N}) \) is an isomorphism. Adjunctive symmetric cells are closed under identities and composition, and thus form a subcategory of the category of modules and symmetric cells. They are also closed under postcomposition; that is, postcomposition with an adjunctive symmetric cell yields another adjunctive symmetric cell. An adjunction between two categories to be studied in the following sections is a special instance of an adjunctive symmetric cell. In Section 7.10, we define an adjoint of an (ordinary) cell using the notion of adjunction between modules.

7.2.1 Definition.
- A right symmetric cell \( \text{A} \xleftarrow{\phi} \text{B} \) is called adjunctive (or an adjunction) if the module morphism
\[
\begin{array}{c}
\text{M}^\phi \xrightarrow{\phi} \text{N}^\phi \\
\text{X} \xrightarrow{P} \text{Y}
\end{array}
\]
\( \phi : (\text{M}) \text{Q} \to \text{P} (\text{N}) : \text{X} \to \text{B} \) is iso. If \( \phi : \text{Q} \to \text{P} : \text{M} \to \text{N} \) is an adjunctive right symmetric cell, then the pair \((\text{Q}, \phi)\), or the functor \( \text{Q} \) itself, is called a right adjoint of \( \text{P} \) along \( \text{M} \) and \( \text{N} \).
- A left symmetric cell \( \text{A} \xleftarrow{\phi} \text{B} \) is called adjunctive (or an adjunction) if the module morphism
\[
\begin{array}{c}
\text{M}^\phi \xleftarrow{\phi} \text{N}^\phi \\
\text{X} \xleftarrow{P} \text{Y}
\end{array}
\]
\( \phi : \text{P} (\text{N}) \to (\text{M}) \text{Q} : \text{X} \to \text{B} \) is iso. If \( \phi : \text{Q} \to \text{P} : \text{M} \to \text{N} \) is an adjunctive left symmetric cell, then the pair \((\text{P}, \phi)\), or the functor \( \text{P} \) itself, is called a left adjoint of \( \text{Q} \) along \( \text{N} \) and \( \text{M} \).

7.2.2 Remark.
1. If \( \phi : \text{Q} \to \text{P} : \text{M} \to \text{N} \) is an adjunctive right symmetric cell, then the inverse \( \phi^{-1} \) of the module isomorphism \( \phi : (\text{M}) \text{Q} \to \text{P} (\text{N}) \) gives the corresponding adjunctive left symmetric cell (and vice versa). Because of this, we often do not care about the direction of the isomorphism \( \phi \), and often write an adjunctive symmetric cell, left or right, just as \( \text{A} \xleftarrow{\phi} \text{B} \).

2. By Proposition 1.1.16, a right \([\text{op. left}]\) symmetric cell in Definition 7.2.1 is adjunctive if and only if all its components
\[
\begin{array}{c}
x(\text{M}) (\text{Q} \cdot \text{b}) \xrightarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b} \\
\text{op.} \quad x(\text{M}) (\text{Q} \cdot \text{b}) \xleftarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b}
\end{array}
\]
(see Remark 7.1.2(1)) are bijective. For each \( \text{M} \)-arrow \( m : x \to \text{Q} \cdot \text{b} \), the corresponding \( \text{N} \)-arrow \( m' : x' \to \text{P} \) is called the left adjunct of \( m \), and for each \( \text{N} \)-arrow \( n : \text{Q} \cdot \text{P} \to \text{b} \), the corresponding \( \text{M} \)-arrow \( n' : x \to \text{Q} \cdot \text{b} \) is called the right adjunct of \( n \).

3. The naturality of the bijections \( x(\phi) \cdot \text{b} \) in (2) above is expressed by the commutativity of
\[
\begin{array}{c}
x(\text{M})(\text{Q} \cdot \text{b}) \xrightarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b} \quad \text{op.} \quad x(\text{M})(\text{Q} \cdot \text{b}) \xleftarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b} \\
\text{op.} \quad x(\text{M})(\text{Q} \cdot \text{b}) \xrightarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b} \quad \text{op.} \quad x(\text{M})(\text{Q} \cdot \text{b}) \xleftarrow{x(\phi) \cdot b} (x \cdot \text{P})(\text{N}) \cdot \text{b}
\end{array}
\]
for any \( \text{X} \)-arrow \( k : x' \to x \) and any \( \text{B} \)-arrow \( h : b \to b' \), and this commutativity is in turn expressed by the identity
\[
(k \circ m) : \phi = (k \cdot \text{P}) \circ (m' \cdot \phi) \quad \text{op.} \quad \phi' : (h \circ n) = (\text{Q} \cdot h) \circ (\phi' \cdot n)
\]
for any \( \text{M} \)-arrow \( m : x \to \text{Q} \cdot \text{b} \) \([\text{op. N} \text{-arrow} n : x' \cdot \text{P} \to \text{b}]\).

4. Recalling Definition 7.1.6 and noting Remark 7.1.7(1), we have the following description of an adjunctive right \([\text{op. left}]\) symmetric cell as an arrow of the module \( \langle \text{M} \mid \text{N} \rangle \) \([\text{op.} \langle \text{M} \mid \text{N} \rangle] \):
• an adjunctive right symmetric cell \(\phi : Q \leadsto P : \mathcal{M} \to \mathcal{N}\) is an \(\langle \mathcal{M} \downarrow \mathcal{N}\rangle\)-arrow \(\phi : Q \leadsto P\) given by an isomorphism \(\phi : \langle \mathcal{M} \rangle Q \to P \langle \mathcal{N}\rangle\) in the category \([X : B]\).

• an adjunctive left symmetric cell \(\phi : Q \leadsto P : \mathcal{M} \to \mathcal{N}\) is an \(\langle \mathcal{M} \downarrow \mathcal{N}\rangle\)-arrow \(\phi : Q \leadsto P\) given by an isomorphism \(\phi : P \langle \mathcal{N}\rangle \to \langle \mathcal{M} \rangle Q\) in the category \([X : B]\).

7.2.3 Notation. We write

\[
\begin{array}{c}
Q \vdash b \\
\downarrow m \downarrow n \\
X \to x \vdash x \vdash P
\end{array}
\]

to express that \(m\) and \(n\) are the adjunct of each other (somewhat unfortunately, \(m\) is the left adjunct of \(n\) and \(n\) is the right adjunct of \(m\)), and call the diagram an adjunct diagram.

7.2.4 Proposition. Given an adjunctive symmetric cell

\[
\begin{array}{c}
A \\
\downarrow \phi \\
X \to Y
\end{array}
\]

\[
\begin{array}{c}
Q \vdash b \\
\downarrow h \\
\downarrow k \\
X \to x \vdash x \vdash P
\end{array}
\]

, if the middle square in

\[
\begin{array}{c}
Q \vdash b' \\
\downarrow h' \\
\downarrow k' \\
x' \to x' \vdash x' \vdash P
\end{array}
\]

is an adjunct diagram, so are all three rectangles in the diagram for any \(X\)-arrow \(k : x' \to x\) and \(B\)-arrow \(h : b \to b'\).

Proof. This is a restatement of Remark 7.2.2(3) in terms of an adjunct diagram, stating the naturality of the bijection \(x(\phi) b\).

7.2.5 Proposition.

(1) The identity right [op. left] symmetric cell is adjunctive.

(2) If two composable right [op. left] symmetric cells are adjunctive, so is their composite.

Proof.

(1) Evident.

(2) Consider a pair of right symmetric cells as in Definition 7.1.3. Then the cell \(\phi \circ \phi' : \mathcal{M} \to \mathcal{M}'\) is defined by the composite module morphism \(\langle \phi \rangle Q' \circ P \langle \phi'\rangle\). Since the module morphisms \(\phi\) and \(\phi'\) are iso by the definition of adjunctiveness, so are the module morphisms \(\langle \phi \rangle Q'\) and \(P \langle \phi'\rangle\) by Proposition 1.1.33. Hence \(\langle \phi \rangle Q' \circ P \langle \phi'\rangle\) is an isomorphism, and the cell \(\phi \circ \phi'\) is adjunctive.

7.2.6 Remark. Modules and adjunctive symmetric cells among them thus constitute a subcategory of the category of modules and symmetric cells (see Proposition 7.1.4).

Note. In the following, we consider the composition of a symmetric cell and a natural transformation defined in Definition 7.1.5.

7.2.7 Proposition.

• If a right symmetric cell \(\phi : Q \leadsto P\) is adjunctive, then for any natural isomorphisms \(\tau : Q' \to Q\) and \(\sigma : P' \to P\), the composite right symmetric cell \(\tau \circ \phi \circ \sigma : Q' \leadsto P'\) is adjunctive.

• If a left symmetric cell \(\phi : Q \leadsto P\) is adjunctive, then for any natural isomorphisms \(\tau : Q \to Q'\) and \(\sigma : P \to P'\), the composite left symmetric cell \(\tau \circ \phi \circ \sigma : Q' \leadsto P'\) is adjunctive.
Proof. Noting the description (see Remark 7.2.2(4)) of an adjunctive symmetric cell as an \( \langle \mathcal{M}, \mathcal{N} \rangle \)-arrow, we see that this is an instance of Proposition 1.1.35.

7.2.8 Proposition. If a symmetric cell

\[
\begin{array}{ccc}
\textbf{A} & \xleftarrow{Q} & \textbf{B} \\
\mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\
\textbf{X} & \xrightarrow{P} & \textbf{Y}
\end{array}
\]

is adjunctive, so is the postcomposition cell

\[
\begin{array}{ccc}
\mathcal{D}, \mathcal{A} & \xrightarrow{[D,Q]} & \mathcal{D}, \mathcal{B} \\
\mathcal{J}, \mathcal{M} & \xrightarrow{\mathcal{J},\phi} & \mathcal{J}, \mathcal{N} \\
\mathcal{E}, \mathcal{X} & \xrightarrow{[E,P]} & \mathcal{E}, \mathcal{Y}
\end{array}
\]

(see Definition 7.1.8) for any module \( \mathcal{J}: \mathcal{E} \rightarrow \mathcal{D} \).

Proof. Like all functors, the functor \( \langle \mathcal{J}, - \rangle: [\mathcal{X} : \mathcal{B}] \rightarrow [[\mathcal{E}, \mathcal{X}]: [\mathcal{D}, \mathcal{B}]] \) (see Remark 1.2.13(3)) preserves isomorphisms.

7.2.9 Remark. The right adjunct of a cell \( \theta: \mathcal{G} \triangleright \mathcal{Q} \circ \mathcal{S} : \mathcal{J} \rightarrow \mathcal{M} \) under \( \langle \mathcal{J}, \phi \rangle \) is given by \( \theta \triangleright \phi: \mathcal{G} \triangleright \mathcal{P} \triangleright \mathcal{S} : \mathcal{J} \rightarrow \mathcal{N} \) (see Remark 7.1.9). Since the cell \( \theta \triangleright \phi \) sends each \( \mathcal{J} \)-arrow \( j: \theta \triangleright \phi \) (i.e. to the right adjunct of \( j: \theta \) under \( \phi \)), a pair of cells \( \theta: \mathcal{G} \triangleright \mathcal{Q} \circ \mathcal{S} : \mathcal{J} \rightarrow \mathcal{M} \) and \( \gamma: \mathcal{G} \triangleright \mathcal{P} \triangleright \mathcal{S} : \mathcal{J} \rightarrow \mathcal{N} \) are the adjunct of each other if and only if the \( \mathcal{M} \)-arrow \( j: \theta \) and the \( \mathcal{N} \)-arrow \( j: \gamma \) are the adjunct of each other for every \( \mathcal{J} \)-arrow \( j \); that is, the square

\[
\begin{array}{ccc}
\mathcal{Q} \circ \mathcal{S} & \xrightarrow{\mathcal{J},\theta \triangleright \phi} & \mathcal{S} \\
\theta & \downarrow & \downarrow \gamma \\
\mathcal{G} & \xrightarrow{\mathcal{J}} & \mathcal{G} \triangleright \mathcal{P}
\end{array}
\]

is an adjunct diagram of \( \langle \mathcal{J}, \phi \rangle \) if and only if the square

\[
\begin{array}{ccc}
\mathcal{Q} \circ \mathcal{S} \circ \mathcal{d} & \xrightarrow{\mathcal{J},\theta \triangleright \phi} & \mathcal{d} \circ \mathcal{S} \\
\theta \circ j & \downarrow & \downarrow j \circ \gamma \\
\mathcal{G} \circ \mathcal{e} & \xrightarrow{\mathcal{J}} & \mathcal{G} \circ \mathcal{P}
\end{array}
\]

is an adjunct diagram of \( \phi \) for every \( \mathcal{J} \)-arrow \( j: e \triangleright d \).

7.2.10 Proposition. If a symmetric cell

\[
\begin{array}{ccc}
\textbf{A} & \xleftarrow{Q} & \textbf{B} \\
\mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\
\textbf{X} & \xrightarrow{P} & \textbf{Y}
\end{array}
\]

is adjunctive, so is the postcomposition cell

\[
\begin{array}{ccc}
\mathcal{E}, \mathcal{A} & \xrightarrow{[E,Q]} & \mathcal{E}, \mathcal{B} \\
\mathcal{E}, \mathcal{M} & \xrightarrow{\mathcal{E},\phi} & \mathcal{E}, \mathcal{N} \\
\mathcal{E}, \mathcal{X} & \xrightarrow{[E,P]} & \mathcal{E}, \mathcal{Y}
\end{array}
\]

(see Definition 7.1.10) for any category \( \mathcal{E} \).

Proof. Like all functors, the functor \( \langle \mathcal{E}, - \rangle: [\mathcal{X} : \mathcal{B}] \rightarrow [[\mathcal{E}, \mathcal{X}]: [\mathcal{E}, \mathcal{B}]] \) (see Remark 4.3.14(3)) preserves isomorphisms.

7.2.11 Remark. The right adjunct of a cylinder \( \alpha: \mathcal{G} \triangleright \mathcal{Q} \circ \mathcal{S} : \mathcal{E} \rightarrow \mathcal{M} \) under \( \langle \mathcal{E}, \phi \rangle \) is given by \( \alpha \circ \phi: \mathcal{G} \circ \mathcal{P} \triangleright \mathcal{S} : \mathcal{E} \rightarrow \mathcal{N} \) (see Remark 7.1.11). Since (see Remark 4.3.12(1)) the component of \( \alpha \circ \phi \)
at $e \in \|E\|$ is given by

$$\left[\alpha \circ \phi\right]_e = \alpha_e \cdot (e \cdot G) \cdot (\phi) \cdot (S \cdot e)$$

(i.e. by the right adjunct of $\alpha_e$ under $\phi$), a pair of cylinders $\alpha : G \Rightarrow Q \circ S : E \Rightarrow \mathcal{M}$ and $\beta : G \circ P \Rightarrow S : E \Rightarrow \mathcal{N}$ are the adjunct of each other if and only if their components $\alpha_e$ and $\beta_e$ are the adjunct of each other for every $e \in \|E\|$; that is, the square

$$\begin{array}{ccc}
Q \circ S & \longrightarrow & S \\
\alpha & \downarrow & \beta \\
G & \longrightarrow & G \circ P
\end{array}$$

is an adjunct diagram of $(E, \phi)$ if and only if the square

$$\begin{array}{ccc}
Q \circ S ; e & \longleftarrow & e ; S \\
\alpha_e & \downarrow & \beta_e \\
G ; e & \longleftarrow & e \cdot G ; P
\end{array}$$

is an adjunct diagram of $\phi$ for every $e \in \|E\|$.

7.2.12 Proposition. If a symmetric cell

$$\begin{array}{ccc}
A & \longleftarrow & Q \\
\mathcal{M} & \phi & \mathcal{N} \\
X & \longleftarrow & Y
\end{array}$$

is adjunctive, so is the postcomposition cell

$$\begin{array}{ccc}
[E, A] & \longleftarrow & [E, B] \\
\langle *E, \mathcal{M} \rangle & \phi & \langle *E, \mathcal{N} \rangle \\
X & \longleftarrow & Y
\end{array}\quad \text{op.} \quad \begin{array}{ccc}
A & \longleftarrow & Q \\
\langle \mathcal{E} + *E, \mathcal{M} \rangle & \phi & \langle \mathcal{E} + *E, \mathcal{N} \rangle \\
[E, X] & \longleftarrow & [E, Y]
\end{array}$$

(see Definition 7.1.12) for any category $E$.

Proof. Like all functors, the functor $\langle *E, - \rangle : [X : B] \to [X : [E, B]]$ [op. $\langle \mathcal{E} + *E, - \rangle : [X : B] \to [[E, X] : B]$] (see Remark 4.6.14(3)) preserves isomorphisms. \qed

7.2.13 Remark.

- The right adjunct of a cone $\alpha : x \Rightarrow Q \circ S : *E \Rightarrow \mathcal{M}$ under $\langle *E, \phi \rangle$ is given by $\alpha \circ \phi : x \cdot P \Rightarrow S : *E \Rightarrow \mathcal{N}$ (see Remark 7.1.13). Since (see Remark 4.6.12) the component of $\alpha \circ \phi$ at an object $e \in \|E\|$ is given by

$$\left[\alpha \circ \phi\right]_e = \alpha_e \cdot x(\phi)(S \cdot e)$$

(i.e. by the right adjunct of $\alpha_e$ under $\phi$), a pair of cones $\alpha : x \Rightarrow Q \circ S : *E \Rightarrow \mathcal{M}$ and $\beta : x \cdot P \Rightarrow S : *E \Rightarrow \mathcal{N}$ are the adjunct of each other if and only if their components $\alpha_e$ and $\beta_e$ are the adjunct of each other for every $e \in \|E\|$; that is, the square

$$\begin{array}{ccc}
Q \circ S & \longleftarrow & S \\
\alpha & \downarrow & \beta \\
x & \longleftarrow & x \cdot P
\end{array}$$

is an adjunct diagram of $\langle *E, \phi \rangle$ if and only if the square

$$\begin{array}{ccc}
Q ; S \cdot e & \longleftarrow & e ; S \\
\alpha_e & \downarrow & \beta_e \\
x & \longleftarrow & x \cdot P
\end{array}$$

is an adjunct diagram of $\phi$ for every $e \in \|E\|$.

- The left adjunct of a cone $\alpha : T \circ P \Rightarrow b : E \Rightarrow \mathcal{N}$ under $\langle E^*, \phi \rangle$ is given by $\alpha \circ \phi : T \Rightarrow Q \cdot b : E \Rightarrow \mathcal{M}$ (see Remark 7.1.13). Since (see Remark 4.6.12) the component of $\alpha \circ \phi$ at an object $e \in \|E\|$ is given by

$$\left[\alpha \circ \phi\right]_e = \alpha_e \cdot (e \cdot T) \cdot (\phi) \cdot b$$
7.3. Adjunctions between categories

In this section, we develop the basic theory of adjunctions between categories. An adjunction \( \phi: G \dashv F: X \to A \) between two categories \( X \) and \( A \) consists of a pair of functors \( X \xrightarrow{G} A \) and a module isomorphism \( \phi: (X)G \to F(A): X \to A \), and is the same thing as an adjunctive symmetric cell \( \begin{array}{ccc} X & \xrightarrow{G} & A \\ \phi & = & \phi \\ \end{array} \) between the hom-modules of \( X \) and \( A \). Some notions and results thus come from Section 7.2.

The isomorphisms in Theorem 5.3.24 allow the definition of the counit and unit of an adjunction. It is immediately seen that the counit and unit of an adjunction \( G \dashv F: X \to A \) are the same thing as a counit—in the sense defined in Section 6.4—of the representable module \( F(A) \) and a unit of the corepresentable module \( (X)G \). If a module \( \mathcal{M}: X \to A \) is corepresented by \( G: A \to X \) and represented by \( F: X \to A \), the corepresentation \( (X)G \cong \mathcal{M} \) and the representation \( \mathcal{M} \cong F(A) \) yield an adjunction \( G \dashv F \); Theorem 7.3.15 shows how the counit and unit of \( G \dashv F \) is derived from the counit and unit of \( \mathcal{M} \). (If we were to show that a pair of functors \( X \xrightarrow{G} A \) constitute an adjunction, sometimes it would be easier and beneficial to construct a module \( \mathcal{M}: X \to A \) independently of \( F \) and \( G \), and show that \( \mathcal{M} \) is represented by \( F \) and corepresented by \( G \); indeed, this is what we will do in Chapter 11.)

Lastly, we show that an adjunction between two categories yields an adjunction between two modules of cones by postcomposition. Later in Section 8.11, we use this device to describe an interaction between an adjunction and a limit.

7.3.1 Definition. Given a pair of functors \( X \xrightarrow{G} A \),

- an adjunction \( \phi: G \dashv F: X \to A \) is defined by a module isomorphism
  \( \phi: (X)G \to F(A): X \to A \)
  ; if \( \phi: G \dashv F: X \to A \) is an adjunction, then the pair \((G, \phi)\), or the functor \( G \) itself, is called a right adjoint of \( F \).
- an adjunction \( \phi: G \dashv F: X \to A \) is defined by a module isomorphism
  \( \phi: F(A) \to (X)G: X \to A \)
  ; if \( \phi: G \dashv F: X \to A \) is an adjunction, then the pair \((F, \phi)\), or the functor \( F \) itself, is called a left adjoint of \( G \).
7.3.2 Remark.

(1) The two forms of adjunctions, $X \rightarrow A$ and $X \rightarrow A$, are referred to as the right and left forms. If $\phi : G \vdash F : X \rightarrow A$ is the right form of an adjunction, then the inverse $\phi^{-1}$ of the module isomorphism $\phi : (X) G \rightarrow F(A)$ gives the corresponding left form of the adjunction (and vice versa). Because of this, we often do not care about the direction of the isomorphism $\phi$, and often regard $\phi : X \rightarrow A$ and $\phi^{-1} : X \rightarrow A$ as the same thing.

(2) Recalling the definition of representation (Definition 2.3.13), we see that

- a right adjoint $G : A \rightarrow X$ of a functor $F : X \rightarrow A$ gives a corepresentation of the representable module $F(A) : X \rightarrow A$.
- a left adjoint $F : X \rightarrow A$ of a functor $G : A \rightarrow X$ gives a representation of the corepresentable module $(X) G : X \rightarrow A$.

(3) By (2) above and by the uniqueness of representation (Corollary 6.4.9), an adjoint, if exists, is unique up to isomorphism.

(4) An adjunction between categories $X$ and $A$ is a special case of an adjunction between modules defined in Definition 7.2.1 where $M$ and $N$ are given by the hom-modules of $X$ and $A$: an adjunction

$$
\phi : G \vdash F : X \rightarrow A \quad \text{op.} \quad \phi : G \vdash F : X \rightarrow A
$$

is the same thing as an adjunctive symmetric cell

$$
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\vdash_{(X)\downarrow} & \phi \downarrow_{(X)\downarrow} & \vdash_{(X)\downarrow} \\
\xrightarrow{F} & A
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\vdash_{(X)\downarrow} & \phi \downarrow_{(X)\downarrow} & \vdash_{(X)\downarrow} \\
\xrightarrow{F} & A
\end{array}
$$

(5) Because of (4) above, adjunctions between categories inherit the notion of adjunct (see Remark 7.2.2(2)) from adjunctions between modules. For a pair of objects $x \in \|X\|$ and $a \in \|A\|$, 

- the component of an adjunction $\phi : G \vdash F : X \rightarrow A$ at $(x, a)$ is the bijection

$$
x(X)(G : a) = x((X) G) a \xrightarrow{\phi a} x(F(A)) a = (x : F)(A) a
$$

which sends each $X$-arrow $g : x \rightarrow G : a$ to the $A$-arrow $g : x : F \rightarrow a$.

- the component of an adjunction $\phi : G \vdash F : X \rightarrow A$ at $(x, a)$ is the bijection

$$
x(X)(G : a) = x((X) G) a \xleftarrow{\phi a} x(F(A)) a = (x : F)(A) a
$$

which sends each $A$-arrow $f : x : F \rightarrow a$ to the $X$-arrow $\phi : x \rightarrow G : a$. For each $X$-arrow $g : x \rightarrow G : a$, the corresponding $A$-arrow $g : x : F \rightarrow a$ is called the left adjunct of $g$, and for each $A$-arrow $f : x : F \rightarrow a$, the corresponding $X$-arrow $\phi : x \rightarrow G : a$ is called the right adjunct of $f$. The adjunct diagram

$$
\begin{array}{c}
G : a \leftarrow a \\
g \downarrow & & \downarrow f \\
x \rightarrow x : F
\end{array}
$$

expresses that $f$ and $g$ are the adjunct of each other (cf. Notation 7.2.3).

(6) An adjunction $\phi : G \vdash F : X \rightarrow A$ is expressed diagrammatically as

$$
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\vdash_{F} & \phi \downarrow_{G} & \vdash_{F} \\
\xrightarrow{F} & A
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{F} & A \\
\vdash_{G} & \phi \downarrow_{F} & \vdash_{G} \\
\xrightarrow{G} & A
\end{array}
$$

with $\phi$ often omitted if it is understood or unimportant.
7.3.3 Proposition. Given an adjunction \( \phi : G \dashv F : X \to A \), if the middle square in
\[
\begin{array}{c}
G \colon a' & \to a \\
\alpha \downarrow & \downarrow \beta \\
\text{a} & \to \text{a}
\end{array}
\]
is a commutative diagram, so are all three rectangles in the diagram for any \( X \)-arrow \( k : x' \to x \) and \( A \)-arrow \( h : a \to a' \).

Proof. This is a special instance of Proposition 7.2.4, stating the naturality of the bijection \( x(\phi) a \) in Remark 7.3.2(5).

Note. Theorem 5.3.24 justifies the following definition.

7.3.4 Definition. 
- The counit of an adjunction \( \phi : G \dashv F : X \to A \) is the natural transformation \( \epsilon : G \circ F \to 1_A \) such that \( X \downarrow \epsilon = \phi \).
- The unit of an adjunction \( \phi : G \dashv F : X \to A \) is the natural transformation \( \eta : 1_X \to G \circ F \) such that \( \eta \downarrow A = \phi \).

7.3.5 Remark. 
(1) Theorem 5.3.24 shows how the counit [op. unit] of an adjunction is obtained and how an adjunction is recovered from its counit [op. unit].

a) The component \( \epsilon_a : a : G \circ F \to a \) of the counit \( \epsilon : G \circ F \to 1_A \) at \( a \in \|A\| \) is given by the left adjunct \( 1_{(G ; a)} \phi \) of the identity \( X \)-arrow \( G \colon a \to G \colon a \), and dually the component \( \eta_x : x \to G \circ F \colon x \) of the unit \( \eta : 1_X \to G \circ F \) at \( x \in \|X\| \) is given by the right adjunct \( \phi : 1_{(x ; F)} \) of the identity \( A \)-arrow \( x : F \to x : F \), as shown in the adjunct diagrams:

\[
\begin{array}{c}
G \colon a & \to a \\
\epsilon_a \downarrow & \downarrow \eta_a \\
G \colon a & \to a : G \circ F \\
\end{array}
\]

\[
\begin{array}{c}
G \colon a & \to a : G \circ F \\
\eta_x \downarrow & \downarrow 1 \\
x & \to x : F
\end{array}
\]

b) Conversely, for an \( X \)-arrow \( g : x \to G \colon a \), its left adjunct \( g \phi : x : F \to a \) is given by the composite \( (g : F) \circ \epsilon_a \), and dually for an \( A \)-arrow \( f : x : F \to a \), its right adjunct \( \phi f : x \to G \colon a \) is given by the composite \( \eta_x \circ (G \colon f) \), as shown in the commutative diagrams:

\[
\begin{array}{c}
G \colon a & \to a : G \circ F \\
g \downarrow & \downarrow \epsilon_a \\
x & \to x : F
\end{array}
\]

\[
\begin{array}{c}
G \colon a & \to a : G \circ F \\
\phi f \downarrow & \downarrow 1 \\
x \eta_x & \to x : F
\end{array}
\]

(2) The adjunct diagram in Remark 7.3.2(5) is depicted more elaborately as

\[
\begin{array}{c}
G \colon a & \to a : G \circ F \\
g \downarrow & \downarrow \epsilon_a \\
x & \to x : F
\end{array}
\]
(3) An adjunction is also denoted using its unit or counit (or both). For example, if \( \eta \) is the unit and \( \epsilon \) is the counit, we write the right \([\text{op. left}]\) form of an adjunction as \((\eta, \epsilon) : G \dashv F : X \rightarrow A\) \([\text{op.} \ (\eta, \epsilon) : G \dashv F : X \rightarrow A]\), or diagrammatically as

\[
\begin{array}{ccc}
X & \xrightarrow{(\eta, \epsilon)} & A \\
G & \xrightarrow{F} & G \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{(\eta, \epsilon)} & A \\
F & \xleftarrow{G} & F \\
\end{array}
\quad
\begin{array}{ccc}
A & \xleftarrow{\epsilon} & X \\
G & \xrightarrow{\eta} & G \\
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\epsilon} & X \\
F & \xleftarrow{G} & F \\
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{(\eta, \epsilon)} & A \\
F & \xleftarrow{G} & F \\
\end{array}
\]

7.3.6 Example. If a functor \( K : D \rightarrow E \) is an isomorphism of categories, then it is both left and right adjoint to its inverse \( K^{-1} : E \rightarrow D \) with the unit and counit given by the identity natural transformations.

7.3.7 Proposition.

- The counit \( \epsilon \) of an adjunction \( \phi : G \dashv F : X \rightarrow A \) makes the diagram

\[
\begin{array}{ccc}
G(X) & \xleftarrow{(G)} & (A) \\
\downarrow{G(\phi)} & & \downarrow{\epsilon(A)} \\
G(F(A)) & = [G \circ F](A) \\
\end{array}
\]

commute, yielding the adjunct diagram

\[
\begin{array}{ccc}
G : \cdot a' & \longleftarrow & \cdot a' \\
\uparrow{h} & & \uparrow{\epsilon_a} \\
G : \cdot a & \longleftarrow & \cdot G : \cdot F \\
\end{array}
\]

for each \( A \)-arrow \( h : a \rightarrow a' \).

- The unit \( \eta \) of an adjunction \( \phi : G \dashv F : X \rightarrow A \) makes the diagram

\[
\begin{array}{ccc}
(X) & \xrightarrow{(F)} & F(A) \\
\downarrow{(X)\eta} & & \downarrow{(\phi)F} \\
(X) [G \circ F] & = ((X) G) F \\
\end{array}
\]

commute, yielding the adjunct diagram

\[
\begin{array}{ccc}
G : \cdot F : \cdot x & \longleftarrow & \cdot x : F \\
\downarrow{\eta_x} & & \downarrow{1} \\
\cdot x & \longleftarrow & \cdot x : F \\
\downarrow{k} & & \downarrow{k : F} \\
\cdot x' & \longleftarrow & \cdot x' : F \\
\end{array}
\]

for each \( X \)-arrow \( k : x' \rightarrow x \).

Proof. The commutativity of the diagram follows from Proposition 5.3.7, and implies that

\[
G(\phi) : (G : \cdot h) = (G(\phi) \circ G) : \cdot h = \epsilon(\cdot A) : \cdot h = h \circ \epsilon_a
\]

for any \( A \)-arrow \( h : a \rightarrow a' \).

7.3.8 Remark. The adjunct diagrams in Proposition 7.3.7 are also derived from the adjunct diagrams in Remark 7.3.5(1a): by Proposition 7.3.3, the outer rectangles of

\[
\begin{array}{ccc}
G : \cdot a' & \longleftarrow & \cdot a' \\
\downarrow{G : \cdot h} & & \downarrow{1} \\
\cdot x & \longleftarrow & \cdot x : F \\
\downarrow{k} & & \downarrow{k : F} \\
\cdot x' & \longleftarrow & \cdot x' : F \\
\end{array}
\quad
\begin{array}{ccc}
G : \cdot F : \cdot x & \longleftarrow & \cdot x : F \\
\downarrow{\eta_x} & & \downarrow{1} \\
\cdot x & \longleftarrow & \cdot x : F \\
\downarrow{k} & & \downarrow{k : F} \\
\cdot x' & \longleftarrow & \cdot x' : F \\
\end{array}
\]

are adjunct diagrams.

7.3.9 Proposition. Consider a pair of functors \( X \xrightarrow{G} A \).
7.3. Adjunctions between categories

For a natural transformation \( \epsilon : G \circ F \to 1_A \), the following conditions are equivalent:

1. the module morphism \( \mathbf{X} \uparrow \epsilon : (\mathbf{X})_G \to F(A) : \mathbf{X} \to A \) is iso;
2. \( \epsilon \) is the counit of an adjunction \( G \dashv F : \mathbf{X} \to A \);
3. \( \epsilon \) regarded as a right cylinder \( \xymatrix{ \mathbf{X} \ar[r]^G \ar[d]_\epsilon & F(A) \ar[d]^\eta_A} \) is a counit of the representable module \( F(A) \);
4. \( \epsilon \) regarded as a two-sided cylinder \( \xymatrix{ \mathbf{X} \ar[r]^G \ar[d]^\epsilon & \ar[l]_F(A) \ar[d]^\eta_A} \) is a pointwise lift of the identity \( 1_A \) along the representable module \( F(A) \);
5. \( \epsilon \) is a pointwise right Kan lift of the identity \( 1_A \) along \( F \);
6. each component \( \epsilon_a : G \cdot F(a) \to a \) is universal from \( F \) to \( a \).

- For a natural transformation \( \eta : 1_X \to G \circ F \), the following conditions are equivalent:
1. the module morphism \( \eta \uparrow A : F(A) \to (\mathbf{X})_G : \mathbf{X} \to A \) is iso;
2. \( \eta \) is the unit of an adjunction \( G \dashv F : \mathbf{X} \to A \);
3. \( \eta \) regarded as a left cylinder \( \xymatrix{ \mathbf{X} \ar[r]^\eta \ar[d]_G & A \ar[d]^{\mathbf{X}}_F} \) is a unit of the corepresentable module \( (\mathbf{X})_G \);
4. \( \eta \) regarded as a two-sided cylinder \( \xymatrix{ \mathbf{X} \ar[r]^\eta \ar[d]_G & \ar[l]_F(A) \ar[d]^\eta_A} \) is a pointwise colift of the identity \( 1_X \) along the corepresentable module \( (\mathbf{X})_G \).
5. \( \eta \) is a pointwise left Kan lift of the identity \( 1_X \) along \( G \);
6. each component \( \eta_a : x \to G \cdot F \cdot x \) is universal from \( x \) to \( G \).

Proof. (1)\( \iff \) (2) By Definition 7.3.1 and Definition 7.3.4.
(1)\( \iff \) (3) By Definition 6.4.1.
(3)\( \iff \) (4) By Proposition 6.5.7.
(4)\( \iff \) (5) By Definition 6.6.1.
(5)\( \iff \) (6) By Remark 6.6.2(4).

Note. By Remark 7.3.2(2), Theorem 7.3.10 and Corollary 7.3.12 are special cases of Theorem 6.4.10 and Corollary 6.4.11 where \( \mathcal{M} \) is given by a representable [op. corepresentable] module.

7.3.10 Theorem.

- Given a functor \( F : \mathbf{X} \to A \), suppose that there is a family of \( A \)-arrows \( \epsilon_a : r_a : F \to a \), one for each object \( a \in |A| \), universal from \( F \) to \( a \). Then there is a unique functor \( G : A \to \mathbf{X} \) with \( G \cdot a = r_a \) such that \( \epsilon := (\epsilon_a)_{a \in |A|} \) forms a natural transformation \( \epsilon : G \circ F \to 1_A \), and \( G \) is a right adjoint of \( F \) with \( \epsilon \) the counit of the adjunction.
- Given a functor \( G : A \to \mathbf{X} \), suppose that there is a family of \( \mathbf{X} \)-arrow \( \eta_x : x \to G \cdot r_x \), one for each object \( x \in |\mathbf{X}| \), universal from \( x \) to \( G \). Then there is a unique functor \( F : \mathbf{X} \to A \) with \( r_x = x : F \) such that \( \eta := (\eta_x)_{x \in |\mathbf{X}|} \) forms a natural transformation \( \eta : 1_{\mathbf{X}} \to G \circ F \), and \( F \) is a left adjoint of \( G \) with \( \eta \) the unit of the adjunction.

Proof. As noted above, this follows as a special case of Theorem 6.4.10 with \( \mathcal{M} \) given by the representable module \( F(A) \) [op. corepresentable module \( (\mathbf{X})_G \)].

7.3.11 Remark. Noting the equivalence of (2) and (5) in Proposition 7.3.9, we see that Theorem 7.3.10 is also a special case of Theorem 6.6.7 where \( \mathcal{L} \) is an identity.

7.3.12 Corollary.

- The following conditions are equivalent for a functor \( F : \mathbf{X} \to A \):
  1. \( F \) has a right adjoint;
  2. for every object \( a \in |\mathbf{X}| \), the right module \( F(A) a : \mathbf{X} \to * \) is representable;
  3. for every object \( a \in |\mathbf{X}| \), the right module \( F(A) a : \mathbf{X} \to * \) is corepresentable;
  4. for every object \( a \in |\mathbf{X}| \), the right module \( F(A) a : \mathbf{X} \to * \) is a representable module.
(3) for every object \( a \in [A] \), there is an object \( r_a \in [X] \) and an \( A \)-arrow \( \epsilon_a : r_a \rightarrow a \) universal from \( F \) to \( a \).

- The following conditions are equivalent for a functor \( G : A \rightarrow X \):
  (1) \( G \) has a left adjoint;
  (2) for every object \( x \in [X] \), the left module \( x(X)G : * \rightarrow A \) is representable;
  (3) for every object \( x \in [X] \), there is an object \( r_x \in [A] \) and an \( X \)-arrow \( \eta_x : x \rightarrow G \circ r_x \) universal from \( x \) to \( G \).

Proof. Since \( F \) has a right adjoint \( G \) is corepresentable \( \iff \) \( F \) has a counit, the assertion is a special case of Corollary 6.4.11 where \( M \) is given by the representable module of \( F \).

7.3.13 Theorem. Given functors \( X \xrightarrow{g} F \xleftarrow{f} A \), a pair of natural transformations \( \eta : 1_X \rightarrow G \circ F \) and \( \epsilon : G \circ F \rightarrow 1_A \) form an adjunction \( (\eta, \epsilon) : G \dashv F : X \rightarrow A \) if and only if the pasting compositions

\[
\begin{array}{ccc}
\xymatrix{ X & \ar[l]_1 & X \\
A & \ar[l]_1 & A }
\end{array}
\]

and

\[
\begin{array}{ccc}
\xymatrix{ X & \ar[l]_1 & X \\
A & \ar[l]_1 & A }
\end{array}
\]

yield the identity natural transformations \( F \rightarrow F \) and \( G \rightarrow G \); that is, if and only if the triangles

\[
\begin{array}{ccc}
\xymatrix{ G \circ F \circ \eta & \ar[l]_1 & G \\
F \circ \epsilon \circ F & \ar[l]_1 & F }
\end{array}
\]

and

\[
\begin{array}{ccc}
\xymatrix{ G \circ F \circ \eta & \ar[l]_1 & G \\
F \circ \epsilon \circ F & \ar[l]_1 & F }
\end{array}
\]

commute.

Proof. By Corollary 5.3.27, the triangles

\[
\begin{array}{ccc}
\xymatrix{ (X)G & \ar[l]_{\eta[A]} & (X)G \\
F(A) & \ar[l]_{\eta[A]} & (X)G }
\end{array}
\]

and

\[
\begin{array}{ccc}
\xymatrix{ F(A) & \ar[l]_{\eta[A]} & (X)G \\
F(A) & \ar[l]_{\eta[A]} & (X)G }
\end{array}
\]

commute. Since the generalized Yoneda functor is fully faithful (see Corollary 5.3.15), \([G \circ \eta] \circ [\epsilon \circ G] = 1_G\) and \([\eta \circ F] \circ [F \circ \epsilon] = 1_F\) \iff \( (X)[[G \circ \eta] \circ [\epsilon \circ G]]\) and \([\eta \circ F] \circ [F \circ \epsilon]\) are the identities; that is, \( \eta[A] \) and \( X[A] \) are the inverse of each other; but this is the case \( \iff \eta \) and \( \epsilon \) are the unit and counit of a adjunction \( G \dashv F : X \rightarrow A \).

7.3.14 Theorem.

- For an adjunction \( \Phi : G \dashv F : X \rightarrow A \), the following conditions are equivalent:
  (1) the functor \( G : A \rightarrow X \) is fully faithful;
  (2) the counit \( \epsilon : G \circ F \rightarrow 1_A \) is a natural isomorphism;
  (3) the module isomorphism \( \Phi : (X)G \rightarrow F(A) \) preserves isomorphisms; that is, for any adjunction diagram

\[
\begin{array}{ccc}
\xymatrix{ G \circ a & \ar[l]_g & a \\
x & \ar[l]_f & x \circ F }
\end{array}
\]

, if \( g \) is an isomorphism, so is \( f \).

- For an adjunction \( \Phi : G \dashv F : X \rightarrow A \), the following conditions are equivalent:
  (1) the functor \( F : X \rightarrow A \) is fully faithful;
  (2) the unit \( \eta : 1_X \rightarrow G \circ F \) is a natural isomorphism;
  (3) the module isomorphism \( \Phi : F(A) \rightarrow (X)G \) preserves isomorphisms; that is, for any adjunction diagram

\[
\begin{array}{ccc}
\xymatrix{ G \circ a & \ar[l]_g & a \\
x & \ar[l]_f & x \circ F }
\end{array}
\]

, if \( f \) is an isomorphism, so is \( g \).
7.3. Adjunctions between categories

Proof. (1)⇒(2) Consider the commutative diagram

\[
\begin{array}{ccc}
G \langle X \rangle & \xrightarrow{(G)} & \langle A \rangle \\
\downarrow \phi & & \downarrow \epsilon(A) \\
G(F(A)) & = & [G \circ F](A)
\end{array}
\]

in Proposition 7.3.7. Since \( \phi : (X) \rightarrow F(A) \) is an isomorphism, so is \( G(\phi) \) by Proposition 1.1.33. Hence \( (G) \) is an isomorphism iff \( \epsilon(A) \) is an isomorphism. But \( (G) \) is an isomorphism iff \( G \) is fully faithful (Proposition 1.2.31), and, since the generalized Yoneda functor is fully faithful (Corollary 5.3.15), \( \epsilon(A) \) is an isomorphism iff \( \epsilon \) is an isomorphism.

(2)⇒(3) Since \( f = (g : F) \circ \epsilon_a \) (see Remark 7.3.5(2)) and any functor preserves isomorphisms, if \( g \) is an isomorphism, so is \( f \).

(3)⇒(2) Immediate because the component \( \epsilon_a \) of the counit at \( a \in \parallel A \parallel \) is given by the left adjunct of the identity \( 1_{(G:a)} \) (see Remark 7.3.5(1a)). \( \square \)

7.3.15 Theorem. Suppose that a module \( \mathcal{M} : X \rightarrow A \) has a counit \( \xrightarrow{\rho} \mathcal{M} \rightarrow A \) and a unit \( \xrightarrow{\lambda} A \rightarrow X \). Then there is an adjunction \( \phi : G \vdash F : X \rightarrow A \) with the module isomorphism \( \phi : X(G) \rightarrow F(A) : X \rightarrow A \) defined by the composition

\[
X(G) \xrightarrow{X(\rho \downarrow \lambda)} \mathcal{M} \xrightarrow{\lambda(A)^{-1}} F(A)
\]

, and the counit \( \epsilon : G \circ F \rightarrow 1_A \) [op. unit \( \eta : 1_X \rightarrow G \circ F \)] of the adjunction are given by the compositions

\[
\begin{array}{ccc}
X & \xrightarrow{\rho \downarrow \lambda} & A \\
\downarrow \mathcal{M} & & \downarrow 1 \\
\mathcal{M} & \xrightarrow{(\lambda A)^{-1}} & A
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
X & \xrightarrow{\lambda(A)^{-1}} & A \\
\downarrow \mathcal{M} & & \downarrow G \\
\mathcal{M} & \xrightarrow{\lambda(A)^{-1}} & A
\end{array}
\]

(see Remark 4.3.16(3)); that is, the component of \( \epsilon \) at \( a \in \parallel A \parallel \) is given by the adjunct of \( \rho_a \) along \( \lambda(a : G) \) and the component of \( \eta \) at \( x \in \parallel X \parallel \) is given by the adjunct of \( \lambda_x \) along \( \rho(F : x) \), as shown below:

\[
\begin{array}{ccc}
a : G & \xrightarrow{\lambda(a : G)} & F \cdot a \\
\downarrow \rho_a & & \downarrow \rho_a \\
a & \xrightarrow{\lambda(a : G)} & F \cdot a
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
x : F \cdot x & \xrightarrow{\rho(F : x)} & G \\
\downarrow \lambda_x & & \downarrow \lambda_x \\
x & \xrightarrow{\lambda(F : x)} & F \cdot x
\end{array}
\]

Proof. The first assertion is obvious since \( X \downarrow \rho \) and \( \lambda \downarrow A \) are isomorphisms by the definition of units. The counit and unit are given by

\[
\begin{align*}
\epsilon &= X \downarrow \phi \\
&= [X \downarrow (X \downarrow \rho)] \circ (\lambda \downarrow A)^{-1} \\
&= \rho \circ (\lambda \downarrow A)^{-1}
\end{align*}
\]

\[
\begin{align*}
\eta &= \phi^{-1} \downarrow A \\
&= ([X \downarrow \rho]^{-1} \circ (\lambda \downarrow A)) \downarrow A \\
&= (X \downarrow \rho)^{-1} \circ [(\lambda \downarrow A) \downarrow A] \\
&= (X \downarrow \rho)^{-1} \circ \lambda
\end{align*}
\]

\( \star_1 \) by Proposition 5.3.22. \( \square \)

Note. Since an adjunction \( \phi \) between two categories is a special instance of a symmetric cell (see Remark 7.3.2(4)), \( \phi \) yields the postcomposition symmetric cell as in Definition 7.1.12.

7.3.16 Definition. Let \( E \) be a category. Given an adjunction

\[
\phi : G \vdash F : X \rightarrow A \quad \text{op.} \quad \phi : G \vdash F : X \rightarrow A
\]
The symmetric cell, the adjunctive right (i.e. by the adjunct of $\alpha$)—the composition of a frame $E$ sends each cone (i.e. by the adjunct of $\phi$) since (see Remark 4.6.12) the component of is an adjunct diagram of $\phi$ sends each cone $\alpha : x \rightsquigarrow G \cdot S : *E \to X$ to the cone $\alpha \circ \phi : x : F \rightsquigarrow S : *E \to A$ defined by

\[
\alpha \circ \phi = \alpha \circ x(\phi) S
\]

—the composition of a frame $\alpha$ of the left module $x(\langle X \rangle) [G \cdot S] : * \to E$ and the module morphism

\[
x(\langle X \rangle) [G \cdot S] = x(\langle X \rangle) G \cdot S \xrightarrow{x(\phi)S} x(\langle F(\langle A \rangle) \rangle S = (x : F) \langle A \rangle S
\]

; since (see Remark 4.6.12) the component of $\alpha \circ \phi$ at an object $e \in \|E\|$ is given by

\[
[a \circ \phi]_e = a_e : x(\phi) (S \cdot e)
\]

(i.e. by the adjunct of $a_e$ under $\phi$), a pair of cones $\alpha : x \rightsquigarrow G \cdot S : *E \to X$ and $\beta : x : F \rightsquigarrow S : *E \to A$ are the adjunct of each other if and only if their components $a_e$ and $\beta_e$ are the adjunct of each other for every $e \in \|E\|$; that is, the square

\[
\begin{array}{ccc}
G \cdot S & \xleftarrow{\alpha} & S \\
\downarrow{\alpha} & & \downarrow{\beta} \\
x & \xrightarrow{x : F} & x : F
\end{array}
\]

is an adjunct diagram of $\langle \alpha(\phi) \rangle$ if and only if the square

\[
\begin{array}{ccc}
G \cdot S & \xleftarrow{a_e} & e : S \\
\downarrow{a_e} & & \downarrow{\beta_e} \\
x & \xrightarrow{x : F} & x : F
\end{array}
\]

is an adjunct diagram of $\phi$ for every $e \in \|E\|$.

* Given an object $a \in \|A\|$ and a functor $T : E \to X$, the adjunctive left symmetric cell $(E^*, \phi)$ sends each cone $\alpha : T \circ F \rightsquigarrow a : E^* \to A$ to the cone $\alpha \circ \phi : T \rightsquigarrow G^* : a : E^* \to X$ defined by

\[
\alpha \circ \phi = \alpha \circ T(\phi) a
\]

—the composition of a frame $\alpha$ of the right module $[T \circ F] \langle A \rangle : a : E \to *$ and the module morphism

\[
T(\langle X \rangle) (G^* : a) = T(\langle X \rangle) G \cdot a \xrightarrow{(\phi) a} T(\langle F(\langle A \rangle) \rangle a = [T \circ F] \langle A \rangle a
\]

; since (see Remark 4.6.12) the component of $\alpha \circ \phi$ at an object $e \in \|E\|$ is given by

\[
[a \circ \phi]_e = a_e : (e : T)(\phi) a
\]

(i.e. by the adjunct of $a_e$ under $\phi$), a pair of cones $\alpha : T \circ F \rightsquigarrow a : E^* \to A$ and $\beta : T \rightsquigarrow G^* : a : E^* \to X$ are the adjunct of each other if and only if their components $a_e$ and $\beta_e$ are the adjunct
of each other for every \( e \in \|E\| \); that is, the square

\[
\begin{array}{ccc}
G \cdot a & \xleftarrow{\beta} & a \\
\downarrow{\alpha} \\
T & \xleftarrow{T \circ F} & F
\end{array}
\]

is an adjunct diagram of \((E^*, \phi)\) if and only if the square

\[
\begin{array}{ccc}
G \cdot a & \xleftarrow{\beta e} & a \\
\downarrow{\alpha e} \\
T \cdot e & \xleftarrow{T \circ e} & F
\end{array}
\]

is an adjunct diagram of \(\phi\) for every \( e \in \|E\| \).

7.3.18 Proposition. Let \( \eta : 1_X \to G \circ F \) and \( \epsilon : G \circ F \to 1_A \) be the unit and the counit of an adjunction

\[
\begin{array}{c}
X \\
\xleftarrow{\phi} \xrightarrow{\blacksquare} A
\end{array}
\]

- Given an object \( x \in \|X\| \) and a functor \( S : E \to A \), the left adjunct of a cone \( \alpha : x \to G \circ S : *E \to X \) under the adjunction \((*E, \phi)\) is given by the composite \((\alpha \circ F) \circ (S \circ \epsilon)\), and the right adjunct of a cone \( \beta : x : F \to S : *E \to A \) under \((*E, \phi)\) is given by the composite \((G \circ \beta) \circ \eta_X\), all as shown in an adjunct diagram

\[
\begin{array}{ccc}
G \circ \beta & \xleftarrow{S \circ \epsilon} & S \\
\downarrow{\eta_X} \\
\beta & \xleftarrow{\alpha \circ F} & \alpha \circ F
\end{array}
\]

- Given an object \( a \in \|A\| \) and a functor \( T : E \to X \), the right adjunct of a cone \( \alpha : T \circ F \to a : E^* \to A \) under the adjunction \((E^*, \phi)\) is given by the composite \((\beta \circ F) \circ \epsilon_a\), all as shown in an adjunct diagram

\[
\begin{array}{ccc}
G \circ \alpha & \xleftarrow{\epsilon_a} & a \\
\downarrow{\eta_X} \\
\beta & \xleftarrow{\beta \circ F} & \beta \circ F
\end{array}
\]

Proof. Since the component of the composite \((G \circ \beta) \circ \eta_X\) (resp. \((\alpha \circ F) \circ (S \circ \epsilon)\)) at \( e \in \|E\| \) is given by the composite \((G \circ \beta_e) \circ \eta_X\) (resp. \((\alpha_e \circ F) \circ (S \circ \epsilon)\)), and since \( \alpha \) and \( \beta \) are the adjunct of each other iff their components \( \alpha_e \) and \( \beta_e \) are the adjunct of each other for every \( e \in \|E\| \) (see Remark 7.3.17(2)), the assertion is reduced to the family of adjunct diagrams

\[
\begin{array}{ccc}
G \circ \beta_e & \xleftarrow{\epsilon(a \circ S)} & S \\
\downarrow{\eta_X} \\
\beta_e & \xleftarrow{\alpha_e \circ F} & \alpha_e \circ F
\end{array}
\]

, one for each \( e \in \|E\| \), of \( \phi \).

7.3.19 Remark. This device is used later in Theorem 8.11.5 to state an interaction between an adjunction and a limit.

7.4 Morphisms of adjunctions

In this section, we define and study morphisms between adjunctions. We first define a lax morphism, then define a pseudo morphism and a (strict) morphism as special cases of a lax morphism.
(what is defined in [ML98] Section IV.7 is a strict morphism). A pseudo morphism is defined so that it preserves the structure of the adjunction up to isomorphism; we use pseudo morphisms in Section 9.4 to describe Frobenius reciprocity and define cartesian closed functors. The definition of a lax morphism also introduces the notion of mates and conjugates; we study conjugates later in Section 7.6.

7.4.1 Definition. Given a pair of adjunctions $\phi : G \dashv F : X \to A$ and $\phi' : G' \dashv F' : X' \to A'$, a lax morphism $\phi \to \phi'$ is defined by a pair of functors $T : X \to X'$ and $S : A \to A'$ and a pair of natural transformations $\tau : G \circ T \to S \circ G'$ and $\sigma : T \circ F' \to F \circ S$, all as in

\[
\begin{array}{cc}
X & \xrightarrow{G} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{G'} & A'
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{cc}
X & \xrightarrow{F} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{F'} & A'
\end{array}
\]

, which make the diagram

\[
\begin{array}{ccc}
(X) G & \xrightarrow{\phi} & F(A) \\
\downarrow (T) \circ \tau & \Downarrow \sigma \circ S & \downarrow (S) \\
(X') G' & \xrightarrow{\phi'} & F'(A')
\end{array}
\]

commute, where $(T) \circ \tau$ and $\sigma \circ S$ are the pasting composites (see Definition 1.2.36) as shown in:

\[
\begin{array}{cc}
X & \xrightarrow{G} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{G'} & A'
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{cc}
X & \xrightarrow{F} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{F'} & A'
\end{array}
\]

7.4.2 Proposition. Given adjunctions $\phi : G \dashv F : X \to A$, $\phi' : G' \dashv F' : X' \to A'$, and $\phi'' : G'' \dashv F'' : X'' \to A''$, consider lax morphisms $(\tau, \sigma) : \phi \to \phi'$ and $(\tau', \sigma') : \phi' \to \phi''$ as in:

\[
\begin{array}{cc}
X & \xrightarrow{G} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{G'} & A'
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{cc}
X & \xrightarrow{F} & A \\
\downarrow T & \Downarrow \Downarrow \Downarrow \Downarrow S & \downarrow S \\
X' & \xrightarrow{F'} & A'
\end{array}
\]

Then their pasting compositions yield the lax morphism $(\tau \circ \tau', \sigma \circ \sigma') : \phi \to \phi''$.

Proof. By Theorem 1.2.38, the two commutative squares

\[
\begin{array}{ccc}
(X) G & \xrightarrow{\phi} & F(A) \\
\downarrow (T) \circ \tau & \Downarrow \sigma \circ S & \downarrow (S) \\
(X') G' & \xrightarrow{\phi'} & F'(A')
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
(X) G & \xrightarrow{\phi} & F(A) \\
\downarrow (T) \circ \tau' & \Downarrow \sigma' \circ S' & \downarrow (S') \\
(X') G'' & \xrightarrow{\phi''} & F''(A'')
\end{array}
\]

yields the commutative square

\[
\begin{array}{ccc}
(X) G & \xrightarrow{\phi} & F(A) \\
\downarrow (T \circ T') \circ [\tau \circ \tau'] & \Downarrow [\sigma \circ \sigma'] \circ (S \circ S') & \downarrow (S \circ S') \\
(X') G'' & \xrightarrow{\phi''} & F''(A'')
\end{array}
\]

7.4.3 Remark. By this composition, adjunctions and lax morphisms among them assemble into a category.
7.4.4 Proposition. Let \( \phi : G \dashv F : X \to A \) and \( \phi' : G' \dashv F' : X' \to A' \) be adjunctions with unit and counit \((\eta, \epsilon)\) and \((\eta', \epsilon')\) respectively, and let \( T : X \to X' \) and \( S : A \to A' \) be functors. Then for a pair of natural transformations \( \tau : G \circ T \to S \circ G' \) and \( \sigma : T \circ F' \to F \circ S \), the following conditions are equivalent (each of (3), (4), and (5) states two equivalent conditions dual to one another).

1. The diagram

\[
\begin{array}{c}
\xymatrix{
X & G F(A) \\
X \\
T & G F'(A') \\
T & F'(A')
\end{array}
\]

commutes, i.e. the pair \((\tau, \sigma)\) forms a lax morphism \( \phi \to \phi' \).

2. If

\[
\begin{array}{c}
G : a \to a \\
\eta \downarrow \\
x \to x : F
\end{array}
\]

is an adjunct diagram of \( \phi \), then

\[
\begin{array}{c}
G' : a \to a : S \\
\eta' \downarrow \\
x : F : S
\end{array}
\]

is an adjunct diagram of \( \phi' \).

3. The diagram

\[
\begin{array}{c}
G \circ T \circ F' \quad \tau \circ F' \\
G \circ F \circ S \quad \epsilon \circ S \\
G \circ T \circ F \circ S \\
\end{array}
\]

\[
\begin{array}{c}
T \quad \eta \circ T \\
T \circ F' \circ G' \quad \sigma \circ G' \\
T \circ F' \circ S \circ G' \\
\end{array}
\]

commutes; that is, the two pasting compositions

\[
\begin{array}{c}
X \xrightarrow{T} X' \\
A \xrightarrow{S} A'
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{T} X' \\
A \xrightarrow{S} A'
\end{array}
\]

yield the same natural transformation \( G \circ T \circ F' \to S \) \([\text{op. } T \to F \circ S \circ G']\).

4. The diagram

\[
\begin{array}{c}
G \circ T \circ F' \quad \tau \circ F' \\
G \circ T \circ F' \circ G' \\
G \circ T \circ F' \circ S \circ G' \\
\end{array}
\]

\[
\begin{array}{c}
T \quad \eta \circ T \\
T \circ F' \circ G' \\
T \circ F' \circ S \circ G' \\
\end{array}
\]

commutes; that is, the pasting composition

\[
\begin{array}{c}
X \xrightarrow{T} X' \\
A \xrightarrow{S} A'
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{T} X' \\
A \xrightarrow{S} A'
\end{array}
\]

yields \( \tau : G \circ T \to S \circ G' \) \([\text{op. } \sigma : T \circ F' \to F \circ S]\).
(5) For each \( a \in \| A \| \) \([op. \ x \in \| X \|]\),
\[
\begin{array}{c}
G' \circ S \cdot a \ar[r] & a \circ S \\
\tau \circ a \ar[u] & a \circ G \circ F \circ S \\
T' \circ G \circ a \ar[r] & a \circ G \circ T \circ F' \\
\end{array}
\]
\[
\begin{array}{c}
G' \circ S \cdot x \ar[r] & x \circ F \circ S \\
\tau \circ F \circ x \ar[u] & x \circ \sigma \\
T' \circ x \ar[r] & x \circ T \circ F' \\
\end{array}
\]
is an adjunct diagram of \( \phi' \).

Proof. (1) \( \iff \) (2) By Remark 1.2.37(1), \( \{ T \} \circ \tau \circ \{ X \} G \to \{ X' \} G' \) sends \( x \overset{G}{\to} G' \circ a \) to the composite \( T \circ x \overset{\eta}{\to} T \circ G \circ a \overset{T \circ \tau}{\to} G' \circ S \circ a \), and \( \sigma \circ (S) \) sends \( x \circ T \circ F' \overset{x \circ \sigma}{\to} a \circ G \circ F \circ S \) to \( a \circ S \). Hence (2) is just a componentwise description of (1).

(4) \( \iff \) (5) By Remark 7.3.5(1b), the diagram in (5) is an adjunct diagram iff the diagram
\[
\begin{array}{c}
G' \circ S \cdot a \ar[r] & G' \circ S \cdot F \circ G \circ a \\
\tau \circ a \ar[u] & G' \circ S \cdot G \circ F \circ G \circ a \\
T' \circ G \circ a \ar[r] & G' \circ F' \circ T \circ G \circ a \\
\end{array}
\]
commutes. Hence (5) is just a componentwise description of (4).

(3) \( \Rightarrow \) (4) By (3) and Theorem 7.3.13,

(4) \( \Rightarrow \) (3) By (4) and Theorem 7.3.13,

(2) \( \Rightarrow \) (5) By Remark 7.3.5(1a),
\[
\begin{array}{c}
G \circ a \ar[r] & a \circ G \circ F \\
\tau \circ a \ar[u] & a \circ \eta \\
G \circ a \ar[r] & a \circ G \circ F \\
\end{array}
\]
is an adjunct diagram of \( \phi \), and by (2) mapped to the adjunct diagram
\[
\begin{array}{c}
G' \circ S \cdot a \ar[r] & a \circ S \\
\tau \circ a \ar[u] & a \circ \eta \\
T' \circ G \circ a \ar[r] & a \circ G \circ F \circ S \\
\end{array}
\]
of \( \phi' \).

(3) \( \Rightarrow \) (2) By Remark 7.3.5(1b), the diagrams in (2) are adjunct diagrams iff the diagrams
\[
\begin{array}{c}
a \circ G \circ F \ar[r] & a \circ S \\
\tau \circ a \ar[u] & a \circ \eta \\
a \circ G \circ T \circ F' \ar[r] & a \circ S \\
\end{array}
\]
commute. Hence we need to show that the commutativity of the triangle on the left implies the
commutativity of the square on the right. Divide the square into three parts as in

\[
\begin{array}{ccc}
\begin{array}{c}
\text{a} : S : G : F' \\
\text{a} : T : F'
\end{array}
& \xrightarrow{\phi} & \begin{array}{c}
\text{a} : S' : G' : F' \\
\text{a} : T : F'
\end{array} \\
\begin{array}{c}
\text{a} : \tau : F'
\end{array}
& \xrightarrow{\sigma} & \begin{array}{c}
\text{a} : \tau : S \\
\text{a} : \tau : T
\end{array} \\
\begin{array}{c}
\text{g} : T : F'
\end{array}
& \xrightarrow{\delta} & \begin{array}{c}
\text{g} : F' : S
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{x} : T : F'
\end{array}
\xrightarrow{\chi : \sigma}
\begin{array}{c}
\text{x} : T : F'
\end{array}
\]

; since the upper trapezoid commutes by (3) and the lower trapezoid commutes by the naturality of \(\sigma\), the commutativity of the outer square is reduced to that of the triangle; we are thus done by the functoriality of \(S\).

**7.4.5 Definition.** A pair of natural transformations \(\sigma\) and \(\tau\) satisfying the equivalent conditions in Proposition 7.4.4 are called mates with respect to the adjunctions \(\phi\) and \(\phi'\).

**7.4.6 Remark.**

1. [CGR12] and the other literature define mates by the condition (4): given \(\sigma\) (or \(\tau\)) its mate is defined by the pasting composition in (4).

2. Natural transformations \(\sigma\) and \(\tau\) are mates with respect to the adjunctions \(\phi\) and \(\phi'\) precisely when they form a lax morphism \(\phi \to \phi'\). Since \(\sigma\) and \(\tau\) define each other, a lax morphism \(\phi \to \phi'\) is in fact determined by either \(\tau\) or \(\sigma\).

3. For the special case where the functors \(T\) and \(S\) in Proposition 7.4.4 are identities, mates are called conjugates. Conjugates are studied in Section 7.6.

**7.4.7 Definition.** A lax morphism in Definition 7.4.1 is called a pseudo morphism if \(\tau\) and \(\sigma\) are natural isomorphisms.

**7.4.8 Remark.**

1. By the conditions (2) and (3) in Proposition 7.4.4 with \(\tau\) and \(\sigma\) isomorphisms, a pseudo morphism preserves adjunct diagrams, counits and units up to isomorphism.

2. Note that \(\tau\) (or \(\sigma\)) being an isomorphism need not imply that its mate is an isomorphism. To see this, consider the adjunctions \(1 \xrightarrow{\tau} 1\) and \(2 \xrightarrow{\Delta_1 : \Delta_0} 2\) between the terminal category and the interval category (each of \(1 \xrightarrow{\tau} 1\) and \(2 \xrightarrow{\Delta_1 : \Delta_0} 2\) admits one and only one adjunction); we have mates

\[
\begin{array}{ccc}
1 & \xrightarrow{\tau} & 1 \\
0 & \xrightarrow{\zeta} & 0 \\
2 & \xrightarrow{\Delta_1} & 2 \\
0 & \xrightarrow{\eta} & 0 \\
2 & \xrightarrow{\Delta_0} & 2
\end{array}
\]

between them with the sole component of \(\tau\) given by the non-invertible arrow \(0 \to 1\) and the sole component of \(\sigma\) by the identity \(0 \to 0\). (We will see a different story for conjugates in Proposition 7.6.6.)

**7.4.9 Definition.** Given a pair of adjunctions \(\phi : G \dashv F : X \to A\) and \(\phi' : G' \dashv F' : X' \to A'\), a (strict) morphism \(\phi \to \phi'\) is defined by a pair of functors \(S : A \to A'\) and \(T : X \to X'\) such that both squares

\[
\begin{array}{ccc}
X & \xrightarrow{G} & A \\
\tau \downarrow & & \downarrow \delta \\
X' & \xrightarrow{G'} & A'
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\tau \downarrow & & \downarrow \delta \\
X' & \xrightarrow{F'} & A'
\end{array}
\]

commute, and such that the diagram

\[
\begin{array}{ccc}
\langle X \rangle & \xrightarrow{\phi} & F \langle A \rangle \\
\langle T \rangle \downarrow & & \downarrow F \langle S \rangle \\
\langle X' \rangle & \xrightarrow{\phi'} & F' \langle A' \rangle
\end{array}
\]

is a total morphism in proposition 7.4.4.
commutes, where \((T)G\) and \(F(S)\) are the pasting composites (see Definition 1.2.33) as shown in:

\[
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{(X)} X \\
\text{and}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
G \xrightarrow{\phi} F(A) \\
\text{and}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{commutes, i.e. the pair} \ (T, S) \ \text{forms a morphism} \ \phi \rightarrow \phi'.
\end{array}
\end{array}
\]

(2) If

\[
\begin{array}{c}
\begin{array}{c}
G \vdash a \rightarrow a \\
x \rightarrow x \vdash F
\end{array}
\end{array}
\]

is an adjunct diagram of \(\phi\), then

\[
\begin{array}{c}
\begin{array}{c}
G' \vdash S \vdash a \rightarrow a \vdash S \\
\text{and}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{is an adjunct diagram of} \ \phi'.
\end{array}
\end{array}
\]

(3) The diagram

\[
\begin{array}{c}
\begin{array}{c}
G \ast T \ast F' \xrightarrow{\text{op.}} S \ast G' \ast F' \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{commutes.}
\end{array}
\end{array}
\]

(4) The diagram

\[
\begin{array}{c}
\begin{array}{c}
G \ast T \xrightarrow{\text{op.}} S \ast G' \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{commutes; that is, the pasting composition}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
yields the identity \ G \ast T = S \ast G' \ [\text{op.} \ T \ast F' = F \ast S].
\end{array}
\end{array}
\]

Proof. This is a special case of Proposition 7.4.4 where \(\tau\) and \(\sigma\) are identities.

7.4.10 Remark. Noting Remark 1.2.37(2), we see that Definition 7.4.9 is a special case of Definition 7.4.1 where \(\tau\) and \(\sigma\) are identities.

7.4.11 Proposition. Given a pair of adjunctions \(\phi: G \dashv F: X \rightarrow A\) and \(\phi': G' \dashv F': X' \rightarrow A'\) with unit and counit \((\eta, \epsilon)\) and \((\eta', \epsilon')\) respectively, consider a pair of functors \(S: A \rightarrow A'\) and \(T: X \rightarrow X'\) making the squares in Definition 7.4.9 commute. Then the following conditions are equivalent (each of (3) and (4) states two equivalent conditions dual to one another).

(1) The diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{(X) G} \xrightarrow{\phi} F\langle A \rangle \\
\text{(T)G} \xrightarrow{\text{F(S)}} \text{(X') G'} \xrightarrow{\phi'} F'\langle A' \rangle \\
\end{array}
\end{array}
\]

(2) If

\[
\begin{array}{c}
\begin{array}{c}
G \vdash a \rightarrow a \\
x \rightarrow x \vdash F
\end{array}
\end{array}
\]

is an adjunct diagram of \(\phi\), then

\[
\begin{array}{c}
\begin{array}{c}
G' \vdash S \vdash a \rightarrow a \vdash S \\
\text{is an adjunct diagram of} \ \phi'.
\end{array}
\end{array}
\]

7.4.12 Remark.

(1) The condition (2) is a pointwise description of the condition (1) and says that an adjunction morphism preserves adjunct diagrams strictly.
The condition (3) says that an adjunction morphism preserves counits and unis strictly.

The condition (4) may be seen as an extension of the triangular identity in Theorem 7.3.13.

7.5 Adjunctions as universal arrows

We will see that an adjunction between categories \( \mathbf{X} \) and \( \mathbf{A} \) is a two-way universal arrow of the module \( (\mathbf{X} \downarrow \mathbf{A}) \) defined in Definition 7.5.1.

Note. The following definition is a special case of Definition 7.1.6 where \( \mathcal{M} \) and \( \mathcal{N} \) are given by hom-modules; the right and left generalized Yoneda functors defined in Definition 2.3.10 allow the following definition.

7.5.1 Definition. Given a pair of categories \( \mathbf{X} \) and \( \mathbf{A} \),

- the module

\[
(\mathbf{X} \uparrow \mathbf{A}) : [\mathbf{A}, \mathbf{X}] \to [\mathbf{X}, \mathbf{A}]^-
\]

is defined by the composition

\[
[\mathbf{A}, \mathbf{X}] \xrightarrow{X \downarrow \mathbf{A}} [\mathbf{X} : \mathbf{A}] \xrightarrow{(\mathbf{X} \downarrow \mathbf{A})} [\mathbf{X} : \mathbf{A}] \xrightarrow{X \downarrow \mathbf{A}} [\mathbf{X}, \mathbf{A}]^-
\]

, where \( X \downarrow \mathbf{A} \) is the right generalized Yoneda functor for the functor category \( [\mathbf{A}, \mathbf{X}] \) and \( X \uparrow \mathbf{A} \) is the left generalized Yoneda functor for the functor category \( [\mathbf{X}, \mathbf{A}] \).

- the module

\[
(\mathbf{X} \downarrow \mathbf{A}) : [\mathbf{A}, \mathbf{X}]^- \to [\mathbf{X}, \mathbf{A}]
\]

is defined by the composition

\[
[\mathbf{A}, \mathbf{X}]^- \xrightarrow{X \downarrow \mathbf{A}} [\mathbf{X} : \mathbf{A}]^- \xrightarrow{(\mathbf{X} \downarrow \mathbf{A})^-} [\mathbf{X} : \mathbf{A}]^- \xrightarrow{X \downarrow \mathbf{A}} [\mathbf{X}, \mathbf{A}]
\]

, where \( X \downarrow \mathbf{A} \) is the right generalized Yoneda functor for the functor category \( [\mathbf{A}, \mathbf{X}] \) and \( X \uparrow \mathbf{A} \) is the left generalized Yoneda functor for the functor category \( [\mathbf{X}, \mathbf{A}] \).

7.5.2 Remark.

(1) The module \( (\mathbf{X} \uparrow \mathbf{A}) \) [op. \( (\mathbf{X} \downarrow \mathbf{A}) \)] is a special case of the module \( (\mathcal{M} \uparrow \mathcal{N}) \) [op. \( (\mathcal{M} \downarrow \mathcal{N}) \)] defined in Definition 7.1.6 where \( \mathcal{M} \) and \( \mathcal{N} \) are given by the hom-module of \( \mathbf{X} \) and that of \( \mathbf{A} \); that is,

\[
(\mathbf{X} \uparrow \mathbf{A}) = \langle (\mathbf{X}) \uparrow (\mathbf{A}) \rangle \quad \text{op.} \quad (\mathbf{X} \downarrow \mathbf{A}) = \langle (\mathbf{X}) \downarrow (\mathbf{A}) \rangle
\]

; for any categories \( \mathbf{X} \) and \( \mathbf{A} \),

\[
(\mathbf{X} \uparrow \mathbf{A})^- \cong (\mathbf{A}^- \downarrow \mathbf{X}^-)
\]

(cf. Remark 7.1.7(3)).

(2) For each pair of functors \( \mathbf{X} \xrightarrow{G} \mathbf{A} \), the set

\[
(G \langle \mathbf{X} \uparrow \mathbf{A} \rangle) (\mathbf{F} \langle \mathbf{A} \rangle) = \langle (\mathbf{X}) G \rangle \langle \mathbf{X} : \mathbf{A} \rangle (\mathbf{F} \langle \mathbf{A} \rangle)
\]

is defined by the composition

\[
(\mathbf{X}) G \to \mathbf{F} \langle \mathbf{A} \rangle : \mathbf{X} \to \mathbf{A} \quad \text{op.} \quad \mathbf{F} \langle \mathbf{A} \rangle \to (\mathbf{X}) G : \mathbf{X} \to \mathbf{A}.
\]

(3) Given an \( \langle \mathbf{X} \uparrow \mathbf{A} \rangle \)-arrow \( \psi : G \to \mathbf{F} \) (i.e. a module morphism \( \psi : \langle \mathbf{X} \rangle \mathbf{G} \to \mathbf{F} \langle \mathbf{A} \rangle : \mathbf{X} \to \mathbf{A} \)) and natural transformations \( \tau : G' \to G : \mathbf{A} \to \mathbf{X} \) and \( \sigma : F' \to F : \mathbf{X} \to \mathbf{A} \), their composite

\[
\begin{array}{ccc}
\mathbf{G} & \xrightarrow{\psi} & \mathbf{F} \\
\downarrow\tau & & \downarrow\sigma^* \\
\mathbf{G}' & \xrightarrow{\psi \circ \sigma^*} & \mathbf{F}'
\end{array}
\]
in the module \( \langle X \mid A \rangle \) is the module morphism \( \tau \circ \psi \circ \sigma : \langle X \rangle G' \to F'(A) : X \to A \) given by the composition
\[
\langle X \rangle G' \xrightarrow{(X)\tau} \langle X \rangle G \xrightarrow{\psi} F(A) \xrightarrow{\sigma(A)} F'(A) .
\]
- Given an \( \langle X \mid A \rangle \)-arrow \( \psi : G \to F \) (i.e. a morphism \( \psi : F(A) \to \langle X \rangle G : X \to A \)) and natural transformations \( \tau : G \to G' : A \to X \) and \( \sigma : F \to F' : X \to A \), their composite
\[
G \xrightarrow{\tau} F \xrightarrow{\sigma} G' \xrightarrow{\psi} F'
\]
in the module \( \langle X \mid A \rangle \) is the module morphism \( \tau \circ \psi \circ \sigma : \langle X \rangle G' : X \to A \) given by the composition
\[
\langle X \rangle G' \xleftarrow{(X)\tau} \langle X \rangle G \xleftarrow{\psi} F(A) \xleftarrow{\sigma(A)} F'(A) .
\]

(4) Noting (2) above, we have the following description of an adjunction as an arrow of the module \( \langle X \mid A \rangle \) [op. \( \langle X \mid A \rangle \)]:
- an adjunction \( \phi : G \dashv F : X \to A \) is an \( \langle X \mid A \rangle \)-arrow \( \phi : G \to F \) given by an isomorphism \( \phi : \langle X \rangle G \to F(A) \) in the category \( \mathcal{X} : \mathcal{A} \).
- an adjunction \( \phi : G \dashv F : X \to A \) is an \( \langle X \mid A \rangle \)-arrow \( \phi : G \to F \) given by an isomorphism \( \phi : F(A) \to \langle X \rangle G \) in the category \( \mathcal{X} : \mathcal{A} \).

(5) The bijection in Theorem 5.3.24 is now written as
\[
(G \circ F)(A, A)(1_A) = (G \circ F)(F(A)) \cong (G \circ F)(1_A)(F)
\]
\[\text{op.}\]
\[
(1_X)(X, X)(G \circ F) = ((X)G \circ F)(A)(F) \cong (G \circ F)(1_A)(F)
\]
, while the module isomorphism in Remark 5.3.25 is written as
\[
\langle X \mid A \rangle : \langle X \circ A \rangle : [X \circ A] \to \langle X \mid A \rangle : [A, X] \to [X, A]^-
\]
\[\text{op.}\]
\[
\langle X \circ A \rangle : \langle X \mid A \rangle : [X \circ A] \to \langle X \mid A \rangle : [A, X]^- \to [X, A]
\]
; the inverse of the module isomorphism \( \langle X \mid A \rangle : [X \circ A] \to \langle X \mid A \rangle : [A, X]^- \to [X, A] \) is represented in the form of a fully faithful cell
\[
[A, X] \xrightarrow{X \mid A} [X, A]^-
\]
\[\text{op.}\]
\[
[A, X] \xrightarrow{X \mid A} [X, A]^-
\]
\[\text{op.}\]
\[
[X \circ A] \xrightarrow{X \mid A} [X, A]^-
\]
\[\text{op.}\]
\[
[X \circ A] \xrightarrow{X \mid A} [X, A]^-
\]
, where \( \langle X \mid A \rangle \) [op. \( \langle X \mid A \rangle \)] denotes the inverse of the generalized Yoneda representation [op. corepresentation]. By (4) above and recalling Definition 7.3.4, we see that
- the cell \( \langle X \mid A \rangle : [X \circ A] \) sends each adjunction \( \phi : G \dashv F : X \to A \) to its counit.
- the cell \( \langle X \circ A \rangle : [X, A] \) sends each adjunction \( \phi : G \dashv F : X \to A \) to its unit.

Note. By Remark 7.5.2(1), the following is a special case of Proposition 7.2.7.

7.5.3 Proposition.
- If \( \phi : G \dashv F : X \to A \) is an adjunction and \( \tau : G' \to G : A \to X \) and \( \sigma : F' \to F : X \to A \) are natural isomorphisms, then their composite in the module \( \langle X \mid A \rangle \) is an adjunction \( \tau \circ \phi \circ \sigma : G' \dashv F' : X \to A \).
- If \( \phi : G \dashv F : X \to A \) is an adjunction and \( \tau : G \to G' : A \to X \) and \( \sigma : F \to F' : X \to A \) are natural isomorphisms, then their composite in the module \( \langle X \mid A \rangle \) is an adjunction \( \tau \circ \phi \circ \sigma : G' \dashv F' : X \to A \).

Proof. We see that this is an instance of Proposition 1.1.35, noting the description (see Remark 7.5.2(4)) of an adjunction as an \( \langle X \mid A \rangle \)-arrow. \(\square\)
7.5.4 Proposition. Let $X$ and $A$ be categories.

- Any adjunction $\phi: G \dashv F: X \rightarrow A$ is a two-way universal $(X \uparrow A)$-arrow.
- Any adjunction $\phi: G \dashv F: X \rightarrow A$ is a two-way universal $(X \downarrow A)$-arrow.

Proof. First recall the construction of the module $(X \uparrow A)$ and the description of an adjunction in Remark 7.5.2(4). Then since the generalized Yoneda functors $X \triangleright A$ and $X \lhd A$ are fully faithful (see Corollary 5.3.15) and $\phi$ is an isomorphism in $[X : A]$, the assertion follows from Theorem 6.2.21. 

7.5.5 Remark. The converse does not hold in general: a two-way universal $(X \uparrow A)$-arrow need not be an adjunction $X \rightarrow A$. To see this, consider the module

$$\langle 2 \uparrow 2 \rangle: [2, 2] \rightarrow [2, 2]$$

made from the discrete category $2 = \{0, 1\}$ and the interval category $2$. The functor category $[2, 2] \cong 2$ consists of the two constant functors $\Delta 0$ and $\Delta 1$, and their corepresentable modules look like

```
0 -> 0
1 -> 1
```

respectively. On the other hand, the opposite of the functor category $[2, 2]$ is depicted as

```
[01]        [11]
↓          ↓
[00]        [10]
```

, where each $[ij]$ denotes the functor $2 \rightarrow 2$ sending $0$ to $i$ and $1$ to $j$; the representable modules of the functors $[00], [01], [10], \text{ and } [11]$ look like

```
0 -> 0
1 -> 1
```

respectively. We now see that the module $\langle 2 \uparrow 2 \rangle$ consists of the following arrows

```
\Delta 0 \xrightarrow{\phi} [01] \xrightarrow{\downarrow} [10],
```

As we can read off from the diagram above, $\phi: \langle 2 \rangle [\Delta 0] \rightarrow [01] \langle 2 \rangle$ is a two-way universal $\langle 2 \uparrow 2 \rangle$-arrow, but it is not an adjunction because the function $1(\phi) 1: 1 \langle 2 \rangle 0 \rightarrow 1 \langle 2 \rangle 1$, the component of $\phi$ at $(1, 1)$, is not bijective; note that $\phi$ corresponds to the non-pointwise Kan lift in Example 6.6.5(1) under the module isomorphism $\langle 2 \uparrow 2 \rangle \langle 2 \triangleright 2 \rangle \langle 2 \lhd 2 \rangle \rightarrow \langle 2 \uparrow 2 \rangle$ (see Remark 7.5.2(5)).

7.6 Conjugation for adjunctions

The notion of conjugates is a special case of the notion of mates introduced in Section 7.4. However we will look at conjugates from a different point of view, namely, that of conjugation; since, as we saw in Section 7.5, adjunctions $X \rightarrow A$ constitute two-way universal arrows of the module $(X \uparrow A)$, we can apply the notion of conjugation presented in Section 6.3 to adjunctions.

Note. Given categories $X$ and $A$, conjugation along a pair of adjunctions $X \rightarrow A$ [op. $X \rightarrow A$] is defined below using the module $(X \uparrow A)$ [op. $(X \downarrow A)$] defined in Definition 7.5.1.
7.6.1 Definition.

- Given two adjunctions $\phi : G \dashv F : X \to A$ and $\phi' : G' \dashv F' : X \to A$, a pair of natural transformations $\tau : G \to G' : A \to X$ and $\sigma : F' \to F : X \to A$ are called conjugate (or conjugates) if the square

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & F \\
\downarrow & & \downarrow \sigma \\
G' & \xrightarrow{\phi'} & F'
\end{array}
\]

commutes; that is, if the square

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & F \\
\downarrow & & \downarrow \sigma \\
G' & \xrightarrow{\phi'} & F'
\end{array}
\]

is a two-way conjugation along the module $(X | A)$ in the sense of Definition 6.3.1.

- Given two adjunctions $\phi : G \dashv F : X \to A$ and $\phi' : G' \dashv F' : X \to A$, a pair of natural transformations $\tau : G \to G' : A \to X$ and $\sigma : F' \to F : X \to A$ are called conjugate (or conjugates) if the square

\[
\begin{array}{ccc}
G & \xleftarrow{\phi} & F \\
\downarrow & & \downarrow \sigma \\
G' & \xleftarrow{\phi'} & F'
\end{array}
\]

commutes; that is, if the square

\[
\begin{array}{ccc}
G & \xleftarrow{\phi} & F \\
\downarrow & & \downarrow \sigma \\
G' & \xleftarrow{\phi'} & F'
\end{array}
\]

is a two-way conjugation along the module $(X | A)$ in the sense of Definition 6.3.1.

7.6.2 Remark.

(1) The commutativity of the conjugation squares in the above definition is expressed component-wise by the commutativity of

\[
\begin{array}{ccc}
G \cdot a & \xrightarrow{\phi} & x(\phi)(a) \\
\tau_a & \xrightarrow{\tau(a)} & x(\tau)(\phi)(a) \\
G' \cdot a & \xrightarrow{\phi'} & x(\phi')\cdot a
\end{array}
\]

for every pair of objects $x \in [X]$ and $a \in [A]$, and these commutativities are in turn expressed by the identities

\[
(g \circ \tau_a) \cdot \phi' = \sigma_x \circ (g \cdot \phi) \quad \phi' \cdot (f \circ \sigma_x) = \tau_a \circ (\phi \cdot f)
\]

for any $X$-arrow $g : x \to G \cdot a$ and any $A$-arrow $x : F \to a$.

(2) Given subcategories $X$ and $A$, the category of adjunctions $X \to A$, denoted by $\text{Adj}[X \uparrow A]$, is given by the full subcategory of the comma category $[X \uparrow A]$ consisting of all adjunctions $X \to A$; given two adjunctions $\phi : G \dashv F : X \to A$ and $\phi' : G' \dashv F' : X \to A$, an arrow $\phi \to \phi'$ is a conjugate pair of natural transformations $\tau : G \to G' : A \to X$ and $\sigma : F' \to F : X \to A$. Two fully faithful forgetful functors $\text{Adj}[X \uparrow A] \to [A, X]$ and $\text{Adj}[X \uparrow A] \to [X, A]$ are given by restricting the functors $(X \uparrow A)^{1}_0 : [X \uparrow A] \to [A, X]$, and $(X \uparrow A)^{1}_1 : [X \uparrow A] \to [X, A]$ (the comma fibration) to $\text{Adj}[X \uparrow A]$; they send each conjugation

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & F \\
\downarrow & & \downarrow \sigma \\
G' & \xrightarrow{\phi'} & F'
\end{array}
\]

to the natural transformations $\tau : G \to G'$ and $\sigma : F' \to F$ respectively. Note that the category of
adjunctions \( X \to A \) is a full subcategory of the category of universal arrows (see Remark 6.3.2) of the module \((X | X)\). The category \( \text{Adj}[X | A] \) is defined dually using the module \((X | A)\) instead of \((X | A)\), and associated with fully faithful forgetful functors \( \text{Adj}[X | A] \to [A, X] \) and \( \text{Adj}[X | A] \to [X, A] \).

(3) We say that two adjunctions \( \phi : G \dashv F : X \to A \) and \( \phi' : G' \dashv F' : X \to A \) are isomorphic if they are so in the category \( \text{Adj}[X | A] \); that is, when there is a conjugate pair \( (\tau, \sigma) \) of natural isomorphisms for \( \phi \) and \( \phi' \). The composite \( \tau \circ \phi \circ \sigma : G' \dashv F' : X \to A \) in Proposition 7.6.3 is an adjunction isomorphic to \( \phi : G \dashv F : X \to A \), and \( \tau^{-1} : G \to G' \) is conjugate to \( \sigma : F' \to F \). Conversely, if adjunctions \( \phi : G \dashv F : X \to A \) and \( \phi' : G' \dashv F' : X \to A \) are isomorphic with a conjugate pair \( (\tau, \sigma) \) of natural isomorphisms, then \( \phi' \) is given by the composite \( \tau^{-1} \circ \phi \circ \sigma \).

### 7.6.3 Proposition.
A conjugate pair \( (\tau, \sigma) \) of natural transformations along adjunctions \( \phi : G \dashv F : X \to A \) and \( \phi' : G' \dashv F' : X \to A \) form a lax morphism \( \phi \to \phi' \), a special case of Definition 7.4.1 where \( T \) and \( S \) are identities \( X \to X \) and \( A \to A \).

#### Proof.
This follows from Remark 1.2.37(4). \( \square \)

### 7.6.4 Proposition.
Consider a pair of adjunctions \( \phi : G \dashv F : X \to A \) and \( \phi' : G' \dashv F' : X \to A \) with unit and counit \( (\eta, \epsilon) \) and \( (\eta', \epsilon') \) respectively. Then for a pair of natural transformations \( \tau : G \to G' : A \to X \) and \( \sigma : F' \to F : X \to A \), the following condition are equivalent (each of (3), (4), and (5) states two equivalent conditions dual to one another).

1. The diagram
   \[
   \begin{array}{ccc}
   \langle X \rangle G & \xrightarrow{\phi} & \langle X \rangle F (A) \\
   \downarrow{\langle X \rangle \tau} & & \downarrow{\sigma (A)} \\
   \langle X \rangle G' & \xleftarrow{\phi'} & \langle X \rangle F' (A)
   \end{array}
   \]

   commutes, i.e. \( \tau \) and \( \sigma \) are conjugate.

2. If the inner parallelogram in
   \[
   \begin{array}{ccc}
   \tau \circ a & \xleftarrow{\epsilon} & a \\
   \downarrow{\tau \circ a} & & \downarrow{\epsilon} \\
   G' \circ a & \xleftarrow{x \circ \sigma} & x \circ \sigma
   \end{array}
   \]

   is an adjunct diagram of \( \phi \), the outer rectangle is an adjunct diagram of \( \phi' \).

3. The square
   \[
   \begin{array}{ccc}
   G \circ F' & \xrightarrow{\tau \circ F'} & G' \circ F' \\
   G \circ F & \xrightarrow{\tau \circ F} & 1_A \\
   \end{array}
   \]

   commutes.

4. The square
   \[
   \begin{array}{ccc}
   G \circ \tau & \xrightarrow{F'} & G' \circ \tau \\
   G \circ \tau & \xrightarrow{\eta \circ \sigma \circ G'} & G \circ F \circ \sigma \\
   \end{array}
   \]

   commutes; that is, the pasting composition
   \[
   \begin{array}{ccc}
   X & \xrightarrow{1} & X \\
   \downarrow{G} & \searrow{\eta \circ G'} & \nearrow{\eta \circ G'} \\
   A & \xrightarrow{1} & A
   \end{array}
   \]

   yields \( \tau : G \to G' \) [op. \( \sigma : F' \to F \)].
(5) For each \( a \in \|A\| \) [op. \( x \in \|X\| \)],

\[
\begin{array}{c}
\begin{array}{c}
G' \cdot a \rightarrow a \quad \text{op.} \quad G' \cdot F : x \rightarrow x : F \\
\tau : a \Downarrow \left[ a : e \right] \\
G' \cdot a \rightarrow a : G' : F \\
\end{array}
\end{array}
\]

is an adjunct diagram of \( \phi' \).

**Proof.** This is a special case of Proposition 7.4.4 where \( T : X \rightarrow X' \) and \( S : A \rightarrow A' \) are identities (see Proposition 7.6.3).

7.6.5 Remark.
(1) A conjugate pair \( (\tau, \sigma) \) is thus a special case of mates defined in Definition 7.4.5.
(2) The condition (5) is just a componentwise description of the condition (4).

7.6.6 Proposition. Consider a conjugate pair \( (\tau, \sigma) \) of natural transformations as in Definition 7.6.1. If one of \( (\tau, \sigma) \) is an isomorphism, so is the other (and thus adjunctions \( \phi \) and \( \phi' \) are isomorphic).

**Proof.** This is an instance of Proposition 6.3.3(2).

**Note.** We saw the uniqueness of adjoint in Remark 7.3.2(3). The following gives a more direct and detailed view of this fact.

7.6.7 Proposition.
- If \( \phi : G' \rightarrow F : X \rightarrow A \) and \( \phi' : G' \rightarrow F : X \rightarrow A \) are adjunctions, then there is a unique natural transformation \( \tau : G \rightarrow G' : A \rightarrow X \) such that \( \phi = \tau \circ \phi' \). Moreover, \( \tau \) is an isomorphism (any two right adjoints \( G \) and \( G' \) of a functor \( F : X \rightarrow A \) are thus isomorphic) and commutes with the units and counit of the adjunctions as shown in

\[
\begin{array}{c}
\begin{array}{c}
G \circ F \xrightarrow{\tau \circ F} G' \circ F \\
\epsilon \Downarrow \left[ e' \right] \\
1 \circ A \\
\end{array}
\end{array}
\]

, where \( \epsilon \) and \( \eta \) are the counit and unit of \( \phi \), and \( e' \) and \( \eta' \) are the counit and unit of \( \phi' \).
- If \( \phi : G \rightarrow F : X \rightarrow A \) and \( \phi' : G \rightarrow F : X \rightarrow A \) are adjunctions, then there is a unique natural transformation \( \sigma : F' \rightarrow F : X \rightarrow A \) such that \( \phi = \phi' \circ \sigma \). Moreover, \( \sigma \) is an isomorphism (any two left adjoints \( F \) and \( F' \) of a functor \( G : A \rightarrow X \) are thus isomorphic) and commutes with the units and counit of the adjunctions as shown in

\[
\begin{array}{c}
\begin{array}{c}
G \circ F' \xrightarrow{G \circ \sigma} G \circ F \\
\epsilon' \Downarrow \left[ \epsilon \right] \\
1 \circ A \\
\end{array}
\end{array}
\]

, where \( \epsilon \) and \( \eta \) are the counit and unit of \( \phi \), and \( e' \) and \( \eta' \) are the counit and unit of \( \phi' \).

**Proof.** A unique natural transformation \( \tau : G \rightarrow G' \) is given by the adjunct of \( \phi \) along \( \phi' \) as shown in

\[
\begin{array}{c}
\begin{array}{c}
G' \xrightarrow{\phi'} F \\
\tau : G \xrightarrow{\phi} F \\
\end{array}
\end{array}
\]

; that is; by the conjugate of the identity \( F \rightarrow F \) as shown in

\[
\begin{array}{c}
\begin{array}{c}
G' \xrightarrow{\phi'} F \\
\tau : G \xrightarrow{\phi} F \\
\end{array}
\end{array}
\]
7.7. Remark.

Proposition 7.6.4(3) for the conjugation above shrink to the commutative triangles in the assertion. □

7.7 Adjunctions with parameters

Since, as we saw in Section 7.5, adjunctions $X \rightarrow A$ constitute two-way universal arrows of the module $(X \upharpoonright A)$, we can define a parameterized adjunction as a pointwise lift along $(X \upharpoonright A)$.

7.7.1 Definition.

- Given a covariant functor $F : E \rightarrow [X, A]$ and a contravariant functor $G : E^\rightarrow \rightarrow [A, X]$, an $E$-parameterized adjunction $\phi : G \dashv F : X \rightarrow A$ is defined by a cylinder

$$
\begin{array}{ccc}
G & \xrightarrow{E^-} & F \\
\downarrow{\phi} & & \downarrow{F} \\
[X, A] & \xrightarrow{f} & [X, A] \\
\end{array}
$$

such that each component is an adjunction $\phi_e : G(e) \dashv F(e) : X \rightarrow A$.

- Given a covariant functor $G : E \rightarrow [A, X]$ and a contravariant functor $F : E^\rightarrow \rightarrow [X, A]$, an $E$-parameterized adjunction $\phi : G \dashv F : X \rightarrow A$ is defined by a cylinder

$$
\begin{array}{ccc}
G & \xleftarrow{E^-} & F \\
\uparrow{\phi} & & \uparrow{F} \\
[X, A] & \xleftarrow{f} & [X, A] \\
\end{array}
$$

such that each component is an adjunction $\phi_e : G(e) \dashv F(e) : X \rightarrow A$.

7.7.2 Remark.

1. By Proposition 7.5.4, if $\phi : G \dashv F : X \rightarrow A$ [op. $\phi : G \dashv F : X \rightarrow A$] is an $E$-parameterized adjunction, then the cylinder $\phi : G \dashv F : E^\rightarrow \rightarrow (X \upharpoonright A)$ [op. $\phi : G \dashv F : E^\rightarrow \rightarrow (X \upharpoonright A)$] is pointwise two-way universal.

2. Given a pair of functors $F : E \rightarrow [X, A]$ and $G : E^\rightarrow \rightarrow [A, X]$, a family of adjunctions $\phi_e : G(e) \dashv F(e) : X \rightarrow A$, one for each object $e \in |E|$, forms an $E$-parameterized adjunction $\phi : G \dashv F : X \rightarrow A$ if and only if the square

$$
\begin{array}{ccc}
G(e) & \xrightarrow{\langle X \rangle [G(e)]} & [F(e)](A) \\
\downarrow{G(h)} & & \downarrow{\{F(h)\}(A)} \\
G(e') & \xrightarrow{\langle X \rangle [G(e')]^{\phi_e}} & [F(e')](A) \\
\end{array}
$$

commutes (i.e. $G(h)$ and $F(h)$ are conjugate) for every $E$-arrow $h : e' \rightarrow e$.

- Given a pair of functors $G : E \rightarrow [A, X]$ and $F : E^\rightarrow \rightarrow [X, A]$, a family of adjunctions $\phi_e : G(e) \dashv F(e) : X \rightarrow A$, one for each object $e \in |E|$, forms an $E$-parameterized adjunction $\phi : G \dashv F : X \rightarrow A$ if and only if the square

$$
\begin{array}{ccc}
G(e) & \xrightarrow{\langle X \rangle [G(e)]} & [F(e)](A) \\
\downarrow{G(h)} & & \downarrow{\{F(h)\}(A)} \\
G(e') & \xrightarrow{\langle X \rangle [G(e')]^{\phi_e}} & [F(e')](A) \\
\end{array}
$$

commutes (i.e. $G(h)$ and $F(h)$ are conjugate) for every $E$-arrow $h : e' \rightarrow e$.

3. For each object $e \in |E|$, the counit $\epsilon_e : G(e) \circ F(e) \rightarrow 1_A$ of the adjunction $\phi_e : G(e) \dashv F(e) : X \rightarrow A$
7.7. Adjunctions with parameters 242

\( X \rightarrow A \) is given by the component of the composite cylinder
\[
\begin{array}{c}
\xymatrix{
\text{G} 
\ar[r]^E 
\ar[dr]_{\phi} & F 
\ar[d]^{\parallel} \\
[X,A]^{-} \ar[r]_{X[A]} & [X,A]^{-}
}
\end{array}
\]

at \( e \in \|E\| \) (see Remark 7.5.2(5) for the cell \( \langle X \rightharpoonup A \rangle (X \times A) \)). The family of counits \( \epsilon_e \) is thus natural in \( e \in \|E\| \); that is, the square
\[
\begin{array}{c}
\xymatrix{
G(e) \circ F(e') 
\ar[r]^{G(h) \circ F(e')} 
\ar[d]^{G(e) \circ F(h)} & G(e') \circ F(e') 
\ar[d]^{G(e) \circ F(h)} \\
G(e) \circ F(e) 
\ar[r]_{\epsilon_e} & 1_A
}
\end{array}
\]

commutes for every \( E \)-arrow \( h : e' \rightarrow e \).

For each object \( e \in \|E\| \), the unit \( \eta_e : 1_X \rightarrow G(e) \circ F(e) \) of the adjunction \( \phi_e : G(e) \dashv F(e) : X \rightarrow A \) is given by the component of the composite cylinder
\[
\begin{array}{c}
\xymatrix{
\text{G} 
\ar[r]^E 
\ar[dr]_{\phi} & F 
\ar[d]^{\parallel} \\
[X,A]^{-} \ar[r]_{X[A]} & [X,A]^{-}
}
\end{array}
\]

at \( e \in \|E\| \) (see Remark 7.5.2(5) for the cell \( [X : A] (X \times A) \)). The family of units \( \eta_e \) is thus natural in \( e \in \|E\| \); that is, the square
\[
\begin{array}{c}
\xymatrix{
1_X 
\ar[r]_{\eta_e} & G(e) \circ F(e) 
\ar[d]^{G(e) \circ F(h)} \\
G(e') \circ F(e') 
\ar[r]_{\parallel} & G(e) \circ F(e')
}
\end{array}
\]

commutes for every \( E \)-arrow \( h : e' \rightarrow e \).

(4) The functors \( F : E \rightarrow [X,A] \) and \( G : E^\rightarrow \rightarrow [A,X] \) in an \( E \)-parameterized adjunction \( \phi : G \dashv F : X \rightarrow A \) are often presented as bifunctors \( F : X \times E \rightarrow A \) and \( G : E^\rightarrow \times A \rightarrow X \) and identified with their exponential transposes. We say “adjunction \( \phi_e : G(e,-) \dashv F(-,e) \) is parameterized by \( e \)" when a family of adjunctions \( \phi_e : G(e,-) \dashv F(-,e) \), one for each object \( e \in \|E\| \), forms an \( E \)-parameterized adjunction \( \phi : G \dashv F \).

7.7.3 Proposition. Let \( F : X \times E \rightarrow A \) and \( G : E^\rightarrow \times A \rightarrow X \) be bifunctors, and suppose that there is an adjunction \( \phi_e : G(e,-) \dashv F(-,e) : X \rightarrow A \) parameterized by \( e \) (see Remark 7.7.2(4)). Then for any \( E \)-arrow \( h : e' \rightarrow e \), if the inner parallelogram in
\[
\begin{array}{c}
\xymatrix{
G(e',a) 
\ar[r]_{G(h,a)} 
\ar[d]_{g} & a 
\ar[d]^{f} \\
G(e,a) 
\ar[r]_{F(x,e)} & F(x,e)
}
\end{array}
\]
is an adjunct diagram of \( \phi_e \), the outer rectangle is an adjunct diagram of \( \phi_{e'} \).

Proof. Apply the condition (2) in Proposition 7.6.4 to the conjugate pair \( (G(h), F(h)) \) in Remark 7.7.2(2).

7.7.4 Remark. Proposition 7.7.3 states the naturality of the bijection
\[
x(X)(G(e,a)) \xrightarrow{x(\phi_e)a} (F(x,e))(A)a
\]
in \( e \), i.e. the commutativity of the diagram

\[
\begin{array}{ccc}
e & x(X)(G(e,a)) & e' \\
\downarrow h & \downarrow (F(x,h))(A)a & \downarrow h \\
e' & x(X)(G(e',a)) & e'
\end{array}
\]

for an \( E \)-arrow \( h : e' \to e \).

7.7.5 Theorem. Given a functor \( F : E \to [X, A] \), suppose that there is a family of adjunctions \( \phi_e : G_e \vdash F(e) : X \to A \), one for each object \( e \in |E| \). Then there is a unique functor \( G : E^\sim \to [X, A] \) with \( G(e) = G_e \) such that \( \phi := (\phi_e)_{e \in |E|} \) forms a cylinder \( G \triangleright F : E^\sim \triangleright (X \uparrow A) \), and \( \phi \) is an \( E \)-parameterized adjunction.

7.7.6 Theorem. Let \( \phi : G \vdash F : X \to A \) and \( \phi' : G' \vdash F' : X \to A \) be \( E \)-parameterized adjunctions as in Definition 7.7.1. If \( \tau : G \to G' \) is a natural transformation, then there exists a unique natural transformation \( \tau : G \to G' \) making the square

\[
\begin{array}{ccc}
G & \sim \triangleright F & G' \\
\downarrow \tau & \downarrow \sigma \sim & \downarrow \tau' \sim F'
\end{array}
\]

commute in the module \( (E^\sim, X \uparrow A) : [E^\sim, [X, A]] \to [E^\sim, [X, A]^\sim] \).

Proof. Recalling from Remark 7.7.2(1) that \( \phi \) and \( \phi' \) are (pointwise) two-way universal cylinders \( E^\sim \triangleright (X \uparrow A) \), we see that the assertion is an instance of Theorem 6.5.26 where \( M \) is given by the module \( (X \uparrow A) \).

7.7.7 Remark. The square in Theorem 7.7.6 commutes if and only if the square

\[
\begin{array}{ccc}
G(e) & \sim \triangleright F(e) & G'(e) \\
\downarrow \tau_e & \downarrow \sigma_e \sim & \downarrow \tau'_e \sim F'(e)
\end{array}
\]

commutes in the module \( (X \uparrow A) : [A, X] \to [X, A]^\sim \) for every object \( e \in |E| \) (cf. Remark 6.5.27). Hence the components of \( \tau \) and \( \sigma \) at each \( e \in |E| \) are conjugate in the sense of Definition 7.6.1; Theorem 7.7.6 says that if \( \sigma_e \) \( \text{[op. } \tau_e \text{]} \) is natural in \( e \), so will be \( \tau_e \) \( \text{[op. } \sigma_e \text{]} \).

7.8 Composition of adjunctions

Since an adjunction between two categories is a special instance of a symmetric cell, we can define the composition of two adjunctions as in Section 7.1.

7.8.1 Definition. Given two adjunctions as in

\[
X \xrightarrow{\phi_0 \tau} C \xleftarrow{\phi_1 \tau} A
\]
, their composite is the adjunction
\[ \begin{array}{c}
\xymatrix{X \ar[r]^{G_0 \circ G_1} & \ar[l]_{F_0 \circ F_1} A}
\end{array} \]
defined by the module isomorphism \( \Phi_0 \circ \Phi_1 : (X) [G_0 \circ G_1] \rightarrow [F_0 \circ F_1] (A) : X \rightarrow A \) given by the composition
\[
\langle X \rangle [G_0 \circ G_1] = \langle \langle X \rangle G_0 \rangle G_1 (\Phi_0 G_1) = \langle F_0 (C) \rangle G_1 = F_0 \langle \langle C \rangle G_1 \rangle F_0 \langle F_1 (A) \rangle = [F_0 \circ F_1] (A) .
\]

7.8.2 Remark.
(1) The composition of adjunctions is the special case of the composition of symmetric cells (see Definition 7.1.3). The results we had in Section 7.1 Section 7.2 for the composition of symmetric cells thus also holds for the composition of adjunctions. First of all, the composite \( \Phi_0 \circ \Phi_1 \) is indeed an adjunction by Proposition 7.2.5(2).

(2) By this composition, categories and adjunctions among them form a category, fully embeddable into the category of modules and adjunctions among them (see Remark 7.2.6). For any category \( C \), the identity adjunction \( 1 : 1_C \rightarrow 1_C : C \rightarrow C \) is given by the identity module morphism \( 1 : X \rightarrow X \).

(3) Given a pair of objects \( x \in \| X \| \) and \( a \in \| A \| \), the components of the composite adjunction \( \Phi_0 \circ \Phi_1 \) at \( (x, a) \) is given by the composite function
\[
x (X) (G_0 : G_1 \cdot a) \xrightarrow{x (\Phi_0 (G_1 \cdot a))} (X : F_0) (C) (G_1 : a) \xrightarrow{(x : F_0) (\Phi_1 a)} (x : F_0 : F_1) (A) a
\]; it first sends an \( X \)-arrow \( f : x \rightarrow G_0 : G_1 \cdot a \) to the \( C \)-arrow \( f : \Phi_0 : x : F_0 \rightarrow G_1 : a \), the left adjunct of \( f \) under \( \Phi_0 \), and then sends this \( C \)-arrow to the \( A \)-arrow \( f : \Phi_0 : \Phi_1 : x : F_0 : F_1 \rightarrow a \), the left adjunct of \( f : \Phi_0 \) under \( \Phi_1 \), all as shown in the adjunct diagram
\[
\xymatrix{G_0 : G_1 : a \ar[r] & G_1 : a & a \ar[l] \\
x \ar[r] & x : F_0 \ar[r] & x : F_0 : F_1}
\]

7.8.3 Proposition. Consider
\[
\xymatrix{G_0 : G_1 : a & G_1 : a \ar[l] & a \\
x \ar[r] & x : F_0 \ar[r] & x : F_0 : F_1}
\]
in the situation of Definition 7.8.1. If the left-hand square is an adjunct diagram of \( \Phi_0 \) and the right-hand square is an adjunct diagram of \( \Phi_1 \), then the outer rectangle is an adjunct diagram of \( \Phi_0 \circ \Phi_1 \). Conversely, every adjunct diagram of \( \Phi_0 \circ \Phi_1 \) is given by a composite of adjunct diagrams of \( \Phi_0 \) and \( \Phi_1 \) as above.

Proof. Obvious by Remark 7.8.2(3). \( \square \)

7.8.4 Proposition. Consider
\[
\xymatrix{G_0 : G_1 : a & G_1 : a \ar[l] & a \\
G_0 : k \ar[u] & k \ar[u] & \ar[u] k \\
G_0 : c & c \ar[l] & c : F_1 \ar[l] \\
x \ar[r] & x : F_0 \ar[r] & x : F_0 : F_1}
\]
in the situation of Definition 7.8.1. If the bottom left square is an adjunct diagram of \( \Phi_0 \) and the top right square is an adjunct diagram of \( \Phi_1 \), then the outer square is an adjunct diagram of \( \Phi_0 \circ \Phi_1 \).

Proof. The left-hand (resp. right-hand) side rectangle is an adjunct diagram of \( \Phi_0 \) (resp. \( \Phi_1 \)) by Proposition 7.3.3. Hence the outer square is an adjunct diagram of \( \Phi_0 \circ \Phi_1 \) by Proposition 7.8.3. \( \square \)
7.8.5 Proposition. Consider two adjunctions as in Definition 7.8.1.

- If \( \epsilon_0 : G_0 \circ F_0 \to 1_C \) and \( \epsilon_1 : G_1 \circ F_1 \to 1_A \) are the counits of the adjunctions \( \Phi_0 \) and \( \Phi_1 \), then the counit \( \epsilon : G_1 \circ G_0 \circ F_0 \circ F_1 \to 1_A \) of the composite adjunction \( \Phi_0 \circ \Phi_1 \) is given by the composite

\[
G_1 \circ G_0 \circ F_0 \circ F_1 \xrightarrow{G_1 \circ G_0 \circ F_0 \circ F_1} G_1 \circ 1_C \circ F_1 = G_1 \circ F_1 \to 1_A.
\]

- If \( \eta_0 : 1_X \to G_0 \circ F_0 \) and \( \eta_1 : 1_C \to G_1 \circ F_1 \) are the units of the adjunctions \( \Phi_0 \) and \( \Phi_1 \), then the unit \( \eta : 1_X \to G_0 \circ G_1 \circ F_0 \circ F_1 \) of the composite adjunction \( \Phi_0 \circ \Phi_1 \) is given by the composite

\[
1_X \xrightarrow{\eta_0} G_0 \circ F_0 = G_0 \circ 1_C \circ F_0 \xrightarrow{G_0 \circ G_0 \circ F_0} G_0 \circ G_1 \circ F_1 \circ F_0.
\]

Proof. By Remark 7.3.5(1a), the component of \( \epsilon \) at \( a \in \mathcal{A} \) is given by

\[
a \cdot \epsilon = 1_{(a \cdot G_1 \cdot G_0)} \cdot (\Phi_0 \circ \Phi_1)
\]

but

\[
1_{(a \cdot G_1 \cdot G_0)} \cdot (\Phi_0 \circ \Phi_1) = (1_{(a \cdot G_1 \cdot G_0)} \cdot \Phi_0) \circ \Phi_1
= (a \cdot G_1 \cdot \epsilon_0) \circ \Phi_1
= (a \cdot G_1 \cdot \epsilon_0 \circ F_1) \circ (a \cdot \epsilon_1)
\]

\((\ast') \) by Remark 7.8.2(3); \((\ast') \) by Remark 7.3.5(1a); \((\ast') \) by Remark 7.3.5(1b)) as shown in the adjunct diagram

![Adjunct Diagram](attachment:image.png)

; hence we have

\[
a \cdot \epsilon = (a \cdot G_1 \cdot \epsilon_0 \circ F_1) \circ (a \cdot \epsilon_1) = a \cdot ((G_1 \circ \epsilon_0 \circ F_1) \circ \epsilon_1)
\]

as required.

Note. A composable pair of adjunctions gives rise to a lax morphism (see Definition 7.4.1) of adjunctions.

7.8.6 Theorem. Given a composable pair of adjunctions

\[
\begin{array}{ccc}
X & \xrightarrow{G_0} & C \\
\Phi_0 \circ \Phi_1 & \xleftarrow{F_0} & \Phi_1 \circ \Phi_0 & \xrightarrow{F_1} & A
\end{array}
\]

- the natural transformations

\[
\begin{array}{ccc}
C & \xrightarrow{G_1} & A \\
G_0 \downarrow & 1 \downarrow & 1 \downarrow \\
X & \xrightarrow{G_0 \circ G_1} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C & \xrightarrow{F_1} & A \\
G_0 \downarrow & \epsilon_0 \circ F_1 \downarrow & 1 \downarrow \\
X & \xrightarrow{G_0 \circ G_1} & A
\end{array}
\]

, where \( \epsilon_0 \) is the counit of \( \Phi_0 \), form a lax morphism \( \Phi_1 \to \Phi_0 \circ \Phi_1 \).

- the natural transformations

\[
\begin{array}{ccc}
X & \xrightarrow{G_0} & C \\
1 \downarrow & G_0 \circ \eta_1 \downarrow & 1 \downarrow \\
X & \xrightarrow{G_0 \circ G_1} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & \xrightarrow{F_0} & C \\
1 \downarrow & \epsilon_0 \circ F_1 \downarrow & 1 \downarrow \\
X & \xrightarrow{F_0 \circ F_1} & A
\end{array}
\]

, where \( \eta_1 \) is the units of \( \Phi_1 \), form a lax morphism \( \Phi_0 \to \Phi_0 \circ \Phi_1 \).

Proof. Let \( \epsilon \) be the counit of the composite adjunction \( \Phi_0 \circ \Phi_1 \). Then by Proposition 7.8.5, the
square
\[
\begin{array}{ccc}
G_1 & G_0 & F_0 & F_1 \\
\downarrow & & & \downarrow 1 \\
G_1 & G_0 & F_0 & F_1 \\
\end{array}
\]
commutes. Hence the identity \( G_1 \circ G_0 \to G_1 \circ G_0 \) and the natural transformation \( \epsilon_0 \circ F_1 : G_0 \circ F_0 \circ F_1 \to F_1 \) satisfy the condition (3) in Proposition 7.4.4.

\[\square\]

7.8.7 Corollary. Given a composable triple of adjunctions

\[
\begin{array}{ccc}
X' & \overset{G_0}{\to} & X & \overset{G}{\to} & A & \overset{\phi_1 \circ \tau}{\to} & A' \\
F_0 & \overset{\phi_0 \circ \tau}{\to} & F & \overset{F_1}{\to} & \end{array}
\]

, the natural transformations

\[
\begin{array}{ccc}
X & \overset{G}{\to} & A & \overset{\phi_1 \circ \tau}{\to} & A' \\
G_0 & \overset{G_0 \circ G \circ \eta_1}{\to} & F_0 & \overset{F_1}{\to} & A'. \\
\end{array}
\]

, where \( \eta_1 \) is the unit of \( \phi_1 \) and \( \epsilon_0 \) is the counit of \( \phi_0 \), form a lax morphism \( \phi \to \phi_0 \circ \phi \circ \phi_1 \).

\[\text{Proof.} \quad \text{Since the natural transformations } G_0 \circ G \circ \eta_1 \text{ and } \epsilon_0 \circ G \circ F_1 \text{ are decomposed into}
\]

\[
\begin{array}{ccc}
X & \overset{G}{\to} & A & \overset{\phi_1 \circ \tau}{\to} & A' \\
G_0 & \overset{1}{\to} & 1 & \overset{F_1}{\to} & A'. \\
\end{array}
\]

respectively, the assertion follows from Theorem 7.8.6 on recalling Proposition 7.4.2.

\[\square\]

7.8.8 Corollary. If functors \( P : X \to Y \) and \( Q : A \to B \) are isomorphisms of categories, then any adjunction \( \phi : G \dashv F : X \to A \), if exists, induces an adjunction \( \phi' : P \circ G \circ Q^{-1} \dashv P^{-1} \circ F \circ Q : Y \to B \) such that \( P \) and \( Q \) form a strict morphism (see Definition 7.4.9) \( \phi \to \phi' \).

\[\text{Proof.} \quad \text{Noting Example 7.3.6, we have a composable triple of adjunctions}
\]

\[
\begin{array}{ccc}
Y & \overset{\phi_0 \circ \tau}{\to} & X & \overset{G}{\to} & A & \overset{Q^{-1}}{\to} & B \\
\overset{p^{-1}}{\overset{p^{-1}}{\to}} & \overset{F}{\to} & \overset{F_1}{\to} & \overset{Q}{\to} & \end{array}
\]

with the unit and counit of \( \phi_0 \) and \( \phi_1 \) given by the identity natural transformations. The assertion thus follows from Corollary 7.8.7.

\[\square\]

Note. The following result involves mates (see Definition 7.4.5) and conjugates (see Definition 7.6.1).

7.8.9 Theorem. Given adjunctions

\[
\begin{array}{ccc}
X & \overset{\phi_0 \circ \tau}{\to} & C & \overset{G}{\to} & A \\
F_0 & \overset{F_0 \circ \tau}{\to} & F_1 & \overset{G_0 \circ \tau}{\to} & A \\
\end{array}
\]

and natural transformations

\[
\begin{array}{ccc}
C & \overset{C}{\to} & A \\
G_0 & \overset{G_0 \circ \tau}{\to} & G_1 & \overset{G_0 \circ \tau}{\to} & G_1' & \overset{G_0 \circ \tau}{\to} & \end{array}
\]

, suppose that \( (\alpha, \tau) \) are mates with respect to the adjunctions \( \phi_1 \) and \( \phi'_0 \) and that \( (\alpha, \sigma) \) are mates with respect to the adjunctions \( \phi_0 \) and \( \phi'_1 \). Then \( (\tau, \sigma) \) are conjugate along the composite adjunctions.
\( \phi_0 \circ \Phi_1 \) and \( \Phi_0' \circ \Phi'_1 \) as shown in the commutative diagram
\[
\begin{array}{ccc}
G_1 \circ G_0 & \xrightarrow{\tau} & G'_1 \circ G'_0 \\
\circ G_0 \circ \Phi_0 & \xrightarrow{\phi_1} & F_0 \circ F_1 \\
\end{array}
\]
\[
\begin{array}{ccc}
G'_1 \circ G'_0 & \xrightarrow{\tau'} & F'_1 \circ F'_0 \\
\circ G'_0 \circ \Phi'_1 & \xrightarrow{\phi'_1} & F'_0 \circ F'_1 \\
\end{array}
\]

**Proof.** Let \((\eta_i, \epsilon_i)\) and \((\eta'_i, \epsilon'_i)\) be the unit and counit of \(\Phi_i\) and \(\Phi'_i\) respectively. By the condition (4) in Proposition 7.4.4, the diagrams
\[
\begin{array}{ccc}
G_1 \circ G_0 & \xrightarrow{\tau} & G'_1 \circ G'_0 \\
\circ G_0 \circ \Phi_0 & \xrightarrow{\phi_1} & F_0 \circ F_1 \\
\end{array}
\]
commute, so we have the commutative diagram
\[
\begin{array}{ccc}
G_1 \circ G_0 & \xrightarrow{\tau} & G'_1 \circ G'_0 \\
\circ G_0 \circ \Phi_0 & \xrightarrow{\phi_1} & F_0 \circ F_1 \\
\end{array}
\]
for \((\eta, \epsilon)\) and \((\eta', \epsilon')\) being the unit and counit of the composite adjunctions \(\Phi_0 \circ \Phi_1\) and \(\Phi'_0 \circ \Phi'_1\), and recalling Proposition 7.8.5, we then have the commutative diagram
\[
\begin{array}{ccc}
G_1 \circ G_0 & \xrightarrow{\tau} & G'_1 \circ G'_0 \\
\circ G_0 \circ \Phi_0 & \xrightarrow{\phi_1} & F_0 \circ F_1 \\
\end{array}
\]
in the condition (4) in Proposition 7.6.4.

**7.8.10 Theorem.** Consider adjunctions
\[
\begin{array}{ccc}
X \xrightarrow{\phi_0 \circ \tau} C \xrightarrow{\Phi_1 \circ \tau} A \\
\circ \Phi_0 \circ \phi_0 \circ \tau_0 \xrightarrow{F_0} F'_0 \\
\end{array}
\]
and conjugate pairs of natural transformations
\[
\left( G_0 \xrightarrow{\tau_0} G_0, F_0 \xrightarrow{\sigma_0} F_0 \right) \quad \left( G_1 \xrightarrow{\tau_1} G_1, F_1 \xrightarrow{\sigma_1} F_1 \right)
\]
making the squares
\[
\begin{array}{ccc}
G_0 \xrightarrow{\phi_0} G_0' \\
\circ \phi_0 \xrightarrow{\tau_0} \sigma_0 \\
\end{array} \quad \begin{array}{ccc}
G_1 \xrightarrow{\phi_1} \Phi_1 \\
\circ \phi_1 \xrightarrow{\tau_1} \sigma_1 \\
\end{array}
\]
commute. Then the composite natural transformations \(\tau_0 \circ \tau_1\) and \(\sigma_0 \circ \sigma_1\) yield a conjugate pair for the composite adjunctions as shown in the commutative square
\[
\begin{array}{ccc}
G_0 \circ G_1 \xrightarrow{\phi_0 \circ \Phi_1} F_0 \circ F_1 \\
\circ G_0 \circ \phi_0 \circ \Phi_1 \xrightarrow{\tau_0 \circ \tau_1} \sigma_0 \circ \sigma_1 \\
\end{array}
\]

**Proof.** From the commutative squares
\[
\begin{array}{ccc}
G_0 \xrightarrow{\phi_0} F_0 \circ (C) \\
\circ \phi_0 \circ \phi_0 \circ \tau_0 \xrightarrow{\tau_0} \sigma_0 \\
\end{array} \quad \begin{array}{ccc}
G_1 \xrightarrow{\phi_1} F_1 \circ (A) \\
\circ \phi_1 \circ \tau_1 \xrightarrow{\tau_1} \sigma_1 \\
\end{array}
\]
\[
\begin{array}{ccc}
G'_0 \xrightarrow{\phi'_0} F'_0 \circ (C) \\
\circ \phi'_0 \circ \phi'_0 \circ \tau_0 \xrightarrow{\tau_0} \sigma_0 \\
\end{array} \quad \begin{array}{ccc}
G'_1 \xrightarrow{\phi'_1} F'_1 \circ (A) \\
\circ \phi'_1 \circ \tau_1 \xrightarrow{\tau_1} \sigma_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
G_0 \circ G_1 \xrightarrow{\phi_0 \circ \Phi_1} F_0 \circ F_1 \\
\circ G_0 \circ \phi_0 \circ \Phi_1 \xrightarrow{\tau_0 \circ \tau_1} \sigma_0 \circ \sigma_1 \\
\end{array}
\]
of the given conjugate pairs, we can construct the commutative diagram

\[
\begin{array}{ccc}
G_0 \circ G_1 & \rightarrow & F_0 \circ F_1 \\
(\phi_0 \circ \phi_1) & \rightarrow & [\sigma_0 \circ \sigma_1]
\end{array}
\]

for all vertical arrows are the forgetful functors. Hence, the composition of adjunctions makes \( C \rightarrow A \) defines a bifunctor

\[
\text{Adj}[X \uparrow C] \times \text{Adj}[C \uparrow A] \rightarrow \text{Adj}[X \uparrow A]
\]

on the adjunction categories (see Remark 7.6.2(2)) such that the diagram

\[
\begin{array}{ccc}
[A, C] \times [C, X] & \rightarrow & [A, X] \\
\uparrow & & \uparrow \\
\text{Adj}[X \uparrow C] \times \text{Adj}[C \uparrow A] & \rightarrow & \text{Adj}[X \uparrow A] \\
\downarrow & & \downarrow \\
[X, C]^\ast \times [C, A]^\ast & \rightarrow & [X, A]^\ast
\end{array}
\]

commutes, where the top and bottom horizontal arrows are the bifunctors on the functor categories and all vertical arrows are the forgetful functors.

**7.8.12 Corollary.** Given a category \( C \), the composition of adjunctions makes \( \text{Adj}[C \uparrow C] \) a strict monoidal category with the unit given by the identity adjunction \( 1 : 1_C : C \rightarrow C \), and the forgetful functors \( \text{Adj}[C \uparrow C] \rightarrow [C, C] \) and \( \text{Adj}[C \uparrow C] \rightarrow [C, C]^\ast \) [op. \( \text{Adj}[C \uparrow C] \rightarrow [C, C]^\ast \) and \( \text{Adj}[C \uparrow C] \rightarrow [C, C]^\ast \)] will be monoidal.

*Proof.* Setting \( X = A = C \) in Remark 7.8.11, we see that the composition of adjunctions \( X \rightarrow C \) defines a bifunctor \( \text{Adj}[C \uparrow C] \times \text{Adj}[C \uparrow C] \rightarrow \text{Adj}[C \uparrow C] \) and the forgetful functors \( \text{Adj}[C \uparrow C] \rightarrow [C, C] \) and \( \text{Adj}[C \uparrow C] \rightarrow [C, C]^\ast \) preserves the composition. Now noting Remark 7.8.2(2), we see that the composition of adjunctions makes \( \text{Adj}[C \uparrow C] \) a strict monoidal category with the unit given by the identity adjunction. Clearly, the forgetful functors preserve the unit. \( \square \)

**7.9 Adjoint monads**

In this section, we look at Eilenberg and Moore’s theorem on adjoint monads as an application of conjugation for adjunctions and composition studied in Section 7.6 and Section 7.8 respectively.

*Note.* In the following, we consider a monoid on the monoidal category \( \text{Adj}[C \uparrow C] \) (see Corollary 7.8.12).
7.9.1 Theorem. Given an adjunction $\phi : G \dashv F : C \rightarrow C$ between two endofunctors, consider a pair of conjugations

\[
\begin{array}{c}
G^2 \xrightarrow{\phi^2} F^2 \\
\downarrow \delta^2 \\
G \xrightarrow{\phi} F \\
\downarrow \epsilon \\
1 \xrightarrow{1} 1
\end{array}
\]

along the module $(C \downarrow C) : [C, C]^* \rightarrow [C, C]$, where $\delta^* : G^2 \rightarrow G$ and $\epsilon^* : 1 \rightarrow G$ denote the opposite of $\delta : G \rightarrow G^2$ and the opposite of $\epsilon : G \rightarrow 1$ respectively. Then the following conditions are equivalent:

1. $(\phi, (\delta, \eta), (\epsilon, \mu))$ is a monoid on $\text{Adj}(C \downarrow C)$.
2. $(G, \delta^*, \epsilon^*)$ is a monoid on $[C, C]^*$, i.e. $(G, \delta, \epsilon)$ is a comonad on $C$;
3. $(F, \eta, \mu)$ is a monoid on $[C, C]$, i.e. $(F, \eta, \mu)$ is a monad on $C$.

Proof. Since the forgetful functors $\text{Adj}(C \downarrow C) \rightarrow [C, C]^*$ and $\text{Adj}(C \downarrow C) \rightarrow [C, C]$ are monoidal (see Corollary 7.8.12) and faithful (in fact, fully faithful), they preserve and reflect monoid structures. Hence $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (3)$.

7.9.2 Theorem. Given an adjunction $\phi : G \dashv F : C \rightarrow C$ between two endofunctors, consider a monad $(F, \eta, \mu)$ and a comonad $(G, \delta, \epsilon)$ corresponding to each other by conjugation as in Theorem 7.9.1. Then there is the canonical isomorphism $L$ from the category $C^F$ of $F$-algebras to the category $C_G$ of $G$-coalgebras, which commutes with the respective forgetful functors, as indicated in the commutative diagram

\[
\begin{array}{ccc}
C^F & \xrightarrow{L} & C_G \\
\downarrow & & \downarrow \\
C & & C
\end{array}
\]

Proof. By the lemma below, $L$ is defined such that it sends each $F$-algebra $(c, h)$ to the $G$-coalgebra $(c, k)$ with $k : c \rightarrow G \cdot c$ given by the adjunct of $h : c : F \rightarrow c$ and sends each $F$-algebra morphism $f : (c, h) \rightarrow (c', h')$ given by $f : c \rightarrow c'$ to the $G$-coalgebra morphism $f : (c, k) \rightarrow (c', k')$ given by $f : c \rightarrow c'$.

Lemma.

1. Consider an adjunct diagram

\[
\begin{array}{cc}
G \cdot c & \xleftarrow{c} \\
| & \downarrow h \\
k | & \downarrow \delta \\
c & \xleftarrow{\phi} c : F
\end{array}
\]

of $\phi$. Then $(c, h)$ is an $F$-algebra if and only if $(c, k)$ is a $G$-coalgebra.

2. Consider a pair of $F$-algebras $(c, h)$ and $(c', h')$ and the corresponding $G$-coalgebras $(c, k)$ and $(c', k')$, then a $C$-arrow $f : c \rightarrow c'$ is an $F$-algebra morphism $(c, h) \rightarrow (c', h')$ if and only if it is a $G$-coalgebra morphism $(c, k) \rightarrow (c', k')$.

Proof.

1. The outer squares of

\[
\begin{array}{c}
G^2 \cdot c \xleftarrow{\phi^2} c \\
\downarrow \delta^2 \\
G \cdot c \xleftarrow{\phi} c : F \\
\downarrow k \\
e \cdot \mu \\
\end{array}
\] 

\[
\begin{array}{c}
G^2 \cdot c \xleftarrow{\phi^2} c \\
\downarrow \delta^2 \\
G \cdot c \xleftarrow{\phi} c : F \\
\downarrow k \\
e \cdot \mu \\
\end{array}
\]

are adjoint diagrams of $\phi^2$ by the condition (2) in Proposition 7.6.4 and Proposition 7.8.4 re-
7.10 Adjoints of a cell

We define adjoints of a cell using the notion of adjunction between modules. Theorem 7.10.6 has many applications in the sequel including RAPL (right adjoints preserve limits).

Note. An adjoint of a cell \( \psi : \mathcal{M} \to \mathcal{N} \) is defined below by an adjoint (in the sense of Definition 7.2.1) of the collage functor \([\psi] : [\mathcal{M}] \to [\mathcal{N}]\) (see Definition 3.1.9) along the collage envelopes (see Definition 3.1.18) of \(\mathcal{M}\) and \(\mathcal{N}\).

7.10.1 Definition. Given a cell \(\begin{array}{c} \mathcal{A} \xrightarrow{\psi} \mathcal{B} \\ \mathcal{M} \xrightarrow{\psi} \mathcal{N} \end{array}\) (an ordinary cell defined in Definition 1.2.1), a right [op. left] adjoint \((R, \phi)\) of the collage functor \([\psi]\) along the left [op. right] collage envelopes of
A right adjoint of a functor $\mathcal{M}$ and $\mathcal{N}$ (shown in

$$
\begin{align*}
\mathcal{M} & \xrightarrow{\phi} \mathcal{N} \\
[M] & \xrightarrow{[\psi]} [N]
\end{align*}
$$

) is called a right (op. left) adjoint of $\psi$.

7.10.2 Remark.

(1) If $(R, \phi)$ is a right adjoint of $\psi$, then the restrictions of $\phi$ to $X$ and $A$ yield right adjoints

$$
\begin{align*}
A & \xleftarrow{R} B \\
\mathcal{M} & \xrightarrow{\phi} \mathcal{N} \\
X & \xleftarrow{p} Y
\end{align*}
$$

of $P$ and $Q$. Conversely, a pair of right adjoints

$$
\begin{align*}
A & \xleftarrow{R} B \\
\mathcal{M} & \xrightarrow{\phi_0} \mathcal{N} \\
X & \xleftarrow{p} Y
\end{align*}
$$

form a right adjoint of $\psi$ if they satisfy the following naturality condition for each object $b \in \mathcal{B}$ and each $M$-arrow $k : x \sim a$: for any $M$-arrow $m : a \rightarrow R \cdot b$ as in

$$
\begin{align*}
\xymatrix{a & m \ar[r] & R \cdot b \\
& x \ar[ur]_{k} &}
\end{align*}
$$

, the triangle

$$
\begin{align*}
\xymatrix{a \cdot Q & m : \phi_1 \ar[r] & b \\
k : \psi \ar[ur]_{(k \cdot m) : \phi_0} & x : P}
\end{align*}
$$

commutes (cf. Remark 7.2.2(3)).

(2) If $(R, \phi)$ is a left adjoint of $\psi$, then the restrictions of $\phi$ to $X$ and $A$ yield left adjoints

$$
\begin{align*}
X & \xrightarrow{p} Y \\
\mathcal{M} & \xleftarrow{\phi} \mathcal{N} \\
A & \xleftarrow{q} B
\end{align*}
$$

of $P$ and $Q$. Conversely, a pair of left adjoints

$$
\begin{align*}
X & \xrightarrow{p} Y \\
\mathcal{M} & \xleftarrow{\phi_0} \mathcal{N} \\
A & \xleftarrow{q} B
\end{align*}
$$

form a left adjoint of $\psi$ if they satisfy the following naturality condition for each object $y \in \mathcal{Y}$ and each $M$-arrow $k : x \sim a$: for any $M$-arrow $n : y : R \rightarrow x$ as in

$$
\begin{align*}
\xymatrix{a & n \cdot k \ar[r] & R \cdot y \\
& x \ar[ur]_{n} &}
\end{align*}
$$

, the triangle

$$
\begin{align*}
\xymatrix{a \cdot Q & n : \phi_1 \ar[r] & y \\
k : \phi \ar[ur]_{(n \cdot k) : \phi_0} & x : P}
\end{align*}
$$

commutes (cf. Remark 7.2.2(3)).

(2) A right adjoint of a functor $Q : A \rightarrow B$ is the same thing as a right adjoint of the hom-cell
(Q) : (A) → (B) (to see this, replace \[ \xrightarrow{\mathbb{M}_1^A} \xrightarrow{\psi} \xrightarrow{\mathbb{N}} \] with \[ \xrightarrow{A} \xrightarrow{\psi} \xrightarrow{B} \) in the argument of (1) above). Dually, a left adjoint of a functor P : X → Y is the same thing as a left adjoint of the hom-cell (P) : (X) → (Y).

**7.10.3 Theorem.** If a functor R : B → A [op. R : Y → X] is a right [op. left] adjoint of a cell \[ \xrightarrow{A} \xrightarrow{\psi} \xrightarrow{B} \), then the postcomposition functor \[ [D, R] : [D, B] \rightarrow [D, A] [op. [E, R] : [E, Y] \rightarrow [E, X]] \] gives a right [op. left] adjoint of the postcomposition cell

\[
\begin{align*}
[D, A] & \xrightarrow{[D, Q]} [D, B] \\
(E, X) & \xrightarrow{[E, P]} [E, Y]
\end{align*}
\]

(see Definition 1.2.25) for any module \( \mathcal{J} : E \rightarrow D \).

**Proof.** Suppose that a cell \( \psi \) has a right adjoint \((R, \phi)\). The postcompositions with the adjunctive symmetric cells \( X(\phi) \) and \( A(\phi) \) in Remark 7.10.2(1) yield the adjunctive symmetric cells

\[
\begin{align*}
[D, A] & \xrightarrow{[D, R]} [D, B] \\
(E, X) & \xrightarrow{[E, P]} [E, Y]
\end{align*}
\]

by Proposition 7.2.8 and Proposition 7.2.10. The proof will be complete if we show that this pair or right adjoints satisfy the naturality condition in Remark 7.10.2(1) for each functor S : D → B and each cell \( \theta : G \rightarrow F : \mathcal{J} \rightarrow \mathcal{M} \), i.e. if we show that for any natural transformation \( \sigma : F \rightarrow R \circ S : D \rightarrow A \) as in

\[
\begin{array}{c}
F \xrightarrow{\sigma} R \circ S \\
\downarrow \phi \\
G \xrightarrow{\theta \circ \phi}
\end{array}
\]

, the triangle

\[
\begin{array}{c}
F \circ Q \xrightarrow{\sigma \circ \phi} S \\
\downarrow \theta \circ \phi \\
G \circ P
\end{array}
\]

commutes. But this is reduced to the commutativity of the triangle

\[
\begin{array}{c}
d : [F \circ Q] \xrightarrow{\sigma \circ \phi} S \circ d \\
j : [\theta \circ \phi] \\
e : [G \circ P]
\end{array}
\]

, i.e.

\[
\begin{array}{c}
d : F \circ Q \xrightarrow{\sigma \circ \phi} S \circ d \\
j : \theta \circ \phi \\
e : G \circ P
\end{array}
\]

(see Remark 1.2.7 and Remark 4.3.16(1)) for each \( \mathcal{J} \)-arrow \( j : e \rightarrow d \), i.e. to the naturality of \( \phi \).

**7.10.4 Corollary.** If a cell \( \psi \) has a right [op. left] adjoint, so does the postcomposition cell \( \langle \mathcal{J}, \psi \rangle \) in Definition 1.4.12.

**Proof.** By the identification in Remark 1.4.13(1), this follows from Theorem 7.10.3. □
7.10.5 Corollary. If a cell \( \psi \) has a right \([\text{op. left}]\) adjoint, each of the following postcomposition cells (see Definition 4.3.7, Definition 4.3.17, and Definition 4.6.17) has a right \([\text{op. left}]\) adjoint as well.

- \( \langle K \cdot \psi \rangle \) and \( \langle K \cdot \psi \rangle \),
- \( \langle E, \psi \rangle \),
- \( \langle \ast E, \psi \rangle \) and \( \langle E \ast, \psi \rangle \).

Proof. By the naturality square in Theorem 5.5.1, the cell \( \langle K \cdot \psi \rangle \) is identified with the cell \( \langle (E)K, \psi \rangle \). Hence \( \langle K \cdot \psi \rangle \) has an adjoint by Theorem 7.10.3. Since \( \langle E, \psi \rangle = \langle 1_E, \psi \rangle \) (see Remark 4.5.8(2)) and \( \langle \ast E, \psi \rangle = \langle 1_E, \psi \rangle \) (see Corollary 4.6.23), \( E, \psi \rangle \) and \( \ast E, \psi \rangle \) have an adjoint as well. The same holds for \( \langle K \cdot \psi \rangle \) and \( \langle E \ast, \psi \rangle \) by duality. \( \square \)

7.10.6 Theorem. If a cell has a left \([\text{op. right}]\) adjoint, then it preserves inverse \([\text{op. direct}]\) universal arrows.

Proof. A left adjoint \( (R, \phi) \) of \( \psi \) in Definition 7.10.1 is depicted more elaborately as

\[
\begin{array}{c}
\xymatrix{
X & (X \downarrow M) \\
\downarrow & \\
[Y(N)] & [M] \\
\downarrow & \\
[N] & [\psi]
}
\end{array}
\]

, and the right exponential transposition yields a natural isomorphism \( \phi \Downarrow \) as in

\[
\begin{array}{c}
\xymatrix{
[X:] & (X \downarrow M) \Downarrow \\
\downarrow & \\
[Y(N)] & [M] \\
\downarrow & \\
[N] & [\psi]
}
\end{array}
\]

by Proposition 2.1.6 and Proposition 2.1.7. Let \( u : r \to a \) be an inverse universal \( M \)-arrow. Then \( [(X \downarrow M) \Downarrow] \cdot u \) is an isomorphism by the very definition of inverse universality, and so is its image \( [R:] \cdot [(X \downarrow M) \Downarrow] \cdot u \) under \( [R:] \) because any functor preserves isomorphisms. Now since \( \phi \Downarrow \) is a natural isomorphism and \( [R:] \cdot [(X \downarrow M) \Downarrow] \cdot u \) is an isomorphism, \( [(Y \uparrow N) \Downarrow] \cdot [\psi] \cdot u \) is an isomorphism as well; that is, \( [\psi] \cdot \cdot u = \psi \cdot u \) is inverse universal. \( \square \)

7.10.7 Corollary. If a cell has a left \([\text{op. right}]\) adjoint, then it preserves lifts \([\text{op. colifts}]\) and pointwise lifts \([\text{op. colifts}]\). Specifically, if a cell \( X \xrightarrow{M} A \) has a left \([\text{op. right}]\) adjoint, then

- if a cylinder \( X \xrightarrow{\mu} A \) is inverse universal (resp. pointwise inverse universal), so is its composite \( Y \xrightarrow{\mu \cdot \psi} B \) with \( \psi \).
- if a cylinder \( X \xrightarrow{\mu} A \) is direct universal (resp. pointwise direct universal), so is its composite \( Y \xrightarrow{\mu \cdot \psi} B \) with \( \psi \).
7.11. Equivalence of categories

Proof. Since an inverse universal cylinder $\mu : E \to \mathcal{M}$ is the same thing as an inverse universal $(E,\mathcal{M})$-arrow, the assertion is equivalent to saying that if a cell $\psi$ has a left adjoint, then the postcomposition cell $(E,\psi)$ preserves inverse universal arrows. But this follows from Theorem 7.10.6 because if $\psi$ has a left adjoint, so does $(E,\psi)$ by Corollary 7.10.5. The pointwise version follows immediately from Theorem 7.10.6 on noting that the component of the composite cylinder $\mu \circ \psi$ at $e \in \|E\|$ is given by $[\mu \circ \psi]_e = \mu_e \circ \psi$.

7.11.1 Definition. A module $M : X \to A$ is called an equivalence provided that
- for each object $a \in \|A\|$ there exist an object $x \in \|X\|$ and a two-way universal $M$-arrow $u : x \to a$, and
- for each object $x \in \|X\|$ there exist an object $a \in \|A\|$ and a two-way universal $M$-arrow $u : x \to a$.

7.11.2 Proposition. If a module $M : X \to A$ is an equivalence, so is any module $X \to A$ isomorphic to $M$.

Proof. Evident since any module isomorphism preserves universal arrows.

7.11.3 Definition. A functor $K : D \to E$ is called an equivalence provided that it is fully faithful and essentially surjective.

7.11.4 Proposition. If a functor $K : D \to E$ is an equivalence, then so is any functor $D \to E$ isomorphic to $K$.

Proof. If a functor $K : D \to E$ is fully faithful (resp. essentially surjective), then so is any functor $D \to E$ isomorphic to $K$.

7.11.5 Proposition. The composite of two equivalence functors is again equivalence.

Proof. The composite of two fully faithful (resp. essentially surjective) functors is again fully faithful (resp. essentially surjective).

7.11.6 Remark. The converse is not the case: two equivalences $D \to E$ need not be isomorphic. To see this, take $D$ and $E$ to be discrete; in this case, any surjective functor $D \to E$ is an equivalence, but two functors $D \to E$ are isomorphic only when they are identical.

7.11.7 Theorem.
- If a functor $G : A \to X$ is an equivalence, then its corepresentable module $(X)G : X \to A$ is an equivalence.
- If a functor $F : X \to A$ is an equivalence, then its representable module $F(A) : X \to A$ is an equivalence.

Proof. By Theorem 6.2.21, for each $a \in \|A\|$, the identity $X$-arrow $a : G \to G \cdot a$ gives a two-way universal $(X)G$-arrow $a : G \to a$, and for each $x \in \|X\|$, an iso $X$-arrow $u : x \to G \cdot a$ gives a two-way universal $(X)G$-arrow $u : x \to a$.

7.11.8 Remark. We will see that the converse also holds in Corollary 7.11.11.
Note. The axiom of choice is used in the proof of the following.

7.11.9 Theorem. Any equivalence module $\mathcal{M} : X \to A$ has
- a counit $\rho : G \to \mathcal{M}$ with $G : A \to X$ an equivalence functor.
- a unit $\lambda : \mathcal{M} \to F$ with $F : X \to A$ an equivalence functor.

Proof. Choose a two-way universal $\mathcal{M}$-arrow $\rho_a : r_a \to a$ for each $a \in |A|$. By Theorem 6.4.10, there is a functor $G : A \to X$ such that $\rho := (\rho_a)_{a \in |A|}$ forms a counit $\rho : G \to \mathcal{M}$. We claim that $G$ is an equivalence. By Theorem 6.4.13, $G$ is fully faithful. It remains to show that $G$ is essentially surjective. Let $x \in |X|$. Since $\mathcal{M}$ is an equivalence, there exist $a \in |A|$ and a two-way universal $\mathcal{M}$-arrow $u : x \to a$, and because $\rho_a : a : G \to a$ is an inverse universal $\mathcal{M}$-arrow by Proposition 6.4.3, we have $x \cong a : G$ by Corollary 6.2.8.

7.11.10 Corollary.
- Suppose that a module $\mathcal{M} : X \to A$ has a counit $X \xrightarrow{G} A$. Then $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\rho$ is two-way universal.
- Suppose that a module $\mathcal{M} : X \to A$ has a unit $X \xrightarrow{F} A$. Then $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence. Moreover, if these equivalent conditions hold, then each component of $\lambda$ is two-way universal.

Proof. Suppose that $\mathcal{M}$ is an equivalence. Then, by Theorem 7.11.9, there is a counit $\rho' : G' \to \mathcal{M}$ with $G'$ an equivalence. Since $G \cong G'$ by Theorem 6.4.8, $G$ is an equivalence by Proposition 7.11.4. Conversely, suppose that $G$ is an equivalence. Then the corepresentable module $(X) \rightarrow G$ is an equivalence by Theorem 7.11.7. Since $\rho$ yields $\mathcal{M} \cong (X) \rightarrow G$ (see Remark 6.4.2), $\mathcal{M}$ is an equivalence by Proposition 7.11.2. The second assertion follows from Theorem 6.4.13.

7.11.11 Corollary.
- Suppose that a module $\mathcal{M} : X \to A$ is corepresented by a functor $G : A \to X$. Then $\mathcal{M}$ is an equivalence if and only if $G$ is an equivalence.
- Suppose that a module $\mathcal{M} : X \to A$ is represented by a functor $F : X \to A$. Then $\mathcal{M}$ is an equivalence if and only if $F$ is an equivalence.

Proof. Since a corepresenting functor has a counit associated with it (see Remark 6.4.2), this is immediate from Corollary 7.11.10.

7.11.12 Definition. An equivalence of categories $(G,F,\eta,\epsilon) : X \cong A$ consists of a pair of functors

$$X \xrightarrow{G} A$$

and a pair of natural isomorphisms

$$\eta : 1_X \to G \circ F : X \to X \quad \epsilon : G \circ F \to 1_A : A \to A$$

in this situation, the functors $G$ and $F$ are said to be quasi-inverse to each other. Two categories $X$ and $A$ are called equivalent, written $X \cong A$, when there is an equivalence of categories between them.

7.11.13 Remark.
(1) Quasi-inverses are unique up to isomorphism: if $(G,F,\eta,\epsilon) : X \cong A$ and $(G,F',\eta',\epsilon') : X \cong A$ are equivalences of categories, then $F$ and $F'$ are isomorphic with a natural isomorphism $F' \to F$ given by the composition

$$F' \xrightarrow{\eta \circ F'} F \circ G \circ F' \xrightarrow{\epsilon \circ F} F.$$
Conversely, if \((G, F, \eta, \epsilon) : X \simeq A\) is an equivalence of categories and \(\sigma : F' \to F\) is a natural isomorphism, there is an equivalence of categories \((G, F', \eta', \epsilon') : X \simeq A\) with the natural isomorphism \(\eta'\) and \(\epsilon'\) given by the compositions

\[
1_X \Rightarrow G \circ F \xrightarrow{G \circ \sigma^{-1}} G \circ F' \quad \text{and} \quad G \circ F \xrightarrow{G \circ \sigma} G \circ F \Rightarrow 1_A
\]

respectively.

7.11.14 Proposition. If \((G, F, \eta, \epsilon) : X \simeq A\) is an equivalence of categories, then \(G\) and \(F\) are equivalences.

Proof. Since \(\eta_x : x \Rightarrow G \circ F \cdot x\) (resp. \(\epsilon_a : a \cdot G \circ F \Rightarrow a\)) is an isomorphism for each \(x \in [X]\) (resp. \(a \in [A]\)), \(G\) (resp. \(F\)) is essentially surjective. The proof is complete if we show that \(G\) and \(F\) are fully faithful. For any arbitrary \(x, y \in [X]\) and \(a, b \in [A]\), consider the commutative diagrams:

\[
\begin{array}{c}
\xymatrix{
x(X) y \ar[r]^{g \cdot G \cdot F} & (x \cdot F) (A) (F \cdot y) \\
(x \cdot F) G (x) (G \cdot F) y \ar[u]^{g-\eta} \ar[r]_{q-\eta} & (a \cdot G) (G \cdot b) \ar[u]^{f-\eta} \ar[d]_{f \cdot G \cdot F} \\
(a \cdot G) F (A) (F \cdot b) \ar[d]^{p \cdot G \cdot F} \ar[r]_{G \cdot G \cdot F} & (a \cdot F) (A) (F \cdot b) \\
}
\end{array}
\]

For any \(X\)-arrow \(g : x \to y\) and any \(A\)-arrow \(f : a \to b\), the squares

\[
\begin{array}{c}
x \xrightarrow{\eta_x} x \cdot F \cdot G \quad \text{and} \quad a \cdot G \cdot F \xrightarrow{\epsilon_a} a \\
\eta^y \downarrow \quad f \cdot G \cdot F \downarrow \\
y \xrightarrow{\eta_y} y \cdot F \cdot G \\
\end{array}
\]

commute by the naturality of \(\eta\) and \(\epsilon\). Since \(\eta_x\) and \(\eta_y\) are isomorphisms, \(g \Rightarrow g \cdot F \cdot G\) is bijective, and hence \(g \Rightarrow g \cdot F\) is injective and \(q \Rightarrow q \cdot G\) is surjective. Symmetrically, since \(\epsilon_a\) and \(\epsilon_b\) are isomorphisms, \(f \Rightarrow f \cdot G\) is bijective and \(q \Rightarrow q \cdot G\) is surjective. Therefore, \(f \Rightarrow f \cdot G\) is injective and \(p \Rightarrow p \cdot F\) is surjective. Since \(x, y, a,\) and \(b\) are arbitrary, the injectivity of \(g \Rightarrow g \cdot F\) and the surjectivity of \(p \Rightarrow p \cdot F\) imply the surjectivity of \(g \Rightarrow g \cdot F\) and \(p \Rightarrow p \cdot F\). Hence \(g \Rightarrow g \cdot F\) and \(p \Rightarrow p \cdot F\) are both injective and surjective, i.e., bijective. The bijectivity of \(f \Rightarrow f \cdot G\) and \(g \Rightarrow g \cdot F\) follows from the bijectivities of the other edges in the diagram, proving the fully faithfulness of \(G\) and \(F\). \(\Box\)

7.11.15 Theorem. If \((G, F, \eta, \epsilon) : X \simeq A\) is an equivalence of categories, then for any category \(E,\)

- the pair of postcomposition functors

\[
\begin{array}{c}
\xymatrix{[E, X] \ar[r]^{[E, G]} & [E, A] \ar[r]_{[E, F]} & [E, X]}
\end{array}
\]

and the pair of postcomposition natural transformations

\[
\begin{array}{c}
\xymatrix{[E, \eta] : 1_{[E, X]} \to [E, G] \circ [E, F] \ar[r] & [E, \epsilon] : [E, G] \circ [E, F] \to 1_{[E, A]}}
\end{array}
\]

(see Preliminary 0.0.2(1)) form an equivalence of categories \([E, X] \simeq [E, A]\).

- the pair of precomposition functors

\[
\begin{array}{c}
\end{array}
\]

and the pair of precomposition natural transformations

\[
\begin{array}{c}
\end{array}
\]

(see Preliminary 0.0.2(1)) form an equivalence of categories \([X, E] \simeq [A, E]\).

Proof. First note that the postcomposition natural transformations

\[
\begin{array}{c}
\xymatrix{[E, \eta] : [E, 1_X] \to [E, G \circ F] \ar[r] & [E, \epsilon] : [E, G \circ F] \to [E, 1_A]}
\end{array}
\]

yield

\[
\begin{array}{c}
\xymatrix{[E, \eta] : 1_{[E, X]} \to [E, G \circ F] \ar[r] & [E, \epsilon] : [E, G \circ F] \to [E, 1_A]}
\end{array}
\]
by the functoriality of \([\mathbf{E}, \cdot] : \mathbf{CAT} \to \mathbf{CAT}\) (see Preliminary 0.0.2(2)). Since (like any other functor) the functors \([\mathbf{E}, \cdot] : [X, X] \to [[\mathbf{E}, X], [\mathbf{E}, X]]\) and \([\mathbf{E}, \cdot] : [A, A] \to [[\mathbf{E}, A], [\mathbf{E}, A]]\) preserve isomorphisms, \([\mathbf{E}, \eta]\) and \([\mathbf{E}, \epsilon]\) are natural isomorphisms. \(\square\)

7.11.16 Remark. Theorem 7.11.15 says that the functor \([\mathbf{E}, \cdot]\) preserves equivalences of categories. In fact, this is immediate if we use the notion of 2-categories and define \([\mathbf{E}, \cdot]\) as a 2-functor.

7.11.17 Definition. An equivalence of categories \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \simeq \mathbf{A}\) is called an adjoint equivalence, written \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\), if \((\eta, \epsilon)\) is the unit and counit of an adjunction \(\mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A}\).

7.11.18 Proposition. For any adjunction \(\mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A}\), the following conditions are equivalent:

(1) the unit \(\eta\) and the counit \(\epsilon\) are natural isomorphisms, i.e. they form an adjoint equivalence \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\);

(2) \(\mathbf{G}\) and \(\mathbf{F}\) are fully faithful;

(3) \(\mathbf{G}\) and \(\mathbf{F}\) are equivalences;

(4) \(\mathbf{G} [\text{op} . \mathbf{F}]\) is an equivalence.

Proof. (1)\(\Rightarrow\)(2) Immediate from Theorem 7.3.14.

(1)\(\Rightarrow\)(3) This is Proposition 7.11.14.

(3)\(\Rightarrow\)(2) Immediate by definition.

(3)\(\Rightarrow\)(4) A well known tautology.

(4)\(\Rightarrow\)(3) By Corollary 7.11.11, showing this is equivalent to showing that if the corepresentable module \((\mathbf{X}) \mathbf{G}\) [op. representable module \(\mathbf{F}(\mathbf{A})\)] is an equivalence, then \((\mathbf{X}) \mathbf{G}\) and \(\mathbf{F}(\mathbf{A})\) are equivalences. But since \((\mathbf{X}) \mathbf{G} \simeq \mathbf{F}(\mathbf{A})\) by the definition of an adjunction, this follows from Proposition 7.11.2. \(\square\)

7.11.19 Proposition. If two adjunctions \((\eta, \epsilon) : \mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A}\) and \((\eta', \epsilon') : \mathbf{G}' \dashv \mathbf{F}' : \mathbf{X} \to \mathbf{A}\) are isomorphic (see Remark 7.6.2(3)), then \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\) is an adjoint equivalence if and only if \((\mathbf{G}', \mathbf{F}', \eta', \epsilon') : \mathbf{X} \to \mathbf{A}\) is.

Proof. Since \(\mathbf{G} \simeq \mathbf{G}'\), by Proposition 7.11.4, \(\mathbf{G}\) is an equivalence iff \(\mathbf{G}'\) is. The assertion thus follows from the equivalence of (1) and (4) in Proposition 7.11.18. \(\square\)

7.11.20 Proposition. Let \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\) be an adjoint equivalence, and suppose that functors \(\mathbf{G}' : \mathbf{A} \to \mathbf{X}\) and \(\mathbf{F}' : \mathbf{X} \to \mathbf{A}\) are isomorphic to \(\mathbf{G}\) and \(\mathbf{F}\) respectively. Then there is an adjoint equivalence \((\mathbf{G}', \mathbf{F}', \eta', \epsilon') : \mathbf{X} \to \mathbf{A}\).

Proof. By Remark 7.6.2(3), there is an adjunction \(\mathbf{G}' \dashv \mathbf{F}' : \mathbf{X} \to \mathbf{A}\) isomorphic to \(\mathbf{G} \dashv \mathbf{F} : \mathbf{X} \to \mathbf{A}\). Let \((\eta', \epsilon')\) be the unit and counit of \(\mathbf{G}' \dashv \mathbf{F}'\). Then \((\mathbf{G}', \mathbf{F}', \eta', \epsilon') : \mathbf{X} \to \mathbf{A}\) is an adjoint equivalence by Proposition 7.11.19. \(\square\)

7.11.21 Proposition. If \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\) is an adjoint equivalence, then so is \((\mathbf{F}, \mathbf{G}, \epsilon^{-1}, \eta^{-1}) : \mathbf{A} \to \mathbf{X}\).

Proof. Reversing the commutative triangles in Theorem 7.3.13, we have the triangular identities

\[
\begin{array}{ccc}
\mathbf{G} & \overset{1}{\leftarrow} & \mathbf{G} \\
\mathbf{G} \circ \eta^{-1} & \mathbf{G} \circ \mathbf{F} \circ \mathbf{G} & \mathbf{G} \circ \epsilon^{-1} \circ \mathbf{G} \\
\mathbf{F} & \overset{1}{\leftarrow} & \mathbf{F} \\
\mathbf{F} \circ \eta^{-1} \circ \mathbf{F} & \mathbf{F} \circ \mathbf{F} \circ \mathbf{F} & \mathbf{F} \circ \epsilon^{-1} \circ \mathbf{F}
\end{array}
\]

of the adjunction \(\mathbf{F} \dashv \mathbf{G}\) whose unit and counit are \(\epsilon^{-1}\) and \(\eta^{-1}\). \(\square\)

7.11.22 Theorem. Given a functor \(\mathbf{G} : \mathbf{A} \to \mathbf{X}\) [op. \(\mathbf{F} : \mathbf{X} \to \mathbf{A}\)], the following conditions are equivalent:

(1) \(\mathbf{G} [\text{op} . \mathbf{F}]\) is an equivalence;

(2) \(\mathbf{G} [\text{op} . \mathbf{F}]\) is a part of an equivalence of categories \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \simeq \mathbf{A}\);

(3) \(\mathbf{G} [\text{op} . \mathbf{F}]\) is a part of an adjoint equivalence \((\mathbf{G}, \mathbf{F}, \eta, \epsilon) : \mathbf{X} \to \mathbf{A}\).
Proof. (1)⇒(3) Since \( G \) is an equivalence, its corepresentable module \( (X)G \) is an equivalence by Theorem 7.11.7, and thus has a unit \( \eta: (X)G \to F \) by Theorem 7.11.9. Hence, by the equivalence of (2) and (3) in Proposition 7.3.9, \( G \) is a part of an adjunction \( (\eta, \epsilon): G \vdash F: X \to A \), in fact an adjoint equivalence because \( G \) is an equivalence.

(3)⇒(2) Immediate by definition.

(2)⇒(1) This is Proposition 7.11.14. \( \square \)

7.11.23 Remark. The equivalence of the conditions (1) and (2) in Theorem 7.11.22 says that two categories are equivalent if and only if there is an equivalence between them.

7.11.24 Corollary. Any equivalence of categories \( (G, F, \eta, \epsilon): X \simeq A \) can be promoted to an adjoint equivalence \( (G, F, \eta', \epsilon'): X \to A \).

Proof. By Theorem 7.11.22, there is an adjoint equivalence \( (G, F', \eta'', \epsilon''): X \to A \). But since \( F' \) is isomorphic to \( F \) by Remark 7.11.13(1), there is an adjoint equivalence \( (G, F, \eta', \epsilon') : X \to A \) by Proposition 7.11.20. \( \square \)

7.11.25 Corollary. If a functor \( K: D \to E \) is an equivalence, so are the postcomposition functor \( [C, K]: [C, D] \to [C, E] \) and the precomposition functor \( [K, C]: [E, C] \to [D, C] \) for any category \( C \).

Proof. By the equivalence of the conditions (1) and (2) in Theorem 7.11.22, this follows from Theorem 7.11.15. \( \square \)

7.11.26 Theorem. An adjunction \( G \vdash F: X \to A \) constitutes an adjoint equivalence if and only if \( G \) is fully faithful and \( F \) is conservative (i.e. isomorphism reflecting), dually, if and only if \( F \) is fully faithful and \( G \) is conservative.

Proof. First recall from Proposition 7.11.18 that an adjunction \( G \vdash F: X \to A \) constitutes an adjoint equivalence if and only if \( G \) and \( F \) are fully faithful. The “only if” part of the assertion is immediate because a fully faithful functor is conservative. To prove the “if” part, assume that \( F \) is conservative; recalling Theorem 7.3.14, it suffices to show that if the counit \( \epsilon \) is an isomorphism, then the unit \( \eta \) is an isomorphism. For this, consider the triangular identity

\[
\begin{array}{ccc}
F & \xrightarrow{\eta \circ F} & F \\
\downarrow{\eta \circ F} & & \downarrow{\epsilon \circ F} \\
F \circ G & \xrightarrow{\eta \circ F} & F \circ G
\end{array}
\]

in Theorem 7.3.13. If \( \epsilon \) is an isomorphism, so is \( F \circ \epsilon \) because any functor preserves isomorphisms, hence so is \( \eta \circ F \) by the commutativity of the triangle above, and so is \( \eta \) by the conservativity of \( F \), as required. \( \square \)

Note. Recall from Definition 1.1.3 that any functor \( K: D \to E \) yields the precomposition functor \( [K]: [E]: = [D:] \).

7.11.27 Theorem. If two categories \( D \) and \( E \) are equivalent, then the categories \( [D:] \) and \( [E:] \) are equivalent: more specifically, if a functor \( K: D \to E \) is an equivalence, so is the precomposition functor \( [K]: [E:] \to [D:] \). \( \square \)

Proof. This is an instance of Corollary 7.11.25.

Note. Recall from Definition 2.3.5 that \( \text{Rep}[X:] \) denotes the full subcategory of the category \( [X:] \) whose objects are representable right modules over \( X \).

7.11.28 Theorem.\(^\dagger\)

- For any category \( X \), \( X \simeq \text{Rep}[X:] \).
- For any category \( A \), \( A^\wedge \simeq \text{Rep}[:A] \).

Proof. Since the Yoneda functor \( [X^\wedge]: X \to [X:] \) is fully faithful and the image of \( X \) under \( X^\wedge \) is isomorphism-dense in \( \text{Rep}[X:] \) (Proposition 2.3.6), the restriction of \( X^\wedge \) to \( \text{Rep}[X:] \) on the codomain is an equivalence. Hence \( X \simeq \text{Rep}[X:] \) by Remark 7.11.23. \( \square \)
7.12 Equivalence of functors

We introduce the notion of conjugation of functors along equivalences of categories. The notion then allows us to define equivalence of functors, which generalizes natural isomorphism.

7.12.1 Lemma. If functors \( P : X \to Y \) and \( Q : B \to A \) are equivalences, so is the functor \([P, Q] : [Y, B] \to [X, A]\).

Proof. Since the functor \([P, Q]\) is given by the diagonal of the commutative square

\[
\begin{array}{ccc}
[Y, B] & \xrightarrow{[P, B]} & [X, B] \\
\downarrow{[Y, Q]} & & \downarrow{[X, Q]} \\
[Y, A] & \xrightarrow{[P, A]} & [X, A]
\end{array}
\]

, the assertion follows from Corollary 7.11.25 by virtue of Proposition 7.11.5. \(\square\)

7.12.2 Definition. Given equivalences \( P : X \to Y \) and \( Q : B \to A \), the equivalence functor \([P, Q] : [Y, B] \to [X, A]\) in Lemma 7.12.1 is called the conjugation along \( P \) and \( Q \), and for a functor \( F : Y \to B \), its image under \([P, Q]\) (i.e. the composite functor \( P \circ F \circ Q : X \to A \)) is called the conjugate of \( F \) along \( P \) and \( Q \).

Note. The definition of “conjugate” in Definition 7.12.2 is too rigid to be useful. We will be less strict and take “up to isomorphism” approach:

7.12.3 Definition. Let \((G, F, \eta, \epsilon) : X \simeq A\) and \((G', F', \eta', \epsilon') : X' \simeq A'\) be equivalences. Then functors \( S : X \to X'\) and \( T : A \to A'\) are said to be conjugate to each other when any (hence all) of the squares

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow{s} & & \downarrow{T} \\
X' & \xleftarrow{F'} & A'
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow{s} & & \downarrow{T} \\
X' & \xleftarrow{F'} & A'
\end{array}
\quad
\begin{array}{ccc}
X & \xleftarrow{G} & A \\
\downarrow{s} & & \downarrow{T} \\
X' & \xleftarrow{G'} & A'
\end{array}
\quad
\begin{array}{ccc}
X & \xleftarrow{G} & A \\
\downarrow{s} & & \downarrow{T} \\
X' & \xleftarrow{G'} & A'
\end{array}
\]

commutes up to isomorphism.

7.12.4 Remark.

(1) A functor \( S : X \to X'\) is thus conjugate to a functor \( T : A \to A'\) along equivalences \( F : X \to A\) and \( G' : A' \to X'\) if \( S \) is isomorphic to the conjugate (in the sense of Definition 7.12.2) of \( T \) along \( F \) and \( G' \).

(2) The notion of conjugate subsumes natural isomorphism as a special case: a functor \( S : X \to X'\) is isomorphic to a functor \( T : X \to X'\) if \( S \) is conjugate to \( T \) along the identities \( X \to X \) and \( X' \to X'\).

7.12.5 Proposition. If \( F : X \to A\) is an equivalence, then for any \( x \in \| X \|\),

- the representable left modules \( x(X) \) and \( (x : F)(A) \) are conjugate to each other along \( F \).
- the representable right modules \( (X)x \) and \( (A)(F : x) \) are conjugate to each other along \( F \).

Proof. Since \( F \) is an equivalence, the home cell

\[
\begin{array}{ccc}
X & \xrightarrow{(X)} & X \\
\downarrow{F} & & \downarrow{F} \\
A & \xrightarrow{(A)} & A
\end{array}
\]

is fully faithful, and thus the left slice \( x(F) \) gives the isomorphism \( x(X) \simeq (x : F)(A) F \) (cf. Proposition 2.1.10). Hence the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{x(X)} & \text{Set} \\
\downarrow{F} & & \downarrow{F} \\
A & \xrightarrow{(x : F)(A)} & \text{Set}
\end{array}
\]

commute up to isomorphism. \(\square\)
7.12.6 Theorem. Let $G_0 : X \to X'$ and $F_1 : A \to A'$ be equivalences, and suppose that $G' : A' \to X'$ and $F' : X' \to A'$ are respectively conjugate to $G : A \to X$ and $F : X \to A$ along $G_0$ and $F_1$; that is, there exist natural isomorphisms

$$
\begin{align*}
X & \xrightarrow{G} A \\
X' & \xrightarrow{G'} A'
\end{align*}
G_0 \downarrow \cong \downarrow F_1
\begin{align*}
X' & \xrightarrow{F} A' \\
X' & \xrightarrow{F'} A'
\end{align*}
G_0 \downarrow \cong \downarrow F_1
$$

then any adjunction $\phi : G \dashv F : X \to A$, if exists, induces an adjunction $\Phi' : G' \dashv F' : X' \to A'$ such that $\tau$ and $\sigma$ form a pseudo morphism (see Definition 7.4.7) $\Phi \to \Phi'$.

Proof. By Theorem 7.11.22, there are adjoint equivalences

$$(G_0, F_0, \eta_0, \epsilon_0) : X' \to X \quad \text{and} \quad (G_1, F_1, \eta_1, \epsilon_1) : A \to A'$$

the composition

$$
\begin{align*}
X' & \xrightarrow{\Phi_0 \tau} X \\
& \xrightarrow{\Phi \tau} A \\
& \xrightarrow{\Phi_1 \tau} A'
\end{align*}
F_0 \downarrow \cong \downarrow F_1
$$

then yields the adjunction

$$(\Phi_0 \circ \Phi : X' \to A')$$

while we have natural isomorphisms

$$\lambda : G_0 \circ G \circ G_1 \to G' \quad \text{and} \quad \rho : F' \to F_0 \circ F \circ F_1$$
given respectively by the compositions

$$G_0 \circ G \circ G_1 \xrightarrow{\tau \circ G_1} G' \circ F_1 \circ G_1 \xrightarrow{G'_1 \circ \epsilon_1} G'$$

and

$$F' \xrightarrow{\eta \circ F'} F_0 \circ F \circ F' \xrightarrow{F_0 \circ \epsilon} F_0 \circ F \circ F_1$$

the adjunction $\Phi' : G' \dashv F' : X' \to A'$ is now defined by the composition as shown in the commutative diagram

$$
\begin{array}{ccc}
G_0 \circ G \circ G_1 & \xrightarrow{\lambda} & (X') \circ [G_0 \circ G \circ G_1] \xrightarrow{\Phi_0 \circ \Phi \circ \Phi_1} [F_0 \circ F \circ F_1] \circ (A') \\
\downarrow & & \downarrow & & \downarrow
\langle X \rangle \lambda \uparrow & & \langle X \rangle \lambda \uparrow & & \langle X \rangle \lambda \uparrow
\end{array}
\begin{array}{ccc}
G' & \xrightarrow{\Phi'} & (X') \circ G' \xrightarrow{\Phi'} (A') \\
\downarrow & & \downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
F_0 \circ F \circ F_1 & \xrightarrow{\rho} & F' \circ (A') \\
\end{array}
$$

($\lambda$ and $\rho$ are conjugate by Remark 7.6.2(3)).

Since by Corollary 7.8.7 the composite natural transformations $G_0 \circ G \circ \eta_1$ and $\epsilon_0 \circ F \circ F_1$ form a pseudo morphism $\Phi \to \Phi_0 \circ \Phi \circ \Phi_1$, and by Proposition 7.6.3 $\lambda$ and $\rho$ form a pseudo morphism $\Phi_0 \circ \Phi \circ \Phi_1 \to \Phi'$, their pasting composites

$$
\begin{align*}
X & \xrightarrow{G} A \\
X' & \xrightarrow{G} A' \\
X' & \xrightarrow{G'} A'
\end{align*}
G_0 \downarrow \cong \downarrow F_1
\begin{align*}
X & \xrightarrow{F} A \\
X' & \xrightarrow{F} A' \\
X' & \xrightarrow{F} A'
\end{align*}
G_0 \downarrow \cong \downarrow F_1
$$

form a pseudo morphism $\Phi \to \Phi'$ by Proposition 7.4.2; these pasting composites are given respectively by the composites

$$
G_0 \circ G \xrightarrow{G_0 \circ G \circ \eta_1} G_0 \circ G \circ G_1 \xrightarrow{\tau \circ G_1} G' \circ F_1 \circ G_1 \xrightarrow{G'_1 \circ \epsilon_1 \circ F_1}
$$

and

$$
G_0 \circ F' \xrightarrow{G_0 \circ F' \circ \eta_0 \circ G_1} G_0 \circ F_0 \circ F \circ G_1 \xrightarrow{\tau \circ G_1} G' \circ F_1 \circ G_1 \xrightarrow{G'_1 \circ \epsilon_1 \circ F_1}
$$

but by the interchange law of natural transformation composition and by the triangular identities (Theorem 7.3.13) of adjunctions, they yield $\tau$ and $\sigma$.

7.12.7 Remark. Corollary 7.8.8 is a special case of Theorem 7.12.6 where $G_0$ and $F_1$ are isomorphisms and $\tau$ and $\sigma$ are identities.
7.12.8 Definition. Two functor $S : X \to X'$ and $T : A \to A'$ are said to be equivalent if there are equivalences $F : X \to A$ and $F' : X' \to A'$ such that $S : X \to X'$ and $T : A \to A'$ are conjugate to each other along $F$ and $F'$.

7.12.9 Remark. A property of functors is “non-evil” if it is invariant under equivalence. Non-evil properties of functors include
- fully faithfulness,
- adjointness (Theorem 7.12.6), and
- preservation and reflection of limits (Corollary 8.2.18).

We list below some “evil” properties of functors with the corresponding non-evil version:

<table>
<thead>
<tr>
<th>evil</th>
<th>non-evil</th>
</tr>
</thead>
<tbody>
<tr>
<td>bijective on objects</td>
<td>essentially surjective</td>
</tr>
<tr>
<td>isomorphism between categories</td>
<td>equivalence between categories</td>
</tr>
<tr>
<td>creation of limits</td>
<td>creation of limits up to isomorphism</td>
</tr>
</tbody>
</table>

7.13 Equivalence of modules

In this section, we extend the notions of equivalence of categories and equivalence functor to those of equivalence of modules and equivalence cell. It is shown, as expected, that an equivalence cell preserves and reflects universal arrows. At the end of the section, we will see that an equivalence module $M : X \to A$ (see Section 7.11) yields the equivalences $(X) \cong M \simeq \{A\}$ of $M$ and the hom-modules of $X$ and $A$.

Note. Recall from Remark 1.2.2(1) that $\psi_0 : M_0 \to N_0$ and $\psi_1 : M_1 \to N_1$ denote the left and right components of a cell $\psi : M \to N$.

7.13.1 Definition. A cell $\psi : M \to N$ is called an equivalence if $\psi$ is fully faithful and the functors $\psi_0 : M_0 \to N_0$ and $\psi_1 : M_1 \to N_1$ (the left and right components of $\psi$) are equivalences.

7.13.2 Remark. As we will see below in Proposition 7.13.3 and Proposition 7.13.4, the notion of an equivalence cell and that of an equivalence functor subsume each other.

7.13.3 Proposition. A functor $K : D \to E$ is an equivalence if and only if the hom-cell $\psi : M \to N$ is an equivalence.

Proof. Immediate from the definitions. \hfill \Box

7.13.4 Proposition. A cell $\psi : M \to N$ is an equivalence if and only if the collage functor $[\psi] : [M] \to [N]$ (see Definition 3.1.9) is an equivalence.

Proof. By the construction of $[\psi]$, we can easily see that $[\psi]$ is fully faithful iff so are $\psi_0$, $\psi_1$, and $\psi$, and that $[\psi]$ is essentially surjective iff so are $\psi_0$ and $\psi_1$. \hfill \Box

7.13.5 Definition. An equivalence of modules $(\varphi, \psi, \eta, \epsilon) : M \simeq N$ consists of a pair of cells

$$
\begin{array}{c}
\varphi : M \to N \\
\psi \downarrow \\
\epsilon : N \to N
\end{array}
$$

and a pair of cell isomorphisms

$$
\eta : 1_M \to \varphi \circ \psi : M \to M \quad \epsilon : \varphi \circ \psi : 1_N : N \to N
$$

; in this situation, the cells $\varphi$ and $\psi$ are said to be quasi-inverse to each other. Two modules $M$ and $N$ are called equivalent, written $M \simeq N$, if there is an equivalence of modules between them.

7.13.6 Remark. As we will see below in Proposition 7.13.7 and Proposition 7.13.8, the notion of equivalence of modules and that of equivalence of categories subsume each other.
7.13.7 Proposition. A pair of natural transformations
\[ \eta : 1_X \to G \circ F : X \to X \] and \[ \epsilon : G \circ F \to 1_A : A \to A \]
as in Definition 7.11.12 form an equivalence of categories \((G, F, \eta, \epsilon) : X \simeq A\) if and only if their hom-cell morphisms
\[ (\eta) : 1_X \to (G) \circ (F) : (X) \to (X) \] and \[ (\epsilon) : (G) \circ (F) \to 1_A(X) : (A) \to (A) \]
(see Definition 1.3.5) form an equivalence of modules \(((G, F, \eta), (\epsilon)) : (X) \simeq (A)\).

Proof. Immediate from Theorem 1.3.6. \(\square\)

7.13.8 Proposition. For a pair of cell morphisms
\[ \eta : 1_M \to \phi \circ \psi : M \to M \] and \[ \epsilon : \phi \circ \psi \to 1_N : N \to N \]
as in Definition 13.7, the following conditions are equivalent:
1. \(\eta\) and \(\epsilon\) form an equivalence of modules \((\phi, \psi, \eta, \epsilon) : M \simeq N\);
2. the collage natural transformations
\[ [\eta] : 1_M \to [\phi] \circ [\psi] : [M] \to [M] \] and \[ [\phi] \circ [\psi] \to 1_N : [N] \to [N] \]
of \(\eta\) and \(\epsilon\) (see Definition 3.1.7) form an equivalence of categories \(([\phi], [\psi], [\eta], [\epsilon]) : [M] \simeq [N]\);
3. the left and right components
\[ \eta_i : 1_{M_i} \to \phi_i \circ \psi_i : M_i \to M_i \] and \[ \phi_i \circ \psi_i \to 1_{N_i} : N_i \to N_i \]
for \(\eta\) and \(\epsilon\) (see Definition 1.3.1) form an equivalence of categories \((\phi_i, \psi_i, \eta_i, \epsilon_i) : M_i \simeq N_i\) \((i = 0, 1)\).

Proof. (1)\(\Leftrightarrow\) (2) First note that the collage natural transformations
\[ [\eta] : 1_M \to [\phi \circ \psi] \]
yield
\[ [\eta] : 1_M \to [\phi] \circ [\psi] \]
by the functoriality of the collage operation \([-]\). Since (like any other full embedding) the full embeddings \([-] : [M] : [M] \to [\{\}, [\{\}]\]) and \([-] : [N] : [N] \to [\{\}, [\{\}]\]) (see Remark 3.1.5(4)) preserve and reflect isomorphisms, the two conditions are equivalent.

(1)\(\Leftrightarrow\) (3) Immediate from Proposition 1.3.4. \(\square\)

Note. The axiom of choice is used in the proof of the following.

7.13.9 Theorem. Given a cell \(\phi : N \to M [op. \psi : M \to N]\), the following conditions are equivalent:
1. \(\phi \ [op. \psi]\) is an equivalence;
2. \(\phi \ [op. \psi]\) is a part of an equivalence of modules \((\phi, \psi, \eta, \epsilon) : M \simeq N\).

Proof. (1)\(\Rightarrow\) (2) We depict \(\phi : N \to M\) as
\[ N_0 \leftarrow N \rightarrow M_0 \]
\[ \phi \downarrow \quad \psi \]
\[ N_1 \rightarrow M_1 \]
First note that if \(\phi\) is an equivalence, then the functors \(\phi_0\) and \(\phi_1\) and the collage functor \([\phi]\) are equivalences by Definition 7.13.1 and Proposition 7.13.4. Now since \(\phi_0\) and \(\phi_1\) are equivalences and thus essentially surjective, we may choose, for each object \(x \in [M_0]\), an object \(r_x \in [N_0]\) and an iso \(M_0\)-arrow \(\epsilon_x : x \to \phi_0 \circ r_x\), and for each object \(x' \in [M_1]\), an object \(r_{x'} \in [N_1]\) and an iso \(M_1\)-arrow \(\epsilon_{x'} : x' \to \phi_1 \circ r_{x'}\). Since \([\phi]\) is fully faithful, by Corollary 6.2.12, each \(\epsilon_x\) is universal from
Theorem 7.13.10. If \((\varphi, \psi, \eta, \epsilon) : \mathcal{M} \simeq \mathcal{N}\) is an equivalence of modules, then

1. for any module \(\mathcal{J}\), the pair of postcomposition cells

\[
\langle \mathcal{J}, \mathcal{M} \rangle \xrightarrow{(\mathcal{J}, \varphi)} \langle \mathcal{J}, \mathcal{N} \rangle
\]

(see Definition 1.2.25) and the pair of postcomposition cell morphisms

\[
\langle \mathcal{J}, \eta \rangle : 1_{\langle \mathcal{J}, \mathcal{M} \rangle} \to \langle \mathcal{J}, \varphi \rangle \circ \langle \mathcal{J}, \psi \rangle \quad \text{and} \quad \langle \mathcal{J}, \epsilon \rangle : \langle \mathcal{J}, \varphi \rangle \circ \langle \mathcal{J}, \psi \rangle \to 1_{\langle \mathcal{J}, \mathcal{N} \rangle}
\]

(see Definition 1.3.7) form an equivalence of modules \(\langle \mathcal{J}, \mathcal{M} \rangle \simeq \langle \mathcal{J}, \mathcal{N} \rangle\).

2. for any category \(\mathcal{E}\), the pair of postcomposition cells

\[
\langle \mathcal{E}, \mathcal{M} \rangle \xrightarrow{(\mathcal{E}, \varphi)} \langle \mathcal{E}, \mathcal{N} \rangle
\]

(see Definition 4.3.17) and the pair of postcomposition cell morphisms

\[
\langle \mathcal{E}, \eta \rangle : 1_{\langle \mathcal{E}, \mathcal{M} \rangle} \to \langle \mathcal{E}, \varphi \rangle \circ \langle \mathcal{E}, \psi \rangle \quad \text{and} \quad \langle \mathcal{E}, \epsilon \rangle : \langle \mathcal{E}, \varphi \rangle \circ \langle \mathcal{E}, \psi \rangle \to 1_{\langle \mathcal{E}, \mathcal{N} \rangle}
\]

(see Definition 4.3.24) form an equivalence of modules \(\langle \mathcal{E}, \mathcal{M} \rangle \simeq \langle \mathcal{E}, \mathcal{N} \rangle\).

3. for any category \(\mathcal{E}\),

   - the pair of postcomposition cells

\[
\langle \mathcal{E}^*, \mathcal{M} \rangle \xrightarrow{(\mathcal{E}^*, \varphi)} \langle \mathcal{E}^*, \mathcal{N} \rangle
\]

(see Definition 4.6.17) and the pair of postcomposition cell morphisms

\[
\langle \mathcal{E}^*, \eta \rangle : 1_{\langle \mathcal{E}^*, \mathcal{M} \rangle} \to \langle \mathcal{E}^*, \varphi \rangle \circ \langle \mathcal{E}^*, \psi \rangle \quad \text{and} \quad \langle \mathcal{E}^*, \epsilon \rangle : \langle \mathcal{E}^*, \varphi \rangle \circ \langle \mathcal{E}^*, \psi \rangle \to 1_{\langle \mathcal{E}^*, \mathcal{N} \rangle}
\]

(see Definition 4.6.25) form an equivalence of modules \(\langle \mathcal{E}^*, \mathcal{M} \rangle \simeq \langle \mathcal{E}^*, \mathcal{N} \rangle\).

Proof. All are proved in the same way as Theorem 7.11.15.

1. The postcomposition cell morphisms

\[
\langle \mathcal{J}, \eta \rangle : \langle \mathcal{J}, 1_{\mathcal{M}} \rangle \to \langle \mathcal{J}, \varphi \circ \psi \rangle \quad \langle \mathcal{J}, \epsilon \rangle : \langle \mathcal{J}, \varphi \circ \psi \rangle \to \langle \mathcal{J}, 1_{\mathcal{N}} \rangle
\]

yield

\[
\langle \mathcal{J}, \eta \rangle : 1_{\langle \mathcal{J}, \mathcal{M} \rangle} \to \langle \mathcal{J}, \varphi \circ \psi \rangle \quad \langle \mathcal{J}, \epsilon \rangle : \langle \mathcal{J}, \varphi \circ \psi \rangle \to 1_{\langle \mathcal{J}, \mathcal{N} \rangle}
\]
by the functoriality of \( \langle J, - \rangle : \text{MOD} \to \text{MOD} \) (see Remark 1.2.28), and since the functors \( \langle J, - \rangle : [M : M] \to [\langle J, M \rangle : \langle J, M \rangle] \) and \( \langle J, - \rangle : [N : N] \to [\langle J, N \rangle : \langle J, N \rangle] \) (see Remark 3.8(2)) preserve isomorphisms, \( \langle J, \eta \rangle \) and \( \langle J, \epsilon \rangle \) are cell isomorphisms.

(2) The postcomposition cell morphisms

\[
\langle E, \eta \rangle : (E, 1_M) \to (E, \varphi \circ \psi) \quad \text{and} \quad \langle E, \epsilon \rangle : (E, \varphi \circ \psi) \to (E, 1_N)
\]

yield

\[
\langle E, \eta \rangle : 1_{\langle E, M \rangle} \to (E, \varphi \circ \psi) \quad \text{and} \quad \langle E, \epsilon \rangle : (E, \varphi \circ \psi) \to 1_{\langle E, N \rangle}
\]

by the functoriality of \( \langle E, - \rangle : \text{MOD} \to \text{MOD} \) (see Remark 4.3.22), and since the functors \( \langle E, - \rangle : [M : M] \to [\langle E, M \rangle : \langle E, M \rangle] \) and \( \langle E, - \rangle : [N : N] \to [\langle E, N \rangle : \langle E, N \rangle] \) (see Remark 4.3.25) preserve isomorphisms, \( \langle E, \eta \rangle \) and \( \langle E, \epsilon \rangle \) are cell isomorphisms.

(3) The postcomposition cell morphisms

\[
\langle \ast E, \eta \rangle : (\ast E, 1_M) \to (\ast E, \varphi \circ \psi) \quad \text{and} \quad \langle \ast E, \epsilon \rangle : (\ast E, \varphi \circ \psi) \to (\ast E, 1_N)
\]

yield

\[
\langle \ast E, \eta \rangle : 1_{\langle \ast E, M \rangle} \to (\ast E, \varphi \circ \psi) \quad \text{and} \quad \langle \ast E, \epsilon \rangle : (\ast E, \varphi \circ \psi) \to 1_{\langle \ast E, N \rangle}
\]

by the functoriality of \( \langle \ast E, - \rangle : \text{MOD} \to \text{MOD} \) (see Remark 4.6.21), and since the functors \( \langle \ast E, - \rangle : [M : M] \to [\langle \ast E, M \rangle : \langle \ast E, M \rangle] \) and \( \langle \ast E, - \rangle : [N : N] \to [\langle \ast E, N \rangle : \langle \ast E, N \rangle] \) (see Remark 4.6.26) preserve isomorphisms, \( \langle \ast E, \eta \rangle \) and \( \langle \ast E, \epsilon \rangle \) are cell isomorphisms.

\[\square\]

7.13.11 Remark. Theorem 7.13.10 says that the functor \( \langle J, - \rangle \) (resp. \( \langle E, - \rangle \), \( \langle E*, - \rangle \)) preserves equivalences of modules (cf. Remark 7.11.16).

7.13.12 Corollary. If a cell \( \psi \) is an equivalence, so are the postcomposition cells \( \langle J, \psi \rangle \), \( \langle E, \psi \rangle \), \( \langle \ast E, \psi \rangle \), and \( \langle \ast E*, \psi \rangle \).

\[\text{Proof.} \text{ By the equivalence of (1) and (2) in Theorem 7.13.9, this follows from Theorem 7.13.10.} \quad \square\]


\[\text{Proof.} \text{ Suppose that a cell } X \xrightarrow{M} A \text{ is an equivalence and let } u : r \sim a \text{ be an } M \text{-arrow. We } Y \xleftarrow{N} B \text{ need to show that } u \text{ is inverse universal iff so is } \psi \circ u; \text{ that is, } X \uparrow u \text{ is iso iff } Y \uparrow (\psi \circ u) \text{ is iso. By Example 5.2.7(1), the diagram}
\]

\[
\begin{array}{c}
\Diamond \langle X \rangle \uparrow uu \quad P \downarrow u \quad P \downarrow u = P \langle Y \rangle (P \circ r)
\end{array}
\]

\[
\begin{array}{c}
\langle P \langle N \rangle Q \rangle \uparrow \quad a \quad P \langle N \rangle (Q \circ a)
\end{array}
\]

commutes. Since \( P \) is fully faithful, \( (P \circ r) \) is iso. Hence, by the commutativity of the diagram above, \( X \uparrow (\psi \circ u) \) is is iso iff \( P \langle Y \rangle (\psi \circ u) \) is iso. Since \( \psi \) is fully faithful (i.e. \( \psi : M \to P \langle N \rangle Q \) is iso), \( X \uparrow u \) is iso iff \( X \uparrow (\psi \circ u) \) is iso. Since \( P \) is essentially surjective, by Proposition 1.1.33, \( Y \uparrow (\psi \circ u) \) is iso iff \( P \langle Y \rangle (\psi \circ u) \) is iso. Hence \( X \uparrow u \) is iso iff \( Y \uparrow (\psi \circ u) \) is iso as required. \[\square\]

7.13.14 Corollary. Any equivalence cell preserves and reflects lifts [op. colifts] and pointwise lifts [op. colifts]. Specifically, if a cell \( X \xrightarrow{M} A \) is an equivalence, then

\[
\begin{array}{c}
\Diamond \langle P \rangle \uparrow uu \quad P \downarrow u \quad P \downarrow u = P \langle Y \rangle (P \circ r)
\end{array}
\]

\[
\begin{array}{c}
\langle P \langle N \rangle Q \rangle \uparrow \quad a \quad P \langle N \rangle (Q \circ a)
\end{array}
\]
Theorem 7.13.13 because if 
postcomposition cell
Proof. Since an inverse universal cylinder \( \mu : E \rightarrow M \) is the same thing as an inverse universal 
\((E, M)\)-arrow, the assertion is equivalent to saying that if a cell \( \psi \) is an equivalence, then the 
postcomposition cell \( (E, \psi) \) preserves and reflect inverse universal arrows. But this follows from 
Theorem 7.13.13 because if \( \psi \) is an equivalence, so is \( (E, \psi) \) by Corollary 7.13.12. The pointwise
version follows immediately from Theorem 7.13.13 on noting that the component of the composite 
cylinder
Remark. We will see other examples of Theorem 7.13.13 later in Theorem 8.2.10 and Theo-
rem 8.2.17.
7.13.16 Theorem. In the situation of Theorem 7.3.15, suppose that the following equivalent con-
ditions (see Corollary 7.11.10) hold:
1. \( M \) is an equivalence;
2. \( G \) is an equivalence;
3. \( F \) is an equivalence.
Then \( M \) is equivalent to the hom-module of \( X \) [op. \( A \)]. Specifically,
the inverse of the module morphism \( X \uparrow \rho : (X) \rightarrow (X) \), a corepresentation of \( M \) by \( G \),
and the module morphism \( X \uparrow \lambda : (X) \rightarrow (M) \) by \( F \rightarrow X \) (see Example 5.3.5(1)) yield the equivalence cells
\[
\begin{align*}
X \xrightarrow{M} A & \quad \text{and} \quad X \xrightarrow{(X)} X \\
X \xrightarrow{(X)} X & \quad \text{and} \quad X \xrightarrow{F} \xrightarrow{\lambda} A
\end{align*}
\]
quasi-inverse to each other, forming an equivalence
\[
\begin{array}{c}
(1, \eta) : (X) \rightarrow (X) \uparrow \rho \uparrow -1 \circ (X) \uparrow \lambda \\
(1, \epsilon) : (X) \uparrow \rho \uparrow -1 \circ (X) \uparrow \lambda \rightarrow 1_M
\end{array}
\]
, where \( \eta \) and \( \epsilon \) are the unit and counit of the adjunction in Theorem 7.3.15.
the inverse of the module morphism \( \lambda \uparrow A : F(A) \rightarrow M : X \rightarrow A \), a representation of \( M \) by \( F \),
and the module morphism \( \rho \uparrow A : (A) \rightarrow G(M) : A \rightarrow A \) (see Example 5.3.5(1)) yield the equivalence cells
\[
\begin{align*}
X \xrightarrow{M} A & \quad \text{and} \quad A \xrightarrow{(A)} A \\
A \xrightarrow{(A)} A & \quad \text{and} \quad X \xrightarrow{\rho \uparrow A} \xrightarrow{\lambda \uparrow A} A
\end{align*}
\]
quasi-inverse to each other, forming an equivalence
\[
\begin{array}{c}
\rho \uparrow A \\
(\lambda \uparrow A) \uparrow -1
\end{array}
\]
of $\mathcal{M}$ and $(A)$ with the cell isomorphisms

$$(\eta, 1_A) : 1_{\mathcal{M}} \to (\rho \downarrow A) \circ (\lambda \downarrow A)^{-1} \quad (\epsilon, 1_A) : (\rho \downarrow A) \circ (\lambda \downarrow A)^{-1} \to 1_{(A)}$$

where $\eta$ and $\epsilon$ are the unit and counit of the adjunction in Theorem 7.3.15.

Proof. Once we verify the claim below, it remains to show that $(1_X, \eta)$ and $(1_X, \epsilon)$ are cell isomorphisms. For this, we just need to show that $\eta : 1_X \to G \circ F$ and $\epsilon : G \circ F \to 1_{A}$ are natural isomorphisms. But since $G$ and $F$ are equivalences, this follows from the equivalence of (1) and (3) in Proposition 7.11.18. \hfill \Box

Claim.

(1) The pair $(1_X, \eta)$ forms a cell morphism $1(1_X) \to (X \uparrow \rho)^{-1} \circ (X \uparrow \lambda)$; that is, the diagram

\[
\begin{array}{ccc}
  y & \xrightarrow{f} & x \\
  1_y \downarrow & & \downarrow \eta_x \\
  y & \xrightarrow{f \circ (X \uparrow \lambda)^{-1} \circ (X \uparrow \rho)} & G \circ F \circ x
\end{array}
\]

commutes for every $X$-arrow $f : y \to x$.

(2) The pair $(1_X, \epsilon)$ forms a cell morphism $(X \uparrow \rho)^{-1} \circ (X \uparrow \lambda) \to 1_{\mathcal{M}}$; that is, the diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{m \circ (X \uparrow \rho)^{-1} \circ (X \uparrow \lambda)} & F \circ G \circ a \\
  1_x \downarrow & & \downarrow \epsilon_a \\
  x & \xrightarrow{m \circ (X \uparrow \alpha)^{-1} \circ (X \uparrow \rho)} & a
\end{array}
\]

commutes for every $\mathcal{M}$-arrow $m : x \sim a$.

Proof.

(1) By Theorem 7.3.15, the diagram

\[
\begin{array}{ccc}
  y & \xrightarrow{f \circ \lambda_x} & x \\
  f \downarrow & & \downarrow \eta_x \\
  x & \xrightarrow{\lambda_x \circ (X \uparrow \lambda \circ \rho(F \circ x))} & G \circ F \circ x
\end{array}
\]

commutes; that is,

$$f \circ \lambda_x = f \circ \eta_x \circ \rho(F \circ x)$$

but

$$f \circ \lambda_x = f \circ (X \uparrow \lambda)$$

and

$$f \circ \eta_x \circ \rho(F \circ x) = (f \circ \eta_x \circ \rho(F \circ x))$$

hence

$$f \circ (X \uparrow \lambda) = f \circ (X \uparrow \rho)$$

i.e.

$$f \circ (X \uparrow \lambda) \circ (X \uparrow \rho)^{-1} = f \circ \eta_x$$

as required.

(2) By Theorem 7.3.15, the diagram

\[
\begin{array}{ccc}
  a & \xrightarrow{\lambda_{(a \uparrow G)}} & F \circ G \circ a \\
  m/\rho_a \downarrow & & \downarrow \epsilon_a \\
  x & \xrightarrow{\rho_a \circ m} & a
\end{array}
\]

commutes, where $m/\rho_a$ is the adjunct of $m$ along $\rho_a$. Since

$$m/\rho_a \circ \lambda_{(a \uparrow G)} = (m \circ (X \uparrow \rho)^{-1}) \circ \lambda_{(a \uparrow G)} = m \circ (X \uparrow \rho)^{-1} \circ (X \uparrow \lambda)$$

we have

$$m = m/\rho_a \circ \lambda_{(a \uparrow G)} \circ \epsilon_a = (m \circ (X \uparrow \rho)^{-1} \circ (X \uparrow \lambda)) \circ \epsilon_a$$

as required. \hfill \Box
8 Limits

8.1 Limits

A limit is defined as a universal cone; specifically, a limit of a functor $E \to A$ along a module $M : X \to A$ is defined by a universal arrow of the module $(E,M)$ defined in Section 4.6, and, as a special case, a limit of a functor $E \to C$ in $C$ is defined by a universal arrow of the module $(E,C)$ defined in Section 4.9. Limits defined here are sometimes referred to as conical limits to distinguish them from cellular limits (or weighted limits) to be introduced in Section 12.3.

Note. In the following, we consider a cone $E \to M$ as an arrow of the module $(E,M)$ defined in Definition 4.6.7.

8.1.1 Definition. Let $E$ be a category and $M : X \to A$ be a module.

- A cone $\star \leftarrow E \quad \text{is called universal if it is an inverse universal} \quad (E,M)\text{-arrow.}$
  
  Given a functor $F : E \to A$, a universal cone $\mu : r \sim F : E \to M$ or the pair $(r, \mu)$, or the object $r$ itself, is called a limit of $F$ along $M$, with the object $r$ denoted by $\prod E F$.

- A cone $G : E \to X$, a universal cone $\mu : G \sim r : E \to M$ or the pair $(r, \mu)$, or the object $r$ itself, is called a colimit of $G$ along $M$, with the object $r$ denoted by $\sqcup E G$.

8.1.2 Remark.

1. A cone $\mu : r \sim F : E \to M$ is universal if and only if to every cone $\alpha : x \sim F : E \to M$ there is a unique $X$-arrow $\alpha/\mu : x \to r$ (the adjunct of $\alpha$ along $\mu$) such that $\alpha = \alpha/\mu \circ \mu$. Dually, a cone $\mu : G \sim r : E \to M$ is universal if and only if to every cone $\alpha : G \sim a : E \to M$ there is a unique $A$-arrow $\alpha \circ \mu/\alpha : r \to a$ such that $\alpha = \mu \circ \mu/\alpha$.

2. A limit $\prod E F$ [op. colimit $\sqcup E G$] is called a product [op. coproduct] when $E$ is discrete.

3. Limits are unique up to isomorphism by Corollary 6.2.8.

8.1.3 Proposition.

- If $\tau : F \to F'$ is a natural isomorphism, then a cone $\mu : r \sim F : E \to M$ is universal if and only if its composite $\mu \circ \tau : r \sim F' \quad \text{with} \quad \tau$ is.

- If $\tau : G' \to G$ is a natural isomorphism, then a cone $\mu : G \sim r : E \to M$ is universal if and only if its composite $\tau \circ \mu : G' \sim r \quad \text{with} \quad \tau$ is.

Proof. This is an instance of Corollary 6.2.17.

8.1.4 Theorem. Given a module and functors as in

\[
\begin{array}{cccccc}
X \xrightarrow{\mu} A & \xleftarrow{\star} E & \xrightarrow{F} K & D & \xrightarrow{G} X \xrightarrow{M} A \\
\end{array}
\]

\[
\begin{array}{cccccc}
\star \xleftarrow{\mu} D & \xrightarrow{F} K & E & \xrightarrow{G} X & \xrightarrow{M} A \\
\end{array}
\]

, a cone

\[
\begin{array}{cccccc}
\star \xleftarrow{\mu} D & \xrightarrow{F} K & E & \xrightarrow{G} X & \xrightarrow{M} A \\
\end{array}
\]

\[
\begin{array}{cccccc}
\star \xleftarrow{\mu} D & \xrightarrow{F} K & E & \xrightarrow{G} X & \xrightarrow{M} A \\
\end{array}
\]

\[
\begin{array}{cccccc}
\star \xleftarrow{\mu} D & \xrightarrow{F} K & E & \xrightarrow{G} X & \xrightarrow{M} A \\
\end{array}
\]

\[
\begin{array}{cccccc}
\star \xleftarrow{\mu} D & \xrightarrow{F} K & E & \xrightarrow{G} X & \xrightarrow{M} A \\
\end{array}
\]
is a limit [op. colimit] of the composite functor $F \circ K$ [op. $K \circ G$] along $\mathcal{M}$ if and only if $\mu$ depicted as

$$
\begin{array}{ccc}
\star & \leftarrow & D \\
\downarrow^{r} & \mu & \downarrow^{i} \\
X & \xrightarrow{\langle M \rangle F} & E \\
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
D & \rightarrow & \star \\
F \downarrow^{\mu} \downarrow^{i} & K \downarrow^{\mu} \downarrow^{i} \\
E & \xrightarrow{G \langle M \rangle} & A \\
\end{array}
$$

is a limit [op. colimit] of $K$ along the composite module $(\mathcal{M}) F$ [op. $G(\mathcal{M})$].

**Proof.** By Proposition 6.2.25, the cell

$$
\begin{array}{ccc}
X & \xrightarrow{\langle (\mathcal{M}) F \rangle} [D, E] \\
\downarrow^{1} & \downarrow^{1} & \downarrow^{[D, F]} \\
X & \xrightarrow{\langle \pi_{D, M} \rangle} [D, A] \\
\end{array}
$$

(see Proposition 4.6.9) preserves and reflects inverse universal arrows; that is, an $(\star D, \langle \mathcal{M} \rangle F)$-arrow $\mu: r \leadsto K$ is inverse universal iff so is the $(\star D, \mathcal{M})$-arrow $\mu: r \leadsto [D, F] \circ K$; that is, a cone $\mu: r \leadsto K: \star D \leadsto \langle \mathcal{M} \rangle F$ is universal iff so is the cone $\mu: r \leadsto F \circ K: \star D \leadsto \mathcal{M}$.

**8.1.5 Definition.** A module $\mathcal{M}: X \rightarrow A$ is said to

- have limits over a category $E$ if every functor $E \rightarrow A$ has a limit along $\mathcal{M}$.
- have colimits over a category $E$ if every functor $E \rightarrow X$ has a colimit along $\mathcal{M}$.

**8.1.6 Proposition.** A module $\mathcal{M}: X \rightarrow A$ has limits [op. colimits] over a category $E$ if and only if the module $(\star E, \mathcal{M})$ [op. $(E^*, \mathcal{M})$] has a counit [op. unit] as in

$$
\begin{array}{ccc}
X & \xrightarrow{\mu} & [E, A] \\
\downarrow^{\Pi_{E}} & \downarrow^{\Pi_{\langle E, \mathcal{M} \rangle}} & \downarrow^{\Pi_{E}} \\
[E, X] & \xrightarrow{\mu} & A \\
\end{array}
$$

**Proof.** Since a limit $\mu: \Pi_{E} F \leadsto F: \star E \leadsto \mathcal{M}$ of a functor $F: E \rightarrow A$ is defined as an inverse universal $(\star E, \mathcal{M})$-arrow to $F$, the assertion follows from the equivalence of (2) and (4) in Corollary 6.4.11.

**8.1.7 Definition.** A module $\mathcal{M}: X \rightarrow A$, locally small or not, is called complete [op. cocomplete] if it has limits [op. colimits] over any small category $E$.

**8.1.8 Proposition.**

- A module $\mathcal{M}$ is complete if and only if for any small category $E$ the module $(\star E, \mathcal{M})$ has a counit.
- A module $\mathcal{M}$ is cocomplete if and only if for any small category $E$ the module $(E^*, \mathcal{M})$ has a unit.

**Proof.** Immediate from Proposition 8.1.6.

**Note.** In the following, we consider a cone $\star E \rightarrow C$ as an arrow of the module $(\star E, C)$ defined in Definition 4.9.3.

**8.1.9 Definition.**

- A cone $\star \xrightarrow{1} E$ is called universal if it is an inverse universal $(\star E, C)$-arrow. Given a functor $\mathcal{L}: E \rightarrow C$, a universal cone $\mu: r \rightarrow \mathcal{L}: \star E \rightarrow C$ or the pair $(r, \mu)$, or the object $r$ itself, is called a limit of $\mathcal{L}$ in $C$, with the object $r$ denoted by $\Pi_{E} \mathcal{L}$.
- A cone $E \xrightarrow{1} \star$ is called universal if it is a direct universal $(E^*, C)$-arrow. Given a functor $\mathcal{L}: E \rightarrow C$, a universal cone $\mu: \mathcal{L} \rightarrow r: E^* \rightarrow C$ or the pair $(r, \mu)$, or the object $r$ itself, is called a colimit of $\mathcal{L}$ in $C$, with the object $r$ denoted by $\bigcup_{E} \mathcal{L}$. 
8.1.10 Remark.  
(1) A cone \( \mu : r \Rightarrow L : \ast E \rightarrow C \) is universal if and only if to every cone \( \alpha : c \Rightarrow L : \ast E \rightarrow C \) there is a unique \( C \)-arrow \( \alpha / \mu : c \rightarrow r \) (the adjunct of \( \alpha \) along \( \mu \)) such that \( \alpha = \alpha / \mu \circ \mu \). Dually, a cone \( \mu : L \Rightarrow r : E \ast \rightarrow C \) is universal if and only if to every cone \( \alpha : L \Rightarrow c : E \ast \rightarrow C \) there is a unique \( C \)-arrow \( \mu \backslash \alpha : r \rightarrow c \) such that \( \alpha = \mu \backslash \mu \circ \alpha \).
(2) Limits and colimits are unique up to isomorphism by Corollary 6.2.8.
(3) Proposition 8.1.3 also holds for limits and colimits in a category.

8.1.11 Theorem. Given functors \( D \xrightarrow{K} E \xrightarrow{L} C \), a cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{\iota} & D \\
\mu \downarrow & \xrightarrow{L \circ \kappa} & C \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\iota} & \ast \\
\mu \downarrow & \xrightarrow{\kappa \circ \mu} & C \\
C & \xrightarrow{\langle C \rangle L} & E
\end{array}
\]

is a limit [op. colimit] of the composite functor \( K \circ \mu \) if and only if \( \mu \) depicted as

\[
\begin{array}{ccc}
\ast & \xleftarrow{\iota} & D \\
\mu \downarrow & \xrightarrow{\kappa} & E \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\iota} & \ast \\
\mu \downarrow & \xrightarrow{\kappa} & C \\
C & \xrightarrow{\langle C \rangle} & C
\end{array}
\]

is a limit [op. colimit], in the sense of Definition 8.1.1, of \( K \) along the composite module \( \langle C \rangle L \) [op. \( L(C) \)].

Proof. This is a special case of Theorem 8.1.4 where \( M \) is given by the hom-module of \( C \).

8.1.12 Definition. A category \( C \) is said to have limits [op. colimits] over a category \( E \) if every functor \( E \rightarrow C \) has a limit [op. colimits] in \( C \).

Note. The following is a special case of Proposition 8.1.6 where \( M \) is given by the hom-module of a category.

8.1.13 Proposition. A category \( C \) has limits [op. colimits] over a category \( E \) if and only if the module \( \langle E, C \rangle \) [op. \( \langle E \ast, C \rangle \)] has a counit [op. unit] as in

\[
\begin{array}{ccc}
C & \xrightarrow{\mu} & \langle E, C \rangle \\
\xrightarrow{\langle C \rangle} & \xrightarrow{\Pi E} & \xrightarrow{\mu} & \langle E, C \rangle & \xrightarrow{\Pi E} & C
\end{array}
\]

Proof. Since a limit \( \mu : \Pi E L \Rightarrow L : \ast E \rightarrow C \) of a functor \( L : E \rightarrow C \) is defined as an inverse universal \( \langle E, C \rangle \)-arrow to \( L \), the assertion follows from the equivalence of (2) and (4) in Corollary 6.4.11.

8.1.14 Remark. Since the module \( \langle E, C \rangle \) [op. \( \langle E \ast, C \rangle \)] is represented [op. corepresented] by the diagonal functor \( \langle 1_E, C \rangle \) (see Remark 4.9.4(2)), noting Remark 7.3.2(2), Proposition 8.1.13 is alternatively stated as below in terms of adjunctions.

\( \triangleright \) A category \( C \) has limits over a category \( E \) if and only if the diagonal functor \( \langle 1_E, C \rangle \) has a right adjoint.

\( \triangleright \) A category \( C \) has colimits over a category \( E \) if and only if the diagonal functor \( \langle 1_E, C \rangle \) has a left adjoint.

8.1.15 Definition. A category \( C \) is called complete [op. cocomplete] if it has limits [op. colimits] over any small category \( E \).

8.1.16 Remark. A category \( C \) is complete [op. cocomplete] if and only if the hom-module \( \langle C \rangle \) is complete [op. cocomplete] in the sense of Definition 8.1.7.

Note. The following is a special case of Proposition 8.1.8 where \( M \) is given by the hom-module of a category.
8.2. Preservation of limits

In Section 8.1, a limit of a functor $E \to A$ along a module $M : X \to A$ is defined as a universal arrow of the module $(E, M)$. Now we say that a cell $\psi : M \to N$ preserve limits over $E$ when the postcomposition cell $(E, \psi) : (E, M) \to (E, N)$ (see Definition 4.6.17) preserves universal arrows. Similarly, as a special case, we say that a functor $H : C \to B$ preserve limits over $E$ when the postcomposition cell $(E, H) : (E, C) \to (E, B)$ (see Definition 4.9.5) preserves universal arrows.

8.2.1 Definition. A cell $X \xrightarrow{\psi} A$ is said to preserve (resp. reflect, create) limits [op. colimits] over a category $E$ if the postcomposition cell

$$
\begin{array}{c}
X \xrightarrow{(\psi, M)} [E, A] \\
\downarrow \int_{\psi} \downarrow \int_{Q} \\
Y \xrightarrow{(\psi, N)} [E, B]
\end{array}
$$

preserves (resp. reflects, creates) inverse [op. direct] universal arrows.

8.2.2 Remark.

1) Recalling the definition of the postcomposition cell, Definition 8.2.1 can be stated in elementary terms as follows: $\psi$ is said to preserve

(a) limits over $E$ if each universal cone $\mu : r \sim F : E \sim M$ yields by composition with $\psi$ a universal cone $\mu \circ \psi : r \cdot P \sim Q \circ F : E \sim N$.

(b) colimits over $E$ if each universal cone $\mu : G \sim r : E* \sim M$ yields by composition with $\psi$ a universal cone $\mu \circ \psi : G \circ P \sim Q \circ r : E* \sim N$.

(c) create

(i) limits over $E$ if for every functor $F : E \to A$ and for every universal cone $\kappa : s \sim Q \circ F : E \sim N$ there is exactly one cone $\mu : r \sim F : E \sim M$ with $\mu \circ \psi = \kappa$, and if this $\mu$ is universal.

(ii) colimits over $E$ if for every functor $G : E \to X$ and for every universal cone $\kappa : G \circ P \sim s : E* \sim N$ there is exactly one cone $\mu : G \sim r : E* \sim M$ with $\mu \circ \psi = \kappa$, and if this $\mu$ is universal.

2) A cell $\psi$ is said to preserve (resp. reflect, create) limits [op. colimits] if it does so over any category, and said to preserve (resp. reflect, create) small limits [op. colimits] if it does so over any small category.

3) We also say that a cell $\psi$ is

(a) continuous if $M$ is complete and $\psi$ preserves small limits.
8.2.3 Proposition. Consider a cell $\psi$ as in Definition 8.2.1.

- If $\psi$ is fully faithful and $P$ is iso, then $\psi$ preserves, reflects, and creates limits.
- If $\psi$ is fully faithful and $Q$ is iso, then $\psi$ preserves, reflects, and creates colimits.

Proof. The assertion is equivalent to saying that if $\psi$ is fully faithful and $P$ is iso, then for any category $E$, the postcomposition cell $\langle *E, \psi \rangle$ preserves, reflects, and creates inverse universal arrows. But since the fully faithfulness of $\psi$ implies the fully faithfulness of $\langle E, \psi \rangle$ (Proposition 4.6.19), this follows from Proposition 6.2.25.

8.2.4 Proposition. Consider a cell $\psi$ as in Definition 8.2.1.

- If $P$ and $\psi$ are fully faithful, then $\psi$ reflects limits.
- If $Q$ and $\psi$ are fully faithful, then $\psi$ reflects colimits.

Proof. This is equivalent to saying that if $P$ and $\psi$ are fully faithful, then for any category $E$, the postcomposition cell $\langle *E, \psi \rangle$ reflects inverse universal arrows. But since the fully faithfulness of $\psi$ implies the fully faithfulness of $\langle E, \psi \rangle$ (Proposition 4.6.19), this follows from Proposition 6.2.26.

8.2.5 Proposition. If two cells $\psi : M \to N$ and $\phi : M \to N$ are isomorphic (see Definition 1.3.3), then

- $\psi$ preserves (resp. reflects) limits if and only if $\phi$ preserves (resp. reflects) limits.
- $\psi$ preserves (resp. reflects) colimits if and only if $\phi$ preserves (resp. reflects) colimits.

Proof. This is equivalent to saying that if $\psi$ and $\phi$ are isomorphic, the postcomposition cell $\langle *E, \psi \rangle$ preserves (resp. reflects) limits if and only if the postcomposition cell $\langle *E, \phi \rangle$ preserves (resp. reflects) limits. But since the functor $\langle *E, \cdot \rangle$ (see Remark 4.6.26(2)) preserve isomorphisms, this follows from Proposition 6.2.28.

8.2.6 Proposition. Let $E$ be a category and let $\psi$ be a cell as in Definition 8.2.1.

- If $N$ has limits over $E$ and $\psi$ creates limits from them, then $M$ has limits over $E$ as well and $\psi$ preserves them.
- If $N$ has colimits over $E$ and $\psi$ creates colimits from them, then $M$ has colimits over $E$ as well and $\psi$ preserves them.

Proof. Noting Proposition 8.1.6, we see that this is an instance of Theorem 6.4.12 where $\psi$ is given by the postcomposition cell $\langle *E, \psi \rangle$.

8.2.7 Proposition. Consider a cell $\psi$ as in Definition 8.2.1.

- If $N$ is complete and $\psi$ creates small limits, then $M$ is complete as well and $\psi$ is continuous.
- If $N$ is cocomplete and $\psi$ creates small colimits, then $M$ is cocomplete as well and $\psi$ is cocontinuous.

Proof. Immediate from Proposition 8.2.6.

8.2.8 Theorem. Let $M : X \to A$ be a module.

- Any corepresentation $X \xrightarrow{\phi} A$ of $M$ preserves, reflects, and creates limits.
- Any representation $X \xrightarrow{\phi} A$ of $M$ preserves, reflects, and creates colimits.

Proof. Immediate from Proposition 8.2.3.
8.2.9 Corollary. Let $\mathcal{M}: \mathbf{X} \to \mathbf{A}$ be a module.

- If $\mathcal{M}$ is corepresentable and $\mathbf{X}$ is complete, then $\mathcal{M}$ is complete as well.
- If $\mathcal{M}$ is representable and $\mathbf{A}$ is cocomplete, then $\mathcal{M}$ is cocomplete as well.

Proof. Since a corepresentation of $\mathcal{M}$ creates limits by Theorem 8.2.8, the assertion follows from Proposition 8.2.7. $$\square$$

8.2.10 Theorem. Any equivalence cell (see Definition 7.13.1) preserves and reflects limits [op. colimits].

Proof. By Definition 8.2.1, the assertion is equivalent to saying that if a cell $\psi$ is an equivalence, then for any category $\mathbf{E}$, the postcomposition cell $(\ast \mathbf{E}, \psi)$ preserves and reflects inverse universal arrows. But this follows from Theorem 7.13.13 because if $\psi$ is an equivalence, so is $(\ast \mathbf{E}, \psi)$ by Corollary 7.13.12. $$\square$$

Note. Since a limit is defined by a universal $(\ast \mathbf{E}, \mathbf{C})$-arrow, we can describe the preservation of limits using the postcomposition cell $(\ast \mathbf{E}, \mathbf{H})$ defined in Definition 4.9.5.

8.2.11 Definition. A functor $\mathbf{H}: \mathbf{C} \to \mathbf{B}$ is said to preserve (resp. reflect, create) limits [op. colimits] over a category $\mathbf{E}$ if the postcomposition cell

$$
\begin{array}{ccc}
\mathbf{C} \ar[r]^{(\ast \mathbf{E}, \mathbf{C})} & [\mathbf{E}, \mathbf{C}] & \text{op.} \ar[l] \ar[d]^H \\
\mathbf{B} \ar[r]_{(\ast \mathbf{E}, \mathbf{B})} & [\mathbf{E}, \mathbf{B}] & [\mathbf{E}, \mathbf{C}] \ar[d]_{(\ast \mathbf{E}, \mathbf{H})} \\
& [\mathbf{E}, \mathbf{H}] & [\mathbf{E}, \mathbf{H}] \ar[l]_H
\end{array}
$$

preserves (resp. reflects, creates) inverse [op. direct] universal arrows.

8.2.12 Remark. (1) Recalling the definition of the postcomposition cell, Definition 8.2.11 can be stated in elementary terms as follows: $\mathbf{H}$ is said to

- a) preserve
  - limits over $\mathbf{E}$ if each universal cone $\mu: r \rightsquigarrow \mathbf{L}: * \mathbf{E} \to \mathbf{C}$ yields by composition with $\mathbf{H}$ a universal cone $\mu \circ \mathbf{H}: r \circ \mathbf{H} \rightsquigarrow \mathbf{H} \circ \mathbf{F}: * \mathbf{E} \to \mathbf{B}$.
  - colimits over $\mathbf{E}$ if each universal cone $\mu: \mathbf{L} \rightsquigarrow \mathbf{r}: \mathbf{E}^\ast \to \mathbf{C}$ yields by composition with $\mathbf{H}$ a universal cone $\mu \circ \mathbf{H}: \mathbf{L} \circ \mathbf{H} \rightsquigarrow \mathbf{H} \circ \mathbf{r}: \mathbf{E}^\ast \to \mathbf{B}$.

- b) reflect
  - limits over $\mathbf{E}$ if a cone $\mu: r \rightsquigarrow \mathbf{L}: * \mathbf{E} \to \mathbf{C}$ is universal whenever the cone $\mu \circ \mathbf{H}: r \circ \mathbf{H} \rightsquigarrow \mathbf{H} \circ \mathbf{L}: * \mathbf{E} \to \mathbf{B}$ is universal.
  - colimits over $\mathbf{E}$ if a cone $\mu: \mathbf{L} \rightsquigarrow \mathbf{r}: \mathbf{E}^\ast \to \mathbf{C}$ is universal whenever the cone $\mu \circ \mathbf{H}: \mathbf{L} \circ \mathbf{H} \rightsquigarrow \mathbf{H} \circ \mathbf{r}: \mathbf{E}^\ast \to \mathbf{B}$ is universal.

- c) create
  - limits over $\mathbf{E}$ if for every functor $\mathbf{L}: \mathbf{E} \to \mathbf{C}$ and for every universal cone $\kappa: s \rightsquigarrow \mathbf{H} \circ \mathbf{L}: * \mathbf{E} \to \mathbf{B}$ there is exactly one cone $\mu: r \rightsquigarrow \mathbf{L}: * \mathbf{E} \to \mathbf{C}$ with $\mu \circ \mathbf{H} = \kappa$, and if this $\mu$ is universal.
  - colimits over $\mathbf{E}$ if for every functor $\mathbf{L}: \mathbf{E} \to \mathbf{C}$ and for every universal cone $\kappa: \mathbf{L} \circ \mathbf{H} \rightsquigarrow \mathbf{s}: \mathbf{E}^\ast \to \mathbf{B}$ there is exactly one cone $\mu: \mathbf{L} \rightsquigarrow \mathbf{r}: \mathbf{E}^\ast \to \mathbf{C}$ with $\mu \circ \mathbf{H} = \kappa$, and if this $\mu$ is universal.

(2) Since $(\ast \mathbf{E}, \mathbf{H}) = (\ast \mathbf{E}, \{\mathbf{H}\})$ [op. $(\mathbf{E}^\ast, \mathbf{H}) = (\mathbf{E}^\ast, \{\mathbf{H}\})$], a functor $\mathbf{H}$ preserves (resp. reflects, creates) limits [op. colimits] over a category $\mathbf{E}$ if and only if the hom-cell $(\mathbf{H})$ does so in the sense of Definition 8.2.1.

(3) A functor $\mathbf{H}$ is said to preserve (resp. reflect, create) small limits [op. colimits] if it does so over any category, and said to preserve (resp. reflect, create) small limits [op. colimits] if it does so over any small category.
(4) We also say that functor $H$ is
  - continuous if $C$ is complete and $H$ preserves small limits.
  - cocontinuous if $C$ is cocomplete and $H$ preserves small colimits.

**Note.** The following is a special case of Proposition 8.2.4 where $\psi$ is given by the hom-cell of a functor.

**8.2.13 Proposition.** A fully faithful functor reflects limits [op. colimits].

**Proof.** Let $H : C \to B$ be a fully faithful functor. The assertion say that, for any category $E$, the postcomposition cell $(\ast E, H)$ reflects inverse universal arrows. But this follows from Proposition 6.2.26 because if $H$ is fully faithful, so is $(\ast E, H)$ by Proposition 4.9.7.

**Note.** The following is a special case of Proposition 8.2.5 where $\psi$ is given by the hom-cell of a functor.

**8.2.14 Proposition.** If two functor $G : C \to B$ and $F : C \to B$ are isomorphic (i.e. if there is a natural isomorphism $G \to F$), then
  - $G$ preserves (resp. reflects) limits if and only if $F$ preserves (resp. reflects) limits.
  - $G$ preserves (resp. reflects) colimits if and only if $F$ preserves (resp. reflects) colimits.

**Proof.** This is equivalent to saying that if $G$ and $F$ are isomorphic, the postcomposition cell $(\ast E, G)$ preserves (resp. reflects) limits if and only if the postcomposition cell $(\ast E, F)$ preserves (resp. reflects) limits. But since the functor $(\ast E, \ast)$ (see Remark 4.9.9) preserve isomorphisms, this follows from Proposition 6.2.28.

**Note.** The following is a special case of Proposition 8.2.6 where $\psi$ is given by the hom-cell of a functor.

**8.2.15 Proposition.** Let $H : C \to B$ be a functor.
  - If $B$ has limits over $E$ and $H$ creates limits from them, then $C$ has limits over $E$ as well and $H$ preserves them.
  - If $B$ has colimits over $E$ and $H$ creates colimits from them, then $C$ has colimits over $E$ as well and $H$ preserves them.

**Proof.** Noting Proposition 8.1.13, we see that this is an instance of Theorem 6.4.12 where $\psi$ is given by the postcomposition cell $(\ast E, H)$.

**Note.** The following is a special case of Proposition 8.2.7 where $\psi$ is given by the hom-cell of a functor.

**8.2.16 Proposition.** Let $H : C \to B$ be a functor.
  - If $B$ is complete and $H$ creates small limits, then $C$ is complete as well and $H$ is continuous.
  - If $B$ is cocomplete and $H$ creates small colimits, then $C$ is cocomplete as well and $H$ is cocontinuous.

**Proof.** Immediate from Proposition 8.2.15.

**Note.** The following is a special case of Theorem 8.2.10 where a cell is given by the hom-cell of a functor.

**8.2.17 Theorem.** Any equivalence functor preserves and reflects limits [op. colimits].

**Proof.** Since a functor preserves and reflects limits iff its hom-cell preserves and reflects limits (see Remark 8.2.12(2)), and since a functor is an equivalence iff its hom-cell is an equivalence (see Proposition 7.13.3), the assertion is reduced to Theorem 8.2.10.
8.2.18 Corollary. If functors $S : X \to X'$ and $T : A \to A'$ are conjugate to each other as in Definition 7.12.3, then

- $S$ preserves (resp. reflects) limits if and only if $T$ preserves (resp. reflects) limits.
- $S$ preserves (resp. reflects) colimits if and only if $T$ preserves (resp. reflects) colimits.

Proof. Immediate from Theorem 8.2.17 and Proposition 8.2.14.

Note. Recall from Proposition 8.1.13 that if a category $C$ has limits over a category $E$, the module $(\ast E, C)$ has a counit $(\prod E, \mu)$.

8.2.19 Theorem. Consider a functor $H : C \to B$. If $C$ and $B$ have limits [op. colimits] over a category $E$, then there is the canonical natural transformation

$$
\begin{array}{ccc}
C \xrightarrow{\prod E} [E, C] & \text{op.} & [E, C] \xrightarrow{\prod E} C \\
\downarrow H & \Downarrow \psi & \downarrow H \\
B \xrightarrow{\prod E} [E, B] & & [E, B] \xrightarrow{\prod E} B \\
\end{array}
$$

; this natural transformation is an isomorphism if and only if $H$ preserves limits [op. colimits] over $E$.

Proof. This is an instance of Corollary 6.5.23 where $\psi$ is given by the postcomposition cell $(\ast E, C)$.

8.3 Limits with parameters

A parameterized limit is defined as a pointwise universal wedge or any of its transposes (see Section 4.8). Left exponential transposition transforms a pointwise universal wedge into a pointwise lift along the module of cones, allowing us to apply the results in Section 6.5 to parameterized limits.

Note. In the following, we consider a wedge $E \times \ast D \to M$ (see Definition 4.8.1) as an arrow of the module $(E \times D, M)$ defined in Definition 4.8.3.

8.3.1 Definition. Given a module $M : X \to A$ and categories $E$ and $D$, a wedge

$$
\begin{array}{ccc}
E \xrightarrow{E \times \ast} E \times D & \text{op.} & E \times D \xrightarrow{E \times \ast} E \\
\downarrow \mu & \downarrow \psi & \downarrow \mu \\
X \xrightarrow{\ast M} A & & X \xrightarrow{\ast A} A \\
\end{array}
$$

is called universal if it is an inverse universal $(E \times \ast D, M)$-arrow [op. direct universal $(E \times D \ast, M)$-arrow].

- Given a functor $F : E \times D \to A$, a universal wedge $\mu : R \to F : E \times \ast D \to M$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a limit of $F$ along $M$, with the functor $R$ denoted by $\prod D F$.
- Given a functor $G : E \times D \to X$, a universal wedge $\mu : G \to R : E \times D \to M$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a colimit of $G$ along $M$, with the functor $R$ denoted by $\bigcup D G$.

Note. Recall from Definition 4.8.5 that a wedge is sliced into pieces of cones. In the following, we consider the case where these cones are universal (see Definition 8.1.1).

8.3.2 Definition. Given a module $M : X \to A$ and categories $E$ and $D$, a wedge

$$
\begin{array}{ccc}
E \xrightarrow{E \times \ast} E \times D & \text{op.} & E \times D \xrightarrow{E \times \ast} E \\
\downarrow \mu & \downarrow \psi & \downarrow \mu \\
X \xrightarrow{\ast M} A & & X \xrightarrow{\ast A} A \\
\end{array}
$$

is called universal if it is an inverse universal $(E \times D, M)$-arrow [op. direct universal $(E \times D \ast, M)$-arrow].
is called pointwise universal if each left slice

\[
\begin{array}{ccc}
\ast & \to & D \\
\downarrow^\ast & & \downarrow^G \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
D & \to & \ast \\
\downarrow^G & & \downarrow^R \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\]

is a universal cone, i.e. an inverse universal \((\ast \mathcal{D}, \mathcal{M})\)-arrow [op. direct universal \((\mathcal{D} \ast, \mathcal{M})\)-arrow].

- Given a functor \(F: \mathcal{D} \to \mathcal{A}\), a pointwise universal wedge \(\mu: \mathcal{R} \to F: \mathcal{D} \to \mathcal{M}\) or the pair \((\mathcal{R}, \mu)_\ast\), or the functor \(\mathcal{R}\) itself, is called an \(\mathcal{E}\)-parameterized limit of \(F\) along \(\mathcal{M}\), with the functor \(\mathcal{R}\) denoted by \(\prod \mathcal{D} F\) or just by \(\prod \mathcal{D}\).

- Given a functor \(G: \mathcal{D} \times \mathcal{D} \to \mathcal{X}\), a pointwise universal wedge \(\mu: G \to \mathcal{R}: \mathcal{D} \times \mathcal{D} \to \mathcal{M}\) or the pair \((\mathcal{R}, \mu)_\ast\), or the functor \(\mathcal{R}\) itself, is called an \(\mathcal{E}\)-parameterized colimit of \(G\) along \(\mathcal{M}\), with the functor \(\mathcal{R}\) denoted by \(\prod \mathcal{D} G\) or just by \(\prod \mathcal{D}\).

**Note.** Since a wedge \(\mathcal{D} \times \mathcal{D} \to \mathcal{M}\) is transposed into a cone \(\mathcal{D} \to (\mathcal{E}, \mathcal{M})\) (see Definition 4.8.5), Definition 8.3.2 has the following variation.

### 8.3.3 Definition

Given a module \(\mathcal{M}: \mathcal{X} \to \mathcal{A}\) and categories \(\mathcal{E}\) and \(\mathcal{D}\), a cone

\[
\begin{array}{ccc}
\ast & \to & \mathcal{D} \\
\downarrow^\ast & & \downarrow^G \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
\mathcal{D} & \to & \ast \\
\downarrow^G & & \downarrow^\mathcal{R} \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\]

along the module \((\mathcal{E}, \mathcal{M})\) is called pointwise universal if each slice

\[
\begin{array}{ccc}
\ast & \to & \mathcal{D} \\
\downarrow^\ast & & \downarrow^G \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
\mathcal{D} & \to & \ast \\
\downarrow^G & & \downarrow^\mathcal{R} \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\]

(see Remark 4.8.8(2)) is a universal cone, i.e. an inverse universal \((\ast \mathcal{D}, \mathcal{M})\)-arrow [op. direct universal \((\mathcal{D} \ast, \mathcal{M})\)-arrow].

- Given a functor \(F: \mathcal{D} \to [\mathcal{E}, \mathcal{A}]\), a pointwise universal cone \(\mu: \mathcal{R} \to F: \mathcal{D} \to (\mathcal{E}, \mathcal{M})\) or the pair \((\mathcal{R}, \mu)_\ast\), or the functor \(\mathcal{R}\) itself, is called a pointwise limit of \(F\) along \((\mathcal{E}, \mathcal{M})\), with the functor \(\mathcal{R}\) denoted by \(\prod \mathcal{D} F\) or just by \(\prod \mathcal{D}\).

- Given a functor \(G: \mathcal{D} \times \mathcal{D} \to [\mathcal{E}, \mathcal{X}]\), a pointwise universal cone \(\mu: G \to \mathcal{R}: \mathcal{D} \times \mathcal{D} \to (\mathcal{E}, \mathcal{M})\) or the pair \((\mathcal{R}, \mu)_\ast\), or the functor \(\mathcal{R}\) itself, is called a pointwise colimit of \(G\) along \((\mathcal{E}, \mathcal{M})\), with the functor \(\mathcal{R}\) denoted by \(\prod \mathcal{D} G\) or just by \(\prod \mathcal{D}\).

### 8.3.4 Proposition

The exponential transposition in Definition 4.8.5 preserves and reflects universality and pointwise universality; that is, the following conditions are equivalent;

1. a wedge

\[
\begin{array}{ccc}
\mathcal{E} & \to & \mathcal{E} \times \mathcal{D} \\
\downarrow^\mathcal{R} & & \downarrow^F \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
\mathcal{E} \times \mathcal{D} & \to & \mathcal{E} \\
\downarrow^G & & \downarrow^\mathcal{R} \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\]

is universal (resp. pointwise universal);

2. its right exponential transpose

\[
\begin{array}{ccc}
\ast & \to & \mathcal{D} \\
\downarrow^\ast & & \downarrow^F \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\quad \text{op.} \quad
\begin{array}{ccc}
\mathcal{D} & \to & \ast \\
\downarrow^G & & \downarrow^\mathcal{R} \\
\mathcal{M} & \to & \mathcal{A}
\end{array}
\]

is a universal (resp. pointwise universal) cone;
(3) its left exponential transpose
\[
\begin{array}{ccl}
\begin{array}{c}
\Delta_X \xrightarrow{\mu} \Delta_A \\
\end{array} & \xleftarrow{G} & \begin{array}{c}
\Delta_{[D,X]} \xrightarrow{\mu} \Delta_A \\
\end{array}
\end{array}
\]

is a universal (resp. pointwise universal) cylinder.

Proof. By the isomorphisms in Remark 4.8.6, \(\mu\) is universal iff \(\mu^{\uparrow}\) is universal iff \(\mu^\ast\) is universal. The pointwise version is immediate from Remark 4.8.8(3).

8.3.5 Remark. By Proposition 8.3.4 and noting the bijectivity of exponential transposition, we see that given a module \(M : X \to A\),
- the following mean the same thing:
  1. a limit (resp. \(E\)-parameterized limit) of \(F : D \times E \to A\) along \(M\);
  2. a limit (resp. pointwise limit) of \(\{F, \phi\} : D \to [E, A]\) along the module \(\{E, M\}\);
  3. a lift (resp. pointwise lift) of \(\{\phi, \mu\} : E \to [D, A]\) along the module \(\{E^\ast, M\}\).
- the following mean the same thing:
  1. a colimit (resp. \(E\)-parameterized colimit) of \(G : E \times D \to X\) along \(M\);
  2. a colimit (resp. pointwise colimit) of \(\{G, \phi\} : D \to [E, X]\) along the module \(\{E, M\}\);
  3. a colift (resp. pointwise colift) of \(\{\phi, \mu\} : E \to [D, X]\) along the module \(\{E^\ast, M\}\).

8.3.6 Proposition. A pointwise universal wedge in Definition 8.3.2 is universal in the sense of Definition 8.3.1.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of (1) and (3) in Proposition 8.3.4.

8.3.7 Proposition. The twist transposition in Definition 4.8.7 preserves and reflects universality and pointwise universality; that is, the following conditions are equivalent;
1. a cone
\[
\begin{array}{c}
\begin{array}{c}
\Delta_X \xrightarrow{\mu} \Delta_A \\
\end{array} \xleftarrow{R} \begin{array}{c}
\Delta_{[E,X]} \xrightarrow{\mu} \Delta_A \\
\end{array}
\end{array}
\]
is universal (resp. pointwise universal);
2. its twist transpose
\[
\begin{array}{ccl}
\begin{array}{c}
\Delta_X \xrightarrow{\mu^\ast} \Delta_A \\
\end{array} & \xleftarrow{G^\ast} & \begin{array}{c}
\Delta_{[D,X]} \xrightarrow{\mu^\ast} \Delta_A \\
\end{array}
\end{array}
\]
is a universal (resp. pointwise universal) cylinder.

Proof. By the isomorphism in Remark 4.8.8(1), \(\mu\) is universal iff \(\mu^\ast\) is universal. Since the slice of \(\mu\) at each \(e \in \{E\}\) is given by the component of \(\mu^\ast\) at \(e\), \(\mu\) is pointwise universal iff \(\mu^\ast\) is pointwise universal.

8.3.8 Remark. By Proposition 8.3.7 and noting the bijectivity of twist transposition, we see that given a module \(M : X \to A\),
- the following mean the same thing:
  1. a limit (resp. pointwise limit) of \(F : D \to [E, A]\) along the module \(\{E, M\}\);
  2. a lift (resp. pointwise lift) of \(F^\ast : E \to [D, A]\) along the module \(\{E^\ast, M\}\).
- the following mean the same thing:
  1. a colimit (resp. pointwise colimit) of \(G : D \to [E, X]\) along the module \(\{E, M\}\);
  2. a colift (resp. pointwise colift) of \(G^\ast : E \to [D, X]\) along the module \(\{E^\ast, M\}\).
8.3.9 Proposition. A pointwise universal cone in Definition 8.3.3 is universal in the sense of Definition 8.1.1.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of (1) and (2) in Proposition 8.3.7. □

8.3.10 Proposition.

- If a wedge \( \mu : R \rightharpoonup F : E \times D \rightharpoonup M \) is universal (resp. pointwise universal), then a natural transformation \( \tau : S \to R \) is isomorphism if and only if the wedge \( \tau \circ \mu : S \rightharpoonup F : E \times D \rightharpoonup M \) is universal (resp. pointwise universal).
- If a wedge \( \mu : G \rightharpoonup R : E \times D \rightharpoonup M \) is universal (resp. pointwise universal), then a natural transformation \( \tau : R \to S \) is isomorphism if and only if the wedge \( \mu \circ \tau : G \rightharpoonup S : E \times D \rightharpoonup M \) is universal (resp. pointwise universal).

Proof. This is reduced to Proposition 6.5.11 by the equivalence of (1) and (3) in Proposition 8.3.4: the statement faithfully translates, along the isomorphism \( \cong \) in Remark 4.8.6, into an instance of Proposition 6.5.11 where \( M \) is given by the module \( \langle D, M \rangle \).

8.3.11 Proposition.

- If a functor \( F : D \to [E, A] \) has a pointwise limit along the module \( \langle E, M \rangle \), then every limit of \( F \) along \( \langle E, M \rangle \) is pointwise.
- If a functor \( G : D \to [E, X] \) has a pointwise colimit along the module \( \langle E, M \rangle \), then every colimit of \( G \) along \( \langle E, M \rangle \) is pointwise.

Proof. By the equivalence of (1) and (2) in Proposition 8.3.7, this is reduced to an instance of Proposition 6.5.12 where \( M \) is given by the module \( \langle D, M \rangle \).

8.3.12 Theorem. Let \( E \) be a category and \( M : X \to A \) be a module as in Definition 8.3.3. Then the family of evaluations \( \langle e, M \rangle \), one for each object \( e \in \| E \| \), collectively creates a pointwise limit \( \langle \text{op. colimit} \rangle \) in the following sense: given a functor \( F : D \to [E, A] \) [op. \( G : D \to [E, X] \)], if there is a family of universal cones

\[
\begin{array}{ccc}
D & \xrightarrow{r_e} & \ast \\
\mu_e & \downarrow & \downarrow \mu \\
\ast & \xleftarrow{\mu} & \ast \\
\end{array}
\quad
g_{\langle e, A \rangle} \quad \text{op.}
\begin{array}{ccc}
D & \xrightarrow{r_e} & \ast \\
\mu_e & \downarrow & \downarrow \mu \\
\ast & \xleftarrow{\mu} & \ast \\
\end{array}
\quad
g_{\langle e, X \rangle}
\end{array}
\]

one for each \( e \in \| E \| \), then there is a unique functor \( R : E \to X \) [op. \( R : E \to A \)] with \( e : R = r_e \) such that \( \mu := ([\mu_e]_a)_{(e,d) \in \langle E \times D \rangle} \) forms a cone

\[
\begin{array}{ccc}
\ast & \xleftarrow{1} & \ast \\
\mu & \downarrow & \downarrow \mu \\
\ast & \xleftarrow{\mu} & \ast \\
\end{array}
\quad
g_{\langle e, M \rangle} \quad \text{op.}
\begin{array}{ccc}
\ast & \xleftarrow{1} & \ast \\
\mu & \downarrow & \downarrow \mu \\
\ast & \xleftarrow{\mu} & \ast \\
\end{array}
\quad
g_{\langle e, M \rangle}.
\end{array}
\]

, and \( \mu \) is pointwise universal.

Proof. By the equivalence of (1) and (2) in Proposition 8.3.7, this is reduced to an instance of Theorem 6.5.14 where \( M \) is given by the module \( \langle D, M \rangle \).

Note. By Theorem 8.3.12, if \( M \) has limits \( \langle \text{op. colimits} \rangle \) over \( D \), so does the module \( \langle E, M \rangle \). This fact is stated below in terms of units (cf. Proposition 8.1.6).

8.3.13 Theorem. Let \( E \) and \( D \) be categories and let \( M : X \to A \) be a module.
8.3. Limits with parameters 278

• If \( \mathcal{M} \) has limits over \( \mathcal{D} \); that is (see Proposition 8.1.6), if the module \( (\mathcal{D}, \mathcal{M}) \) has a counit

\[
\begin{array}{c}
\xymatrix{
\prod_{\mathcal{D}} \ar[rr]^{\mu} && \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A} \\
[\mathcal{E}, \mathcal{X}] \ar[rr]^{\mu} && \mathcal{E} \times \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A}
}
\end{array}
\]

, then this counit yields a counit

\[
\begin{array}{c}
\xymatrix{
\prod_{\mathcal{D}} \ar[rr]^{\mu} && \mathcal{E} \times \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A}
}
\end{array}
\]

of the module of wedges \( \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M} \), giving for each functor \( \mathcal{F} : \mathcal{E} \times \mathcal{D} \to \mathcal{A} \), a universal wedge

\[
\mu_{\mathcal{F}} : \prod_{\mathcal{D}} \mathcal{F} \xrightarrow{\sim} \mathcal{F} : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

, in fact a pointwise universal wedge, with each left slice

\[
\left[ e \times \mu_{\mathcal{F}} \right] : (e : \prod_{\mathcal{D}} \mathcal{F}) \xrightarrow{\sim} [e \times \mathcal{F}] : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

giving a limit

\[
\mu_{[e \times \mathcal{F}]} : \prod_{\mathcal{D}} [e \times \mathcal{F}] \xrightarrow{\sim} [e \times \mathcal{F}] : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

of the functor \( [e \times \mathcal{F}] : \mathcal{D} \to \mathcal{A} \).

• If \( \mathcal{M} \) has colimits over \( \mathcal{D} \); that is (see Proposition 8.1.6), if the module \( (\mathcal{D}, \mathcal{M}) \) has a unit

\[
\begin{array}{c}
\xymatrix{
\mathcal{D} \mathcal{X} \ar[rr]^{\mu} && \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A} \\
[\mathcal{E}, \mathcal{X}] \ar[rr]^{\mu} && \mathcal{E} \times \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A}
}
\end{array}
\]

, then this unit yields a unit

\[
\begin{array}{c}
\xymatrix{
\mathcal{D} \mathcal{X} \ar[rr]^{\mu} && \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A}
}
\end{array}
\]

of the module of wedges \( \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M} \), giving for each functor \( \mathcal{G} : \mathcal{E} \times \mathcal{D} \to \mathcal{X} \), a universal wedge

\[
\mu_{\mathcal{G}} : \prod_{\mathcal{D}} \mathcal{G} \xrightarrow{\sim} \mathcal{G} : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

, in fact a pointwise universal wedge, with each left slice

\[
\left[ e \times \mu_{\mathcal{G}} \right] : (e : \prod_{\mathcal{D}} \mathcal{G}) \xrightarrow{\sim} [e \times \mathcal{G}] : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

giving a colimit

\[
\mu_{[e \times \mathcal{G}]} : \prod_{\mathcal{D}} [e \times \mathcal{G}] \xrightarrow{\sim} [e \times \mathcal{G}] : \mathcal{E} \times \mathcal{D} \xrightarrow{\sim} \mathcal{M}
\]

of the functor \( [e \times \mathcal{G}] : \mathcal{D} \to \mathcal{X} \).

Proof. By Corollary 6.5.20, a counit \( \mu \) of the module \( (\mathcal{D}, \mathcal{M}) \) yields a counit

\[
\begin{array}{c}
\xymatrix{
\mathcal{E} \mathcal{X} \ar[rr]^{\mu} && \mathcal{E} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A} \\
[\mathcal{E}, \mathcal{X}] \ar[rr]^{\mu} && \mathcal{E} \times \mathcal{D} \ar@{|-}[r] \ar@{|-}[d] & \ar[l] \mathcal{A}
}
\end{array}
\]

of the module \( (\mathcal{E}, (\mathcal{D}, \mathcal{M})) \), and, from this counit, the iso cells in Remark 4.8.6 create a counit of the module \( (\mathcal{E} \times \mathcal{D}, \mathcal{M}) \). \( \square \)

8.3.14 Theorem. If a module \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) has limits [op. colimits] over a category \( \mathcal{D} \), so is the module \( (\mathcal{E}, \mathcal{M}) \) for any category \( \mathcal{E} \), and every limit [op. colimit] over \( \mathcal{D} \) is given pointwise.

Proof. By Theorem 8.3.13, every functor \( \mathcal{F} : \mathcal{D} \to [\mathcal{E}, \mathcal{A}] \) has a pointwise limit along \( (\mathcal{E}, \mathcal{M}) \), and every limit of \( \mathcal{F} \) is pointwise by Proposition 8.3.11. \( \square \)

8.3.15 Corollary. If a module \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) is complete [op. cocomplete], so is the module \( (\mathcal{E}, \mathcal{M}) \) for any category \( \mathcal{E} \), and all small limits [op. colimits] are given pointwise.

Proof. Immediate from Theorem 8.3.14. \( \square \)
8.3.16 Theorem. If a module \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \) is complete [op. cocomplete], then for any functor \( K : \mathbf{E}' \to \mathbf{E} \) the precomposition cell

\[
\begin{array}{c}
\mathbf{E} \times \mathbf{X} \xrightarrow{(E,M)} \mathbf{E} \times \mathbf{A} \\
\mathbf{K} \mathbf{X} \xrightarrow{K,M} \mathbf{K} \mathbf{A} \\
\mathbf{E}' \times \mathbf{X} \xrightarrow{(E',M)} \mathbf{E}' \times \mathbf{A}
\end{array}
\]

(see Definition 4.3.28) is continuous [op. cocontinuous].

Proof. Assume that \( \mathcal{M} \) is complete. We need to show that for any small category \( \mathbf{D} \), the postcomposition cell \( \langle \mathbf{K}, (\mathbf{K}, \mathcal{M}) \rangle \) preserves inverse universal arrows. Since every universal cone \( \mu : \mathbf{K} \mathbf{D} \to (\mathbf{E}, \mathcal{M}) \) is pointwise universal by Corollary 8.3.15, the problem translates (see Proposition 8.3.7) along the naturality square

\[
\begin{array}{c}
(\mathbf{K}, (\mathbf{E}, \mathcal{M})) \xrightarrow{\mathbf{K}} (\mathbf{E}, (\mathbf{K}, \mathcal{M})) \\
(\mathbf{K}, (\mathbf{K}, \mathcal{M})) \xrightarrow{\mathbf{K} \mathbf{K}} (\mathbf{K}, (\mathbf{K}, \mathcal{M})) \\
(\mathbf{K}, (\mathbf{E}', \mathcal{M})) \xrightarrow{\mathbf{K}} (\mathbf{E}', (\mathbf{K}, \mathcal{M}))
\end{array}
\]

into showing that the precomposition cell \( \langle \mathbf{K}, (\mathbf{E}, \mathcal{M}) \rangle \) preserves pointwise universality. But we have already seen this in Corollary 6.5.16. \( \square \)

8.4 Limits in functor categories

This section deals with a special case of the previous section where a module \( \mathcal{M} \) is given by the hom-module of a category \( \mathbf{C} \). Except for the results starting with Theorem 8.4.16, the section is largely analogous to Section 8.3.

Note. The following is a special case of Definition 8.3.1 where \( \mathcal{M} \) is given by the hom-module of a category; we consider a wedge \( \mathbf{E} \times \mathbf{D} \to \mathbf{C} \) (see Definition 4.9.15) as an arrow of the module \( (\mathbf{E} \times \mathbf{D}, \mathbf{C}) \) defined in Definition 4.9.17.

8.4.1 Definition. Given categories \( \mathbf{E}, \mathbf{D}, \) and \( \mathbf{C} \), a wedge

\[
\begin{array}{c}
\mathbf{E} \xrightarrow{\mathbf{E} \times \mathbf{C}} \mathbf{E} \times \mathbf{D} \\
\mathbf{R} \xrightarrow{\mu} \mathbf{L} \\
\mathbf{C} \xrightarrow{\langle \mathbf{C} \rangle} \mathbf{C}
\end{array}
\]

is called universal if it is an inverse universal \( (\mathbf{E} \times \mathbf{D}, \mathbf{C}) \)-arrow [op. direct universal \( (\mathbf{E} \times \mathbf{D}^+, \mathbf{C}) \)-arrow]. Given a functor \( \mathbf{L} : \mathbf{E} \times \mathbf{D} \to \mathbf{C} \),

- a universal wedge \( \mu : \mathbf{R} \to \mathbf{L} : \mathbf{E} \times \mathbf{D} \to \mathbf{C} \) or the pair \( \langle \mathbf{R}, \mu \rangle \), or the functor \( \mathbf{R} \) itself, is called a limit of \( \mathbf{L} \), with the functor \( \mathbf{R} \) denoted by \( \prod_{\mathbf{D}} \mathbf{R} \).
- a universal wedge \( \mu : \mathbf{L} \to \mathbf{R} : \mathbf{E} \times \mathbf{D}^+ \to \mathbf{C} \) or the pair \( \langle \mathbf{R}, \mu \rangle \), or the functor \( \mathbf{R} \) itself, is called a colimit of \( \mathbf{L} \), with the functor \( \mathbf{R} \) denoted by \( \underset{\mathbf{D}}{\bigcup} \mathbf{R} \).

Note. The following is a special case of Definition 8.3.2 where \( \mathcal{M} \) is given by the hom-module of a category. Recall from Definition 4.9.19 that a wedge is sliced into pieces of cones. In the following, we consider the case where these cones are universal (see Definition 8.1.9).

8.4.2 Definition. Given categories \( \mathbf{E}, \mathbf{D}, \) and \( \mathbf{C} \), a wedge

\[
\begin{array}{c}
\mathbf{E} \xrightarrow{\mathbf{E} \times \mathbf{C}} \mathbf{E} \times \mathbf{D} \\
\mathbf{R} \xrightarrow{\mu} \mathbf{L} \\
\mathbf{C} \xrightarrow{\langle \mathbf{C} \rangle} \mathbf{C}
\end{array}
\]

is called universal if it is an inverse universal \( (\mathbf{E} \times \mathbf{D}, \mathbf{C}) \)-arrow [op. direct universal \( (\mathbf{E} \times \mathbf{D}^+, \mathbf{C}) \)-arrow]. Given a functor \( \mathbf{L} : \mathbf{E} \times \mathbf{D} \to \mathbf{C} \),

- a universal wedge \( \mu : \mathbf{R} \to \mathbf{L} : \mathbf{E} \times \mathbf{D} \to \mathbf{C} \) or the pair \( \langle \mathbf{R}, \mu \rangle \), or the functor \( \mathbf{R} \) itself, is called a limit of \( \mathbf{L} \), with the functor \( \mathbf{R} \) denoted by \( \prod_{\mathbf{D}} \mathbf{R} \).
- a universal wedge \( \mu : \mathbf{L} \to \mathbf{R} : \mathbf{E} \times \mathbf{D}^+ \to \mathbf{C} \) or the pair \( \langle \mathbf{R}, \mu \rangle \), or the functor \( \mathbf{R} \) itself, is called a colimit of \( \mathbf{L} \), with the functor \( \mathbf{R} \) denoted by \( \underset{\mathbf{D}}{\bigcup} \mathbf{R} \).
is called pointwise universal if each left slice
\[
\begin{array}{ccc}
* & \xleftarrow{1} & D \\
\downarrow_{e:R} & & \downarrow_{e \times \mu} \\
\downarrow_{\mathcal{C} \rightarrow \mathcal{C}} & & \downarrow_{\mathcal{C} \rightarrow \mathcal{C}} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
D & \xrightarrow{1} & * \\
\downarrow_{e \times L} & & \downarrow_{e \times \mu} \\
\downarrow_{\mathcal{C} \rightarrow \mathcal{C}} & & \downarrow_{\mathcal{C} \rightarrow \mathcal{C}} \\
\end{array}
\]

is a universal cone, i.e. an inverse universal \((\star D, C)\)-arrow [op. direct universal \((D \ast, C)\)-arrow].

Given a functor \(L : E \times D \rightarrow C\),

- a pointwise universal wedge \(\mu : R \Rightarrow L : E \times \ast D \rightarrow C\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called an \(E\)-parameterized limit of \(L\), with the functor \(R\) denoted by \([\prod^E_D L]\) (or just by \([\prod L]\)).
- a pointwise universal wedge \(\mu : L \Rightarrow R : E \times D \rightarrow C\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called an \(E\)-parameterized colimit of \(L\), with the functor \(R\) denoted by \([\prod^E_D L]\) (or just by \([\prod L]\)).

**Note.** Since a wedge \(E \times \ast D \rightarrow C\) is transposed into a cone \(\ast D \rightarrow \langle E, C \rangle\) (see Definition 4.9.19), Definition 8.4.2 has the following variation.

**8.4.3 Definition.** Given categories \(E, D, \) and \(C\), a cone

\[
\begin{array}{ccc}
\star & \xleftarrow{1} & D \\
\downarrow_{R} & & \downarrow_{L} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
D & \xrightarrow{1} & * \\
\downarrow_{L} & & \downarrow_{R} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\]

in the functor category \([E,C]\) is called pointwise universal if each slice

\[
\begin{array}{ccc}
\star & \xleftarrow{1} & D \\
\downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{L}} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
D & \xrightarrow{1} & * \\
\downarrow_{\mathcal{L}} & & \downarrow_{\mathcal{R}} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\]

(see Remark 4.9.22(2)) is a universal cone, i.e. an inverse universal \((\star D, C)\)-arrow [op. direct universal \((D \ast, C)\)-arrow]. Given a functor \(L : D \rightarrow [E, C]\),

- a pointwise universal cone \(\mu : R \Rightarrow L : \ast D \rightarrow [E, C]\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a pointwise limit of \(L\), with the functor \(R\) denoted by \([\prod^E_D L]\) (or just by \([\prod L]\)).
- a pointwise universal cone \(\mu : L \Rightarrow R : D \ast \rightarrow [E, C]\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a pointwise colimit of \(L\), with the functor \(R\) denoted by \([\prod^E_D L]\) (or just by \([\prod L]\)).

**8.4.4 Proposition.** The exponential transposition in Definition 4.9.19 preserves universality and pointwise universality; that is, the following conditions are equivalent:

1. a wedge

\[
\begin{array}{ccc}
E & \xleftarrow{E \times !} & E \times D \\
\downarrow_{R} & & \downarrow_{L} \\
\downarrow_{\mathcal{C} \rightarrow \mathcal{C}} & & \downarrow_{\mathcal{C} \rightarrow \mathcal{C}} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
E \times D & \xrightarrow{E \times !} & E \\
\downarrow_{L} & & \downarrow_{R} \\
\downarrow_{\mathcal{C} \rightarrow \mathcal{C}} & & \downarrow_{\mathcal{C} \rightarrow \mathcal{C}} \\
\end{array}
\]

is universal (resp. pointwise universal);

2. its right exponential transpose

\[
\begin{array}{ccc}
\star & \xleftarrow{1} & D \\
\downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{L}} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
D & \xrightarrow{1} & * \\
\downarrow_{\mathcal{L}} & & \downarrow_{\mathcal{R}} \\
\downarrow_{[E,C]} & & \downarrow_{[E,C]} \\
\end{array}
\]

is a universal (resp. pointwise universal) cone;

3. its left exponential transpose

\[
\begin{array}{ccc}
E & \xleftarrow{r \mu} & \ast D \\
\downarrow_{\mathcal{R}} & & \downarrow_{\mathcal{L}} \\
\downarrow_{[D,C]} & & \downarrow_{[D,C]} \\
\end{array}
\quad \text{op.} \quad \begin{array}{ccc}
E & \xrightarrow{r \mu} & \ast \\
\downarrow_{\mathcal{L}} & & \downarrow_{\mathcal{R}} \\
\downarrow_{[D,C]} & & \downarrow_{[D,C]} \\
\end{array}
\]
is a universal (resp. pointwise universal) cylinder.

**Proof.** By the isomorphisms in Remark 4.9.20, μ is universal iff μ\( ^\top \) is universal iff \( \mu \) is universal. The pointwise version is immediate from Remark 4.9.22(3).

8.4.5 Remark. By Proposition 8.4.4 and noting the bijectivity of exponential transposition, we see that

- the following mean the same thing:
  1. a limit (resp. \( E \)-parameterized limit) of \( L : E \times D \to C \);
  2. a limit (resp. pointwise limit) of \( [L \times \cdot] : D \to [E, C] \);
  3. a lift (resp. pointwise lift) of \( \langle \cdot, L \rangle : E \to [D, C] \) along the module \( \langle \cdot, D, C \rangle \);
  4. a right Kan lift (resp. pointwise right Kan lift) along the diagonal functor \( !_D, C \).

- the following mean the same thing:
  1. a colimit (resp. \( E \)-parameterized colimit) of \( L : E \times D \to C \);
  2. a colimit (resp. pointwise colimit) of \( [L \times \cdot] : D \to [E, C] \);
  3. a colift (resp. pointwise colift) of \( \langle \cdot, L \rangle : E \to [D, C] \) along the module \( \langle D \cdot, C \rangle \);
  4. a left Kan lift (resp. pointwise left Kan lift) along the diagonal functor \( !_D, C \).

(Recall from Remark 4.9.4(2) that \( \langle D \cdot, C \rangle \) [op. \( \langle D \cdot, C \rangle \)] is the representable [op. corepresentable] module of the diagonal functor \( !_D, C \).)

8.4.6 Proposition. A pointwise universal wedge in Definition 8.4.2 is universal in the sense of Definition 8.4.1.

**Proof.** This is reduced to Proposition 6.5.10 by the equivalence of (1) and (3) in Proposition 8.4.4.

8.4.7 Proposition. The twist transposition in Definition 4.9.21 preserves universality and pointwise universality; that is, , the following conditions are equivalent;

1. a cone

\[
\begin{array}{ccc}
\ast & \langle L \rangle & D \\
\mu & \downarrow & \mu \\
R & \downarrow & L \\
\langle E, C \rangle & \langle E, C \rangle & \langle E, C \rangle \\
\end{array}
\]

is universal (resp. pointwise universal);

2. its twist transpose

\[
\begin{array}{ccc}
E & \longrightarrow & D \\
\mu & \downarrow & \mu \\
\langle D, C \rangle & \langle D, C \rangle & \langle D, C \rangle \\
\end{array}
\]

is a universal (resp. pointwise universal) cylinder.

**Proof.** By the isomorphism in Remark 4.9.22(1), μ is universal iff μ\( ^\top \) is universal. Since the slice of μ at each \( e \in \| E \| \) is given by the component of μ\( ^\top \) at \( e \), μ is pointwise universal iff μ\( ^\top \) is pointwise universal.

8.4.8 Remark. By Proposition 8.4.7 and noting the bijectivity of twist transposition, we see that

- the following mean the same thing:
  1. a limit (resp. pointwise limit) of \( L : D \to [E, C] \);
  2. a lift (resp. pointwise lift) of \( L \times \cdot : E \to [D, C] \) along the module \( \langle \cdot, D, C \rangle \);
  3. a right Kan lift (resp. pointwise right Kan lift) along the diagonal functor \( !_D, C \).

- the following mean the same thing:
  1. a colimit (resp. pointwise colimit) of \( L : D \to [E, C] \);
  2. a colift (resp. pointwise colift) of \( L \times \cdot : E \to [D, C] \) along the module \( \langle D \cdot, C \rangle \);
  3. a left Kan lift (resp. pointwise left Kan lift) along the diagonal functor \( !_D, C \).
8.4.9 Proposition. A pointwise universal cone in Definition 8.4.3 is universal in the sense of Definition 8.1.9.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of (1) and (2) in Proposition 8.4.7. □

8.4.10 Proposition. If a functor \( L : D \to [E, C] \) has a pointwise limit \([op. colimit] \), then every limit \([op. colimit] \) of \( L \) in is pointwise.

Proof. By the equivalence of (1) and (2) in Proposition 8.4.7, this is reduced to an instance of Proposition 6.5.12 where \( M \) is given by the module \((\ast D, C)\). □

Note. The following is a special case of Theorem 8.3.12 where \( M \) is given by the hom-module of a category.

8.4.11 Theorem. Let \( E \) and \( C \) be categories as in Definition 8.4.3. Then the family of evaluations \([e, C] : [E, C] \to C\), one for each object \( e \in |E|\), collectively creates a pointwise limit \([op. colimit] \) in the following sense: given a functor \( L : D \to [E, C] \), if there is a family of universal cones

\[
\begin{array}{ccc}
& \mu_e \downarrow & \\
\ast \downarrow r_e & [e, C] \downarrow & \ast \downarrow r_e \\
C \downarrow (C) & = & C
\end{array}
\]

\[
\begin{array}{ccc}
& \mu \downarrow & \\
\ast \downarrow r_e & [e, C] \downarrow & \ast \downarrow r_e \\
C \downarrow (C) & = & C
\end{array}
\]

, for each \( e \in |E|\), then there is a unique functor \( R : E \to C \) with \( e \cdot R = r_e \) such that \( \mu := ([\mu_e]_e)_{(e,d) \in [E \times D]} \) forms a cone

\[
\begin{array}{ccc}
& \mu \downarrow & \\
\ast \downarrow r_e & [e, C] \downarrow & \ast \downarrow r_e \\
C \downarrow (C) & = & C
\end{array}
\]

\[
\begin{array}{ccc}
& \mu \downarrow & \\
\ast \downarrow r_e & [e, C] \downarrow & \ast \downarrow r_e \\
C \downarrow (C) & = & C
\end{array}
\]

, and \( \mu \) is pointwise universal.

Proof. By the equivalence of (1) and (2) in Proposition 8.4.7, this is reduced to an instance of Theorem 6.5.14 where \( M \) is given by the module \((\ast D, C)\). □

Note. The following is a special case of Theorem 8.3.13 where \( M \) is given by the hom-module of a category.

8.4.12 Theorem. Let \( E, D, \) and \( C \) be categories.

- If \( C \) has limits over \( D\); that is (see Proposition 8.1.13), if the module \((\ast D, C)\) has a counit

\[
\begin{array}{ccc}
& \mu \downarrow & \\
\Pi_D \downarrow & [D, C] \downarrow & \ast \\
C \downarrow (\ast D, C) & = & [D, C]
\end{array}
\]

, then this counit yields a counit

\[
\begin{array}{ccc}
& \mu \downarrow & \\
\Pi_D \downarrow & [E \times \ast D, C] \downarrow & \ast \\
[E, C] \downarrow (E \times \ast D, C) & = & [E \times \ast D, C]
\end{array}
\]

of the module of wedges \( E \times \ast D \to C\), giving for each functor \( L : E \times D \to C\), a universal wedge \( \mu_L : \Pi_D L \Rightarrow L : E \times \ast D \to C \)

, in fact a pointwise universal wedge, with each left slice

\[
[e \cdot \mu_L] : (e \cdot \Pi_D L) \to [e \cdot L] : \ast D \to C
\]
giving a limit

\[ \mu_{[e \times L]} : \prod_D [e \times L] \rightarrow [e \times L] : *D \rightarrow C \]

of the functor \([e \times L] : D \rightarrow C\).

- If \(C\) has colimits over \(D\); that is (see Proposition 8.1.13), if the module \([D, *] \times_C C\) has a unit

\[ [D, C] \xrightarrow{\mu} C \]

, then this unit yields a unit

\[ [E \times D, C] \xrightarrow{\mu} [E, C] \]

of the module of wedges \(E \times D \rightarrow C\), giving for each functor \(L : E \times D \rightarrow C\), a universal wedge

\[ \mu_L : \prod_D L \rightarrow *D \rightarrow C \]

, in fact a pointwise universal wedge, with each left slice

\[ [e \times L] : (e^* \prod_D L) \rightarrow [e \times L] : D \rightarrow C \]

giving a colimit

\[ \mu_{[e \times L]} : \prod_D [e \times L] \rightarrow [e \times L] : D \rightarrow C \]

of the functor \([e \times L] : D \rightarrow C\).

Proof. By Corollary 6.5.20, a counit \(\mu\) of the module \([*D, C]\) yields a counit

\[ [E, C] \xrightarrow{[E, \mu]} [E, [D, C]] \]

of the module \([E, (*D, C)]\), and, from this counit, the iso cells in Remark 4.9.20 create a counit of the module \([E \times *D, C]\).

\[ \square \]

Note. The following is a special case of Theorem 8.3.14 where \(\mathcal{M}\) is given by the hom-module of a category.

8.4.13 Theorem. If a category \(C\) has limits \([\text{op. colimits}]\) over a category \(D\), so is the functor category \([E, C]\) for any category \(E\), and every limit \([\text{op. colimit}]\) over \(D\) is given pointwise.

Proof. By Theorem 8.4.12, every functor \(F : D \rightarrow [E, A]\) has a pointwise limit along \([E, C]\), and every limit of \(F\) is pointwise by Proposition 8.4.10.

\[ \square \]

8.4.14 Corollary. If a category \(C\) is complete \([\text{op. cocomplete}]\), so is the functor category \([E, C]\) for any category \(E\), and all small limits \([\text{op. colimits}]\) are given pointwise.

Proof. Immediate from Theorem 8.4.13.

\[ \square \]

Note. The following is a special case of Theorem 8.3.16 where \(\mathcal{M}\) is given by the hom-module of a category.

8.4.15 Theorem. If a category \(C\) is complete \([\text{op. cocomplete}]\), then for any functor \(K : E' \rightarrow E\) the precomposition functor \([K, C] : [E, C] \rightarrow [E', C]\) is continuous \([\text{op. cocontinuous}]\).

Proof. Assume that \(C\) is complete. We need to show that for any small category \(D\), the postcomposition cell \([*D, (K, C)]\) preserves inverse universal arrows. Since every universal cone \(\mu : *D \approx (E, C)\) is pointwise universal by Corollary 8.4.14, the problem translates (see Proposition 8.4.7) along the naturality square

\[
\begin{array}{ccc}
[*D, (E, C)] & \xrightarrow{i} & (E, [*D, C]) \\
\downarrow & & \downarrow \\
(*D, (K, C)) & \xrightarrow{(K, i)} & (E', (*D, C))
\end{array}
\]
into showing that the precomposition cell \( (K, \{ \ast D, C \} ) \) preserves pointwise universal. But we have already seen this in Corollary 6.5.16.

\[ \]

\textbf{Note.} The following shows that the pasting composition (see Definition 4.9.23) of a universal cone and a pointwise universal wedge yields a universal cone.

\textit{8.4.16 Theorem.}

\begin{itemize}
  \item If a bifunctor \( L : E \times D \to C \) has an \( E \)-parameterized limit \( E \xrightarrow{Exl} E \times D \), then a limit of \( L \) exists
  \( C \xrightarrow{(\text{C})} \)
  if and only if a limit of \( R : E \to C \) exists; specifically, if \( R \) has a limit \( \nu : r \xrightarrow{\sim} R \), then the pasting composite \( \nu \circ \mu : r \xrightarrow{\sim} L \) gives a limit of \( L \), and conversely if \( L \) has a limit \( \kappa : L \xrightarrow{\sim} r \), then there is a unique cone \( \nu : r \xrightarrow{\sim} R \) such that \( \kappa = \nu \circ \mu \), and \( \nu \) gives a limit of \( R \).
  \item If a bifunctor \( L : E \times D \to C \) has an \( E \)-parameterized colimit \( E \times D \xrightarrow{Exl} E \), then a colimit of \( L \)
  \( C \xrightarrow{(\text{C})} \)
  exists if and only if a colimit of \( R : E \to C \) exists; specifically, if \( R \) has a colimit \( \nu : R \xrightarrow{\sim} c \), then the pasting composite \( \mu \circ \nu : R \xrightarrow{\sim} c \) gives a colimit of \( L \), and conversely if \( L \) has a colimit \( \kappa : \mu \xrightarrow{\sim} c \), then there is a unique cone \( \nu : \mu \xrightarrow{\sim} L \) such that \( \kappa = \mu \circ \nu \), and \( \nu \) gives a colimit of \( R \).
\end{itemize}

\textbf{Proof.} This follows from the lemma below.

\textbf{Lemma.} If the wedge \( \mu : R \xrightarrow{\sim} c \) in Definition 4.9.23 is pointwise universal, then the right module morphism in Proposition 4.9.24 is an isomorphism; that is, for any bicone \( \kappa : c \xrightarrow{\sim} L \), there is a unique cone \( \nu : c \xrightarrow{\sim} R \) such that \( \kappa = \nu \circ \mu \).

\textbf{Proof.} Since \( \mu \) is pointwise universal, each left slice \( [e \times \mu] : (e \times R) \xrightarrow{\sim} (e \times L) \) is a universal cone; hence for the left slice \( [e \times \kappa] : c \xrightarrow{\sim} (e \times L) \) of \( \kappa \) at \( e \in [E] \) (see Definition 4.9.13), there is a unique \( C \)-arrow \( \nu_c : c \xrightarrow{\sim} (e \times R) \) such that \( \nu_c \circ [e \times \mu] = [e \times \kappa] \). The proof is thus complete if we show that \( \nu := \nu_c \circ (e \times \mu) \) forms a cone \( c \xrightarrow{\sim} R \). For this, let \( h : e \to e' \) be an \( E \)-arrow and consider the diagram

\[
\begin{array}{ccc}
  & e \times R & \xrightarrow{e \times \mu} & e \times L \\
  c & \downarrow{\nu_c} & \downarrow{\nu_{e'}} & \downarrow{\nu_{e}} \\
  & e' \times R & \xrightarrow{e' \times \mu} & e' \times L
\end{array}
\]

; we need to show that the left-hand triangle commutes. Since the right-hand square commutes by the naturality of \( \times \mu \), and the outer pentagon commutes by the naturality of \( \times \kappa \), simple diagram chasing shows that

\[
\nu_c \circ (h \times R) \circ [e' \times \mu] = \nu_{e'} \circ [e' \times \mu]
\]

; hence

\[
\nu_c \circ (h \times R) = \nu_{e'}
\]

by the universality of \( e' \times \mu \).

\textit{8.4.17 Corollary.}

\begin{itemize}
  \item Suppose that a bifunctor \( L : E \times D \to C \) has an \( E \)-parameterized limit and a \( D \)-parameterized limit as shown below
  \[
  \begin{array}{ccc}
    E & \xrightarrow{Exl} & E \times D \\
    R \downarrow{\mu} & \downarrow{L} & \downarrow{L} \\
    C & \xrightarrow{(\text{C})} & C
  \end{array}
  \quad
  \begin{array}{ccc}
    D & \xrightarrow{lx D} & E \times D \\
    R' \downarrow{\mu'} & \downarrow{L} & \downarrow{L} \\
    C & \xrightarrow{(\text{C})} & C
  \end{array}
  \]
\end{itemize}
; then a limit of $R$ exists if and only if a limit of $R'$ exists; specifically, if $R'$ has a limit $\nu' : r \leadsto R'$, then there exist a unique cone $\nu : r \leadsto R$ making the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\mu} & L \\
\nu' & \downarrow & \mu' \\
r & \xrightarrow{\nu} & R'
\end{array}
\]

commutes, and $\nu$ gives a limit of $R$.
- Suppose that a bifunctor $L : E \times D \to C$ has an $E$-parameterized colimit and a $D$-parameterized colimit as shown below

\[
\begin{array}{ccc}
E \times D & \xrightarrow{E \times \mu} & E \\
L \downarrow \mu & \downarrow \mu_R & \downarrow \mu' \\
C & \xrightarrow{(C)} & C
\end{array}
\quad
\begin{array}{ccc}
E \times D & \xrightarrow{1 \times D - \nu} & E \\
\downarrow \mu' & \downarrow \nu_R & \downarrow \nu' \\
C & \xrightarrow{(C)} & C
\end{array}
\]

; then a colimit of $R$ exists if and only if a colimit of $R'$ exists; specifically, if $R'$ has a colimit $\nu' : R' \leadsto r$, then there exist a unique cone $\nu : R \leadsto r$ making the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\mu} & R \\
\mu' \downarrow & \downarrow \nu \\
R' & \xrightarrow{\nu'} & r
\end{array}
\]

commutes, and $\nu$ gives a colimit of $R$.

Proof. Immediate from Theorem 8.4.16.

8.4.18 Remark. The results in Theorem 8.4.16 and Corollary 8.4.17 are expressed by

\[
\prod_{E \in D} L \cong \prod_{E \times D} \prod_{D \in E} L \cong \prod_{E \times D \in D} L \quad \text{op.}
\]

or, more informatively, by

\[
\prod_{e \in E \times d \in D} L(e, d) \cong \prod_{(e', d) \in E \times D} L(e, d) \cong \prod_{d \in D \in E} \prod_{e \in E} L(e, d) \quad \text{op.}
\]

and called the limits [op. colimit] interchange property or sometimes the Fubini theorem for limits [op. colimit].

8.4.19 Theorem. Suppose that a functor $H : D \to [C, B]$ from a category $D$ to a functor category $[C, B]$ has a pointwise limit $\kappa : R \leadsto H : * \leadsto D \to [C, B]$ [op. colimit $\kappa : H \leadsto R : D \to [C, B]$]. Then if $H(d) : C \to B$ preserves limits [op. colimits] over a category $E$ for every $d \in |D|$, so does $R$.

Proof. Given a cone $\mu : r : *E \to C$, we need to show that if $\mu$ is universal, so is its composite $\mu \circ \check{R} : r : R \leadsto L \circ \check{R} : *E \to B$ with $R$. Note that the composition

\[
D \xrightarrow{\Delta R} [C, B] \xrightarrow{[\Delta r, B]} [C, B]
\]

of $\kappa$ and the precomposition natural transformation $[\mu, B]$ is given by the diagonal of the commutative square

\[
\begin{array}{ccc}
L \circ \check{R} & \xrightarrow{\kappa \circ \check{L} \mu} & H \circ \check{L} B \\
\mu \circ \check{R} & \downarrow & \downarrow \mu \circ \check{L} \mu_B \\
r \circ \check{R} & \xrightarrow{\kappa \circ \check{L} [\Delta r, B]} & H \circ \check{L} [\Delta r, B]
\end{array}
\]

, and this commutative square is described componentwise by

\[
\begin{array}{ccc}
e : L : R & \xrightarrow{e \circ L \circ \kappa d} & e : L : H(d) \\
\mu_e : R & \downarrow & \downarrow \mu_e \circ H(d) \\
r : R & \xrightarrow{r \circ \kappa d} & r : H(d)
\end{array}
\]
for every \((e, d) \in \prod E \times D\). Comparing this commutative square with that in Corollary 8.4.17, we see that the universality of \(\mu \circ R : r : R \Rightarrow L \circ R\) follows if we show that

1. for each \(d \in D\), the family of \(B\)-arrows \(\mu_{e, e} : H(d) : r : H(d) \Rightarrow e : L : H(d)\), one for each \(e \in E\), form a universal cone;
2. for each \(e \in E\), the family of \(B\)-arrows \(e : L : k_d : r : R \Rightarrow e : L : H(d)\), one for each \(d \in D\), form a universal cone;
3. the family of \(B\)-arrows \(r : \kappa_d : r : R \Rightarrow r : H(d)\), one for each \(d \in D\), form a universal cone.

But (1) holds because \(H(d)\) preserves limits, and (2) and (3) hold because \(\kappa : R \Rightarrow H\) is a pointwise limit. \(\square\)

### 8.5 Limits of modules

In this section, we study limits in the category \(\mathbf{Set}\); that is, limits of one-sided modules. We recall from Section 4.2 that a frame-set is a categorical generalization of the notion of a cartesian product; Theorem 8.5.2 states that the frame-set of a small module \(\mathcal{M}\) gives a limit of \(\mathcal{M}\) in \(\mathbf{Set}\), generalizing the fact that the cartesian product gives a product in \(\mathbf{Set}\) for an indexed family of sets. (The notations introduced in Section 4.10 are used in this section.)

#### 8.5.1 Theorem

- A limit of a small left module \(\mathcal{M} : \ast \to E\) (i.e. a functor \(\mathcal{M} : E \to \mathbf{Set}\) with \(E\) small) is given by an equalizer \(\pi\) as in

\[
\begin{array}{ccc}
\Pi_E \mathcal{M} & \xrightarrow{\pi} & \Pi_{e \in [E]} (\mathcal{M}) e \\
\downarrow & & \downarrow \\
\Pi_{e' \in [E]} (\mathcal{M}) e' & \xrightarrow{\pi} & \Pi_{e' \in [E]} (\Delta_{e, e'})_{e \in [E]} \Pi_{e \in [E]} [e(\mathcal{M}) e', (\mathcal{M}) e']
\end{array}
\]

where \(\Gamma_{e, e'} : (\mathcal{M}) e \to [e(\mathcal{M}) e', (\mathcal{M}) e'] ; m \mapsto [h \mapsto m \circ h]\) is the function given by the exponential transpose of the composition between \(\mathcal{M}\)-arrows and \(E\)-arrows, and

\[
\Delta_{e, e'} : (\mathcal{M}) e' \to [(\mathcal{M}) e', (\mathcal{M}) e'] ; m' \mapsto [h \mapsto h \circ m']
\]
is the diagonal function.

- A limit of a small right module \(\mathcal{M} : E \to \ast\) (i.e. a functor \(\mathcal{M} : E \to \mathbf{Set}\) with \(E\) small) is given by an equalizer \(\pi\) as in

\[
\begin{array}{ccc}
\Pi_E \mathcal{M} & \xrightarrow{\pi} & \Pi_{e \in [E]} e(\mathcal{M}) \\
\downarrow & & \downarrow \\
\Pi_{e' \in [E]} e'(\mathcal{M}) & \xrightarrow{\pi} & \Pi_{e' \in [E]} (\Delta_{e, e'})_{e \in [E]} \Pi_{e \in [E]} [e'(\mathcal{M}) e, e'(\mathcal{M})]
\end{array}
\]

where \(\Gamma_{e, e'} : e(\mathcal{M}) \to [e(\mathcal{M}) e', e'(\mathcal{M})] ; m \mapsto [h \mapsto h \circ m]\) is the function given by the exponential transpose of the composition between \(\mathcal{M}\)-arrows and \(E\)-arrows, and

\[
\Delta_{e, e'} : e'(\mathcal{M}) \to [e(\mathcal{M}) e', e'(\mathcal{M})] ; m' \mapsto [h \mapsto h \circ m']
\]
is the diagonal function.

**Proof.** By the claim below, a limit of \(\mathcal{M}\) is given by a universal fork on \(\Pi_{e \in [E]} (\Gamma_{e, e'})_{e' \in [E]}\) and \(\Pi_{e' \in [E]} (\Delta_{e, e'})_{e \in [E]}\), i.e. by an equalizer of them. \(\square\)
Claim. Given a small set $\mathcal{S}$, a family of functions $\xi_e : \mathcal{S} \to \langle \mathcal{M} \rangle e$, one for each $e \in \| \mathcal{E} \|$, forms a cone $\mathcal{S} \to \mathcal{M}$ if and only if the diagram

$$
\begin{array}{c}
\mathcal{S} \\
\downarrow \xi_e \\
\Pi_{e \in \mathcal{E}} \langle \mathcal{M} \rangle e
\end{array}
\xrightarrow{\xi_e \circ e E} 
\begin{array}{c}
\Pi_{e \in \mathcal{E}} \langle \mathcal{M} \rangle e' \\
\downarrow \Pi_{e \in \mathcal{E}} \langle \Delta_{e, e'} \rangle e E \\
\Pi_{e \in \mathcal{E}} \langle \mathcal{E} \rangle e \langle \mathcal{M} \rangle e'
\end{array}
$$

commutes, i.e. if and only if the function $(\xi_e)_{e \in \mathcal{E}} : \mathcal{S} \to \Pi_{e \in \mathcal{E}} \langle \mathcal{M} \rangle e$ forms a fork on the functions $\Pi_{e \in \mathcal{E}} \langle \Gamma_{e, e'} \rangle e'_{e E}$ and $\Pi_{e \in \mathcal{E}} \langle \Delta_{e, e'} \rangle e E$.

Proof. The diagram in the claim commutes iff the diagram

$$
\begin{array}{c}
\mathcal{S} \\
\downarrow \xi_e \\
\Pi_{e \in \mathcal{E}} \langle \mathcal{M} \rangle e
\end{array}
\xrightarrow{\xi_e \circ e E} 
\begin{array}{c}
\Pi_{e \in \mathcal{E}} \langle \mathcal{M} \rangle e' \\
\downarrow \Pi_{e \in \mathcal{E}} \langle \Delta_{e, e'} \rangle e E \\
\Pi_{e \in \mathcal{E}} \langle \mathcal{E} \rangle e \langle \mathcal{M} \rangle e'
\end{array}
$$

commutes for each $e \in \| \mathcal{E} \|$, and this diagram commutes iff the diagram

$$
\begin{array}{c}
\mathcal{S} \\
\downarrow \xi_e \\
\langle \mathcal{M} \rangle e
\end{array}
\xrightarrow{\xi_e \circ e E} 
\begin{array}{c}
\langle \mathcal{M} \rangle e' \\
\downarrow \Gamma_{e, e'} \\
\langle \mathcal{E} \rangle e' \langle \mathcal{M} \rangle e'
\end{array}
$$

commutes for each $e' \in \| \mathcal{E} \|$, and this diagram commutes iff the triangle

$$
\begin{array}{c}
\mathcal{S} \\
\downarrow \xi_e \\
\langle \mathcal{M} \rangle e
\end{array}
\xrightarrow{\xi_e \circ e E} 
\begin{array}{c}
\langle \mathcal{M} \rangle h \\
\downarrow \Gamma_{e, e'} \\
\langle \mathcal{E} \rangle e' \langle \mathcal{M} \rangle e'
\end{array}
$$

commutes for each $\mathcal{E}$-arrow $h : e \to e'$ (because for any $s \in \mathcal{S}$,

$$s : [\xi_e \circ \langle \mathcal{M} \rangle h] = (s : \xi_e) \circ h = h : [s : \Delta_{e, e'}] = h : [s : \xi_e \circ \langle \mathcal{E} \rangle e']$$

and

$$s : \xi_e = h : [(s : \xi_e) \circ \langle \mathcal{E} \rangle e'] = h : [s : \xi_{e'} \circ \Delta_{e, e'}]$$

by the definitions of $\Gamma_{e, e'}$ and $\Delta_{e, e'}$.)

Note. Just like the cartesian product $\prod_{i \in I} \mathcal{S}_i$ gives a product in the category Set for an indexed family of sets $\{\mathcal{S}_i\}_{i \in I}$, the frame-set $\prod_{\mathcal{E}} \mathcal{M}$ (see Definition 4.2.1) gives a limit of a small left module $\mathcal{M} : * \to \mathcal{E}$.

8.5.2 Theorem. For a small left module $\mathcal{M} : * \to \mathcal{E}$ [op. right module $\mathcal{M} : \mathcal{E} \to *$], the frame-set of $\mathcal{M}$ gives a limit of $\mathcal{M}$ with the universal cone

$$\pi_M : \prod_{\mathcal{E}} \mathcal{M} \to \mathcal{M} : * \to * \mathcal{E} \quad \text{op.} \quad \pi_M : \prod_{\mathcal{E}} \mathcal{M} \to \mathcal{M} : * \to *$$

defined such that its component

$$\langle \pi_M \rangle e : \prod_{\mathcal{E}} \mathcal{M} \to \langle \mathcal{M} \rangle e \quad \text{op.} \quad e \langle \pi_M \rangle : \prod_{\mathcal{E}} \mathcal{M} \to e \langle \mathcal{M} \rangle$$

at $e \in \| \mathcal{E} \|$ maps each frame $\alpha$ of $\mathcal{M}$ to its component at $e$; that is,

$$\alpha : \langle \pi_M \rangle e := \alpha_e \quad \text{op.} \quad \alpha : e \langle \pi_M \rangle := \alpha_e.$$  

Proof. This follows from the observation that an element $\alpha$ of the product $\prod_{\mathcal{E} \in \| \mathcal{E} \|} \langle \mathcal{M} \rangle e$ forms a frame of $\mathcal{M}$ iff it lies in the equalizer in Theorem 8.5.1.

8.5.3 Remark. For a small left module $\mathcal{M} : * \to \mathcal{E}$, the notation $\prod_{\mathcal{E}} \mathcal{M}$ thus denotes both the frame-set of $\mathcal{M}$ and a limit of $\mathcal{M}$. 
Note. Recall from Proposition 4.2.9 that the assignment of the frame-set $\prod_E M$ to a small left module $M : E \to \text{Set}$ extends to the functor $\prod_E : [: E] \to \text{Set}$.

8.5.4 Corollary. Let $E$ be a small category. The functor

$$\prod_E : [: E] \to \text{Set} \quad \text{op.} \quad \prod_{E^*} : [E^*] \to \text{Set}$$

and the family of universal cones

$$\pi_M : \prod_E M \to M : * \to *E \quad \text{op.} \quad \pi_M : \prod_{E^*} M \to M : E^* \to *$$

one for each left module $M : * \to E$ [op. right module $M : E \to *$], forms a counit

$$\text{Set} \xrightarrow{\pi} [: E] \quad \text{op.} \quad \text{Set} \xrightarrow{\pi} [E^*]$$

of the module $\langle *E \rangle$ [op. $\langle *E^* \rangle$].

Proof. We just need to verify that $\pi$ satisfies the naturality condition, i.e. that the square

$$\prod_E M \xrightarrow{\pi_M} M \quad \prod_{E^*} M \xrightarrow{\pi_M} M$$

commutes for any left module morphism $\psi : M \to N$. But for any frame $\alpha \in \prod_E M$ and any object $e \in \|E\|$,

$$\alpha : \langle \pi_M \circ \psi \rangle e = \alpha : \langle \pi_M \rangle e : \langle \psi \rangle e = \alpha : \langle \psi \rangle e$$

and

$$\alpha : \langle \prod_E \psi \circ \pi_N \rangle e = \alpha : \prod_E \psi : \langle \pi_N \rangle e = \alpha : \langle \psi \rangle e$$

by Theorem 8.5.2 and Definition 4.2.6. □

8.5.5 Corollary. Let $E$ be a small category. Given a category $X$ [op. $A$], the module $\langle X : *E \rangle$ [op. $\langle *E : A \rangle$] has a counit

$$[X:] \xrightarrow{\pi} [: E] \quad \text{op.} \quad [A] \xrightarrow{\pi} [E^*]$$

, giving for each module $M : X \to E$ [op. $M : E \to A$] a universal wedge

$$\pi_M : \prod_E M \to M : X \to *E \quad \text{op.} \quad \pi_M : \prod_{E^*} M \to M : *E \to A$$

, in fact a pointwise universal wedge, with each slice

$$\langle \pi_M \rangle : \langle \prod_E M \rangle \to \langle M \rangle : * \to *E \quad \text{op.} \quad \langle \pi_M \rangle : \langle \prod_{E^*} M \rangle \to \langle M \rangle : * \to *$$

giving a limit

$$\pi_{\langle M \rangle} : \prod_E \langle M \rangle \to \langle \langle M \rangle \rangle : * \to *E \quad \text{op.} \quad \pi_{\langle M \rangle} : \prod_{E^*} \langle M \rangle \to \langle \langle M \rangle \rangle : * \to *$$

of the left module $\langle X \rangle : * \to E$ [op. right module $\langle M \rangle : E \to *$].

Proof. Apply Theorem 8.4.12 to the counit in Corollary 8.5.4. □

8.5.6 Remark.

(1) By Remark 4.3.4(4), the counit

$$[: A] \xrightarrow{\pi} [E : A]$$

is the same thing as the unit

$$[E : A] \xrightarrow{\pi} [: A]$$.
(2) Given a small category $\mathsf{E}$ and a module $\mathsf{M} : \mathsf{X} \to \mathsf{A}$, the triangle

$$
\begin{array}{ccc}
\{ \mathsf{E}, \mathsf{A} \} & \xrightarrow{\mathsf{M} \times \mathsf{E}} & \{ \mathsf{X} : \mathsf{E} \} \\
\{ \mathsf{X} : \mathsf{E} \} & \xleftarrow{\prod_{\mathsf{E}}} & \{ \mathsf{X} : \mathsf{E} \}
\end{array}
$$

commutes, where $\{ \mathsf{E}, \mathsf{M} \} \times$ is the right exponential transpose of the module $\{ \mathsf{E}, \mathsf{M} \} : \mathsf{X} \to \{ \mathsf{E}, \mathsf{A} \}$ of cones $\mathsf{E} \sim \mathsf{M}$ and $\mathsf{M} \times \mathsf{E}$ is the right action of $\mathsf{M}$ on the functor category $\{ \mathsf{E}, \mathsf{A} \}$. Indeed, for any $F \in \{ \mathsf{E}, \mathsf{A} \}$ and any $x \in \mathsf{X}$,

$$(x) \{ \mathsf{E}, \mathsf{M} \} (F) = \prod_{\mathsf{E}} x (\mathsf{M}) F = x (\prod_{\mathsf{E}} (\mathsf{M}) F)$$

; dually, the triangle

$$
\begin{array}{ccc}
\{ \mathsf{E}, \mathsf{X} \} & \xleftarrow{\mathsf{E} \times \mathsf{M}} & \{ \mathsf{X} : \mathsf{E} \} \\
\{ \mathsf{E} : \mathsf{A} \} & \xrightarrow{\prod_{\mathsf{E}}} & \{ : \mathsf{A} \}
\end{array}
$$

commutes, where $\mathsf{E} \times (\mathsf{M}, \mathsf{E})$ is the left exponential transpose of the module $(\mathsf{E}, \mathsf{M}) : \{ \mathsf{E}, \mathsf{X} \} \to \mathsf{A}$ of cones $\mathsf{E} \sim \mathsf{M}$ and $\mathsf{E} \times \mathsf{M}$ is the left action of $\mathsf{M}$ on the functor category $\{ \mathsf{E}, \mathsf{X} \}$.

8.5.7 Theorem. The category $\mathbf{Set}$ is complete.

Proof. By Theorem 8.5.2. □

8.5.8 Corollary. For any categories $\mathsf{X}$ and $\mathsf{A}$, the category $\{ \mathsf{X} : \mathsf{A} \}$ of right modules over $\mathsf{X}$, the category $\{ : \mathsf{A} \}$ of left modules over $\mathsf{A}$, and the category $\{ \mathsf{X} : \mathsf{A} \}$ of modules $\mathsf{X} \to \mathsf{A}$ are complete, and all small limits are given pointwise.

Proof. Since $\mathbf{Set}$ is complete, so is the category $\{ \mathsf{X} : \mathsf{A} \} := \{ \mathsf{X} \times \mathsf{A}, \mathbf{Set} \}$ by Corollary 8.4.14. □

8.5.9 Corollary. For any functors $\mathsf{G} : \mathsf{E} \to \mathsf{X}$ and $\mathsf{F} : \mathsf{D} \to \mathsf{A}$, the precomposition functors $[\mathsf{G} :] : \{ \mathsf{X} : \mathsf{E} \} \to \{ : \mathsf{D} \}$ and $[\mathsf{G} : \mathsf{F}] : \{ \mathsf{X} : \mathsf{A} \} \to \{ \mathsf{E} : \mathsf{D} \}$ are continuous.

Proof. Since $\mathbf{Set}$ is complete, the precomposition functor $[\mathsf{G} : \mathsf{F}] := [\mathsf{G} \times \mathsf{F}, \mathbf{Set}]$ is continuous by Theorem 8.4.15. □

8.6 Colimits of modules

In this section, we study colimits in the category $\mathbf{Set}$; that is, colimits of one-sided modules. We recall from Section 4.11 that the set of orbits of a one-sided module is a categorical generalization of the notion of the disjoint union of an indexed family of sets; Theorem 8.6.2 states that the set of orbits of a small module $\mathsf{M}$ gives a colimit of $\mathsf{M}$ in $\mathbf{Set}$, generalizing the fact that the disjoint union gives a coproduct in $\mathbf{Set}$ for an indexed family of sets. (The notations introduced in Section 4.10 are used in this section.)

8.6.1 Theorem.

A colimit of a small left module $\mathsf{M} : \star \to \mathsf{E}$ (i.e. a functor $\mathsf{M} : \mathsf{E} \to \mathbf{Set}$ with $\mathsf{E}$ small) is given by a coequalizer $\pi$ as in

$$
\begin{array}{ccc}
\bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} \bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} (\mathsf{M}) \mathsf{e} \times \mathsf{e} (\mathsf{E}) \mathsf{e} & \xrightarrow{\bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} (\Gamma_{\mathsf{e}}, \mathsf{e}) \mathsf{e} : \mathsf{e} \mathsf{e}'} & \bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} (\mathsf{M}) \mathsf{e}' \\
\bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} \bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} (\Delta_{\mathsf{e}}, \mathsf{e}) \mathsf{e} : \mathsf{e} \mathsf{e} & \xrightarrow{\pi} & \bigsqcup_{\mathsf{e} \in \{ \mathsf{E} \}} \mathsf{M}
\end{array}
$$

, where

$$
\Gamma_{\mathsf{e}, \mathsf{e}'} : (\mathsf{M}) \mathsf{e} \times \mathsf{e} (\mathsf{E}) \mathsf{e}' \to (\mathsf{M}) \mathsf{e}' ; (m, h) \mapsto m \circ h
$$
is the function given by the composition between $\mathcal{M}$-arrows and $E$-arrows, and

$$\Delta_{e,e'} : \langle \mathcal{M} \rangle e \times e(\langle E \rangle) e' \to \langle \mathcal{M} \rangle e; (m,h) \mapsto m$$

is the projection.

- A colimit of a small right module $\mathcal{M} : E \to \ast$ (i.e. a functor $\mathcal{M} : E^\ast \to \text{Set}$ with $E$ small) is given by a coequalizer $\pi$ as in

$$\begin{array}{c}
\coprod_{e \in [E]} \coprod_{e' \in [E]} e(\langle \mathcal{M} \rangle) \times e'(\langle E \rangle) e \xrightarrow{\coprod_{e' \in [E]} (\Gamma_{e,e'} \circ e)_{e \in [E]}} \coprod_{e' \in [E]} e'(\langle \mathcal{M} \rangle) e' \\
\coprod_{e \in [E]} (\Delta_{e,e'})_{e' \in [E]} \downarrow \pi \\
\coprod_{e \in [E]} \langle \mathcal{M} \rangle e \xrightarrow{\pi} \prod_{e \in [E]} \langle E \rangle \mathcal{M}
\end{array}$$

where

$$\Gamma_{e,e'} : e(\langle \mathcal{M} \rangle) \times e'(\langle E \rangle) e \to e'(\langle \mathcal{M} \rangle) ; (m,h) \mapsto m \circ h$$

is the function given by the composition between $\mathcal{M}$-arrows and $E$-arrows, and

$$\Delta_{e,e'} : e(\langle \mathcal{M} \rangle) \times e'(\langle E \rangle) e \to e(\langle \mathcal{M} \rangle) ; (m,h) \mapsto m$$

is the projection.

**Proof.** By the claim below, a colimit of $\mathcal{M}$ is given by a universal fork on $\coprod_{e \in [E]} (\Delta_{e,e'})_{e' \in [E]}$ and $\coprod_{e' \in [E]} (\Gamma_{e,e'})_{e \in [E]}$, i.e. by a coequalizer of them. $\square$

**Claim.** Given a small set $S$, a family of functions $\xi_e : \langle \mathcal{M} \rangle e \to S$, one for each $e \in [E]$, forms a cone $\mathcal{M} \to S$ if and only if the diagram

$$\begin{array}{c}
\coprod_{e \in [E]} \coprod_{e' \in [E]} \langle \mathcal{M} \rangle e \times e(\langle E \rangle) e' \xrightarrow{\coprod_{e' \in [E]} (\Gamma_{e,e'} \circ e)_{e \in [E]}} \coprod_{e' \in [E]} e'(\langle \mathcal{M} \rangle) e' \\
\coprod_{e \in [E]} (\Delta_{e,e'})_{e' \in [E]} \downarrow \pi \\
\coprod_{e \in [E]} \langle \mathcal{M} \rangle e \xrightarrow{\pi} \prod_{e \in [E]} \langle E \rangle \mathcal{M}
\end{array}$$

commutes, i.e. if and only if the function $(\xi_e)_{e \in [E]} : \coprod_{e \in [E]} \langle \mathcal{M} \rangle e \to S$ forms a fork on the functions $\coprod_{e \in [E]} (\Delta_{e,e'})_{e' \in [E]}$ and $\coprod_{e' \in [E]} (\Gamma_{e,e'})_{e \in [E]}$.

**Proof.** The diagram in the claim commutes iff the diagram

$$\begin{array}{c}
\coprod_{e' \in [E]} \langle \mathcal{M} \rangle e \times e(\langle E \rangle) e' \xrightarrow{\coprod_{e' \in [E]} (\Gamma_{e,e'} \circ e)_{e \in [E]}} \coprod_{e' \in [E]} e'(\langle \mathcal{M} \rangle) e' \\
(\Delta_{e,e'})_{e' \in [E]} \downarrow \pi \\
\langle \mathcal{M} \rangle e \xrightarrow{\pi} S
\end{array}$$

commutes for each $e \in [E]$, and this diagram commutes iff the diagram

$$\begin{array}{c}
\langle \mathcal{M} \rangle e \times e(\langle E \rangle) e' \xrightarrow{\coprod_{e' \in [E]} (\Gamma_{e,e'} \circ e)_{e \in [E]}} \coprod_{e' \in [E]} e'(\langle \mathcal{M} \rangle) e' \\
\Delta_{e,e'} \downarrow \Pi_{e'} \\
\langle \mathcal{M} \rangle e \xrightarrow{\Pi_{e'}} S
\end{array}$$

commutes for each $e' \in [E]$, and this diagram commutes iff the triangle

$$\begin{array}{c}
\langle \mathcal{M} \rangle e \\
\langle \mathcal{M} \rangle h \downarrow \Pi_{e'} \\
\langle \mathcal{M} \rangle e' \xrightarrow{\Pi_{e'}} S
\end{array}$$

commutes for each $E$-arrow $h : e \to e'$ (because for any $\mathcal{M}$-arrow $m : \ast \to e$,

$$m : [\langle \mathcal{M} \rangle h \circ \xi_{e'}] = (m \circ h) : \xi_{e'} = (m,h) : \Gamma_{e,e'} : \xi_{e'} = (m,h) : [\Gamma_{e,e'} \circ \xi_{e'}]$$
and

\[ m \cdot \xi_e = (m, h) \cdot \Delta_{e,e'} \cdot \xi_e = (m, h) \cdot [\Delta_{e,e'} \cdot \xi_e] \]

by the definitions of \( \Gamma_{e,e'} \) and \( \Delta_{e,e'} \).

\[ \square \]

**Note.** Just like the disjoint union \( \bigsqcup_{i \in I} S_i \) gives a coproduct in the category \( \text{Set} \) for an indexed family of sets \( \{S_i\}_{i \in I} \), the set \( \bigsqcup \mathcal{E} \mathcal{M} \) of orbits of a small left module \( \mathcal{M} : * \to \mathcal{E} \) (see Definition 4.11.15) gives a colimit of \( \mathcal{M} \):

8.6.2 **Theorem.** For a small left module \( \mathcal{M} : * \to \mathcal{E} \) [op. right module \( \mathcal{M} : \mathcal{E} \to * \)], the set of orbits of \( \mathcal{M} \) gives a colimit of \( \mathcal{M} \) with the universal cone

\[ \pi_\mathcal{M} : \mathcal{M} \to \bigsqcup \mathcal{E} \mathcal{M} : * \to \mathcal{E} * \text{ op.} \quad \pi_\mathcal{M} : \mathcal{M} \to \bigsqcup \mathcal{E} \mathcal{M} : \mathcal{E} * \to * \]

defined by the natural projection such that

\[ m : \pi_\mathcal{M} := m^\prime \]

for each \( \mathcal{M} \)-arrow \( m \).

**Proof.** The coequalizer in Theorem 8.6.1 is given by the quotient of \( \bigsqcup_{e \in \mathcal{E}} (\mathcal{M}) e \) by the equivalence relation generated by the pairs \((m, m \circ h)\) for every \( \mathcal{M} \)-arrow \( m : * \to e \) and every \( \mathcal{E} \)-arrow \( h : e \to e' \); that is, (see Remark 4.11.13), by the equivalence relation defined in Definition 4.11.10.

8.6.3 **Remark.** For a small left module \( \mathcal{M} : * \to \mathcal{E} \), the notation \( \bigsqcup \mathcal{E} \mathcal{M} \) thus denotes both the set of components of \( \mathcal{M} \) and a colimit of \( \mathcal{M} \).

**Note.** Recall from Definition 4.11.18 that the assignment of the set \( \bigsqcup \mathcal{E} \mathcal{M} \) of orbits to a left module \( \mathcal{M} : \mathcal{M} : * \to \mathcal{E} \) extends to the functor \( \bigsqcup \mathcal{E} : [\cdot : \mathcal{E}] \to \text{Set} \).

8.6.4 **Corollary.** Let \( \mathcal{E} \) be a small category. The functor

\[ \bigsqcup \mathcal{E} : [\cdot : \mathcal{E}] \to \text{Set} \text{ op.} \quad \bigsqcup \mathcal{E} : [\cdot : \mathcal{E}] \to \text{Set} \]

and the family of universal cones

\[ \pi_\mathcal{M} : \mathcal{M} \to \bigsqcup \mathcal{E} \mathcal{M} : * \to \mathcal{E} * \text{ op.} \quad \pi_\mathcal{M} : \mathcal{M} \to \bigsqcup \mathcal{E} \mathcal{M} : \mathcal{E} * \to * \]

, one for each left module \( \mathcal{M} : * \to \mathcal{E} \) [op. right module \( \mathcal{M} : \mathcal{E} \to * \)], forms a unit

\[ [\cdot : \mathcal{E}] \overset{\pi_\mathcal{E}}{\longrightarrow} \text{Set} \text{ op.} \quad [\cdot : \mathcal{E}] \overset{\pi_\mathcal{E}}{\longrightarrow} \text{Set} \]

of the module \( \langle \cdot : \mathcal{E} \rangle \) [op. \( \langle \mathcal{E} \mathcal{E} \rangle \)].

**Proof.** We just need to verify that \( \pi \) satisfies the naturality condition, i.e. that the square

\[ \begin{array}{ccc}
\mathcal{M} & \xrightarrow{\pi_\mathcal{M}} & \bigsqcup \mathcal{E} \mathcal{M} \\
\psi \downarrow & & \downarrow \bigsqcup \psi \\
\mathcal{N} & \xrightarrow{\pi_\mathcal{N}} & \bigsqcup \mathcal{E} \mathcal{N}
\end{array} \]

commutes for any left module morphism \( \psi : \mathcal{M} \to \mathcal{N} \). But for any \( \mathcal{M} \)-arrow \( m \),

\[ m : (\psi \circ \pi_\mathcal{N}) = m : \psi : \pi_\mathcal{N} = (m : \psi)^\prime \]

and

\[ m : (\pi_\mathcal{M} \circ \bigsqcup \psi) = m : \pi_\mathcal{M} : \bigsqcup \psi = m^\prime : \bigsqcup \psi = (m : \psi)^\prime \]

by Theorem 8.6.2 and Definition 4.11.18.

8.6.5 **Corollary.** Let \( \mathcal{E} \) be a small category. Given a category \( \mathcal{X} \) [op. \( \mathcal{A} \)], the module \( \langle \mathcal{X} : \mathcal{E} \rangle \) [op. \( \langle \mathcal{E} \mathcal{E} \rangle \)] has a unit

\[ \begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi_\mathcal{X}} & [\cdot : \mathcal{X}] \\
\mathcal{E} & \xrightarrow{\pi_\mathcal{E}} & [\cdot : \mathcal{A}]
\end{array} \]
8.7. Yoneda morphisms and limits

In this section, we give a characterization of a limit in terms of limits in \( \mathbf{Set} \). Recall from Section 5.4 that the Yoneda morphism transforms a cone into a wedge in \( \mathbf{Set} \). We will show that a cone is universal precisely when this wedge in \( \mathbf{Set} \) is pointwise universal; that is, when it forms a parameterized limit (see Section 8.4) in \( \mathbf{Set} \). (The notations introduced in Section 4.10 are used in this section.)

Note: Corollary 8.5.5, Remark 8.5.6, and Theorem 6.5.18 allow the following definition.

8.7.1 Definition. Given a small category \( \mathbf{E} \) and a module \( \mathcal{M} : \mathbf{X} \to \mathbf{A} \), the pointwise inverse \([\mathbf{op. direct}]\) universal cylinder

\[
\xymatrix{
\langle \mathbf{E}, \mathcal{M} \rangle & [\mathbf{E}, \mathbf{A}] \ar[ll]_{\times \mathcal{M} \to \mathbf{E}} \ar[rr]^{\times \mathcal{M} \to \mathbf{E}} & & \times \mathcal{M} \to \mathbf{E} \\
[\mathbf{X}] & & \ar[ll]_{\times \mathcal{M} \to \mathbf{E}} [\mathbf{M} \times \mathbf{E}] & [\mathbf{E}, \mathbf{A}] \\
[\mathbf{X} \times \mathbf{E}] & & \ar[ll]_{\times \mathcal{M} \to \mathbf{E}} [\mathbf{M} \times \mathbf{E}] & [\mathbf{E}, \mathbf{A}] \\
} \]

is defined by the composition

\[
\xymatrix{
[\mathbf{X}] & [\mathbf{X} \times \mathbf{E}] \ar[ll]_{\Pi \mathcal{M}} & [\mathbf{E}, \mathbf{A}] \\
[\mathbf{X} \times \mathbf{E}] \ar[rr]^\times_{\times \mathcal{M} \to \mathbf{E}} & & [\mathbf{E}, \mathbf{A}] \\
[\mathbf{X} \times \mathbf{E}] \ar[ll]_{\Pi \mathcal{M}} & [\mathbf{E}, \mathbf{A}] \\
} \]

8.7.2 Remark. By Theorem 6.5.18, the cylinder \([\mathbf{M} \times \mathbf{E}] \ [\mathbf{op. direct}] \times \mathcal{M} \mathbf{E} \) is pointwise inverse \([\mathbf{op. direct}]\) universal;
the component of \([\mathcal{M} \ast \mathbf{E}]\) at a functor \(F : \mathbf{E} \to \mathbf{A}\) is the pointwise universal wedge
\[
[\mathcal{M} \ast \mathbf{E}]_F : \ast \mathbf{E}, \mathcal{M})(F) \to \langle \mathcal{M} \rangle F : \mathbf{X} \to \ast \mathbf{E}
\]
given by \([\mathcal{M} \ast \mathbf{E}]_F = \pi_{\langle \mathcal{M} \rangle F}\) (the component of the counit \(\pi\) at \(\langle \mathcal{M} \rangle F\)), and the slice of \([\mathcal{M} \ast \mathbf{E}]_F\) at \(x \in \mathbf{X}\) is the universal cone
\[
x ((\mathcal{M} \ast \mathbf{E})_F) : (x) \ast \mathbf{E}, \mathcal{M})(F) \to x \langle \mathcal{M} \rangle F : \ast \to \ast \mathbf{E}
\]
given by \(x ((\mathcal{M} \ast \mathbf{E})_F) = x (\pi_{\langle \mathcal{M} \rangle F}) = \pi_{\langle \mathcal{M} \rangle F}\) (see Corollary 8.5.5).

- The component of \([\mathbf{E} \ast \mathcal{M}]\) at a functor \(G : \mathbf{E} \to \mathbf{X}\) is the pointwise universal wedge
\[
[\mathbf{E} \ast \mathcal{M}]_G : (G) \langle \mathbf{E} \ast \mathbf{,} \mathcal{M} \rangle \to G \langle \mathcal{M} \rangle : \mathbf{E} \ast \to \mathbf{A}
\]
given by \([\mathbf{E} \ast \mathcal{M}]_G = \pi_{G\langle \mathcal{M} \rangle}\) (the component of the unit \(\pi\) at \(G \langle \mathcal{M} \rangle\)), and the slice of \([\mathbf{E} \ast \mathcal{M}]_G\) at \(a \in \mathbf{A}\) is the universal cone
\[
[(\mathbf{E} \ast \mathcal{M})_G] a : (G) \langle \mathbf{E} \ast \mathbf{,} \mathcal{M} \rangle (a) \to G \langle \mathcal{M} \rangle a : \mathbf{E} \ast \to \ast
\]
given by \([(\mathbf{E} \ast \mathcal{M})_G] a = (\pi_{G\langle \mathcal{M} \rangle}) a = \pi_{G\langle \mathcal{M} \rangle} a\) (see Corollary 8.5.5).

Note. Given a category \(\mathbf{E}\) and a module \(\mathcal{M} : \mathbf{X} \to \mathbf{A}\), we have two Yoneda morphisms for \((\ast \mathbf{E}, \mathcal{M})\),

\[
\begin{array}{c}
\mathbf{X} - \longrightarrow [\mathbf{E}, \mathbf{A}] \\
\xrightarrow{x} \langle \mathbf{X} \ast \mathbf{E}, \mathcal{M} \rangle \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} \langle \mathcal{M} \rangle \xrightarrow{\mathbf{E} \ast \mathbf{E}} \mathbf{X}
\end{array}
\]

, one defined in Definition 5.2.3 and the other in Definition 5.4.1. The pointwise inverse universal cylinder \([\mathcal{M} \ast \mathbf{E}]\) defined in Definition 8.7.1 reconciles these two Yoneda morphisms.

8.7.3 Theorem. Let \(\mathbf{E}\) be a small category and \(\mathcal{M} : \mathbf{X} \to \mathbf{A}\) be a module.

- The right generalized Yoneda morphism for \((\ast \mathbf{E}, \mathcal{M})\) is obtained by “pasting” the pointwise inverse universal cylinder \([\mathcal{M} \ast \mathbf{E}]\) to the right Yoneda morphism for \((\ast \mathbf{E}, \mathcal{M})\) as shown in

\[
\begin{array}{c}
\mathbf{X} - \longrightarrow (\ast \mathbf{E}, \mathcal{M}) \\
\xrightarrow{x} \langle \mathbf{X} \ast \mathbf{E}, \mathcal{M} \rangle \xrightarrow{(\ast \mathbf{E}, \mathcal{M})} \langle \mathcal{M} \rangle \xrightarrow{\mathbf{E} \ast \mathbf{E}} \mathbf{X}
\end{array}
\]

; that is, given a cone \(\alpha : \mathbf{X} \to F : \ast \mathbf{E} \to \mathcal{M}\), the wedge

\[
\langle \mathbf{X} \ast \mathcal{M} \rangle \alpha : \langle \mathbf{X} \rangle \mathbf{X} \to \langle \mathcal{M} \rangle F : \mathbf{X} \to \ast \mathbf{E}
\]

is obtained by the composition of the right module morphism

\[
\langle \mathbf{X} \ast \mathbf{E}, \mathcal{M} \rangle \alpha : \langle \mathbf{X} \rangle \mathbf{X} \to (\ast \mathbf{E}, \mathcal{M})(F) : \mathbf{X} \to \ast
\]

and the wedge

\[
[\mathcal{M} \ast \mathbf{E}]_F : (\ast \mathbf{E}, \mathcal{M})(F) \to \langle \mathcal{M} \rangle F : \mathbf{X} \to \ast \mathbf{E}
\]

- The left generalized Yoneda morphism for \((\ast \mathbf{E}, \mathcal{M})\) is obtained by “pasting” the pointwise direct universal cylinder \([\mathbf{E} \ast \mathcal{M}]\) to the left Yoneda morphism for \((\ast \mathbf{E}, \mathcal{M})\) as shown in

\[
\begin{array}{c}
\mathbf{E} \ast \mathcal{M} \\
\xrightarrow{(\ast \mathbf{E}, \mathcal{M})} \langle \mathbf{E} \ast \mathcal{M} \rangle \mathbf{A} \xrightarrow{\leftarrow} \mathbf{A}
\end{array}
\]

; that is, given a cone \(\alpha : \mathbf{G} \to \mathbf{A} : \ast \mathbf{E} \to \mathcal{M}\), the wedge

\[
\alpha \langle \mathbf{M} \rangle \mathbf{A} : \mathbf{a} \mathbf{A} \to \langle \mathcal{M} \rangle \mathbf{E} : \mathbf{E} \ast \to \mathbf{A}
\]
is obtained by the composition of the left module morphism
\[ \alpha \langle (E^*, M) \mid A \rangle : a(A) \to (G)(E^*, M) : * \to A \]
and the wedge
\[ [E^* M]_G : (G)(E^*, M) \twoheadrightarrow G(M) : E^* \to A. \]

**Proof.** We need to show that
\[ x' \langle (X \mid M) \alpha \rangle e = x' \langle (X \mid (E^*, M)) \alpha \rangle x' \langle [M \ast E]_f \rangle e \]
for \( x' \in |X| \) and \( e \in |E| \). But for an \( X \)-arrow \( h : x' \to x \),
\[ h : x' \langle (X \mid M) \alpha \rangle e = h \circ \alpha_e \]
and by Remark 8.7.2 and Theorem 8.5.2,
\[ h : x' \langle (X \mid (E^*, M)) \alpha \rangle x' \langle [M \ast E]_f \rangle e = (h \circ \alpha)_e = h \circ \alpha_e. \]

\[ \square \]

**Note.** The following characterizes a universal cone in terms of pointwise universal wedge (see Definition 8.3.2) in \( \mathcal{S}et \).

**8.7.4 Theorem.** Given a category \( E \) and a module \( M : X \to A \), the right generalized Yoneda morphism for \((E^*, M)\) [op. left generalized Yoneda morphism for \((E^*, M)\)] preserves and reflects inverse [op. direct] universal arrows in the following sense: a cone
\[ \mu : r \to F : *E \to M \quad \text{op.} \quad \mu : G \to r : E^* \to M \]
is universal if and only if the wedge
\[ (X \mid M) \mu : (X \mid r \to F \mid X \to E^*) \quad \text{op.} \quad \mu (M \mid A) : r (A) \to G(M) : E^* \to A \]
is pointwise universal; that is, if and only if the cone
\[ x \langle X \mid M \rangle \mu : x \langle X \rangle r \to x \langle F \rangle : * \to *E \quad \text{op.} \quad \mu (M \mid A) : r (A) \to G(M) : A : E^* \to * \]
is universal for every \( x \in |X| \) [op. \( a \in |A| \)].

**Proof.** First enlarge the universe if necessary so that \( E \) becomes small. For a cone \( \mu : r \to F : *E \to M \), by Theorem 8.7.3, \( (X \mid M) \mu \) is given by the composition of \( (X \mid (E^*, M)) \mu \) and \([M \ast E]_f \). Since \([M \ast E]_f \) is a pointwise universal wedge (see Remark 8.7.2), by Proposition 8.3.10, \((X \mid M) \mu \) is pointwise universal iff \((X \mid (E^*, M)) \mu \) is iso, i.e. iff \( \mu \) is inverse universal.

**Note.** The following characterizes a universal cone in terms of a pointwise universal cone (see Definition 8.3.3) in \( \mathcal{S}et \).

**8.7.5 Corollary.** The right [op. left] Yoneda morphism for a module \( M : X \to A \) preserves and reflects limits [op. colimits] in the following sense: a cone
\[ \mu : r \to F : *E \to M \quad \text{op.} \quad \mu : G \to r : E^* \to M \]
is universal if and only if its composite
\[ \begin{array}{ccc}
E & \xrightarrow{t} & E \\
\mu & \downarrow & \downarrow \mu \\
X \to M & \to & A \\
\end{array} \quad \text{op.} \quad \begin{array}{ccc}
F & \xrightarrow{t} & *E \\
\mu & \downarrow & \downarrow \mu \\
X & \to M & A \\
\end{array} \]
with the right [op. left] Yoneda morphism for \( M \) is pointwise universal.

**Proof.** Since the right exponential transpose of the wedge \((X \mid M) \mu \) is given by the cone \((X \mid M \rangle \text{op.} \delta \mu \) (see Remark 5.4.5), by the equivalence of (1) and (2) in Proposition 8.3.4, the assertion is reduced to Theorem 8.7.4 (and vice versa).

\[ \square \]
Note. The following is a special case of Theorem 8.7.4 where \( M \) is given by the hom-module of a category.

**8.7.6 Theorem.** Given categories \( C \) and \( E \), the right generalized Yoneda morphism for \((\ast E, C)\) \([\text{op. left generalized Yoneda morphism for } (E^\ast, C)\] (see Definition 5.4.6) preserves and reflects inverse \([\text{op. direct}]\) universal arrows in the following sense: a cone

\[
\mu : r \rightsquigarrow L : \ast E \to C \quad \text{op.} \quad \mu : L \rightsquigarrow r : E^\ast \to C
\]
is universal if and only if the wedge

\[
(C)\mu : (C)\mu \rightrightarrows (C)L : C \to \ast E \quad \text{op.} \quad \mu(C) : r(C) \rightrightarrows L(C) : E^\ast \to C
\]
is pointwise universal; that is, if and only if the cone

\[
c(C)\mu : c(C) \rightrightarrows c(C)L : \ast \to \ast E \quad \text{op.} \quad \mu(C)c : r(C)c \rightrightarrows L(C)c : E^\ast \to \ast
\]
is universal for every \( c \in \|C\| \).

**Proof.** By Remark 5.4.7(2), this is a special case of Theorem 8.7.4 where \( M \) is given by the hom-module of \( C \).

Note. The following is a special case of Corollary 8.7.5 where \( M \) is given by the hom-module of a category.

**8.7.7 Corollary.** The right \([\text{op. left}]\) Yoneda functor for a category \( C \) preserves and reflects limits \([\text{op. colimits}]\) in the following sense: a cone

\[
\mu : r \rightsquigarrow L : \ast E \to C \quad \text{op.} \quad \mu : L \rightsquigarrow r : E^\ast \to C
\]
is universal if and only if its composite

\[
\ast \leftarrow \overset{1}{\bullet} \overset{1}{\ast E} \quad \text{op.} \quad \overset{1}{E} \overset{1}{\to} \ast
\]

with the right \([\text{op. left}]\) Yoneda functor for \( C \) is pointwise universal.

**Proof.** Since the right exponential transpose of the wedge \((C)\mu\) is given by the cone \([C\rightharpoonup] \delta \mu \) (see Remark 5.4.10), by the equivalence of (1) and (2) in Proposition 8.3.4, the assertion is reduced to Theorem 8.7.6 (and vice versa).

Note. As shown in the proof of Corollary 8.7.7, Theorem 8.7.6 and Corollary 8.7.7 are restatements of each other. They may be further restated more concisely as follows.

**8.7.8 Corollary.** Let \( C \) and \( E \) be categories.

- A cone \( \mu : r \Rightarrow L : \ast E \to C \) is universal if and only if so is its composite \( c(C)\mu : c(C) \rightrightarrows c(C)L : \ast \to \ast E \) with the representable left module \( c(C) \) for every \( c \in \|C\| \).
- A cone \( \mu : L \Rightarrow r : E^\ast \to C \) is universal if and only if so is its composite \( \mu(C)c : r(C)c \rightrightarrows L(C)c : E^\ast \to \ast \) with the representable right module \( (C)c \) for every \( c \in \|C\| \).

**Proof.** Immediate from Theorem 8.7.6.

**8.7.9 Corollary.**

- A functor \( H : C \to B \) preserves limits over a category \( E \) if and only if for every \( b \in \|B\| \) the left module \( b(B)H : \ast \to C \) does.
- A functor \( H : C \to B \) preserves colimits over a category \( E \) if and only if for every \( b \in \|B\| \) the right module \( H(B)b : C \to \ast \) does.
8.8. Limits in comma categories

In this section, we look for conditions for a comma category to have limits and study if limits are universal for every \( b \in \parallel B \parallel \).

8.8.1 Theorem. In this section, we look for conditions for a comma category to have limits and study if limits are universal. Let \( M \) be a module and the corresponding collage be given the same name and identify with each other.

8.8.2 Corollary. A representable left module \( M : * \to A \) preserves limits over a category \( E \) if and only if the inclusion \( M : A \to [M] \) does.

8.8.3 Corollary. A representable right module \( M : X \to * \) preserves colimits over a category \( E \) if and only if the inclusion \( M : X \to [M] \) does.

8.8.4 Remark. Suppose that a left module \( M : * \to A \) preserves limits over a category \( E \). Under this condition, \( M \) is representable.

In this section, we look for conditions for a comma category to have limits and study if limits are universal for every \( b \in \parallel B \parallel \).

8.8.1 Theorem. Consider the comma and collage

\[
\begin{array}{ccc}
X & \xleftarrow{M_0} & [M] & \xrightarrow{M_1} & A \\
& \parallel M_0 \parallel & \parallel [M] \parallel & \parallel M_1 \parallel & \\
\end{array}
\]

of a module \( M : X \to A \).

- If the inclusion \( M_1 \) preserves limits over a category \( E \), then the pair of functors \( M_0 \) and \( M_1 \) creates limits over \( E \) in the following sense: given a functor \( H : E \to [M] \), if \( M_0 \circ H \) and \( M_1 \circ H \) have limits \( \rho : r \sim M_0 \circ H \) and \( \sigma : s \sim M_1 \circ H \), then there is a unique cone \( \mu : m \sim H \) in \([M]\) such that \( \mu \circ M_0 = \rho \) and \( \mu \circ M_1 = \sigma \), and moreover \( \mu \) is universal.

Proof. By Corollary 8.7.8, for any cone \( \mu : *E \to C \), \( H \circ \mu \) is universal if \( b(B)[H \circ \mu] = (b(B)H) \mu \) is universal for every \( b \in \parallel B \parallel \). 

8.7.10 Corollary.

- A representable left module \( M : * \to A \) preserves limits; that is, if a cone \( \mu : r \sim F : *E \to A \) is universal, so is the cone \( \langle M \rangle \mu : \langle M \rangle r \sim \langle M \rangle F : * \to *E \).

- A representable right module \( M : X \to * \) preserves colimits; that is, if a cone \( \mu : G \sim r : E* \to X \) is universal, so is the cone \( \mu \langle M \rangle : r \langle M \rangle \sim G \langle M \rangle : E* \to * \).

Proof. Since \( M \) is representable, \( M \cong \alpha(A) \) for some \( \alpha \in \parallel A \parallel \). But by Corollary 8.7.8, if a cone \( \mu : r \sim F : *E \to A \) is universal, so is the cone \( \alpha(A) \mu : \alpha(A) r \sim \alpha(A) F : * \to *E \).

Note. In the following, a module and the corresponding collage are given the same name and identified with each other.

8.7.11 Corollary. A representable left module \( M : * \to A \) preserves limits over a category \( E \) if and only if the inclusion \( M : A \to [M] \) does.

Proof. Since the objects of the collage category \([M]\) consists of all objects of \( A \) and the object \(*\), by Corollary 8.7.9, the inclusion \( M : A \to [M] \) preserves limits over \( E \) iff the following conditions hold:

1. for every \( a \in \parallel A \parallel \), \( a([M])M = a(A) \) preserves limits over \( E \);
2. \( *([M])M = M \) preserves limits over \( E \).

Since the first condition always holds by Corollary 8.7.10, the assertion follows.
• If the inclusion $\mathcal{M}_0$ preserves colimits over a category $\mathcal{E}$, then the pair of functors $\mathcal{M}_0^1$ and $\mathcal{M}_1^1$ creates colimits over $\mathcal{E}$ in the following sense: given a functor $H : \mathcal{E} \to [\mathcal{M}]$, if $H \circ \mathcal{M}_0^1$ and $H \circ \mathcal{M}_1^1$ have colimits $\rho : \mathcal{M}_0^1 \circ r \to s$ and $\sigma : \mathcal{M}_1^1 \circ s$, then there is a unique cone $\mu : H \to m$ in $[\mathcal{M}]$ such that $\mu \circ \mathcal{M}_0^1 = \rho$ and $\mu \circ \mathcal{M}_1^1 = \sigma$, and moreover $\mu$ is universal.

Proof. We use the commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{M}_0^1} & [\mathcal{M}] \\
\downarrow & & \downarrow \\
[\mathcal{M}] & \xrightarrow{\mathcal{M}_1^1} & [\mathcal{M}]
\end{array}
\]

in Definition 3.4.21(1). We first observe that, by the construction of a collage, $\mathcal{M}_0$ preserves limits. Now suppose that $\mathcal{M}_0^1 \circ \delta \mathcal{H}$ and $\mathcal{M}_1^1 \circ \delta \mathcal{H}$ have limits $\rho : r \to [0, [\mathcal{M}]] \circ \delta \mathcal{H}$ and $\sigma : s \to [1, [\mathcal{M}]] \circ \delta \mathcal{H}$. Since $\mathcal{M}_0$ and $\mathcal{M}_1$ preserve limits, they yield universal cones $\rho : r \to [0, [\mathcal{M}]] \circ \delta \mathcal{H}$ and $\sigma : s \to [1, [\mathcal{M}]] \circ \delta \mathcal{H}$. Hence, by Theorem 8.4.11, there is a unique cone $\mu : m \to \mathcal{H}$ in $[2, [\mathcal{M}]]$ such that $\mu \circ [0, [\mathcal{M}]] = \rho$ and $\mu \circ [1, [\mathcal{M}]] = \sigma$, and $\mu$ is universal. Clearly, $\mu$ is in $[\mathcal{M}]$, and the assertion follows. \[\square\]

8.8.2 Corollary.
• If a left module $\mathcal{M} : * \to A$ preserves limits over a category $\mathcal{E}$, then the left comma fibration $\mathcal{M}^1 : [\mathcal{M}] \to \mathcal{A}$ creates limits over $\mathcal{E}$.

• If a right module $\mathcal{M} : X \to *$ preserves colimits over a category $\mathcal{E}$, then the right comma fibration $\mathcal{M}^1 : [\mathcal{M}] \to \mathcal{X}$ creates colimits over $\mathcal{E}$.

Proof. Since the inclusion $\mathcal{M} : \mathcal{A} \to [\mathcal{M}]$ preserves limits over $\mathcal{E}$ by Corollary 8.7.11, the assertion follows as a special case of Theorem 8.8.1 where $X$ is the terminal category. \[\square\]

8.8.3 Corollary.
• Given a left module $\mathcal{M} : * \to A$, if $A$ is complete and $\mathcal{M}$ is continuous, then the comma category $[\mathcal{M}]$ is complete and the left comma fibration $\mathcal{M}^1 : [\mathcal{M}] \to \mathcal{A}$ is continuous.

• Given a right module $\mathcal{M} : X \to *$, if $X$ is cocomplete and $\mathcal{M}$ is cocontinuous, then the comma category $[\mathcal{M}]$ is cocomplete and the right comma fibration $\mathcal{M}^1 : [\mathcal{M}] \to \mathcal{X}$ is cocontinuous.

Proof. Since $\mathcal{M}$ preserves small limits, the left comma fibration $\mathcal{M}^1 : [\mathcal{M}] \to \mathcal{A}$ creates small limits by Corollary 8.8.2. The assertion thus follows from Proposition 8.2.16. \[\square\]

8.8.4 Corollary. Let $\mathcal{C}$ be a category and $c$ be an object of $\mathcal{C}$.
• The forgetful functor $\Sigma^c : c\backslash \mathcal{C} \to \mathcal{C}$ of the coslice category creates limits.

• The forgetful functor $\Sigma_{c} : \mathcal{C}/c \to \mathcal{C}$ of the slice category creates colimits.

Proof. First recall from Remark 3.3.16 that the forgetful functor $\Sigma^c : c\backslash \mathcal{C} \to \mathcal{C}$ is the same thing as the comma fibration of the left comma $\mathcal{(c(C))^1 : * \to c}$ of the representable left module $\mathcal{(c(C))^1 : * \to c}$. Since any representable left module preserves limits (Corollary 8.7.10), the assertion follows from Corollary 8.8.2. \[\square\]

8.8.5 Corollary. Let $\mathcal{C}$ be a category and $c$ be an object of $\mathcal{C}$.
• If $\mathcal{C}$ is complete, then the coslice category $c\backslash \mathcal{C}$ is complete and the forgetful functor $\Sigma^c : c\backslash \mathcal{C} \to \mathcal{C}$ is continuous.

• If $\mathcal{C}$ is cocomplete, then the slice category $\mathcal{C}/c$ is cocomplete and the forgetful functor $\Sigma_{c} : \mathcal{C}/c \to \mathcal{C}$ is cocontinuous.

Proof. Since the forgetful functor $\Sigma^c : c\backslash \mathcal{C} \to \mathcal{C}$ creates limits by Corollary 8.8.4, the assertion follows from Proposition 8.2.16. \[\square\]

Note. The following is a restatement of Theorem 8.8.1 in terms of a comma $\mathcal{K} : X \to A$ instead of a module $\mathcal{M} : X \to A$. 
8.8.6 Theorem. Consider a comma $\mathbb{K} : \mathbf{X} \to \mathbf{A}$ and its corresponding collage $\mathbb{K}^\dagger : \mathbf{X} \to \mathbf{A}$ (see Remark 3.4.29(2)):

\[
\begin{array}{ccc}
\mathbf{X} & \xleftarrow{\mathbb{K}_0} & \mathbb{K} & \xrightarrow{\mathbb{K}_1} & \mathbf{A}
\end{array}
\]

- If the inclusion $\mathbb{K}^\dagger_1$ preserves limits over a category $\mathbf{E}$, then the pair of functors $\mathbb{K}_0$ and $\mathbb{K}_1$ creates limits over $\mathbf{E}$ in the following sense: given a functor $\mathbb{H} : \mathbf{E} \to \mathbb{K}$, if $\mathbb{K}_0 \circ \mathbb{H}$ and $\mathbb{K}_1 \circ \mathbb{H}$ have limits $\rho : r \leadsto \mathbb{K}_0 \circ \mathbb{H}$ and $\sigma : s \leadsto \mathbb{K}_1 \circ \mathbb{H}$, then there is a unique cone $\mu : m \leadsto \mathbb{H}$ in $\mathbb{K}$ such that $\mu \circ \mathbb{K}_0 = \rho$ and $\mu \circ \mathbb{K}_1 = \sigma$, and moreover $\mu$ is universal.
- If the inclusion $\mathbb{K}^\dagger_0$ preserves colimits over a category $\mathbf{E}$, then the pair of functors $\mathbb{K}_0$ and $\mathbb{K}_1$ creates colimits over $\mathbf{E}$ in the following sense: given a functor $\mathbb{H} : \mathbf{E} \to \mathbb{K}$, if $\mathbb{H} \circ \mathbb{K}_0$ and $\mathbb{H} \circ \mathbb{K}_1$ have colimits $\rho : H \circ \mathbb{K}_0 \leadsto r$ and $\sigma : H \circ \mathbb{K}_1 \leadsto s$, then there is a unique cone $\mu : H \leadsto m$ in $\mathbb{K}$ such that $\mu \circ \mathbb{K}_0 = \rho$ and $\mu \circ \mathbb{K}_1 = \sigma$, and moreover $\mu$ is universal.

Proof. In Theorem 8.8.1, replace $\mathbb{M} : \mathbf{X} \to \mathbf{A}$ by the module $\mathbb{K}^\dagger : \mathbf{X} \to \mathbf{A}$; this yields Theorem 8.8.6 by the isomorphism $\mathbb{K} \cong (\mathbb{K}^\dagger)^\dagger$ (see Theorem 3.4.30).

8.8.7 Corollary.
- For any right comma $\mathbb{K} : \mathbf{X} \to \ast$, the comma fibration $\mathbb{K} : \mathbb{K} \to \mathbf{X}$ creates limits over a connected category.
- For any left comma $\mathbb{K} : \ast \to \mathbf{A}$, the comma fibration $\mathbb{K} : \mathbb{K} \to \mathbf{A}$ creates colimits over a connected category.

Proof. The assertion follows as a special case of Theorem 8.8.6 where $\mathbf{A}$ is the terminal category, observing that the inclusion $\mathbb{K}^\dagger_1 : \ast \to \mathbb{K}$ preserves limits over a connected category (an immediate consequence of [ML98] p90 Exercise 8).

8.8.8 Remark. Pullbacks and equalizers are examples of limits over a connected category. The right comma fibration $\mathbb{K} : \mathbb{K} \to \mathbf{X}$ thus creates pullbacks and equalizers. Hence, by Proposition 8.2.15, if $\mathbf{X}$ has pullbacks (resp. equalizers), so does the comma category $\mathbb{K}$ and the right comma fibration $\mathbb{K} : \mathbb{K} \to \mathbf{X}$ preserves them.

8.8.9 Corollary. Let $\mathbf{C}$ be a category and $c$ be an object of $\mathbf{C}$.
- The forgetful functor $\Sigma_c : \mathbf{C}/c \to \mathbf{C}$ of the slice category creates limits over a connected category.
- The forgetful functor $\Sigma^c : \mathbf{c}/\mathbf{C} \to \mathbf{C}$ of the coslice category creates colimits over a connected category.

Proof. Recalling from Remark 3.3.16 that the forgetful functor $\Sigma_c : \mathbf{C}/c \to \mathbf{C}$ is the same thing as the comma fibration of the right comma $(\mathbb{C}/c)^\dagger : \mathbb{C} \to \ast$, we see that this is an instance of Corollary 8.8.7.

8.8.10 Remark. For example, the forgetful functor $\Sigma_c : \mathbf{C}/c \to \mathbf{C}$ creates pullbacks and equalizers (cf. Remark 8.8.8); hence, if $\mathbf{C}$ has pullbacks (resp. equalizers), so does the slice category $\mathbf{C}/c$ and the forgetful functor $\Sigma_c : \mathbf{C}/c \to \mathbf{C}$ preserves them.

8.8.11 Corollary. Let $\mathbf{C}$ be a category and $c$ be an object of $\mathbf{C}$.
- If $\mathbf{C}$ is complete, so is the slice category $\mathbf{C}/c$.
- If $\mathbf{C}$ is cocomplete, so is the coslice category $\mathbf{c}/\mathbf{C}$.

Proof. Suppose that $\mathbf{C}$ is complete. Then $\mathbf{C}/c$ has equalizers as we saw in Remark 8.8.10. Now since all limits can be constructed from products and equalizers (see [ML98] p113 Theorem 1), we are done if we show that $\mathbf{C}/c$ has products. But this is immediate because products in $\mathbf{C}/c$ are the same thing as (multiple) pullbacks in $\mathbf{C}$.
8.8.12 Remark. We saw in Remark 8.8.10 that the forgetful functor $\Sigma_c : C/c \to C$ preserves pullbacks and equalizers, but this is not the case for products (limits over a discrete category); because products in the slice category $C/c$ are given by pullbacks in $C$, we cannot expect the forgetful functor $\Sigma_c : C/c \to C$ to preserve products.

8.9 Split idempotents

Isomorphisms are seen as limits of a diagram of the one element monoid (trivial group). In this section, we look at limits of a slightly less trivial diagram, that of the two element monoid.

8.9.1 Definition. An arrow $h : x \to x$ in a category is said to be idempotent (or an idempotent) if $h \circ h = h$.

8.9.2 Remark. Let $\ast_2$ denote the two element monoid consisting of the identity and an idempotent. Then an idempotent $h$ in a category $C$ is identified with a functor $h : \ast_2 \to C$.

8.9.3 Proposition. If $h : x \to x$ and $k : y \to y$ are idempotents, then for any $f : x \to y$ the commutativity of any of

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  h & \downarrow & h \\
  x & \xrightarrow{f} & y \\
\end{array}
\quad
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  h & \downarrow & h \\
  x & \xrightarrow{f} & y \\
\end{array}
\quad
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  h & \downarrow & h \\
  x & \xrightarrow{f} & y \\
\end{array}
\]

implies the commutativity of the others.

Proof. The only non-trivial part is to show that the commutativity of the third diagram implies that of the first and second. But if $h \circ f \circ k = f$, then

\[ h \circ f = h \circ h \circ f \circ k = h \circ f \circ k = f = h \circ f \circ k = h \circ f \circ k \circ k = f \circ k. \]

\[ \square \]

8.9.4 Definition. A retract of an object $x$ consists of an object $r$ and a pair of arrows $x \xrightarrow{\sigma} r \xleftarrow{\rho} x$ such that $\sigma \circ \rho = 1_r$.

8.9.5 Remark.

(1) If $x \xrightarrow{\sigma} r$ is a retract of $x$, then $\sigma$ is a split monomorphism, being a section of $\rho$, while $\rho$ is a split epimorphism, being a retraction of $\sigma$.

(2) We say that two retracts $x \xrightarrow{\sigma} r$ and $x \xrightarrow{\sigma'} r'$ of an object $x$ are isomorphic when there is an isomorphism $r \to r'$ making the diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{\rho} & r \\
  \rho & \downarrow & \sigma \\
  x & \xleftarrow{\sigma'} & r' \\
\end{array}
\]

commute. Two isomorphic retracts give the same idempotent $\rho \circ \sigma = \rho' \circ \sigma'$. We will see in Theorem 8.9.15 that the converse is also the case.

8.9.6 Definition. Let $h : x \to x$ be an idempotent. We say that $h$ splits, or $h$ is a split idempotent, if there is a retract $x \xrightarrow{\sigma} r$ of $x$ such that $h = \rho \circ \sigma$.

8.9.7 Remark. The notions of an idempotent, a retract, and a split idempotent are autodual.

8.9.8 Proposition. Any functor preserves idempotents, retracts, and split idempotents.

Proof. Any functor preserves the identities $h \circ h = h$, $\sigma \circ \rho = 1$, and $\rho \circ \sigma = h$. \[ \square \]
8.9.9 Proposition. If an idempotent \( h : x \to x \) splits as \( h = \rho \circ \sigma \), then

- for any arrow \( f : x' \to x \), \( f \circ h = h \) if and only if \( f \circ \rho = \rho \).
- for any arrow \( g : x \to x' \), \( h \circ g = h \) if and only if \( \sigma \circ g = \sigma \).

Proof. This is because \( \sigma \) [op. \( \rho \)] is monic [op. epic]. \( \square \)

8.9.10 Proposition. Let \( *_2 \) be the two element monoid (see Remark 8.9.2) and \( h : x \to x \) be an idempotent in a category \( C \).

- For any \( C \)-arrow \( \rho : x \to r \), the following conditions are equivalent:
  1. \( \rho \) is a colimit of \( h \), i.e. a colimit of the functor \( h : *_2 \to C \).
  2. \( \rho \) is a coequalizer of \( \xymatrix{ x \ar[r]^h & x } \).
  3. \( \rho \) has a (necessarily unique) section \( \sigma \) such that \( h \) splits as \( h = \rho \circ \sigma \).
- For any \( C \)-arrow \( \sigma : r \to x \), the following conditions are equivalent:
  1. \( \sigma \) is a limit of \( h \), i.e. a limit of the functor \( h : *_2 \to C \).
  2. \( \sigma \) is an equalizer of \( \xymatrix{ x \ar[r]^h & x } \).
  3. \( \sigma \) has a (necessarily unique) retraction \( \rho \) such that \( h \) splits as \( h = \rho \circ \sigma \).

Proof. (1)\( \Rightarrow \) (2) Immediate by recalling the definition of a coequalizer and a colimit.

(2)\( \Rightarrow \) (3) Since \( \xymatrix{ x \ar[r]^h & x \ar[r]^\rho & r } \) forms a fork, if \( \xymatrix{ x \ar[r]^h & x \ar[r]^\rho & r } \) is a universal fork, there exists a unique \( \sigma : r \to x \) such that \( h = \rho \circ \sigma \). Since \( \rho \circ \sigma \circ \rho = h \circ \rho = \rho \), we have \( \sigma \circ \rho = 1 \) by the epicity of \( \rho \).

(3)\( \Rightarrow \) (2) If \( h \) splits as \( h = \rho \circ \sigma \), then \( \xymatrix{ x \ar[r]^h & x \ar[r]^\rho & y } \) forms a fork, and, given another fork \( \xymatrix{ x \ar[r]^h & x \ar[r]^f & y } \), the composite \( \xymatrix{ r \ar[r]^\sigma & x \ar[r]^f & y } \) gives a unique (because \( \rho \) is epic) \( r \to y \) making the diagram commute. \( \square \)

8.9.11 Definition. We say that \( \rho : x \to r \) [op. \( \sigma : r \to x \)] is a coequalizer [op. equalizer] of an idempotent \( h : x \to x \) when it satisfies the equivalent conditions in Proposition 8.9.10.

8.9.12 Remark. An idempotent \( h \) splits precisely when it has a coequalizer [op. equalizer]:

- if \( h \) splits as \( h = \rho \circ \sigma \), then \( \rho \) [op. \( \sigma \)] is a coequalizer [op. equalizer] of \( h \), and conversely
- if \( h \) has a coequalizer \( \rho \) [op. equalizer \( \sigma \)], then \( h \) splits as \( h = \rho \circ \sigma \).

8.9.13 Proposition. If an idempotent \( h : x \to x \) is factored as \( x \xrightarrow{\rho} r \xrightarrow{\sigma} x \), then the following conditions are equivalent:

1. \( \rho \) and \( \sigma \) splits \( h \), i.e. \( x \xrightarrow{\sigma} r \xrightarrow{\rho} x \) is a retract of \( x \);
2. \( \rho \) is a coequalizer of \( h \);
3. \( \sigma \) is an equalizer of \( h \).

Proof. Immediate from Proposition 8.9.10. \( \square \)


Proof. Any functor preserves the condition (3) in Proposition 8.9.10 (cf. Proposition 8.9.8). \( \square \)

8.9.15 Theorem. If an idempotent \( h : x \to x \) splits as \( \rho \circ \sigma = \rho' \circ \sigma' \), then retracts \( x \xrightarrow{\sigma} r \) and \( x \xrightarrow{\sigma'} r' \) are isomorphic in the sense of Remark 8.9.5(2).
Proof. Since \( \rho \) and \( \rho' \) are colimits of \( h \), there exists an isomorphism \( r \to r' \) making the left-hand triangle of

\[
\begin{array}{c}
\ x \\
\rho \downarrow\\
\rho' \\
\end{array}
\begin{array}{c}
r \\
\downarrow\sigma
\end{array}
\begin{array}{c}
x
\end{array}
\]

commute; but since \( \rho \) is epic, this isomorphism also makes the right-hand triangle commute.

8.9.16 Definition. If \( C \) is a full subcategory of \( D \), then the retract-closure of \( C \) in \( D \), denoted by \( \text{Ret}_D[C] \), is the full subcategory of \( D \) whose objects are retracts of some object in \( C \).

8.9.17 Remark:
(1) \( C \subseteq \text{Ret}_D[C] \) because the identity \( x \xrightarrow{1} x \) is a retract of \( x \).
(2) \( \text{Ret}_D[\text{Ret}_D[C]] = \text{Ret}_D[C] \) because the composition of retracts \( x \xrightarrow{\sigma} r \) and \( r \xrightarrow{\sigma'} r' \) yields the retract \( x \xrightarrow{\rho \sigma \rho' \sigma'} r' \).

8.9.18 Definition. A full subcategory \( C \) of a category \( D \) is called
(1) retract-closed if every retract of an object in \( C \) is again in \( C \); that is, if \( C = \text{Ret}_D[C] \).
(2) retract-dense if every object of \( D \) is a retract of an object in \( C \); that is, if \( D = \text{Ret}_D[C] \).

8.9.19 Remark. If \( C \) is retract-dense in \( D \), we assume that each object \( x \in |D| \) is assigned an object \( x^* \in |C| \) and a pair of \( D \)-arrows \( x^* \xrightarrow{\rho_x} x \) making \( x \) a retract of \( x^* \), with the following convention and notation:
(1) for \( x \in |C| \), \( x^* = x \) and \( \rho_x = \sigma_x = 1_x \);
(2) for each object \( x \in |D| \), the idempotent \( C \)-arrow given by the retract \( x^* \xrightarrow{\rho_x} x \) is denoted by \( \iota_x \); that is, \( \iota_x = \rho_x \circ \sigma_x : x^* \to x^* \).
(3) for each \( D \)-arrow \( f : x \to y \), the \( C \)-arrow given by the composition \( \rho_x \circ f \circ \sigma_y \) is denoted by \( f^* \) as indicated in the commutative diagram

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\rho_x \downarrow & & \downarrow \sigma_y \\
x^* & \xrightarrow{f^*} & y^*
\end{array}
\]

8.9.20 Proposition. The assumption in Remark 8.9.19 yields the following:
(1) for any \( x \in |D| \), \( 1_x^* = \iota_x \);
(2) for any \( f : x \to y \) and \( g : y \to z \), \( (f \circ g)^* = f^* \circ g^* \);
(3) for any \( f : x \to y \), \( \iota_x \circ f^* = f^* \), \( f^* \circ \iota_y = f^* \), and the diagrams

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\sigma_x \downarrow & & \downarrow \rho_y \\
x^* & \xrightarrow{f^*} & y^*
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\rho_x \downarrow & & \downarrow \sigma_y \\
x^* & \xrightarrow{f^*} & y^*
\end{array}
\]

commute;
(4) for any \( x \in |C| \), \( \iota_x = 1_x \), and for any \( C \)-arrow \( f : x \to y \), \( f^* = f \);
(5) if \( D \)-arrow \( h : x \to x \) is idempotent, so is the \( C \)-arrow \( h^* : x^* \to x^* \).

Proof. (1) By Remark 8.9.19(3) and (2).
(2), (3) These are read off from the following commutative diagrams:

\[
\begin{array}{c}
x \xrightarrow{f} y \\
\rho_x \downarrow \sigma_x \\
x^* \xrightarrow{f^*} y^*
\end{array}
\quad\text{and}\quad
\begin{array}{c}
x \xrightarrow{f} y \\
\sigma_x \sigma_y \downarrow \rho_y \rho_y \\
x^* \xrightarrow{f^*} y^*
\end{array}
\]

\[
\begin{array}{c}
x \xrightarrow{f} y \\
\sigma_x \sigma_y \downarrow \rho_y \rho_y \\
x^* \xrightarrow{f^*} y^*
\end{array}
\quad\text{and}\quad
\begin{array}{c}
x \xrightarrow{f} y \\
\rho_x \sigma_x \rho_y \sigma_y \downarrow \rho_y \rho_y \\
x^* \xrightarrow{f^*} y^*
\end{array}
\]

\[
\begin{array}{c}
x \xrightarrow{f} y \\
\rho_x \sigma_x \rho_y \sigma_y \downarrow \rho_y \rho_y \\
x^* \xrightarrow{f^*} y^*
\end{array}
\quad\text{and}\quad
\begin{array}{c}
x \xrightarrow{f} y \\
\rho_x \sigma_x \rho_y \sigma_y \downarrow \rho_y \rho_y \\
x^* \xrightarrow{f^*} y^*
\end{array}
\]
(4) By Remark 8.9.19(1).
(5) By (2) above, \( h^* \circ h^* = (h \circ h)^* = h^* \).

8.9.21 Theorem. If \( C \) is retract-dense in \( D \), then given a parallel pair of functors \( F, G : D \to E \), any natural transformation \( \tau : C \circ F \to G \circ C : C \to E \) uniquely extends to a natural transformation \( \tau^+ : F \to G : D \to E \). Moreover, if \( \tau : C \circ F \to G \circ C \) is a natural isomorphism, so is its extension \( \tau^+ : F \to G \).

Proof. Let \( x \in \| D \| \) and let \( x^* \xrightarrow{\sigma_x} x \) be as in Remark 8.9.19. Then \( \rho_x \) is a coequalizer of \( \iota_x = \rho_x \circ \sigma_x \), and by Proposition 8.9.14, \( \rho_x : F \) and \( \rho_x : G \) are respectively coequalizers of \( \iota_x : F \) and \( \iota_x : G \). Hence there exists a unique \( E \)-arrow \( \tau^+_x : x^* : F \to G : x \) (the conjugate of \( \tau^+_x : \iota_x : F \to G : \iota_x \)) making the square

\[
\begin{array}{ccc}
x^* : F & \xrightarrow{\rho_x} & G : x \\
\downarrow{\sigma_x} & & \downarrow{\iota_x} \\
x : F & \xrightarrow{\tau^+_x} & G : x \\
\end{array}
\]

commute. We will show that the family of \( E \)-arrows \( \tau^+_x : x^* : F \to G : x \), one for each \( x \in \| D \| \), form a unique natural transformation \( F \to G \) extending \( \tau : C \circ F \to G \circ C \). Since any such natural transformation \( \tau^+ \) makes the above square commute, \( \tau^+ \) is unique. For \( x \in \| C \| \), since \( \rho_x = 1_x \) by convention, \( \tau^+_x = \tau_x \). To verify \( \tau^+ \) satisfies the naturality condition, given a \( D \)-arrow \( f : x \to y \), consider the diagram

\[
\begin{array}{ccc}
x : F & \xrightarrow{\tau^+_x} & G : x \\
\downarrow{\rho_x} & & \downarrow{\iota_x} \\
x^* : F & \xrightarrow{\tau^+_x} & G : x \\
\end{array}
\]

; the bottom face commutes by the naturality of \( \tau \), the left and right faces commute by Proposition 8.9.20(3), and the front and back faces commute by the definition of \( \tau^+ \). Hence the top face commutes by the epicity of \( \rho_x : F \).

The second assertion holds because, for each \( x \in \| D \| \), if \( \tau^+_x \) an isomorphism, so is \( \tau^+_x \) by Proposition 6.3.3(2).

8.9.22 Definition. A category is called idempotent complete if every idempotent splits.

8.9.23 Remark. Let \( D \) be an idempotent complete category and \( C \) be an isomorphism-closed full subcategory of \( D \). Then \( C \) is idempotent complete if and only if \( C \) is retract-closed; that is, if and only if \( C = \text{Ret}_D [C] \).

8.9.24 Proposition. If \( C \) is retract-dense in \( D \), then \( D \) is idempotent complete if and only if every idempotent in \( C \) splits in \( D \).

Proof. Let \( h : x \to x \) be an idempotent in \( D \) and let \( x^* \xrightarrow{\sigma_x} x \) be as in Remark 8.9.19. Since the \( C \)-arrow \( h^* : x^* \to x^* \) is an idempotent by Proposition 8.9.20(5), \( h^* \) has a splitting \( x^* \xrightarrow{\sigma} r \) by assumption. We will show that the composition \( x \xrightarrow{\rho_x} x^* \xrightarrow{\sigma} r \) yields a retract of \( x \) splitting \( h \). First note that \( h = (\sigma \circ \rho) \circ (\rho \circ \sigma) \) as can be read off from

\[
\begin{array}{ccc}
x & \xrightarrow{h} & x \\
\downarrow{\sigma_x} & & \downarrow{\rho_x} \\
x^* & \xrightarrow{h^*} & x^* \\
\end{array}
\]

\[
\begin{array}{ccc}
x & \xrightarrow{1} & x \\
\downarrow{\sigma_x} & & \downarrow{\rho_x} \\
x^* & \xrightarrow{h^*} & x^* \\
\end{array}
\]

\[
\begin{array}{ccc}
x & \xrightarrow{1} & x \\
\downarrow{\sigma_x} & & \downarrow{\rho_x} \\
x^* & \xrightarrow{h^*} & x^* \\
\end{array}
\]

\[
\begin{array}{ccc}
x & \xrightarrow{1} & x \\
\downarrow{\sigma_x} & & \downarrow{\rho_x} \\
x^* & \xrightarrow{h^*} & x^* \\
\end{array}
\]
; hence, by Proposition 8.9.13, it suffices to show that $\sigma_x \circ \rho$ is a coequalizer of $h$. For this, consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{r} & \xrightarrow{1} & \mathbb{r} \\
\rho \downarrow & & \downarrow \rho \\
\mathbb{x}^* & \xrightarrow{\iota_x} & \mathbb{x}^* \\
\h^* \downarrow & & \downarrow \h \\
\mathbb{x}^* & \xrightarrow{\iota_x} & \mathbb{x}^* \\
\end{array}
$$

(the inner four squares commute by Proposition 8.9.20(3) and Proposition 8.9.9). In the diagram, $\rho$ is a coequalizer of $h^*$, $\rho_x$ is a coequalizer of $\iota_x$, and 1 is a coequalizer of 1. Now regarding the left bottom square as a bifunctor $\ast_2 \times \ast_2 \to \mathbb{D}$ and applying Corollary 8.4.17, we see that $\sigma_x \circ \rho$ is a coequalizer of $h$.

8.9.25 Theorem. If $C$ is retract-dense in $\mathbb{D}$ and $F : C \to E$ is a functor from $C$ to an idempotent complete category $E$, then $F$ extends uniquely up to isomorphism to a functor $F^* : \mathbb{D} \to E$.

Proof. Extensions, if exist, are unique up to isomorphism by Theorem 8.9.21. Now assume that each $x \in |\mathbb{D}|$ is assigned $x^* \xrightarrow{\sigma_x \rho_x} x$ as in Remark 8.9.19 and for each idempotent $h : a \to a$ in $C$, chose a retract $a : F \xrightarrow{\sigma(h) \rho(h)} \mathbb{r}$ that splits the idempotent $h : F$ (for an identity $1_a$ in $C$, let $\rho(1_a) = \sigma(1_a) = 1_{a:F}$). Then $F^* : \mathbb{D} \to E$ is defined such that $x^* : F^* = r(\iota_x)$ for any $x \in |\mathbb{D}|$ and $\iota_F : F^* = \sigma(\iota_x) \circ (f^* : F) \circ \sigma(\iota_y)$ for any $\mathbb{D}$-arrow $f : x \to y$. $F^*$ extends $F$ because, for any $\mathbb{C}$-arrow $f : a \to b$,

$$
\begin{align*}
\sigma(\iota_a) \circ (f^* : F) \circ \rho(\iota_b) &= \sigma(\iota_a) \circ (f^* : F) \circ \rho(\iota_b) \\
&= \sigma(\iota_a) \circ (f^* : F) \circ \rho(\iota_b) \\
&= 1_a : F \circ \rho(\iota_b) \\
&= f : F
\end{align*}
$$

(*1 by Proposition 8.9.20(4), and $F^*$ is functorial because, for any $f : x \to y$ and $g : y \to z$,

$$
\begin{align*}
(f \circ g) : F^* &= \sigma(\iota_x) \circ ((f \circ g)^* : F) \circ \sigma(\iota_z) \\
&= \sigma(\iota_x) \circ ((f \circ g)^* : F) \circ \sigma(\iota_z) \\
&= \sigma(\iota_x) \circ (f^* : F) \circ (g^* : F) \circ \sigma(\iota_z) \\
&= \sigma(\iota_x) \circ (f^* : F) \circ \rho(\iota_y) \circ \sigma(\iota_y) \circ (g^* : F) \circ \sigma(\iota_z) \\
&= \sigma(\iota_x) \circ (f^* : F) \circ \rho(\iota_y) \circ \sigma(\iota_z)
\end{align*}
$$

(*1 by Proposition 8.9.20(2) and (3)), and

$$
\begin{align*}
1_x : F^* &= \sigma(\iota_x) \circ (1_x^* : F) \circ \rho(\iota_x) \\
&= \sigma(\iota_x) \circ (1_x^* : F) \circ \rho(\iota_x) \\
&= \sigma(\iota_x) \circ \rho(\iota_x) \circ \sigma(\iota_x) \circ \rho(\iota_x) \\
&= 1_x : F^*
\end{align*}
$$

(*1 by Proposition 8.9.20(1)).

8.9.26 Theorem. If a category $C$ is idempotent complete, so is the functor category $[E, C]$ for any category $E$, and every idempotent natural transform $\alpha : L \to L : E \to C$ splits pointwise.

Proof. Noting Remark 8.9.12, we see that this is an instance of Theorem 8.4.13 where $\mathbb{D}$ is given by the two element monoid.

8.9.27 Definition. An idempotent completion of a category $C$ is a full embedding $C \to \mathbb{D}$ into an idempotent complete category $\mathbb{D}$ such that $C$ is retract-dense in $\mathbb{D}$.
8.9.28 Remark. If $C$ is a full subcategory of an idempotent complete category $D$, the inclusion $C \to \text{Ret}_D[C]$ is an idempotent completion (cf. Remark 8.9.23).

8.9.29 Proposition. A full embedding $C \to D$ is an idempotent completion if and only if $C$ is retract-dense in $D$ and every idempotent in $C$ splits in $D$.

Proof. By Proposition 8.9.24, $D$ is idempotent complete. □

8.9.30 Proposition. Given equivalent categories $C$ and $C'$, if $C \to D$ and $C' \to D'$ are idempotent completions, then any equivalence functor $C \to C'$ extends to an equivalence functor $D \to D'$.

Proof. By Theorem 8.9.25 and Theorem 8.9.21, any equivalence of categories $(G, F, \eta, \epsilon) : C \simeq C'$ extends to an equivalence of categories $(G', F', \eta', \epsilon') : D \simeq D'$. □

8.9.31 Remark. In particular, idempotent completion is unique up to equivalence; that is, if $C \to D$ and $C \to D'$ are idempotent completions, then $D$ and $D'$ are equivalent.

8.9.32 Proposition. If $C \to D$ is an idempotent completion, then for any category $E$, the functor $[C, E] : [D, E] \to [C, E]$, restriction to $C$ (see Preliminary 0.0.3), is an equivalence.

Proof. The functor $[C, E] : [D, E] \to [C, E]$ is fully faithful by Theorem 8.9.21, and surjective on objects by Theorem 8.9.25. □

Note. We now give a canonical idempotent completion of a category, called the Karoubi envelope.

8.9.33 Definition. Given a category $C$, the category $\text{Split}[C]$ is defined in the following way.

- The objects of $\text{Split}[C]$ are pairs $(x, h)$ where $x$ is an object of $C$ and $h$ is an idempotent $x \to x$.
- An arrow $f : (x, h) \to (y, k)$ of $\text{Split}[C]$ is a $C$-arrows $f : x \to y$ making any one of the diagrams

  \[
  \begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{h} & \searrow{k} & \\
  x & \xrightarrow{f} & y
  \end{array}
  \]

  in Proposition 8.9.3 commute.
- The composition of $\text{Split}[C]$ is that of $C$; that is, the composition of

  \[
  \begin{array}{ccc}
  x & \xrightarrow{f} & y & \xrightarrow{g} & z \\
  \downarrow{h} & \searrow{k} & \downarrow{g} & \searrow{l} & \\
  x & \xrightarrow{f} & y & \xrightarrow{g} & z
  \end{array}
  \]

  is given by

  \[
  \begin{array}{ccc}
  x & \xrightarrow{f \circ g} & z \\
  \downarrow{h} & \searrow{j} & \\
  x & \xrightarrow{f \circ g} & z
  \end{array}
  \]

  - The identity $(x, h) \to (x, h)$ is given by

    \[
    \begin{array}{ccc}
    x & \xrightarrow{h} & x \\
    \downarrow{h} & \searrow{h} & \\
    x & \xrightarrow{h} & x
    \end{array}
    \]

8.9.34 Remark.

(1) $C$ is fully embedded into $\text{Split}[C]$: each object $x$ of $C$ is identified with the object $(x, 1_x)$ of $\text{Split}[C]$, and each arrow $f : x \to y$ of $C$ is identified with the arrow

  \[
  \begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{1} & \searrow{1} & \\
  x & \xrightarrow{f} & y
  \end{array}
  \]

  of $\text{Split}[C]$. 
(2) Every object \((x, h)\) of \(\text{Split}[\mathcal{C}]\) is a retract of \((x, 1_x)\) with the retraction and the section given as in

\[
\begin{array}{ccc}
X \xrightarrow{h} X \xrightarrow{h} X \\
\downarrow h \quad \downarrow 1 \quad \downarrow h \\
X \xrightarrow{h} X \xrightarrow{h} X
\end{array}
\]

(3) Every idempotent \(h : x \to x\) in \(\mathcal{C}\) splits as in

\[
\begin{array}{ccc}
X \xrightarrow{h} X \xrightarrow{h} X \\
\downarrow 1 \quad \downarrow h \quad \downarrow 1 \\
X \xrightarrow{h} X \xrightarrow{h} X
\end{array}
\]

(4) By its construction, if \(\mathcal{C}\) is small (resp. locally small), so is \(\text{Split}[\mathcal{C}]\).

8.9.35 Proposition. \(\text{Split}[\mathcal{C}]\) is an idempotent completion of \(\mathcal{C}\).

Proof. By Remark 8.9.34(1) and (2), \(\mathcal{C}\) is a retract-dense in \(\text{Split}[\mathcal{C}]\). Hence, by Remark 8.9.34(3) and Proposition 8.9.29, \(\text{Split}[\mathcal{C}]\) is idempotent complete. \(\square\)

8.9.36 Theorem. If \(\mathcal{C} \to \mathcal{D}\) is an idempotent completion of a small category \(\mathcal{C}\), then \(\mathcal{D}\) is essentially small.

Proof. By Proposition 8.9.35 and Remark 8.9.31, \(\mathcal{D} \simeq \text{Split}[\mathcal{C}]\); but \(\text{Split}[\mathcal{C}]\) is small as we noted in Remark 8.9.34(4). \(\square\)

Note. Recall from Definition 2.3.5 that \(\text{Rep}[\mathcal{C}]\) denotes the full subcategory of the category \([\mathcal{C}]\) whose objects are representable right modules over \(\mathcal{C}\).

8.9.37 Theorem. A category \(\mathcal{C}\) is idempotent complete if and only if \(\text{Rep}[\mathcal{C}]\) is retract-closed in \([\mathcal{C}]\); that is, if and only if \(\text{Rep}[\mathcal{C}] \simeq \text{Ret}_{[\mathcal{C}]}[\text{Rep}[\mathcal{C}]]\).

Proof. Since \(\mathcal{C} \simeq \text{Rep}[\mathcal{C}]\) (see Theorem 7.11.28), we just need to show that \(\text{Rep}[\mathcal{C}]\) is idempotent complete iff it is retract-closed in \([\mathcal{C}]\). But since \([\mathcal{C}]\) is cocomplete (a fortiori, idempotent complete) and \(\text{Rep}[\mathcal{C}]\) is isomorphism-closed, this is the case by Remark 8.9.23. \(\square\)

Note. The following shows that an idempotent completion of a category \(\mathcal{C}\) is also obtained under the Yoneda embedding \([\mathcal{C} \to \mathcal{C}]: \mathcal{C} \to [\mathcal{C}]\).

8.9.38 Theorem. If \(\mathcal{C} \to \mathcal{D}\) is an idempotent completion, then \(\mathcal{D}\) is equivalent to \(\text{Ret}_{[\mathcal{C}]}[\text{Rep}[\mathcal{C}]]\).

Proof. Since \([\mathcal{C}]\) is complete (a fortiori, idempotent complete), by Remark 8.9.28 the inclusion \(\text{Rep}[\mathcal{C}] \to \text{Ret}_{[\mathcal{C}]}[\text{Rep}[\mathcal{C}]]\) is an idempotent completion. Hence, by Proposition 8.9.30, the equivalence \(\mathcal{C} \simeq \text{Rep}[\mathcal{C}]\) (see Theorem 7.11.28) extends to the equivalence \(\mathcal{D} \simeq \text{Ret}_{[\mathcal{C}]}[\text{Rep}[\mathcal{C}]]\). \(\square\)

Note. The following is an instance of Proposition 8.9.32 where \(\mathcal{E}\) is given by \(\text{Set}^\to\).

8.9.39 Theorem. If \(\mathcal{C} \to \mathcal{D}\) is an idempotent completion, then the categories \([\mathcal{C}]\) and \([\mathcal{D}]\) are equivalent. \(\square\)

8.10 Absolute colimits

In Section 8.7, we saw that the right Yoneda functor preserves limits. This is not the case for colimits. In fact, the class of colimits preserved by the right Yoneda functor is rather limited: the right Yoneda functor preserves only those colimits which are preserved by any functor.

8.10.1 Definition. A colimit \([\text{op. limit}]\) is called absolute if it is preserved by any functor.
8.10.2 Remark.  
(1) If a cone $\mu$ is an absolute colimit [op. limit], then the composite $\mu \circ H$ with any functor $H$ is again an absolute colimit [op. limit].  
(2) An isomorphism $\mu : x \to y$ [op. $\mu : y \to x$] in a category $\mathbf{C}$ is a colimit [op. limit] of the functor $x : * \to \mathbf{C}$, and an absolute one because any functor preserves isomorphisms.

8.10.3 Proposition. An absolute colimit [op. limit] in a factor category $[\mathbf{E}, \mathbf{C}]$ is pointwise (see Definition 8.4.3).

Proof. This is because each evaluation $[e, \mathbf{C}] : [\mathbf{E}, \mathbf{C}] \to \mathbf{C}$ preserves colimit. □

8.10.4 Theorem. A coequalizer [op. equalizer] of an idempotent is an absolute colimit [op. limit].


Note. Any functor preserves isomorphisms and coequalizers of an idempotent because they are defined by identities involving only composites (i.e. by commutative diagrams). We may ask if this is the case for any absolute colimit. The answer is positive as we will see below.

8.10.5 Theorem.  
- For a cone $\mu : L \to r : E \to C$, the following conditions are equivalent:  
  (1) $\mu$ is an absolute colimit;  
  (2) the composite $\mu \circ [C, r]$ with the right Yoneda functor $[C, r] : \mathbf{C} \to [\mathbf{C}]$ is a pointwise colimit;  
  (3) the wedge $\langle C \rangle \mu : \langle C \rangle L \sim \langle C \rangle r$ is pointwise universal;  
  (4) there exists an object $s \in \mathbf{E}$ such that the component of $\mu$ at $s$ is a retraction, and there is a section $s : L \overset{\mu_s}{\to} r$ of $\mu_s$ such that for every $e \in \mathbf{E}$, the composite $e : L \overset{\mu_s}{\to} r \overset{\sigma}{\to} L : s$ and the identity $e : L \to L : e$ are connected (i.e. belong to the same orbit) in the left module $(e : L) \langle C \rangle L : * \to \mathbf{E}$.

- For a cone $\mu : r \to L : * \mathbf{E} \to \mathbf{C}$, the following conditions are equivalent:  
  (1) $\mu$ is an absolute limit;  
  (2) the composite $\mu \circ [\mathbf{C}, r]$ with the left Yoneda functor $[\mathbf{C}, r] : \mathbf{C} \to [\mathbf{C}]^\mathbf{op}$ is a pointwise colimit;  
  (3) the wedge $\mu \langle C \rangle : L \langle C \rangle \sim r \langle C \rangle$ is pointwise universal;  
  (4) there exists an object $s \in \mathbf{E}$ such that the component of $\mu$ at $s$ is a section, and there is a retraction $s : L \overset{\mu_s}{\to} r$ of $\mu_s$ such that for every $e \in \mathbf{E}$, the composite $s : L \overset{\mu_s}{\to} r \overset{\sigma}{\to} L : e$ and the identity $e : L \to L : e$ are connected (i.e. belong to the same orbit) in the right module $L \langle C \rangle (L : e) : \mathbf{E} \to *$.

Proof. (1)⇒(2) If $\mu$ is an absolute colimit, then $\mu \circ [\mathbf{C}, r]$ is an absolute colimit by Remark 8.10.2(1), and hence a pointwise limit by Proposition 8.10.3.

(2)⇒(3) Since the right exponential transpose of the wedge $\mu \langle C \rangle$ is given by the cone $\mu \circ [\mathbf{C}, r]$ (see Remark 5.4.10), this follows from the equivalence of (1) and (2) in Proposition 8.4.4.

(3)⇒(4) First enlarge the universe if necessary so that $\mathbf{E}$ becomes small. Since the wedge $\mu \langle C \rangle : L \langle C \rangle \sim \langle C \rangle r$ is pointwise universal, there is a unique right module isomorphism $\theta : \langle C \rangle r \to \bigoplus_{\mathbf{E}} \langle C \rangle L$ making the diagram

$$
\begin{array}{ccc}
\langle C \rangle L & \overset{\mu \langle C \rangle}{\longrightarrow} & \langle C \rangle r \\
\pi_{\langle C \rangle L} & \downarrow^\theta & \\
\bigoplus_{\mathbf{E}} \langle C \rangle L
\end{array}
$$

commute (recall from Corollary 8.6.5 that $c (\bigoplus_{\mathbf{E}} \langle C \rangle L) = \bigoplus_{\mathbf{E}} c \langle C \rangle L$ for each $c \in [\mathbf{C}]$ and recall from Theorem 8.6.2 that $\bigoplus_{\mathbf{E}} c \langle C \rangle L$ is given by the set of orbits of the composite left module $c \langle C \rangle L : * \to \mathbf{E}$). By Theorem 5.2.8, $\theta = \mathbf{C} \uparrow \sigma$ for some $\sigma : r \to L (s)$ with $1_r \circ \theta = \sigma$. We thus have

$$
1_r = \sigma \circ \theta^{-1} = \sigma \circ \pi_{\langle C \rangle L} \circ \theta^{-1} = \sigma \circ \langle C \rangle \mu = \sigma \circ \mu_s
$$
and for any $e \in \|E\|$, 

$$(1_L : e)^\ast = 1_{L^\ast : e^\ast} : \pi(C)_L = 1_{L^\ast : e^\ast} : (C) \mu^\ast : \emptyset = \mu : \emptyset = \mu(e^\ast \sigma^\ast) = (\mu \circ \sigma)^\ast.$$ 

(4)$\Rightarrow$(1) To show that $(r, \mu)$ is a colimit of $L$, let $\alpha : L \to c : E^\ast \to C$ be another cone and define $p : r \to c$ by $p = \sigma \circ \alpha$. Then for every $e \in \|E\|$, the diagram

$$
\begin{array}{ccc}
\quad & \mu & \\
\alpha & r & \downarrow p \\
e & L & \downarrow c \\
\end{array}
$$

i.e.

$$
\begin{array}{ccc}
r & \sigma & L^\ast : s \\
\downarrow \mu & \downarrow & \downarrow \\
e & L^\ast : e & \downarrow c \\
\end{array}
$$

commutes by Theorem 4.11.21. The uniqueness of $p$ follows from the epicity of $\mu_s$. $(r, \mu)$ is an absolute colimit because any functor preserves retracts, sections, and connectedness. \(\square\)

8.10.6 Corollary. If $H : C \to B$ is a fully faithful functor, then a cone $\mu$ in $C$ is an absolute colimit [op. limit] if and if the composite $\mu \circ H$ is an absolute colimit [op. limit].

Proof. Any fully faithful functor preserves and reflects the properties in Theorem 8.10.5(4). \(\square\)

8.10.7 Definition. For a full subcategory $C$ of a category $D$, $\text{Abs}_D[C]$ denotes the full subcategory of $D$ whose objects are absolute colimits [op. limits] of some functor $L : E \to C$ for some category $E$.

8.10.8 Remark. We may use separate notations, say, $\text{Abs}_D[C]$ and $\text{Sba}_D[C]$ for absolute limits and colimits; however, it turns out (Proposition 8.10.9) that $\text{Abs}_D[C] = \text{Sba}_D[C]$.

8.10.9 Proposition. For any full subcategory $C$ of $D$, $\text{Abs}_D[C]$ coincides with the retract-closure defined in Definition 8.9.16:

$$\text{Abs}_D[C] = \text{Ret}_D[C].$$

Proof. By Theorem 8.10.4, a retract is an absolute colimit [op. limit]. Conversely, if $r \in \|D\|$ is an absolute colimit [op. limit] of a functor $L : E \to C$, then, by Theorem 8.10.5, $r$ is a retract of $s : L$ for some $s \in \|E\|$. \(\square\)

8.10.10 Definition. A full subcategory $C$ of a category $D$ is said to be closed under absolute colimits [op. limits] when $C = \text{Abs}_D[C]$.

8.10.11 Proposition. A full subcategory of a category is closed under absolute colimits [op. limits] if and only if it is retract-closed.

Proof. Immediate from Proposition 8.10.9. \(\square\)

8.10.12 Remark. The closedness under absolute colimits and the closedness under absolute limits are thus the same thing.

8.10.13 Definition. We say that a category $C$ is closed under absolute colimits (or has absolute colimits), if $\text{Rep}[C]$ (see Definition 2.3.5) is closed under absolute colimits in the category $[C : C]$; that is, if $\text{Rep}[C] = \text{Abs}_{[C : C]}[\text{Rep}[C]]$.

8.10.14 Remark. If a category $C$ is closed under absolute colimits, then the Yoneda functor $[C : C] : C \to [C : C]$ creates (up to isomorphism) absolute colimits in the following sense. Given a functor $L : E \to C$, if the composite $L \circ [C, C]$ has an absolute colimit $\Pi : L \circ [C, C] \to R$, then $R$ is isomorphic to the representable right module $(C)r$ for some $r \in \|C\|$. Since the Yoneda functor is fully faithful, we can lift $\Pi : L \circ [C, C] \to (C)r$ to a cone $\mu : L \to r$ in $C$, and this cone is an absolute colimit by Corollary 8.10.6.
8.10.15 Theorem. A category $C$ is idempotent complete if and only if it is closed under absolute colimits.

Proof. By Proposition 8.10.11, this is reduced to Theorem 8.9.37.

8.11 Adjunctions and limits

In this section, we study the interaction between adjunctions and limits.

8.11.1 Theorem. If a cell has a left \([\text{op. right}]\) adjoint (see Definition 7.10.1), then it preserves limits \([\text{op. colimits}]\).

Proof. By Definition 8.2.1, the assertion is equivalent to saying that if a cell $\psi$ has a left adjoint, then for any category $E$, the postcomposition cell $\ast \circ (\ast E, \psi)$ preserves inverse universal arrows. But this follows from Theorem 7.10.6 because if $\psi$ has a left adjoint, so does $\ast \circ (\ast E, \psi)$ by Corollary 7.10.5. \qed

Note. The following is a special case of Theorem 8.11.1 where a cell is given by the hom-cell of a functor.

8.11.2 Theorem. If a functor has a left \([\text{op. right}]\) adjoint, then it preserves limits \([\text{op. colimits}]\).

Proof. Since a functor preserves limits iff its hom-cell preserves limits (see Remark 8.2.12(2)), and since an adjoint of a functor is the same thing as an adjoint of its hom-cell (see Remark 7.10.2(2)), the assertion is reduced to Theorem 8.11.1.

8.11.3 Corollary. Let $H : C \to B$ be a fully faithful functor.

- If $B$ has limits over a category $E$ and $H$ has a right adjoint, then $C$ has limits over $E$ as well.
- If $B$ has colimits over a category $E$ and $H$ has a left adjoint, then $C$ has colimits over $E$ as well.

Proof. A limit of a functor $L : E \to C$ in $C$ is given as follows. Since $B$ has limits over $E$, the composite $L \circ H$ has a limit $\mu : r \to H \circ L$ in $B$. Now let $G : B \to C$ be a right adjoint of $H$. Since $G$ preserves limits by Theorem 8.11.2, it sends $\mu : r \to H \circ L$ to a limit $\mu \circ G : r \circ G \to G \circ H \circ L$ in $C$. Since $H$ is fully faithful, the unit $\eta : 1_C \to G \circ H$ of the adjunction is a natural isomorphism by Theorem 7.3.14, and hence so is $\eta \circ L : L \to G \circ H \circ L$ (any functorial operation preserve isomorphisms). Now the composition of $\mu \circ G : r \circ G \to G \circ H \circ L$ and the inverse of $\eta \circ L : L \to G \circ H \circ L$ yields a limit of $L$ in $C$ by Proposition 8.1.3. \qed

8.11.4 Corollary. Let $H : C \to B$ be a fully faithful functor.

- If $B$ is complete and $H$ has a right adjoint, then $C$ is complete as well.
- If $B$ is cocomplete and $H$ has a left adjoint, then $C$ is cocomplete as well.

Proof. Immediate from Corollary 8.11.3. \qed

Note. Recall from Definition 7.3.16 that any adjunction $\phi$ yields the adjunctive symmetric cells $(\ast E, \phi)$ and $(\ast E^+, \phi)$ for a given category $E$.

8.11.5 Theorem. Let $\begin{array}{c} \chi \\Downarrow \\chi' \\ \mu \Downarrow \nu \end{array}$ be an adjunction and $E$ be a category.

- If $G$ is fully faithful, then the adjunctive right symmetric cell $(\ast E, \phi)$ preserves limits in the following sense: for any adjunct diagram

$$
\begin{array}{c}
G \circ S & \to & S \\
\mu & \Downarrow \nu \\
\chi & \to & \chi' \circ F
\end{array}
$$

of $(\ast E, \phi)$, if $\mu$ is a universal cone, so is $\nu$. 
8.11. Adjunctions and limits

- If $F$ is fully faithful, then the adjunctive left symmetric cell $(E^*, \phi)$ preserves colimits in the following sense: for any adjunct diagram

$$
\begin{array}{ccc}
G &: & a \\
\nu & \downarrow & \mu \\
F & \circ & \Downarrow T
\end{array}
$$

of $(E^*, \phi)$, if $\mu$ is a universal cone, so is $\nu$.

Proof. By Proposition 7.3.18, the adjunct diagram is depicted more elaborately as

$$
\begin{array}{ccc}
G & \circ & S \\
\mu & \downarrow & \Downarrow \eta_x \\
F & \circ & \Downarrow \xi
\end{array}
$$

with the unit $\eta$ of the adjunction. The assertion now follows from the claim below with the following argument: if $\mu$ is universal and $\eta_x$ is an isomorphism, then the cone $G \circ \nu$ is universal as well by Proposition 8.1.3, and so is $\nu$ because a fully faithful functor reflects universal cones (Proposition 8.2.13).

Claim. If $\mu$ is a universal and $G$ is fully faithful, then $\eta_x$ is an isomorphism.

Proof. Since $\mu$ is universal, there is a unique arrow $h : x : F \cdot G \to x$ making the diagram

$$
\begin{array}{ccc}
G \circ S & \leftarrow & S \\
\mu & \downarrow & \Downarrow \eta_x \\
G \circ F & \circ & \Downarrow \xi
\end{array}
$$

commute. We will show that $h$ is the inverse of $\eta_x$. Since

$$
\eta_x \circ h \circ \mu = \eta_x \circ (G \circ \nu) = \mu
$$

we have $\eta_x \circ h = 1$ by the universality of $\mu$. This identity in turn yields

$$
\eta_x \circ h \circ \eta_x = 1 \circ \eta_x = \eta_x
$$

and we have $h \circ \eta_x = 1$ because the function $\eta_x(X) \cdot (G \circ F \cdot x) : (x : F \cdot G) \cdot \eta_x \cdot (G \cdot F \cdot x) \to x(X) \cdot (G \cdot F \cdot x)$ is bijective by Theorem 6.2.11 (recall from Proposition 7.3.9 that $\eta_x : x \to G \cdot F \cdot x$ is universal from $x$ to $G$).

8.11.6 Corollary. Let $H : C \to B$ be a fully faithful functor.

- If $B$ has limits over a category $E$ and $H$ has a left adjoint, then $C$ has limits over $E$ as well and $H$ preserves them.
- If $B$ has colimits over a category $E$ and $H$ has a right adjoint, then $C$ has colimits over $E$ as well and $H$ preserves them.

Proof. The adjunction involving $H$ “creates” limits over $E$ as in Theorem 8.11.5. Hence if $B$ has limits over $E$, so does $C$. By Theorem 8.11.2, $H$ preserves limits (we can also prove this directly using the adjunct diagram in the proof of Theorem 8.11.5).

8.11.7 Corollary. Let $H : C \to B$ be a fully faithful functor.

- If $B$ is complete and $H$ has a left adjoint, then $C$ is complete as well and $H$ is continuous.
- If $B$ is cocomplete and $H$ has a right adjoint, then $C$ is cocomplete as well and $H$ is cocontinuous.

Proof. Immediate from Corollary 8.11.6.
8.11.8 Theorem. Given an adjunction $\phi : G \dashv F : X \to A$ and conjugate pair $(\tau, \sigma)$ of idempotent natural transformations as in the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & (X)G \\
\downarrow & \downarrow \downarrow & \downarrow \downarrow \\
\tau & \xrightarrow{\phi} & (X)G \\
\end{array}
$$

, if $\tau$ and $\sigma$ split as $G \xrightarrow{\tau''} T \xrightarrow{\tau'} G$ and $F \xrightarrow{\sigma''} S \xrightarrow{\sigma'} F$, then there is an adjunction $\gamma : T \dashv S : X \to A$ making $(\tau'', \sigma')$ and $(\tau', \sigma'')$ conjugate as shown in the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & (X)G \\
\downarrow & \downarrow \downarrow & \downarrow \downarrow \\
\tau'' & \xrightarrow{\phi} & (X)G \\
\end{array}
$$

; specifically, $\gamma : T \dashv S : X \to A$ is defined by the module isomorphism $\gamma : (X)T \to S(A)$ given by the composition

$$
\begin{array}{ccc}
(X)T & \xrightarrow{(X)\tau'} & (X)G \\
& \xrightarrow{\phi} & F(A) \\
& \xrightarrow{\sigma''(A)} & S(A) \\
\end{array}
$$

with the inverse given by the composition

$$
\begin{array}{ccc}
(X)T & \xleftarrow{(X)\tau''} & (X)G \\
& \xleftarrow{\phi^{-1}} & F(A) \\
& \xleftarrow{\sigma''(A)} & S(A) \\
\end{array}
$$

Proof. Indeed,

$$(X)\tau'' \circ \phi \circ \sigma'(A) \circ \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

and

$$(X)\tau'' \circ \sigma''(A) \circ \sigma'(A) \circ \phi \circ \sigma'(A) = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

$$= \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

Proof. Indeed,

$$(X)\tau'' \circ \phi \circ \sigma'(A) \circ \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

and

$$(X)\tau'' \circ \sigma''(A) \circ \sigma'(A) \circ \phi \circ \sigma'(A) = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

Proof. Indeed,

$$(X)\tau'' \circ \phi \circ \sigma'(A) \circ \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

and

$$(X)\tau'' \circ \sigma''(A) \circ \sigma'(A) \circ \phi \circ \sigma'(A) = \sigma''(A) \circ \phi^{-1} \circ (X)\tau'' \circ \phi \circ \sigma'(A)$$

Proof. Indeed,
9 Exponentials

9.1 Cartesian categories

We begin this chapter with the definition of a cartesian category.

9.1.1 Definition. A cartesian monoidal category $\mathbf{A}$ is a category with a designated terminal object $1$ and a designated product diagram $x \leftarrow x \times p \rightarrow p$ for each pair of objects $(x, p)$, together rendering $\mathbf{A}$ a symmetric monoidal category.

9.1.2 Remark.

(1) [ML98] p73 Proposition 1 shows the canonical way how chosen binary product diagrams and terminal object induce a monoidal structure on the category; the proposition also shows that a category with binary products and a terminal object has finite products.

(2) If $\mathbf{A}$ is a cartesian monoidal category equipped with a product $[\cdot \times \cdot] : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$, the partial functors $[- \times p] : \mathbf{A} \to \mathbf{A}$ and $[p \times -] : \mathbf{A} \to \mathbf{A}$ are also written just as $[\times p] : \mathbf{A} \to \mathbf{A}$ and $[p \times ] : \mathbf{A} \to \mathbf{A}$ (square brackets are used for punctuation, they may be omitted if readability is not impaired).

(3) For a pair of objects $(x, p)$, we denote the product diagram by

$$
x \xleftarrow{(x\times p)_0} x \times p \xrightarrow{(x\times p)_1} p
$$

; the diagram

$$
x \xleftarrow{(x\times p)_0} x \times p \xrightarrow{(x\times p)_1} p
$$

commutes for any pair of arrows $(f, g)$.

(4) Since, as easily seen, $x \xleftarrow{x} x \rightarrow 1$ and $1 \xleftarrow{0} p \rightarrow p$ are product diagrams, we have commutative diagrams

$$
x \xleftarrow{(x\times 1)_0} x \times 1 \xrightarrow{(x\times 1)_1} 1
$$

and

$$
1 \xleftarrow{(1\times p)_0} 1 \times p \xrightarrow{(1\times p)_1} p
$$

with $(x, !)$ and $(x \times 1)_0$ (resp. $(1, p)$ and $(1 \times p)_0$) giving canonical isomorphisms $x \times 1 \cong x$ (resp. $1 \times p \cong p$) inverse to each other. The commutativity of

$$
x \xleftarrow{(x\times p)_0} x \times p \xrightarrow{(x\times p)_1} p
$$

and

$$
1 \xleftarrow{(1\times p)_0} 1 \times p \xrightarrow{(1\times p)_1} p
$$

yields the commutative diagram

$$
x \times 1 \xleftarrow{x \times !} x \times p \xrightarrow{! \times p} 1 \times p
$$

; the projections $(x \times p)_0$ and $(x \times p)_1$ thus can be recovered from the arrows $x \times !$ and $! \times p$ by the canonical isomorphism $(x \times 1)_0$ and $(1 \times p)_1$, and can be alternatively written as

$$
x \times ! \quad x \times p \xrightarrow{! \times x} p
$$

under the identification $x \times 1 \cong x$. 

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9.1.3 Example.

(1) The category \( \text{Set} \) is cartesian monoidal with the usual definition of cartesian product and any choice of a singleton set.

(2) The category \( \text{CAT} \) is cartesian monoidal with the usual definition of product category and any choice of a terminal category.

9.1.4 Proposition. If a category \( A \) is cartesian monoidal, so is the functor category \( [E, A] \) for any category \( E \) with the cartesian monoidal structure given pointwise such that the evaluation \( [e, A] : [E, A] \to A \) is strict monoidal for each \( e \in \| E \| \).

**Proof.** This is an instance of Theorem 8.4.13. \( \square \)

9.1.5 Theorem. For any category \( X \), the category \( [X:] \) of right modules over \( X \) is cartesian monoidal with the product \( M \times N \) of modules \( M \) and \( N \) given by the pointwise cartesian product and the terminal object given by the module \( I \) such that \( x(I) a \) is the singleton set \( 1 = \{0\} \) for every \( x \in \| X \| \) and \( a \in \| A \| \).

**Proof.** Since \( [X:] = [X^\ast, \text{Set}] \), this follows from Proposition 9.1.4 and Example 9.1.3(1). \( \square \)

9.1.6 Definition. Let \( A \) be a cartesian monoidal category. For any \( p \in \| A \| \), the functor

\[
p^* : A \to A/p
\]

is defined such that \( p^* \) sends each \( x \in \| A \| \) to \( \left( x \times p \xrightarrow{(x \times p)_0} p \right) \in \| A/p \| \) and sends each \( A \)-arrow \( h : x \to y \) to the \( A/p \)-arrow \( h \times p : x \times p \to y \times p \).

9.1.7 Proposition. The functor \( p^* : A \to A/p \) in Definition 9.1.6 is right adjoint to the forgetful functor \( \Sigma_p : A/p \to A \); the adjunction

\[
\begin{array}{c}
A/p \xrightarrow{p^*} A \\
\Sigma_p \downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

is defined in a canonical way with the component of the counit at \( x \in \| A \| \) given by the projection \( x \leftarrow x \times p \) as shown in the adjunct diagram

\[
\begin{array}{c}
\xymatrix{ (x \times p) \ar[r]^-{(x \times p)_0} & p \ar[d]^f & x \ar[l]_g \\
(a \leftarrow p) & x \times p \ar[d]^f \ar[u]^g & a \ar[l]_t \ar[u]^g}
\end{array}
\]

**Proof.** Given \( f : a \to x \) and \( t : a \to p \), there is a unique \( g : a \to x \times p \) making the diagram

\[
\begin{array}{c}
x \leftarrow (x \times p)_0 \ar[r]^f & x \times p \ar[r]^-{(x \times p)_1} & p
\end{array}
\]

commute. This translates into the adjunct diagram in the assertion. \( \square \)

9.1.8 Remark. There is a converse: in any category \( A \), if the forgetful functor \( \Sigma_p : A/p \to A \) has a right adjoint \( p^* : A \to A/p \), then the product \( x \times p \) exists for any \( x \in \| A \| \). To see this, write the image of \( x \) under \( p^* \) as \( x \times p \xrightarrow{(x \times p)_0} p \), and write the component at \( x \in \| A \| \) of the counit of the adjunction \( p^* \vdash \Sigma_p \) as \( x \leftarrow (x \times p)_0 \times x \times p \). Then we have the adjunct diagram in Proposition 9.1.7, which exhibits the universality of the product diagram \( x \leftarrow (x \times p)_0 \times x \times p \xrightarrow{(x \times p)_1} p \).

9.1.9 Definition. A functor between cartesian monoidal categories is called cartesian monoidal if it preserves finite products.
9.1.10 Remark.
(1) A cartesian monoidal functor is strong monoidal; that is, it preserves the monoidal structure up to isomorphism.
(2) By Theorem 8.2.19, if a functor \( G : \mathcal{A} \to \mathcal{X} \) is cartesian monoidal, then there is the canonical natural isomorphism

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\times} & A \\
G \downarrow \cong & \Downarrow & \Downarrow G \\
X \times X & \xrightarrow{\times} & X
\end{array}
\]

consisting of iso \( \mathcal{X} \)-arrows \( \nu(x,p) : (x \cdot G) \times (p \cdot G) \to (x \times p) \cdot G \) making the diagram

\[
\begin{array}{ccc}
x \cdot G & \xrightarrow{(x \cdot G) \times (p \cdot G)} & (x \times p) \cdot G \\
\| & \Downarrow & \| \\
(x \cdot G) & \xrightarrow{\nu(x,p)} & (x \times p) \cdot G
\end{array}
\]

commute for every \( x \in \mathcal{A} \) and \( p \in \mathcal{A} \).

9.1.11 Definition. A category \( \mathcal{A} \) is called locally cartesian if the slice category \( \mathcal{A}/b \) is cartesian monoidal for every \( b \in \mathcal{A} \).

9.1.12 Proposition. For a category \( \mathcal{A} \), the following conditions are equivalent:
(1) \( \mathcal{A} \) is locally cartesian;
(2) for every \( b \in \mathcal{A} \), each pair of objects \( x \to b \) and \( p \to b \) of the slice category \( \mathcal{A}/b \) has a designated product diagram;
(3) each cospan \( x \to b \leftarrow p \) in \( \mathcal{A} \) has a designated pullback diagram.

Proof. (1)\(\Rightarrow\)(2) Immediate on noting that each slice category \( \mathcal{A}/b \) has the terminal object given by the identity \( b \to b \).
(2)\(\Rightarrow\)(3) A pullback over \( b \) is the same thing as a product in \( \mathcal{A}/b \). \( \square \)

9.1.13 Remark. For any cospan \( x \to b \leftarrow p \), its pullback diagram in \( \mathcal{A} \) (i.e. a product in \( \mathcal{A}/b \)) is written as

\[
\begin{array}{ccc}
& s \times k & p \\
& \| & \| & \Downarrow k \\
& x & s & b
\end{array}
\]

(cf. Remark 9.1.2(3)).

9.1.14 Example.
(1) The category \( \mathbf{Set} \) is locally cartesian with the canonical pullbacks (see [AHS09] 11.12(1)).
(2) The category \( \mathbf{CAT} \) is locally cartesian with the canonical pullbacks (see Preliminary 0.0.9).

9.1.15 Proposition. If a category \( \mathcal{A} \) is locally cartesian, so is the functor category \([\mathcal{E}, \mathcal{A}]\) for any category \( \mathcal{E} \) with the pullbacks given pointwise such that for any functor \( K : \mathcal{E} \to \mathcal{A} \), the slice \([\mathcal{E}, \mathcal{A}]/K : [\mathcal{E}, \mathcal{A}]/K \to \mathcal{A}/(k \cdot K)\) of the evaluation \([e, \mathcal{A}] : [\mathcal{E}, \mathcal{A}] \to \mathcal{A}\) over \( K \) is strict monoidal for each \( e \in \mathcal{E} \).

Proof. This is an instance of Theorem 8.4.13. \( \square \)
9.1.16 **Theorem.** For any category $X$, the category $[X:]$ of right modules over $X$ is locally cartesian.

*Proof.* Since $[X:] = [X^-, Set]$, this follows from Proposition 9.1.15 and Example 9.1.14(1). □

9.1.17 **Definition.** Let $A$ be a locally cartesian category. For any $A$-arrow $k : p \to b$, the pullback functor

$$k^* : A/b \to A/p$$

along $k$ is defined such that $k^*$ sends each $(x \xrightarrow{s} b) \in \|A/b\|$ to $\left(s \times k \xrightarrow{(s \times k)_0} p\right) \in \|A/p\|$ and sends each $A/b$-arrow $h : s \to s'$ to the $A/p$-arrow $h \times k : s \times k \to s' \times k$.

9.1.18 **Remark.** Let $\Sigma_b/k : [A/b]/k \to A/p$ be the isomorphism in Proposition 3.3.10. Then the square

$$\begin{array}{ccc}
[A/b]/k & \xleftarrow{k^*} & A/b \\
\Sigma_b/k & \| & \\
A/p & \xrightarrow{k^*} & A/b
\end{array}$$

commutes, where $k^* : A/b \to [A/b]/k$ is as defined in Definition 9.1.6 with $k$ regarded as an object of the slice category $A/b$.

9.1.19 **Proposition.** If $A$ is a locally cartesian category, then for any $A$-arrow $k : p \to b$ the pullback functor $k^* : A/b \to A/p$ in Definition 9.1.17 is right adjoint to the postcomposition functor $A/k : A/p \to A/b$; the adjunction

$$A/p \xleftarrow{\cong} A/b$$

is defined in a canonical way with the component of the counit at $(x \xrightarrow{s} b) \in \|A/b\|$ given the projection $x \xleftarrow{(s \times k)_0} s \times k$ as shown in the adjoint diagram

$$\begin{array}{ccc}
(s \times k)_1 & \xleftarrow{(s \times k)_0} & s \\
\downarrow{g} & & \uparrow{f} \\
(s \times k)_1 \circ k & & f
\end{array}$$

, where $s$, $t$, $f$, and $g$ are as in the commutative diagram

$$\begin{array}{ccc}
\xrightarrow{f} & a & \xrightarrow{t} \\
\xleftarrow{g} & s \times k & \xrightarrow{p}
\end{array}$$

, where $s$, $t$, $f$, and $g$ are as in the commutative diagram

$$\begin{array}{ccc}
\xrightarrow{f} & a & \xrightarrow{t} \\
\xleftarrow{g} & s \times k & \xrightarrow{p}
\end{array}$$

Proof. The commutative diagram in the assertion translates into the adjunct diagram: since

$$\begin{array}{ccc}
x & \xleftarrow{(s \times k)_0} & s \times k & \xrightarrow{(s \times k)_1} & p \\
\downarrow{s} & & \downarrow{k} & & \downarrow{k}
\end{array}$$

is a pullback diagram (cf. Remark 9.1.113), for any $f : a \to x$ making the diagram left below commute, there exists a unique $g : a \to s \times k$ such that the diagram right below commutes.
9.1.20 Remark. In fact, by the commutativity of the diagrams in Proposition 3.3.10 and Remark 9.1.18, Proposition 9.1.19 is reduced along the isomorphism $[Σ_b/k]^{-1}: \mathcal{A}/p \to [\mathcal{A}/b]/k$ into an instance of Proposition 9.1.7: an adjunction $k^*: \mathcal{A}/k: \mathcal{A}/p \to \mathcal{A}/b$ is induced (see Corollary 7.8.8) from $k^*: [\mathcal{A}/b]/k \to \mathcal{A}/b$ such that the commutative diagram

$$
\begin{array}{ccc}
[A/b]/k & \xleftarrow[k^*]{} & \mathcal{A}/b \\
\downarrow_{[Σ_b/k]} & & \downarrow_{[Σ_k/k]} \\
\mathcal{A}/p & \xleftarrow[k^*]{} & \mathcal{A}/b \\
\end{array}
$$

constitutes a strict morphism of adjunctions, where $k: p \to b$ in the upper adjunction is seen as an object of the slice category $\mathcal{A}/b$.

9.1.21 Definition. A functor between locally cartesian monoidal categories is called locally cartesian if it preserves pullbacks.

9.1.22 Remark. A functor $G: \mathcal{A} \to \mathcal{X}$ between locally cartesian categories is locally cartesian if and only if the slice functor $G/b: \mathcal{A}/b \to \mathcal{X}/(b: G)$ is cartesian monoidal for every $b \in \|\mathcal{A}\|$ (cf. Proposition 9.1.12).

9.1.23 Definition. A cartesian category $\mathcal{A}$ is a category with a designated terminal object $1$ and a designated pullback diagram (see Remark 9.1.13) for each cospan $x \xleftarrow{s} b \xrightarrow{c} p$ of $\mathcal{A}$-arrows.\(^1\)

Note. The following gives alternative definitions of a cartesian category.

9.1.24 Proposition. For a category $\mathcal{A}$, the following conditions are equivalent:

1. $\mathcal{A}$ is cartesian;
2. $\mathcal{A}$ has a designated terminal object $1$ and is locally cartesian;
3. $\mathcal{A}$ is cartesian monoidal and locally cartesian;
4. $\mathcal{A}$ is cartesian monoidal and the slice category $\mathcal{A}/b$ is cartesian monoidal for every $b \in \|\mathcal{A}\|$.

Proof. (1)⇒(2) By the equivalence of (1) and (3) in Proposition 9.1.12.
(1&2)⇒(3) A cartesian category is cartesian monoidal because a product of $x$ and $p$ is the same thing as a pullback of the cospan $x \xrightarrow{1} 1 \xleftarrow{1} p$.
(3)⇒(2) By definition, a cartesian monoidal category has a designated terminal object $1$.
(3)⇒(4) By the definition of a locally cartesian category.

Note. The following gives an alternative definition of a locally cartesian category (cf. Definition 9.1.11).

9.1.25 Proposition. A category $\mathcal{A}$ is locally cartesian if and only if the slice category $\mathcal{A}/b$ is cartesian for every $b \in \|\mathcal{A}\|$.

Proof. ($\Rightarrow$) Immediate because a cartesian category is cartesian monoidal by Proposition 9.1.24.
($\Rightarrow$) Since (see Proposition 3.3.10) there is a canonical isomorphism $[\mathcal{A}/b]/k: [\mathcal{A}/b]/k \to \mathcal{A}/p$ for any object $p \xrightarrow{k} b$ of $\mathcal{A}/b$, if $\mathcal{A}$ is locally cartesian, then the slice category $\mathcal{A}/b$ is again locally cartesian and hence cartesian by Proposition 9.1.24 because $\mathcal{A}/b$ has the terminal object $b \xrightarrow{1} b$.

9.1.26 Proposition. Any cartesian category has finite limits.

Proof. Any category with a terminal object and pullbacks has finite limits.\(^2\)

---

\(^1\)This definition of a cartesian category is after [Joh02]. In our terminology, “cartesian” is a stronger condition than “cartesian monoidal” (see Proposition 9.1.24). Note, however, that many authors use the term “cartesian category” to mean a cartesian monoidal category.

\(^2\)This definition of a cartesian category is after [Joh02]. In our terminology, “cartesian” is a stronger condition than “cartesian monoidal” (see Proposition 9.1.24). Note, however, that many authors use the term “cartesian category” to mean a cartesian monoidal category.
9.1.27 Remark. Roughly speaking, a cartesian category is a category with finite limits. Our convention, however, requires that a cartesian category and any of its slice categories are equipped with a cartesian monoidal structure.

Note. Recall from Proposition 9.1.24 that a category is cartesian precisely when it is cartesian monoidal and locally cartesian.

9.1.28 Example.
(1) By Example 9.1.3(1) and Example 9.1.14(1), the category Set is cartesian. Note that the cartesian product \( S \times T \) coincides with the canonical pullback of the cospan \( S \to 1 \leftarrow T \).
(2) By Example 9.1.3(2) and Example 9.1.14(2), the category CAT is cartesian. Note that the product category \( X \times A \) coincides with the canonical pullback of the cospan \( X \to 1 \leftarrow A \).
(3) By Theorem 9.1.5 and Theorem 9.1.16, for any category \( X \), the category \([X : ]\) of right modules over \( X \) is cartesian.
(4) A meet-semilattice is locally cartesian and a meet-semilattice with a top-element is cartesian (since a pullback is "the same thing as" a product in a preorder, the conditions "cartesian" and "cartesian monoidal" coincide).

9.1.29 Theorem. The category CFR (see Definition 3.2.15) is locally cartesian and, for any category \( X \), the category \([X \downarrow]\) of right commas over \( X \) is cartesian.

Proof. Since CFR is closed under pullbacks (see Remark 3.2.16), the locally cartesianness of CFR follows from the locally cartesianness of CAT (see Example 9.1.14(2)). Since \([X \downarrow]\) is the same thing as the slice category of CFR over \( X \), \([X \downarrow]\) is cartesian by Proposition 9.1.25. \(\square\)

9.1.30 Definition. A functor between cartesian categories is called cartesian if it preserves finite limits.

9.1.31 Remark. A functor \( G : A \to X \) between cartesian categories is cartesian if and only if it is cartesian monoidal and locally cartesian (cf. Proposition 9.1.24).

9.2 Exponentials

In this section, we study exponentials in a cartesian monoidal category as an example of an adjunction.

9.2.1 Definition. Let \( A \) be a cartesian monoidal category. An object \( p \in \| A \| \) is said to be exponential if the functor \([x, p] : A \to A\) has a right adjoint \([p \rightarrow] : A \to A\), and when this is the case, for any \( a \in \| A \| \), the object \( p \rightarrow a \in \| A \| \) is called the exponential of \( a \) by \( p \). An adjunction \([p \rightarrow] \vdash [x, p]\) is called an exponential adjunction at \( p \) and often written as \( \epsilon_p : [p \rightarrow] \vdash [x, p] \), or diagrammatically as \( A \xrightarrow{\epsilon_p} p \), using its counit \( \epsilon_p \).

9.2.2 Remark.
(1) Given an exponential adjunction at \( p \in \| A \| \), the right adjunct of an \( A \)-arrow \( f : x \times p \to a \) is denoted by \( \times f : x \to p \triangleright a \) and the left adjunct of an \( A \)-arrow \( g : x \to p \triangleright a \) is denoted by \( \triangleright g : x \times p \to a \), as in the adjunct diagrams

\[
\begin{array}{ccc}
\begin{array}{c}
\times f \downarrow \downarrow f \\
x \longrightarrow x \times p
\end{array}
& & \begin{array}{c}
g \downarrow \downarrow \triangleright g \\
\begin{array}{c}
\begin{array}{c}
p \triangleright a \longrightarrow a \\
\end{array}
\begin{array}{c}
p \triangleright a \longrightarrow a
\end{array}
\end{array}
\end{array}
\end{array}
\]

; \( \times f \) (resp. \( \triangleright g \)) is called the left exponential transpose of \( f \) (resp. \( g \)). The counit \( \epsilon_p \) of an exponential adjunction is called the evaluation; its component at \( a \in \| A \| \) is an \( A \)-arrow \( \epsilon_{p,a} \).
(p ⊸ a) × p → a, whose universality from the functor [xp] to a is expressed by the commutative diagram
\[
\begin{array}{ccc}
  p ⊸ a & \longrightarrow & a \\
  \uparrow \sigma f & & \downarrow \epsilon_{p,a} \\
  x & \longrightarrow & x \times p
\end{array}
\]
(cf. Remark 7.3.5(2)).

(2) If we use the functor [px]: A → A instead of [xp]: A → A, an exponential adjunction \( \epsilon_p: [p ⊸] \dashv [px] \) maps an A-arrow \( f: p \times x \to a \) to its right exponential transpose \( f^\triangleright: x \to p ⊸ a \) and maps A-arrow \( g: x \to p ⊸ a \) to its right exponential transpose \( g^\triangleright : p \times x \to a \) as shown in the adjunct diagrams
\[
\begin{array}{ccc}
p ⊸ a & \longrightarrow & a \\
\uparrow f & & \downarrow \epsilon_{p,a} \\
x & \longrightarrow & p \times x
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
p ⊸ a & \longrightarrow & a \\
\uparrow g & & \downarrow \epsilon_{p,a} \\
x & \longrightarrow & p \times x
\end{array}
\]
; in this case, the evaluation has the form \( \epsilon_{p,a} : p \times (p ⊸ a) \to a \), and its universality is expressed by the commutative diagram
\[
\begin{array}{ccc}
p ⊸ a & \longrightarrow & p \times (p ⊸ a) \\
\uparrow f & & \downarrow \epsilon_{p,a} \\
x & \longrightarrow & p \times x
\end{array}
\]

(3) If objects p and q in A are exponentiable, the composition
\[
\begin{array}{ccc}
A & \xrightarrow{\sigma_{pq}} & p \times q \\
\epsilon_p & \downarrow & \epsilon_q \\
A & \xrightarrow{\epsilon_{p,q}} & A
\end{array}
\]
of exponential adjunctions yields the adjunction
\[
\begin{array}{ccc}
A & \xrightarrow{p \circ (q \triangleright \cdot)} & A \\
\epsilon_{p,q} & \downarrow & \epsilon_{q,a} \\
A & \xrightarrow{\epsilon_{p,q,a} q} & A
\end{array}
\]
; by Proposition 7.8.5, the component of the counit \( \epsilon_{(p \triangleright q)} \) at \( a \in \|A\| \) is given by the composite
\[
((p ⊸ (q ⊸ a)) \times p) \times q \xrightarrow{\epsilon_{p,q,a} q} (q ⊸ a) \times q \xrightarrow{\epsilon_{q,a}} a
\]
as shown in the adjunct diagram
\[
\begin{array}{ccc}
p ⊸ (q ⊸ a) & \longrightarrow & q ⊸ a \\
\uparrow 1 & & \downarrow \epsilon_{p,q,a} \\
& \longrightarrow & (q ⊸ a) \times q
\end{array}
\quad
\begin{array}{ccc}
p ⊸ (q ⊸ a) & \longrightarrow & p \times (q ⊸ a) \\
\uparrow 1 & & \downarrow \epsilon_{p,q,a} \\
& \longrightarrow & ((p ⊸ (q ⊸ a)) \times p) \times q
\end{array}
\]

(4) If objects p and q in A are exponentiable, then for any A-arrow \( f: p \to q \), the conjugate (see Definition 7.6.1) of the natural transformation \([xf]: [xp] \to [xq]\) along the exponential adjunctions is denoted by \([f]: [q ⊸] \to [p ⊸]\) as indicated in the commutative diagram
\[
\begin{array}{ccc}
[q \triangleright] & \xrightarrow{\epsilon_q} & [xq] \\
[f] & \downarrow & \downarrow \epsilon_{xq} \\
[p \triangleright] & \xrightarrow{\epsilon_p} & [xp]
\end{array}
\]

9.2.3 Proposition. For any A-arrow \( h: a \to a' \), the exponential transpose of a composite A-arrow
\( x \times p \xrightarrow{f} a \to a' \) is given by the composite \( x \xrightarrow{sf} p \triangleright a \xrightarrow{peh} p \triangleright a' \), and for any A-arrow \( k: x' \to x \), the exponential transpose of a composite A-arrow \( x' \times p \xrightarrow{f} x \times p \to a \) is given by the composite
There is a canonical isomorphism \( x' \rightarrow x \rightarrow p \ni a \), as shown in the adjunct diagrams:

\[
\begin{array}{ccc}
\begin{array}{c}
p \triangleright a' \leftarrow a' \\
p \triangleright a \\ p \triangleright a \leftarrow a
\end{array}
\end{array}
\]

Proof. This is an instance of Proposition 7.3.3, stating the naturality of the bijection

\[ x(A)(p \triangleright a) \leftarrow (x \times p)(A) a \]

in \( x \) and \( a \).

9.2.4 Proposition. The terminal object \( 1 \) of \( A \) is exponentiable. Moreover, there is an isomorphism \( a \ni 1 \triangleright a \), natural in \( a \), given by the exponential transpose of the canonical isomorphism \( a \times 1 \cong a \) as shown in the adjunct diagram

\[
\begin{array}{ccc}
\begin{array}{c}
1 \triangleright a \leftarrow a \\
= z \\
a \leftarrow a \times 1
\end{array}
\end{array}
\]

Proof. By Proposition 7.5.3 and Proposition 7.6.7, the canonical natural isomorphism \( \sigma : [-] \cong [- \times 1] \) yields an adjunction \( \epsilon_1 : [1 \triangleright] \times [x] \cong a \) and \( \kappa \cong \) as shown in the conjugation diagram

\[
\begin{array}{c}
[1 \triangleright] \cong [x] \\
\tau \leftarrow \sigma \\
1_A \cong 1_A
\end{array}
\]

By the condition (2) in Proposition 7.6.4, the conjugate pair \( (\tau, \sigma) \) yields the adjunction diagram

\[
\begin{array}{c}
a \leftarrow a \\
\tau a \\
\sigma a^{-1} \\
1 \triangleright a \leftarrow a \times 1 \\
\sigma a \\
\kappa [a^{-1}] \\
\kappa [a^{-1}] \leftarrow a
\end{array}
\]

for each \( a \in [A] \). Since the right adjunct of the identity \( a \rightarrow a \) under the identity adjunction \( 1 : 1_A \leftarrow 1_A \) is the identity \( a \rightarrow a \), we have \( \kappa [a^{-1}] = \tau a^{-1} \).

9.2.5 Proposition. If objects \( p \) and \( q \) in \( A \) are exponentiable, so is the product \( p \times q \). Moreover, there is a canonical isomorphism \( p \triangleright (q \triangleright a) \cong (p \times q) \triangleright a \), natural in \( a \), given by the exponential transpose of the composite

\[
(p \triangleright (q \triangleright a)) \times (p \times q) \cong ((p \triangleright (q \triangleright a)) \times p) \times q \cong (p \triangleright a) \times q \cong a
\]

as shown in the adjunct diagram

\[
\begin{array}{c}
(p \times q) \triangleright a \leftarrow a \\
\epsilon_{q,a} \\
(q \triangleright a) \times q \\
\epsilon_{p \times q \times q} \\
((p \triangleright (q \triangleright a)) \times p) \times q \\
\epsilon_{p \times q \times q} \\
(p \triangleright (q \triangleright a)) \times (p \times q)
\end{array}
\]

Proof. By Proposition 7.5.3 and Proposition 7.6.7, the canonical natural isomorphism \( [- \times (p \times q)] \) yields an adjunction \( \epsilon_{p \times q} : [(p \times q) \triangleright -] \cong [- \times (p \times q)] \) isomorphic to the adjunction...
\[\varepsilon_{(p,q)} : [p \triangleright (q \triangleright -)] \sim [(- \times p) \times q] \text{ in Remark 9.2.2(3) as shown in the commutative diagram}
\]
\[
\begin{array}{ccc}
[p \triangleright (q \triangleright -)] & \sim & [(- \times p) \times q] \\
\downarrow & & \downarrow \\
[(p \times q) \triangleright -] & \sim & [-(p \times q)]
\end{array}
\]

Since the component of the counit \(\varepsilon_{(p,q)}\) at \(a \in \|A\|\) is given by the composite \((\varepsilon_{p,q} \circ a \times q) \circ \varepsilon_{q,a}\) as we saw in Remark 9.2.2(3), the condition (5) in Proposition 7.6.4 yields the second assertion.

**9.2.6 Proposition.** If an object \(p\) in an idempotent complete category \(A\) is exponentiable, then so is any retract of \(p\).

**Proof.** Let \(p \xrightarrow{\sigma} r\) be a retract of \(p\) and let \(h = \rho \circ \sigma\) be the split idempotent defined by the retract. Then the natural transformation \([\times h] : [\times p] \to [\times p]\) is an idempotent and splits as \([\times p] \xrightarrow{\times \sigma} [\times r] \xrightarrow{\times \rho} [\times p]\). Now let \([h \triangleright] : [p \triangleright] \to [p \triangleright]\) be the conjugate of \([\times h]\) as indicated in the commutative diagram

\[
\begin{array}{ccc}
[p \triangleright] & \sim & [\times p] \\
\downarrow & & \downarrow \\
[h \triangleright] & \sim & [\times h]
\end{array}
\]

(cf. Remark 9.2.2(4)). Since \([\times h]\) is an idempotent, so is \([h \triangleright]\) by the functoriality of conjugation (see Proposition 6.3.3(1)), and \([h \triangleright]\) splits because the idempotent completeness of \(A\) implies the idempotent completeness of the functor category \([A, A]\) (see Theorem 8.9.26). Hence, by Theorem 8.11.8, the functor \([\times r]\) has a right adjoint \([r \triangleright]\) (and \([h \triangleright]\) in fact splits as \([p \triangleright] \xrightarrow{\times \rho} [r \triangleright] \xrightarrow{\times \sigma} [p \triangleright]\) making the diagram

\[
\begin{array}{ccc}
[p \triangleright] & \sim & [\times p] \\
\downarrow & & \downarrow \\
[r \triangleright] & \sim & [\times r] \\
\downarrow & & \downarrow \\
[p \triangleright] & \sim & [\times p]
\end{array}
\]

commute). \(\square\)

**Note.** Recall that a cartesian category is cartesian monoidal and admits pullbacks (see Proposition 9.1.24).

**9.2.7 Definition.** Let \(A\) be a cartesian category, and suppose that \(p \in \|A\|\) is exponentiable. Then the functor

\[\Pi_p : A/p \to A\]

is defined such that \(\Pi_p\) sends each \(h : a \to p\) to \(\ast (1 \times p)_1 \times (p \triangleright h)\) as shown in the pullback diagram

\[
\begin{array}{ccc}
\mu_1 & \xrightarrow{\Pi_p h} & p \triangleright a \\
\mu_0 \downarrow & & \downarrow_{p \triangleright h} \\
1 & \xrightarrow{\ast (1 \times p)_1} & p \triangleright p
\end{array}
\]

where \(\ast (1 \times p)_1\) is the exponential transpose of the (isomorphic) projection \((1 \times p)_1 : 1 \times p \to p\).

**9.2.8 Remark.** If \(p\) is given by a set \(P\) and \(h : a \to p\) is given by a function \(\psi : A \to P\), then the pullback diagram in Definition 9.2.7 is read as

\[
\begin{array}{ccc}
\Pi_P \psi & \to & [P, A] \\
\downarrow & & \downarrow_{[P, \psi]} \\
1 & \xrightarrow{\ast (1 \times P)_1} & [P, P]
\end{array}
\]
; noting that \([P, \psi]\) is the postcomposition with \(\psi\) and \(1 \times P\), we see that \(\Pi_P \psi\) is given by the set of sections of \(\psi : A \to P\), i.e. the cartesian product \(\Pi_{\iota \epsilon P} \psi^{-1}(i)\).

9.2.9 Proposition. The functor \(\Pi_P : A/p \to A\) in Definition 9.2.7 is right adjoint to the functor \(p^* : A \to A/p\) in Definition 9.1.6; the adjunction

\[
A/p \xrightarrow{\Pi_P} A
\]

is defined in a canonical way.

Proof. Given an object \(x\) of \(A\) and an object \(a \xrightarrow{h} p\) of \(A/p\), we will show that there is a canonical bijection \((x \cdot p^*) (A/p) h \to (x) (A/p) (\Pi_P h)\). For this, let \(f : x \cdot p^* \to h\) be an arrow of \(A/p\) as shown in

\[
x \times p \xrightarrow{f} a
\]

, and consider the cube

consisting of the adjunct diagrams

(see Proposition 9.2.3) and the pullback diagram in Definition 9.2.7. Since the right face of the cube commutes by Remark 9.1.2(4), so does the left face, and we have a unique arrow \((\mu_1 \cdot f) : x \to \Pi_P h\) making the triangles there commute. Now the assignment \(f \mapsto (\mu_1 \cdot f)\) gives the desired bijection with the inverse given by \(g \mapsto \mu_1(g \delta \mu_1)\). It is straightforward to verify the naturality of the bijection in \(x\) and \(h\).

Note. There is a converse:

9.2.10 Proposition. For any cartesian monoidal category \(A\) and any object \(p \in \|A\|\), if the functor \(p^* : A \to A/p\) in Definition 9.1.6 has an right adjoint \(\Pi_P : A/p \to A\), then \(p\) is exponentiable.

Proof. Note that the functor \([\times p] : A \to A\) is given by the composite

\[
A \xrightarrow{p^*} A/p \xrightarrow{\Sigma_p} A
\]

; since (Proposition 9.1.7) \(p^*\) is right adjoint to \(\Sigma_p\), if \(p^*\) has a right adjoint \(\Pi_P\), the composite \(\Pi_P \delta p^*\) gives a right adjoint of \([\times p]\).

9.3 Cartesian closed categories

In this section, we study exponentials in a cartesian closed category as an example of a parameterized adjunction. In the latter part of the section, preservation of cartesian closedness is defined using the notion of pseudo morphisms of adjunctions introduced in Section 7.4.
9.3.1 Definition. A cartesian monoidal category \( A \) is said to be closed, or cartesian closed, if all objects of \( A \) are exponentiable.

9.3.2 Remark. 
(1) If \( A \) is cartesian closed, by applying Theorem 7.7.5 to the bifunctor \((x,p) \mapsto x \times p : A \times A \rightarrow A\) and a family of exponential adjunctions \( \epsilon_p : [p] \dashv [xp] \), one for each \( p \in \|A\| \), we have a bifunctor \((p,a) \mapsto p \triangleright a : A^\times A \rightarrow A\), “exponentiation bifunctor”, and an \( A \)-parameterized adjunction

\[
\begin{array}{c}
p \mapsto [p] \\
\epsilon_p \downarrow \\
A \\
\downarrow \epsilon_{A\triangleright A} \\
A^\times A \\
\end{array}
\]

, whose component at \( p \in \|A\| \) is the exponential adjunction \( \epsilon_p : [p] \dashv [xp] \). By Remark 9.2.2(3), the evaluation \( \epsilon_{p,a} : (p \triangleright a) \times p \rightarrow a \) (i.e. the counit of the exponential adjunction) is natural in \( p \) as well as in \( a \).

9.3.3 Proposition. If \( A \) is cartesian closed, the assignment \((p,a) \mapsto p \triangleright a \) defines a bifunctor \([\triangleright] : A^\times A \rightarrow A\), and the evaluation \( \epsilon_{p,a} : (p \triangleright a) \times p \rightarrow a \) is natural in \( p \) and \( a \).

Proof. See Remark 9.3.2(1) for the whole story.

9.3.4 Proposition. Let \( A \) be a cartesian closed category. Then for any \( A \)-arrow \( k : p' \rightarrow p \), the exponential transpose of a composite \( A \)-arrow \( x \times p \xrightarrow{xf} x \times p \rightarrow a \) is given by the composite \( x \rightarrow p \xrightarrow{k} p' \rightarrow a \) as shown in the adjunct diagram

\[
\begin{array}{c}
p' \triangleright a \\
\downarrow k \\
p \triangleright a \\
\downarrow \epsilon_{p,a} \\
x \times p \\
\downarrow (x \times k) \\
x \times p'
\end{array}
\]

Proof. This is an instance of Proposition 7.7.3, stating the naturality of the bijection

\[
x(A)(p \triangleright a) \cong (x \times p)(A)a
\]

in \( p \), i.e. the commutativity of the diagram

\[
\begin{array}{c}
p \times x(A)(p \triangleright a) \\
\downarrow \epsilon_{x(A)(p \triangleright a)} \\
x(A)(k \triangleright a) \\
\downarrow (x \times k)(A)a \\
p' \times x(A)(p' \triangleright a) \\
\downarrow \epsilon_{x(A)(p' \triangleright a)} \\
\end{array}
\]

for an \( A \)-arrow \( k : p' \rightarrow p \) (cf. Remark 7.7.4).

9.3.5 Proposition. In a cartesian closed category \( A \), the canonical isomorphism \( p \triangleright (q \triangleright a) \cong (p \times q) \triangleright a \) in Proposition 9.2.5 is natural in \( p \) and \( q \) as well.

Proof. First note that the composition of the bifunctor \((p,q) \mapsto p \times q : A \times A \rightarrow A\) and the parameterized adjunction in Remark 9.3.2(1) yields the adjunction \( \epsilon_{p,q} : [(p \times q) \dashv [- \times (p \times q)] \) parameterized by \( p \) and \( q \) with the bifunctors \((x,(p,q)) \mapsto x \times (p \times q) : A \times (A \times A) \rightarrow A\) and \(((p,q),a) \mapsto (p \times q) \triangleright a : (A \times A)^\times A \rightarrow A\). Consider now the conjugation

\[
[(p \triangleright (q \triangleright a)] \cong [(- \times p) \times q]
\]

\[
\epsilon_{p,q} \downarrow \epsilon_{q,p} \downarrow [(- \times (p \times q)]
\]

for any \( A \)-arrow \( k : p \rightarrow p \) and \( a \) in \( A \).
in the proof of Proposition 9.2.5. The adjunction $\epsilon_{p,q}: [(p \times q) \triangleright -] \dashv [\cdot \times (p \times q)]$ is parameterized by $p$ and $q$ as we have just noted, and so is the adjunction $\epsilon_{(p,q)}: [p \triangleright (q \triangleright -)] \dashv [(- \times p) \times q]$ (Remark 9.3.2(2)). Now since the natural isomorphism $[\cdot \times (p \times q)] \cong [(- \times p) \times q]$ is natural in $p$ and $q$, so will be the natural isomorphism $[p \triangleright (q \triangleright -)] \cong [(p \times q) \triangleright -]$ by Theorem 7.7.6 (see Remark 7.7.7).

9.3.6 Proposition. If a cartesian monoidal category $X$ is closed, so is any cartesian monoidal category equivalent to $X$.

Proof. Let $G: A \to X$ be an equivalence. Given an object $p \in \|A\|$, we need to show that the functor $[\times p]: A \to A$ has a right adjoint $[p \triangleright ]: A \to A$. But since the functor $[\times p]: A \to A$ is conjugate (see Definition 7.12.3) to $[\times (p; G)]$: $X \to X$ along $G$ by Remark 9.1.10(2), $[p \triangleright ]: A \to A$ is given by a conjugate of $[(p; G) \triangleright ]: X \to X$ along $G$ by Theorem 7.12.6.

9.3.7 Definition. A locally cartesian category $A$ is called locally cartesian closed if the slice category $A/b$ is closed for every $b \in \|A\|$.

9.3.8 Remark. Since (see Proposition 3.3.10) there is a canonical isomorphism $\Sigma_b/k: [A/b]/k \rightarrow A/p$ for any object $p \rightarrow^k b$ of $A/p$, if $A$ is locally cartesian closed, then so is the slice category $A/b$ for every $b \in \|A\|$.

9.3.9 Proposition. Let $A$ be a locally cartesian closed category. Then for each $A$-arrow $k: b \rightarrow p$, the pullback functor $k^*: A/b \rightarrow A/p$ in Definition 9.1.17 has the canonical right adjoint $\Pi_k: A/p \rightarrow A/b$.

Proof. By the commutativity of the square in Remark 9.1.18, the assertion translates along the isomorphism $[\Sigma_b/k]^{-1}: A/p \rightarrow [A/b]/k$ into an instance of Proposition 9.2.9: an adjunction $\Pi_k \dashv k^*: A/p \rightarrow A/b$ is induced (see Corollary 7.8.8) from $\Pi_k \dashv k^*: [A/b]/k \rightarrow A/b$ such that the commutative diagram

$$
\begin{array}{ccc}
[A/b]/k & \xrightarrow{\Pi_k} & A/b \\
\Sigma_b/k \downarrow & & \downarrow \\
A/p & \xrightarrow{k^*} & A/b
\end{array}
$$

constitutes a strict morphism of adjunctions, where $k: p \rightarrow b$ in the upper adjunction is seen as an object of the slice category $A/b$.

Note. There is a converse:

9.3.10 Proposition. Let $A$ be a locally cartesian category. If the pullback functor $k^*: A/b \rightarrow A/p$ in Definition 9.1.17 has a right adjoint $\Pi_k: A/p \rightarrow A/b$ for every $A$-arrow $k: p \rightarrow b$, then $A$ is locally cartesian closed.

Proof. Note that the functor $[\times k]: A/b \rightarrow A/b$ is given by the composite

$$
A/b \xrightarrow{k^*} A/p \xrightarrow{\Pi_k} A/b
$$

; since (Proposition 9.1.19) $k^*$ is right adjoint to $A/k$, if $k^*$ has a right adjoint $\Pi_k$, the composite $\Pi_k \circ k^*$ gives a right adjoint of $[\times k]$.

9.3.11 Remark. In fact, by the commutativity of the diagram in Remark 9.1.18, the assertion is reduced along the isomorphism $[\Sigma_p/k]^{-1}: A/p \rightarrow [A/b]/k$ to a special case of Proposition 9.2.10.
9.4 Frobenius reciprocity

We look at Frobenius reciprocity and cartesian closed functors as examples of pseudo morphisms of adjunctions introduced in Section 7.4. The main result is Theorem 9.4.10.

9.4.1 Lemma. Let \( X \) and \( A \) be cartesian monoidal categories and let \( X \xrightarrow{G} A \) be an adjunction with unit and counit \((\eta, \epsilon)\). Then \( G \) is cartesian monoidal by Theorem 8.11.2 and for any \( p \in \|A\| \), there is a canonical lax endomorphism (see Definition 7.4.1) \( \phi \rightarrow \phi \) defined by the canonical natural isomorphism \( \nu_p \) in Remark 9.1.10(2) and its mate (see Definition 7.4.5) \( \sigma_p \) as shown in

\[
\begin{array}{c}
\begin{array}{c}
A \xrightarrow{x \cdot p} A \\
\xrightarrow{G} X \\
\xrightarrow{X \times (p : G)} X
\end{array}
\end{array}
\]

; the component of \( \sigma_p \) at \( x \in \|X\| \) is given by the unique arrow

\[
\sigma_{p,x} : (x \times (p : G)) \cdot F \rightarrow (x : F) \times p
\]

making the diagram

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\times (p : G) : F} \\
\xrightarrow{\eta_{x \times (p : G)}} \\
\xrightarrow{\epsilon_{x : F} \times p : F}
\end{array}
\end{array}
\]

commute.

Proof. By the condition (4) in Proposition 7.4.4, \( \sigma_p \) is given by the composition

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\times (p : G) : F} \\
\xrightarrow{\eta_{x \times (p : G)}} \\
\xrightarrow{\epsilon_{x : F} \times p : F}
\end{array}
\end{array}
\]

, or componentwise by

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\times (p : G) : F} \\
\xrightarrow{\eta_{x \times (p : G)}} \\
\xrightarrow{\epsilon_{x : F} \times p : F}
\end{array}
\end{array}
\]

for each \( x \in \|X\| \). We now see that the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{(x : F) \times (p : G) : F} \\
\xrightarrow{(x \times p : G) : F} \\
\xrightarrow{((x : F) \times p) : F}
\end{array}
\end{array}
\]

(the bottom squares commute by the naturality of \( \epsilon \) and the functoriality of \( F \)) yields the desired one on noting that the composition of the left-hand side vertical arrows yields the identity by Theorem 7.3.13.

\( \Box \)

9.4.2 Definition. An adjunction \( X \xrightarrow{G} A \) between cartesian monoidal categories is said to satisfy Frobenius reciprocity if the natural transformation \( \sigma_p \) in Lemma 9.4.1 is an isomorphism;
that is, if the canonical morphism
\[ \sigma_{p,x} : (x \times (p \cdot G)) \to x \times F \times p \]
is an isomorphism for every \( p \in \|A\| \) and every \( x \in \|X\| \).

9.4.3 Remark. In other words, an adjunction \( \xrightarrow{G} \xleftarrow{F} A \) between cartesian monoidal categories satisfies Frobenius reciprocity if \( \nu_p \) and \( \sigma_p \) form a pseudo morphism (see Definition 7.4.7) \( \Phi \to \Phi \) for every \( p \in \|A\| \).

9.4.4 Proposition. Let \( \xrightarrow{G} \xleftarrow{F} A \) be an adjunction as in Lemma 9.4.1 between cartesian monoidal categories, and suppose that \( F \) is cartesian monoidal. Then \( \Phi \) satisfies Frobenius reciprocity if and only if \( G \) is fully faithful; that is (Theorem 7.3.14), if and only if the counit \( \epsilon \) is a natural isomorphism.

Proof. (\( \Rightarrow \)) Putting \( x = 1 \) in Lemma 9.4.1 and noting that \( 1 \cdot F \cong 1 \), we have a commutative square
\[
\begin{array}{ccc}
(1 \times (p \cdot G)) & \xrightarrow{\sigma_{1,p}} & p \cdot G : F \\
\downarrow^{\sigma_{1,1}} & & \downarrow^{\epsilon_p} \\
1 \cdot x \times p & \xrightarrow{\approx} & p
\end{array}
\]
; hence if \( \sigma_{1,1} \) is an isomorphism, so is \( \epsilon_p \).

(\( \Leftarrow \)) Since \( F \) preserves binary products and \( \epsilon_p \) is an isomorphism, the upper left leg and the upper right leg of the commutative diagram in Lemma 9.4.1 form a product diagram for \( x \cdot F \) and \( p \); hence \( \sigma_{p,x} \) is an isomorphism. \( \square \)

9.4.5 Theorem. Let \( A \) be a locally cartesian category. Then the adjunction
\[
\xrightarrow{k^*} \xleftarrow{A/k} A/b
\]
in Proposition 9.1.19 satisfies Frobenius reciprocity.

Proof. Let \( a \xrightarrow{f} p \) and \( x \xrightarrow{f} b \) be objects of the slice category \( A/p \) and \( A/b \) respectively. Then \( (t \times (f \cdot k^*)) : A/k \) is given by the diagonal \( t \times (f \cdot k) \rightarrow b \) of the commutative diagram
\[
\begin{array}{ccc}
t \times (f \cdot k^*) & \xrightarrow{\epsilon_f} & x \\
\downarrow & & \downarrow^{f} \\
a & \xrightarrow{t \cdot k} & p \times k
\end{array}
\]
consisting of two pullback squares (\( \epsilon_f \) is the component of the counit of the adjunction at \( f \)), and \( (t : A/k) \times f \) is given by the diagonal of the pullback square
\[
\begin{array}{ccc}
(t \circ k) \times f & \xrightarrow{f} & x \\
\downarrow & & \downarrow^{f} \\
a & \xrightarrow{t \circ k} & b
\end{array}
\]
, while the canonical morphism
\[ \sigma_{f,t} : (t \times (f \cdot k^*)) : A/k \rightarrow (t : A/k) \times f \]
is the unique arrow \( t \times (f \cdot k^*) \rightarrow (t \cdot k) \times f \) making the diagram
\[
\begin{array}{ccc}
a & \xrightarrow{t \times (f \cdot k^*)} & f \times k \\
\downarrow & & \downarrow^{f} \\
a & \xrightarrow{t \cdot k} & x
\end{array}
\]

\[ \xrightarrow{\sigma_{f,t}} \]
9.4.6 Lemma. A cartesian monoidal functor $G : A \to X$ between cartesian closed categories yields, for any $p \in |A|$, a canonical lax morphism (see Definition 7.4.1) from the exponential adjunction $\epsilon_p : [p] \dashv \times_p$ at $p$ to the exponential adjunction $\epsilon_{G \cdot p} : ([p] \cdot G) \dashv \times (p \cdot G)$ at $p \cdot G$ defined by the canonical natural isomorphism $\nu_p$ in Remark 9.1.10(2) and its mate (see Definition 7.4.5) $\tau_p$:

\[
\begin{array}{ccc}
A \xrightarrow{\times_p} A & \xrightarrow{\nu_p} & A \\
G & \downarrow \nu_p & \downarrow G \\
X \xrightarrow{\times (p \cdot G)} X & \xrightarrow{\tau_p} & X
\end{array}
\]

Proof. Self explanatory.

9.4.7 Remark. 

(1) The component of $\nu_p$ at $x \in |A|$ is the $X$-arrow $\nu_{p,x} : (x : G) \times (p : G) \to (x \times p) : G$, and by the condition (5) in Proposition 7.4.4, the component of $\tau_p$ at $a \in |A|$ is the $X$-arrow $\tau_{p,a} : (p \cdot a) : G \to (p : G) \cdot (a : G)$ given by the exponential transpose of the composite $\nu_{p \cdot p : a \cdot G}$ as indicated in the adjunct diagram

\[
\begin{array}{ccc}
(p \cdot G) \cdot (a : G) & \xrightarrow{\tau_{p,a}} & (p \cdot a : G) \\
\downarrow^{\epsilon_{p,a} \cdot G} & & \downarrow^{\nu_{p \cdot p \cdot a}} \\
((p \cdot a) \times p : G) & \xrightarrow{\epsilon_{p \cdot p \cdot a} \cdot G} & ((p \cdot a : G) \times (p \cdot G))
\end{array}
\]

(2) By Theorem 7.8.9, the canonical natural transformation $\tau_p$ in Lemma 9.4.6 and the canonical natural transformation $\sigma_p$ in Lemma 9.4.1 are conjugate along the composite adjunctions

\[
\begin{array}{ccc}
X \xrightarrow{G} A & \xrightarrow{\phi_p} & A \\
\downarrow F & & \downarrow F \\
X \xrightarrow{(p \cdot G) \circ \tau_p} A & \xrightarrow{G} & A
\end{array}
\]

9.4.8 Definition. For a cartesian monoidal functor $G : A \to X$ between cartesian closed categories, we say that $G$ preserves cartesian closedness (or $G$ is cartesian closed), if the natural transformation $\tau_p$ in Lemma 9.4.6 is an isomorphism, that is, if $\nu_p$ and $\tau_p$ form a pseudo morphism (see Definition 7.4.7) $\epsilon_p \Rightarrow \epsilon_{(p \cdot G)}$.

9.4.9 Remark. There are non-closed cartesian monoidal functors between cartesian closed categories. For example, the functor $0 : 1 \to 2$ from the terminal category to the interval category sending 0 to 0 (i.e. the order preserving map $0 \Rightarrow 0$ from the degenerate Boolean algebra 1 to the two-element Boolean algebra 2) preserves finite products but not cartesian closedness (1 and 2 are cartesian closed like any other Boolean algebras). Indeed, we can see this by writing the adjunctions $1 \xrightarrow{\top} 1$ and $2 \xrightarrow{\Delta 1 \top \Delta 0} 2$ in Remark 7.4.8(2) as $1 \xrightarrow{0 \cdot \top x_0} 1$ and $2 \xrightarrow{0 \cdot \top x_0} 2$.

9.4.10 Theorem. Let $X \xrightarrow{G} A$ be an adjunction between cartesian closed categories. Then $G$ is cartesian closed if and only if $\Phi$ satisfies Frobenius reciprocity.

Proof. First note that $G$ is cartesian monoidal by Theorem 8.11.2. Now recalling Definition 9.4.8 and Definition 9.4.2, and noting Remark 9.4.7(2), we see that the assertion is reduced to an instance of Proposition 7.6.6.
9.4.11 Remark. Recall that Frobenius reciprocity is defined between cartesian monoidal categories which are not necessarily closed. Theorem 9.4.10 says that the cartesian closedness of a functor (i.e. preservation of exponentials) translates into a property of the underlying cartesian monoidal structure.

9.4.12 Corollary. Let \( \xrightarrow{\phi} A \) be an adjunction between cartesian closed categories, and suppose that \( F \) is cartesian monoidal. Then \( G \) is cartesian closed if and only if \( G \) is fully faithful.

Proof. Immediate from Theorem 9.4.10 and Proposition 9.4.4.

Note. The following is an example of what is noted in Remark 9.4.11: we sometimes know that a functor is cartesian closed just by looking at the underlying cartesian monoidal structure.

9.4.13 Theorem. Let \( A \) be a locally cartesian closed category. Then the pullback functor \( k^* : A/b \to A/p \) in Definition 9.1.17 is cartesian closed for any \( A \)-arrow \( k : p \to b \).

Proof. By Theorem 9.4.10, the assertion is reduced to Theorem 9.4.5.

9.5 Exponentials of modules

We will show that for any category \( X \), the category \([X:]\) of right modules over \( X \) is cartesian closed, and that for any cartesian closed category \( X \), the Yoneda functor \( \mathcal{Y} : X \to [X:] \) preserves cartesian closedness.

Note. Since, as we saw in Theorem 9.1.5, the category \([X:]\) is cartesian monoidal for any category \( X \), it admits the functor \([x \mathcal{P}] : [X:] \to [X:]\) for each right module \( \mathcal{P} : X \to * \). The functor defined below will be proved to right adjoint to \([x \mathcal{P}]\) in Theorem 9.5.3.

9.5.1 Definition. Let \( X \) be a small category. For each right module \( \mathcal{P} : X \to * \), the functor \([\mathcal{P} \triangleright] : [X:] \to [X:]\) is defined by the right exponential transpose (see Definition 2.1.1) of the representable module

\[
[[X^\triangleright] \odot [x \mathcal{P}]](x) : X \to [X:]
\]

of the composite functor

\[
X \xrightarrow{X^\triangleright} [X:] \xrightarrow{x \mathcal{P}} [X:].
\]

9.5.2 Remark.

1. The smallness of \( X \) guarantees the local smallness of the category \([X:]\) and the module \([[X^\triangleright] \odot [x \mathcal{P}]](X) : X \to [X:]\): if \( X \) is small, \([[X^\triangleright] \odot [x \mathcal{P}]](X)\) is a functor \( X^\times [X^-, \text{Set}] \to \text{Set} \) to the category of small set. The right exponential transpose of \([[X^\triangleright] \odot [x \mathcal{P}]](X)\) thus gives a functor \([X^\times, \text{Set}] \to [X^-, \text{Set}]\), i.e. a functor \([X:] \to [X:]\).

2. The object function of \([\mathcal{P} \triangleright] : [X:] \to [X:]\) sends each right module \( \mathcal{M} : X \to * \) to the right module \( \mathcal{P} \triangleright \mathcal{M} : X \to * \) given by

\[
x(\mathcal{P} \triangleright \mathcal{M}) = ([X] x \times \mathcal{P}) (X) (\mathcal{M})
\]

; that is,

a) for an object \( x \in [X] \), \( x(\mathcal{P} \triangleright \mathcal{M}) \) consists of all right module morphisms \((X) x \times \mathcal{P} \to \mathcal{M} : X \to * \), and

b) for \( \theta \in x(\mathcal{P} \triangleright \mathcal{M}) \) and an \( X \)-arrow \( f : x' \to x \), their composite \( f \circ \theta \in x' (\mathcal{P} \triangleright \mathcal{M}) \) is given by the composition \((X) x' \times \mathcal{P} \xrightarrow{\mathcal{Y} f \times \mathcal{P}} \mathcal{Y} (X) x \times \mathcal{P} \xrightarrow{\theta} \mathcal{M} ;

3. The arrow function of \([\mathcal{P} \triangleright] : [X:] \to [X:]\) sends each right module morphism \( \psi : \mathcal{M} \to \mathcal{M}' : X \to * \) to the right module morphism \( \mathcal{P} \triangleright \psi : \mathcal{P} \triangleright \mathcal{M} \to \mathcal{P} \triangleright \mathcal{M}' : X \to * \) which maps each \( \theta \in x(\mathcal{P} \triangleright \mathcal{M}) \) to \( \theta : x(\mathcal{P} \triangleright \psi) \in x(\mathcal{P} \triangleright \psi) \) given by the composition \((X) x \times \mathcal{P} \xrightarrow{\psi} \mathcal{M} \xrightarrow{\psi} \mathcal{M}' \).
9.5.3 Theorem. For any small category \( X \), the category \([X:]\) of right modules over \( X \) is cartesian closed; for each right module \( P : X \to * \), the right adjoint of the functor \([X:] \to [X:]\) is given by the functor \([P \triangleright] : [X:] \to [X:]\) defined in Definition 9.5.1 with the counit \( \epsilon \) and the unit \( \eta \) of the adjunction defined in the following way:

(1) for each right module \( M : X \to * \), the component of the counit at \( M \) is the right module morphism \( \epsilon_M : (P \triangleright M) \times P \to M \) defined by

\[
(\theta, p) : \epsilon_M = (1_x, p) : \theta
\]

for \( x \in [X] \), \( \theta \in x(P \triangleright M) \), and \( p \in x(P) \) \((\theta \in x(P \triangleright M) \) is a right module morphism \( \theta : (X)x \times P \to M \).

(2) for each left module \( N : X \to * \), the component of the unit at \( N \) is the right module morphism \( \eta_N : N \to P \triangleright (N \times P) \) defined by

\[
n : \eta_N = X\!\!\downarrow n \times P
\]

for \( x \in [X] \) and \( n \in x(N) \) \((X)x \times P \to N \times P \), i.e. an element of \( x(P \triangleright (N \times P)) \).

Proof. Below we examine the definitions of \( \epsilon \) and \( \eta \), and show that they satisfy the triangular identities in Theorem 7.3.13.

(1) \( \epsilon_M \) so defined is indeed a right module morphism \( (P \triangleright M) \times P \to M \); that is,

\[
(\phi \circ (\theta, p)) : \epsilon_M = \phi \circ ((\theta, p) : \epsilon_M)
\]

for any \( X \)-arrow \( f : x' \to x \) and any \((\theta, p) \in x(P \triangleright M) \times x(P)\) as shown below:

\[
(\phi \circ (\theta, p)) : \epsilon_M = (\phi \circ \theta, \phi \circ p) : \epsilon_M
\]

\[
= (\{ (X) f \times P \} \circ \theta, \phi \circ p) : \epsilon_M
\]

\[
= (1_x', \phi \circ p) : \{ (X) f \times P \circ \theta \}
\]

\[
= (f, \phi \circ p) : \theta
\]

\[
= (\phi \circ (1_x, p)) : \theta
\]

\[
= \phi \circ ((1_x, p) : \theta)
\]

\[
= \phi \circ ((\theta, p) : \epsilon_M)
\]

(\#1 by Remark 9.5.2(2b)).

(2) \( \eta_N \) so defined is indeed a right module morphism \( N \to P \triangleright (N \times P) \); that is,

\[
(\phi \circ n) : \eta_N = \phi \circ (n : \eta_N)
\]

for any \( X \)-arrow \( f : x' \to x \) and any \( n \in x(N) \) as shown below:

\[
(\phi \circ n) : \eta_N = X\!\!\downarrow (f \circ n) \times P
\]

\[
= \{ (X) f \circ X\!\!\downarrow n \} \times P
\]

\[
= \{ (X) f \circ X\!\!\downarrow n \} \circ \{ (X) f \circ P \}
\]

\[
= f \circ (X\!\!\downarrow n \times P)
\]

\[
= f \circ (n : \eta_N)
\]

(\#1 by Proposition 5.2.18; \#2 by Remark 9.5.2(2b)).

(3) \( \epsilon_M \) is natural in \( M \); that is, the square

\[
\begin{array}{c}
\begin{array}{c}
(P \triangleright M) \times P \\
\downarrow \psi
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\epsilon_M : M \\
\downarrow \psi
\end{array}
\end{array}
\]

commutes for any right module morphism \( \psi : M \to M' \); indeed, for \( x \in [X] \), \( \theta \in x(P \triangleright M) \),
and \( p \in x(\mathcal{P}) \),
\[
(\theta, p) : \langle (\mathcal{P} \triangleright \psi) \times \mathcal{P} \rangle : \epsilon_{\mathcal{M}'_n} = (\theta \circ \psi, p) : \epsilon_{\mathcal{M}'}
\]
\[
= (1_{x_n}, p) : (\theta \circ \psi)
\]
\[
= (1_{x_n}, p) : \theta : \psi
\]
\[
= (\theta, p) : \epsilon_{\mathcal{M}} : \psi
\]
\[\quad \quad (\ast^1 \text{ by Remark 9.5.2(3)}).\]

(4) \( \eta_{\mathcal{N}} \) is natural in \( \mathcal{N} \); that is, the square
\[
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\eta_{\mathcal{N}}} & \mathcal{P} \triangleright (\mathcal{N} \times \mathcal{P}) \\
\psi & & \downarrow \psi'_{(\mathcal{P} \triangleright (\mathcal{N} \times \mathcal{P}))}
\end{array}
\]
commutes for any right module morphism \( \psi : \mathcal{N} \rightarrow \mathcal{N}' \); indeed, for \( x \in \|X\| \) and \( n \in x(\mathcal{N}) \),
\[
n_\psi : \eta_{\mathcal{N}_n} = X \uparrow (n_\psi \times \mathcal{P})
\]
\[
= (X \uparrow n \circ \psi) \times \mathcal{P}
\]
\[
= (X \uparrow n \times \mathcal{P}) \circ (\psi \times \mathcal{P})
\]
\[
= (n \circ \eta_{\mathcal{N}}) \circ (\psi \times \mathcal{P})
\]
\[
= n \circ \eta_{\mathcal{N}} : (\mathcal{P} \triangleright (\psi \times \mathcal{P}))
\]
\[\quad \quad (\ast^1 \text{ by Proposition 5.2.18}; \ast^2 \text{ by Remark 9.5.2(3)}).\]

(5) The triangle
\[
\begin{array}{ccc}
\mathcal{N} \times \mathcal{P} & \xrightarrow{\eta_{\mathcal{N}_n} \times \mathcal{P}} & (\mathcal{P} \triangleright (\mathcal{N} \times \mathcal{P})) \times \mathcal{P} \\
\mathcal{N} \times \mathcal{P} & \xrightarrow{\epsilon_{(\mathcal{N} \times \mathcal{P})}} & \mathcal{N} \times \mathcal{P}
\end{array}
\]
commutes; indeed, for \( x \in \|X\| \) and \( (n, p) \in x(\mathcal{N}) \times x(\mathcal{P}) \),
\[
(n, p) : \eta_{\mathcal{N}_n} \times \mathcal{P} : \epsilon_{(\mathcal{N} \times \mathcal{P})} = (X \uparrow n \times \mathcal{P}, p) : \epsilon_{(\mathcal{N} \times \mathcal{P})}
\]
\[
= (1_{x_n}, p) : (X \uparrow n \times \mathcal{P})
\]
\[
= (n, p).
\]

(6) The triangle
\[
\begin{array}{ccc}
\mathcal{P} \triangleright \mathcal{M} & \xrightarrow{\eta_{(\mathcal{P} \triangleright \mathcal{M})} \times \mathcal{P}} & (\mathcal{P} \triangleright (\mathcal{P} \triangleright \mathcal{M}) \times \mathcal{P}) \\
\mathcal{P} \triangleright \mathcal{M} & \xrightarrow{\epsilon_{(\mathcal{P} \triangleright \mathcal{M})}} & \mathcal{P} \triangleright \mathcal{M}
\end{array}
\]
commutes.

**Proof.** For \( x \in \|X\| \) and \( \theta \in x(\mathcal{P} \triangleright \mathcal{M}) \),
\[
\theta : \eta_{(\mathcal{P} \triangleright \mathcal{M})} : \mathcal{P} \triangleright \mathcal{M} = X \uparrow \theta \times \mathcal{P} : \mathcal{P} \triangleright \mathcal{M}
\]
\[
= (X \uparrow \theta \times \mathcal{P}) \circ \epsilon_{\mathcal{M}}
\]
\[\quad \quad (\ast^1 \text{ by Remark 9.5.2(3)}). \text{ Hence we are done if we show that the triangle}
\]
\[
\begin{array}{ccc}
X \times \mathcal{P} & \xrightarrow{\theta} & \mathcal{M} \\
x \uparrow \theta \times \mathcal{P} & \xrightarrow{\epsilon_{\mathcal{M}}} & (\mathcal{P} \triangleright \mathcal{M}) \times \mathcal{P}
\end{array}
\]

commutes; but for \( x' \in \| X \| \) and \((f, p) \in x'(X) \times x'(P),\)

\[
(f, p) : (X \uparrow \theta \times P) : \varepsilon_M = (f \circ \theta, p) : \varepsilon_M
= ((\{X\} f \times P) \circ \theta, p) : \varepsilon_M
= (1_{x'}, p) : \{(X) f \times P \circ \theta\}
= (f, p) : \theta
\]

\((\ast 1)\) by Remark 9.5.2(2b)). □

9.5.4 Remark.
(1) [Awo06] and [Lei14] contain a slick proof of Theorem 9.5.3 using the Yoneda lemma and the density theorem without constructing the unit and counit of the adjunction.
(2) Since \([\cdot : A] = [A \cdot \cdot],\) the category \([\cdot : A]\) of left modules over \( A \) is cartesian closed as well, and so is the category \([X : A]\) of two-sided modules \( X \rightarrow A \) because \([X : A] = [X \times A^- :].\)
(3) Consider an adjunct diagram

\[
\begin{array}{c}
\begin{array}{ccc}
P \triangleright M & \xrightarrow{p \triangleright \psi} & M \\
\downarrow \psi & & \downarrow \psi \\
N & \xleftarrow{\psi} & N \times P
\end{array}
\end{array}
\]

of the adjunction \([P \triangleright] \dashv [x \times P].\) By Theorem 9.5.3(2) and Remark 9.5.2(2b), \(\times \psi\) (the exponential transpose of \(\psi\)) maps each \( n \in x(N) \) to \( n \times \psi \in x(P \triangleright M) \) given by the composition \((X) x \times P \xrightarrow{x \circ \eta_{\langle P \triangleright \psi \rangle}} N \times \psi \rightarrow M.\)
(4) We will see in Theorem 11.6.10 that the category \([X : \cdot]\) is in fact locally cartesian closed.

Note. Finally, we show that the Yoneda functor preserves cartesian closedness in the sense of Definition 9.4.8.

9.5.5 Theorem. For any small cartesian closed category \( C, \) the right Yoneda functor \([C, \cdot] : C \rightarrow [C : \cdot]\) preserves cartesian closedness.

Proof. Since the right Yoneda functor preserves limits by Corollary 8.7.7, it is cartesian. To show that it is closed, consider the canonical natural isomorphism and its mate

\[
\begin{array}{ccc}
C & \xrightarrow{\times P} & C \\
\|C\| & \xrightarrow{\times C} & \|C\|
\end{array}
\]

between the exponential adjunctions \(\epsilon_p : [P \triangleright] \dashv [x \times \cdot]\) and \(\epsilon_{\langle C \times \cdot \rangle} : [\langle C \times \cdot \rangle \triangleright] \dashv [\times \langle C \times \cdot \rangle]\) for \( p \in \| C \|.\)

We need to show that \(\tau_p\) is also an isomorphism. By Remark 9.4.7(1), the component of \(\tau_p\) at \( a \in \| C \|\) is the right module morphism \(\tau_{p,a} : \langle C \rangle (p \triangleright a) \rightarrow \langle C \rangle p \triangleright \langle C \rangle a\) given by the adjunct of the composite \(\circ \chi_{p,p \triangleright a} \circ \langle C \rangle \epsilon_{p,a} \) as indicated in the adjunct diagram

\[
\begin{array}{ccc}
\langle C \rangle p & \langle C \rangle a & \langle C \rangle a \\
\downarrow \tau_{p,a} & \downarrow \chi_{p,p \triangleright a} & \downarrow \chi_{p,p \triangleright a} \\
\langle C \rangle ((p \triangleright a) \times p) & \langle C \rangle (p \triangleright a) \times \langle C \rangle p
\end{array}
\]

by Remark 9.5.4(3), for each \( x \in \| C \|, \) \( x(\tau_{p,a}) : x(\langle C \rangle (p \triangleright a) \rightarrow x((\langle C \rangle p \triangleright \langle C \rangle a) maps g : x \rightarrow p \triangleright a\) to the right module morphism \(\langle C \rangle x \times (\langle C \rangle p \rightarrow \langle C \rangle a\) given by the composition

\[
\langle C \rangle x \times (\langle C \rangle p \xrightarrow{\langle C \rangle \times (\langle C \rangle p)} \langle C \rangle (p \triangleright a) \times (\langle C \rangle p \xrightarrow{\chi_{p,p \triangleright a}} \langle C \rangle ((p \triangleright a) \times p) \xrightarrow{\chi_{p,p \triangleright a}} \langle C \rangle a
\]
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; but by the commutativity of the diagram

\[
\begin{align*}
\langle C \times (g \times p) \rangle & \xrightarrow{\langle C \rangle (g \times p)} \langle C \rangle (p \triangleright a) \times \langle C \rangle p \\
\langle C \rangle (x \times p) & \xrightarrow{\nu_p, x} \langle C \rangle \langle C \rangle ((g \times p) \circ \epsilon_p, a) \\
\langle C \rangle x \times \langle C \rangle p & \xrightarrow{\langle C \rangle g \times (C)p} \langle C \rangle (p \triangleright a) \times \langle C \rangle p
\end{align*}
\]

(the upper triangle commutes by the functoriality of the Yoneda embedding and the lower square commutes by the naturality of the isomorphism \(\nu_p\)), we have

\[
\langle \langle C \rangle g \times \langle C \rangle p \rangle \circ \nu_{p, p \triangleleft a} \circ \langle C \rangle \epsilon_{p, a} = \nu_{p, x} \circ \langle C \rangle ((g \times p) \circ \epsilon_{p, a})
\]

; hence the bijectivity of \(x(\tau_{p, a})\) is reduced to the bijectivity of the assignment

\[
g \mapsto \nu_{p, x} \circ \langle C \rangle ((g \times p) \circ \epsilon_{p, a})
\]

, which in turn reduced to the bijectivity of the assignment

\[
g \mapsto (g \times p) \circ \epsilon_{p, a}
\]

(the assignment of the adjunct) because \(\nu_{p, x}\) is an isomorphism and the Yoneda functor is fully faithful. \(\square\)
10 Adjoint Functor Theorem

10.1 Cofinality

What we look at in this section is a categorical generalization of the notion of cofinality in order theory. The results in this section are used to prove the general adjoint functor theorem in Section 10.2.

10.1.1 Definition. A left [op. right] module \( M \) is said to be connected if it has at least one \( M \)-arrow and any two \( M \)-arrows are connected in the sense of Definition 4.11.10; that is, if \( M \) consists of exactly one orbit (see Definition 4.11.15).

10.1.2 Remark. By Remark 4.11.11, a left [op. right] module \( M \) is connected if the comma category \( \mathcal{M} \) is connected.

10.1.3 Proposition. A representable left [op. right] module is connected.

Proof. First recall (Remark 6.1.2(1)) that a representable left module has a unit. Now if \( u \) is a unit of a left module \( M \), then every \( M \)-arrow is connected to \( u \) by its adjunct along \( u \).

10.1.4 Definition. (1) A functor \( K : D \to E \) is called
   - left (or downward) cofinal\(^1\) if for every object \( e \in |E| \) the right module \( K(e) : D \to * \) is connected.
   - right (or upward) cofinal if for every object \( e \in |E| \) the left module \( e(K) : * \to D \) is connected.

(2) A subcategory \( D \) of a category \( E \) is called
   - left cofinal in \( E \) if the inclusion \( D \to E \) is left cofinal.
   - right cofinal in \( E \) if the inclusion \( D \to E \) is right cofinal.

(3) A set \( \mathcal{D} \) of objects in a category \( E \) is called
   - left cofinal in \( E \) if the full subcategory of \( E \) generated by \( \mathcal{D} \) is left cofinal.
   - right cofinal in \( E \) if the full subcategory of \( E \) generated by \( \mathcal{D} \) is right cofinal.

10.1.5 Remark. A functor \( K : D \to E \) is cofinal if and only if its image is cofinal in \( E \).

10.1.6 Proposition. An object \( d \) of a category \( E \) is initial [op. terminal] if and only if the set \( \{d\} \) is left [op. right] cofinal in \( E \).

Proof. By definition, the set \( \{d\} \) is left cofinal in \( E \) iff the right module \( d(E)e : * \to * \) is connected for every \( e \in |E| \). But this is the case iff \( d(E)e \) has exactly one arrow.

10.1.7 Proposition. If a functor \( K : D \to E \) has a right [op. left] adjoint, then \( K \) is left [op. right] cofinal.

Proof. Since \( K \) has a right adjoint iff for every object \( e \in |E| \) the right module \( K(E)e : D \to * \) is representable (see Corollary 7.3.12), the assertion follows from Proposition 10.1.3.

10.1.8 Remark. As an immediate consequence, if \( D \) is a coreflective [op. reflective] subcategory of a category \( E \), then \( D \) is left [op. right] cofinal in \( E \).

10.1.9 Proposition. An equivalence functor is both left and right cofinal.

Proof. Since an equivalence functor has both left and right adjoints by Theorem 7.11.22, the assertion follows from Proposition 10.1.7.

\(^1\)Some authors use the term “final” instead of “cofinal”.

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10.1.10 Theorem. Let $\mathcal{M} : \ast \to \mathbb{E}$ be a left module. If $K : \mathbb{D} \to \mathbb{E}$ is a left cofinal functor, then for any frame $\alpha$ of the composite left module $(\mathcal{M})K : \ast \to \mathbb{D}$ there exists a unique frame $\alpha'$ of $\mathcal{M}$ such that $\alpha = \alpha' \circ K$.

Let $\mathcal{M} : \mathbb{E} \to \ast$ be a right module. If $K : \mathbb{D} \to \mathbb{E}$ is a right cofinal functor, then for any frame $\alpha$ of the composite right module $K(\mathcal{M}) : \mathbb{D} \to \ast$ there exists a unique frame $\alpha'$ of $\mathcal{M}$ such that $\alpha = K \circ \alpha'$.

Proof. Let $\alpha$ be a frame of $(\mathcal{M})K$. Given $e \in \mathbb{E}$, choose $d \in \mathbb{D}$ and an $\mathbb{E}$-arrow $f : K \circ d \to e$ (the left cofinality of $K$ allows such choices), and define an $\mathcal{M}$-arrow $\alpha'_e : \ast \to e$ by $\alpha'_e = \alpha_d \circ f$. We claim that

1. the definition of $\alpha'_e$ is independent of the choices of $d$ and $f$;
2. \( \alpha' : (\alpha'_e)_{e \in \mathbb{E}} \) forms a frame of $\mathcal{M}$;
3. $\alpha'$ is the only frame of $\mathcal{M}$ such that $\alpha = \alpha' \circ K$.

Let $d'$ and $f'$ be alternative choices for defining $\alpha'_e$. Then the diagram

\[
\begin{array}{ccc}
\ast & \xrightarrow{\alpha_d} & \ast \\
K \circ d & \xrightarrow{f} & \ast \\
\ast & \xrightarrow{\alpha_d'} & \ast \\
K \circ d' & \xrightarrow{f'} & \ast \\
\end{array}
\]

commutes because, by the left cofinality of $K$, the interior of the diagram is divided into a finite number of commutative diagrams as in

\[
\begin{array}{cccccccc}
\ast & \xrightarrow{\alpha_d} & \ast & \xrightarrow{\alpha_d} & \ast & \xrightarrow{\alpha_d} & \ast \\
K \circ d & \xrightarrow{f} & K \circ d & \xrightarrow{f} & K \circ d & \xrightarrow{f} & K \circ d \\
\ast & \xrightarrow{\alpha_d'} & \ast & \xrightarrow{\alpha_d'} & \ast & \xrightarrow{\alpha_d'} & \ast \\
K \circ d' & \xrightarrow{f'} & K \circ d' & \xrightarrow{f'} & K \circ d' & \xrightarrow{f'} & K \circ d' \\
\end{array}
\]

; hence the definition of $\alpha'_e$ is independent of the choices of $d$ and $f$. Let $\alpha'_e = \alpha_d \circ f$ and $\alpha'_e = \alpha_d' \circ f'$ be two $\mathcal{M}$-arrows defined as above and let $h : e \to e'$ be an $\mathbb{E}$-arrow. Then the diagram

\[
\begin{array}{ccc}
\ast & \xrightarrow{\alpha'_e} & \ast \\
K \circ d & \xrightarrow{f} & \ast \\
\ast & \xrightarrow{\alpha'_e'} & \ast \\
K \circ d' & \xrightarrow{f'} & \ast \\
\end{array}
\]

i.e.

\[
\begin{array}{ccc}
\ast & \xrightarrow{\alpha_d} & \ast \\
K \circ d & \xrightarrow{f} & \ast \\
\ast & \xrightarrow{\alpha_d'} & \ast \\
K \circ d' & \xrightarrow{f'} & \ast \\
\end{array}
\]

commutes by the first claim. $\alpha' : (\alpha'_e)_{e \in \mathbb{E}}$ thus forms a frame of $\mathcal{M}$. By setting $e = K \circ d$ and $f = 1_{K \circ d}$, we have $\alpha_d = \alpha'_e$ for every $d \in \mathbb{D}$; hence $\alpha = \alpha' \circ K$. It is clear that this is the only possible construction of $\alpha'$ with $\alpha = \alpha' \circ K$ if $\alpha'$ is to satisfy the naturality condition. \(\square\)

10.1.11 Corollary. Given a functor $K : \mathbb{D} \to \mathbb{E}$ and a module $\mathcal{M} : \mathbb{X} \to \mathbb{A}$, if $K$ is left [op. right] cofinal, then the precomposition cell

\[
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{\mathbb{X} \circ \mathcal{M}} & \mathbb{E} \circ \mathcal{M} \\
\mathbb{A} & \xrightarrow{\mathbb{A} \circ \mathcal{M}} & \mathbb{A} \\
\mathbb{X} & \xrightarrow{\mathbb{X} \circ \mathcal{M}} & \mathbb{A} \\
\mathbb{X} & \xrightarrow{\mathbb{X} \circ \mathcal{M}} & \mathbb{A} \\
\end{array}
\]

(see Definition 4.6.28) is fully faithful.
Proof. Let \( x \) be an object in \( X \) and \( F \) be a functor \( E \rightarrow A \). By Theorem 10.1.10, for a cone \( \alpha : x \leadsto F \circ K : * \leadsto M \), i.e. a frame \( \alpha \) of the left module \( x(\mathcal{M})[F \circ K] = (x(\mathcal{M})F)K : * \rightarrow D \), there is a unique frame \( \alpha' \) of the left module \( x(\mathcal{M})F : * \rightarrow E \), i.e. a unique cone \( \alpha' : x \leadsto F : * \leadsto M \), such that \( \alpha = \alpha' \circ K \). \( \square \)

10.1.12 Remark.
(1) As a consequence, by Proposition 6.2.25,

- the precomposition cell \((*F, \mathcal{M})\) preserves, reflects, and creates inverse universal arrows:
  a) a cone \( \mu : r \leadsto F : * \rightarrow M \) is universal if and only if the composite cone \( \mu \circ K : r \leadsto F \circ K : * \leadsto M \) is universal;
  b) given a functor \( F : E \rightarrow A \), if the composite functor \( F \circ K : D \rightarrow A \) has a limit \( \mu : r \leadsto F \circ K : * \leadsto M \), then the unique cone \( \mu' : r \leadsto F : * \rightarrow M \) with \( \mu = \mu' \circ K \) is a limit of \( F \).

- the precomposition cell \((K*, \mathcal{M})\) preserves, reflects, and creates direct universal arrows:
  a) a cone \( \mu : G \rightarrow r : * \rightarrow M \) is universal if and only if the composite cone \( K \circ \mu : K \circ G \rightarrow r : D * \rightarrow M \) is universal;
  b) given a functor \( G : E \rightarrow X \), if the composite functor \( K \circ G : D \rightarrow X \) has a colimit \( \mu : K \circ G \rightarrow r : D * \rightarrow M \), then the unique cone \( \mu' : G \rightarrow r : E * \rightarrow M \) with \( \mu = \mu' \circ K \) is a colimit of \( G \).

(2) If \( K \) is an equivalence, so are the precomposition cells \((*K, \mathcal{M})\) and \((K*, \mathcal{M})\) (because \([K, A]\) is an equivalence by Theorem 7.11.15, and \((*K, \mathcal{M})\) and \((K*, \mathcal{M})\) are fully faithful by Proposition 10.1.9 and Corollary 10.1.11).

10.1.13 Theorem.
- Let \( \mathcal{M} : * \rightarrow E \) be a left module. If \( D \) is a left cofinal subcategory of \( E \), then a frame of the left module \((\mathcal{M})D : * \rightarrow D \) (the restriction of \( \mathcal{M} \) to \( D \)) uniquely extends to a frame of \( \mathcal{M} \).

- Let \( \mathcal{M} : E \rightarrow * \) be a right module. If \( D \) is a right cofinal subcategory of \( E \), then a frame of the right module \((\mathcal{M})D : D \rightarrow * \) (the restriction of \( \mathcal{M} \) to \( D \)) uniquely extends to a frame of \( \mathcal{M} \).

Proof. This is a special case of Theorem 10.1.10 where \( K : D \rightarrow E \) is an inclusion. \( \square \)

10.1.14 Corollary. Let \( \mathcal{M} : X \rightarrow A \) be a module and let \( D \) be a subcategory of a category \( E \). If \( D \) is left \([op. \, right]\) cofinal in \( E \) in \( \mathcal{E} \), then the cell

\[
\begin{array}{ccc}
X & \xrightarrow{(E, A)} & [E, A] \\
\downarrow \downarrow & \downarrow & \downarrow \\
(D, X) & \xrightarrow{(D, A)} & [D, A] \\
\end{array}
\]

—restriction to \( D \) (see Example 4.6.30(1))—is fully faithful.

Proof. Let \( x \) be an object in \( X \) and \( F \) be a functor \( E \rightarrow A \). By Theorem 10.1.13, a cone \( \alpha : x \leadsto F \circ D : * \leadsto M \), i.e. a frame \( \alpha \) of the left module \( x(\mathcal{M})[F \circ D] = (x(\mathcal{M})F)D : * \rightarrow D \), uniquely extends to a frame \( \alpha' \) of the left module \( x(\mathcal{M})F : * \rightarrow E \), i.e. a cone \( \alpha' : x \leadsto F : * \leadsto M \). \( \square \)

10.1.15 Remark.
(1) As a consequence, by Proposition 6.2.25,

- the cell \((D, \mathcal{M})\) preserves, reflects, and creates inverse universal arrows:
  a) a cone \( \mu : r \leadsto F : * \rightarrow M \) is universal if and only if its restriction \( \mu \circ D : r \leadsto F \circ D : * \rightarrow M \) to \( D \) is universal;
  b) if the restriction of a functor \( F : E \rightarrow C \) to \( D \) has a limit \( \mu : r \leadsto F \circ D : * \rightarrow M \), then its unique extension \( \mu' : r \leadsto F : * \rightarrow M \) is a limit of \( F \).

- the cell \((D*, \mathcal{M})\) preserves, reflects, and creates direct universal arrows:
  a) a cone \( \mu : G \rightarrow r : * \rightarrow M \) is universal if and only if its restriction \( D \circ \mu : D \circ G \rightarrow r : D* \rightarrow M \) to \( D \) is universal;
  b) if the restriction of a functor \( G : E \rightarrow C \) to \( D \) has a colimit \( \mu : D \circ G \rightarrow r : D* \rightarrow M \), then its unique extension \( \mu' : G \rightarrow r : E* \rightarrow M \) is a colimit of \( G \).
10.1.16 Theorem.

Let \( \mathcal{M} : * \to \mathbb{E} \) be a left module. If \( \mathbf{d} \) is an initial object of \( \mathbb{E} \), then for any \( \mathcal{M} \)-arrow \( \mathbf{m} : * \to \mathbf{d} \) there is a unique frame \( \alpha \) of \( \mathcal{M} \) with \( \alpha_\mathbf{d} = \mathbf{m} \). Specifically, \( \alpha \) is given by the family of \( \mathcal{M} \)-arrows \( \alpha_e = \mathbf{m} \circ \nu_e \), one for each \( e \in \| \mathbb{E} \| \), with \( \nu_e \) the unique \( \mathbb{E} \)-arrow \( \mathbf{d} \to e \).

Let \( \mathcal{M} : \mathbb{E} \to * \) be a right module. If \( \mathbf{d} \) is a terminal object of \( \mathbb{E} \), then for any \( \mathcal{M} \)-arrow \( \mathbf{m} : \mathbf{d} \to * \) there is a unique frame \( \alpha \) of \( \mathcal{M} \) with \( \alpha_\mathbf{d} = \mathbf{m} \). Specifically, \( \alpha \) is given by the family of \( \mathcal{M} \)-arrows \( \alpha_e = \nu_e \circ \mathbf{m} \), one for each \( e \in \| \mathbb{E} \| \), with \( \nu_e \) the unique \( \mathbb{E} \)-arrow \( e \to \mathbf{d} \).

Proof. Evident. \( \square \)

10.1.17 Remark. If \( \mathbf{d} \) is an initial object of \( \mathbb{E} \), the functor \( \mathbf{d} : * \to \mathbb{E} \) is left cofinal by Proposition 10.1.6. Dually, if \( \mathbf{d} \) is a terminal object of \( \mathbb{E} \), the functor \( \mathbf{d} : * \to \mathbb{E} \) is right cofinal. Hence Theorem 10.1.16 is seen as a special case of Theorem 10.1.10 where \( \mathbb{D} \) is the terminal category.

10.1.18 Corollary. Let \( \mathcal{M} : \mathbb{X} \to \mathbb{A} \) be a module and \( \mathbb{E} \) be a category.

If \( \mathbf{d} \) is an initial object of \( \mathbb{E} \), then the cell

\[
\begin{array}{c}
\mathbb{X} \xrightarrow{\{ \mathcal{M} \}} \mathbb{E} \\
\downarrow \mathbf{1} \quad \downarrow \mathcal{M} \\
\mathbb{X} \quad \mathbb{X} \\
\end{array}
\]

—evaluation at \( \mathbf{d} \) (see Example 4.6.30(2))—is fully faithful. Specifically, for an \( \mathcal{M} \)-arrow \( \mathbf{x} : \mathbb{F} : \mathbf{d} \) with \( \mathbb{F} \) a functor \( \mathbb{E} \to \mathbb{A} \), there is a unique cone \( \alpha : \mathbf{x} \to \mathbb{F} : * \mathbf{E} \to \mathcal{M} \) with \( \alpha_\mathbf{d} = \mathbf{m} \), given by the family of \( \mathcal{M} \)-arrows \( \alpha_e = \mathbf{m} \circ (\mathbb{F} : \nu_e) \), one for each \( e \in \| \mathbb{E} \| \), with \( \nu_e \) the unique \( \mathbb{E} \)-arrow \( \mathbf{d} \to e \).
If \( d \) is a terminal object of \( E \), then the cell

\[
\begin{array}{c}
[E, X] \xrightarrow{(E, M)} A \\
(d, X) \downarrow \quad \downarrow \delta \\
X \xrightarrow{=} A
\end{array}
\]

—evaluation at \( d \) (see Example 4.9.12(2))—is fully faithful. Specifically, for an \( M \)-arrow \( m : d \to a \) with \( G \) a functor \( E \to X \), there is a unique cone \( \alpha : G \to a \) such that \( \alpha_d = m \), given by the family of \( M \)-arrows \( \alpha_e = (v_e \cdot G) \circ m \), one for each \( e \in \| E \| \), with \( v_e \) the unique \( E \)-arrow \( e \to d \).

**Proof.** Let \( x \) be an object in \( X \) and \( F \) be a functor \( E \to A \). By Theorem 10.1.16, for an \( M \)-arrow \( m : x \to F \cdot d \), i.e. an arrow \( m : x \to d \) in the left module \( x \cdot (M) F \), there is a unique frame \( \alpha \) of \( x \cdot (M) F \), i.e. a unique cone \( \alpha : x \to F : x \cdot (M) F \), with \( \alpha_d = m \), and the component of \( \alpha \) at \( e \in \| E \| \) is given by the \( x \cdot (M) F \)-arrow \( \alpha_e = m \circ v_e \), i.e. the \( M \)-arrow \( \alpha_e = m \circ (F \cdot v_e) \).

**10.1.19 Remark.**

(1) As a consequence, by Proposition 6.2.25,

- the cell \((d, M)\) preserves, reflects, and creates inverse universal arrows:
  a) a cone \( \mu : r \to F : r \cdot E \to M \) is universal if and only if the \( M \)-arrow \( \mu_d : r \to F \cdot d \) is inverse universal;
  b) given a functor \( F : E \to A \), if its value at \( d \) has an inverse universal \( M \)-arrow \( u : r \to F \cdot d \), then the unique cone \( \mu : r \to F : r \cdot E \to M \) with \( \mu_d = u \) is a limit of \( F \).

- the cell \((d, M)\) preserves, reflects, and creates direct universal arrows:
  a) a cone \( \mu : G \to r : E \cdot r \to M \) is universal if and only if the \( M \)-arrow \( \mu_d : G \to r \cdot d \) is direct universal;
  b) given a functor \( G : E \to X \), if its value at \( d \) has a direct universal \( M \)-arrow \( u : d \cdot G \to r \), then the unique cone \( \mu : G \to r : E \cdot r \to M \) with \( \mu_d = u \) is a colimit of \( G \).

(2) As a special case where \( M \) is given by the hom-module of a category of \( C \),

- if \( d \) is an initial object of \( E \), the cell

\[
\begin{array}{c}
C \xrightarrow{(E, C)} [E, C] \\
\downarrow (d, C) \quad \downarrow (d, C) \\
C \xrightarrow{=} C
\end{array}
\]

(see Example 4.9.12(2)) is fully faithful, and thus preserves, reflects, and creates inverse universal arrows:

a) a cone \( \mu : r \to L : *E \to C \) is universal if and only if the \( C \)-arrow \( \mu_d : r \to L \cdot d \) is iso (see Proposition 6.2.5);

b) for a functor \( L : E \to C \) and an iso \( C \)-arrow \( u : r \to L \cdot d \), there is a unique cone \( \mu : r \to L \cdot d \) with \( \mu_d = u \), given by the family of \( C \)-arrows \( \mu_e = u \circ (L \cdot v_e) \), one for each \( e \in \| E \| \), with \( v_e \) the unique \( E \)-arrow \( d \to e \), and this unique cone is a limit of \( L \).

- if \( d \) is a terminal object of \( E \), the cell

\[
\begin{array}{c}
[E, C] \xrightarrow{(E, C)} C \\
(d, C) \downarrow \quad \downarrow (d, C) \\
C \xrightarrow{=} C
\end{array}
\]

(see Example 4.9.12(2)) is fully faithful, and thus preserves, reflects, and creates direct universal arrows:

a) a cone \( \mu : L \to r : E \cdot r \to C \) is universal if and only if the \( C \)-arrow \( \mu_d : L \to r \cdot d \) is iso (see Proposition 6.2.5);

b) for a functor \( L : E \to C \) and an iso \( C \)-arrow \( u : d \cdot L \to r \), there is a unique cone \( \mu : L \to r \):
10.2. General adjoint functor theorem

In this section, we prove GAFT (General Adjoint Functor Theorem). We first prove the representability theorem\(^2\) (Theorem 10.2.5) and deduce GAFT from it. To prove the representability theorem, we first show the cofinality of a solution set and then apply the results in Section 10.1 (the same approach as is taken by [Bor94]).

10.2.1 Definition. Given a category \(E\), a set \(\mathcal{D}\) of the objects in \(E\) is called

\(\triangleright\) (jointly) weakly initial in \(E\) if to every object \(e \in \|E\|\) there is an \(E\)-arrow \(d \to e\) with \(d \in \mathcal{D}\).

\(\triangleright\) (jointly) weakly terminal in \(E\) if to every object \(e \in \|E\|\) there is an \(E\)-arrow \(e \to d\) with \(d \in \mathcal{D}\).

10.2.2 Proposition. Let \(\mathcal{D}\) be a set of objects in a category \(E\).

\(\triangleright\) If \(\mathcal{D}\) is left cofinal, then \(\mathcal{D}\) is weakly initial. The converse holds if \(E\) is complete.

\(\triangleright\) If \(\mathcal{D}\) is right cofinal, then \(\mathcal{D}\) is weakly terminal. The converse holds if \(E\) is cocomplete.

\(^2\)The name “representability theorem” is taken from [ML98].
10.2. Theorem. \(D \subseteq \|E\|\) is weakly initial. Given \(e \in \|E\|\), we need to show that the right module \(D(E)e : D \to \ast\) is connected, where \(D\) is the full subcategory of \(E\) generated by \(D\). Let \(f_0 : d_0 \to e\) and \(f_1 : d_1 \to e\) be two \(E\)-arrows with \(d_0, d_1 \in D\). Since \(E\) is complete, there is a pullback

\[
\begin{array}{ccc}
  d_0 & \xrightarrow{f_0} & e \\
  \downarrow{u_0} & & \downarrow{u_1} \\
  d_1 & \xrightarrow{f_1} & e
\end{array}
\]

; since \(D\) is weakly initial, there is an arrow \(h : d \to r\) with \(d \in D\), yielding a commutative diagram

\[
\begin{array}{ccc}
  d & \xrightarrow{h} & r \\
  \downarrow{h \circ u_0} & & \downarrow{u_0} \\
  d_0 & \xrightarrow{h \circ u_1} & e
\end{array}
\]

\[
\begin{array}{ccc}
  d & \xrightarrow{h} & r \\
  \downarrow{h \circ u_0} & & \downarrow{u_0} \\
  d_1 & \xrightarrow{h \circ u_1} & e
\end{array}
\]

; hence \(f_0\) and \(f_1\) are connected in \(D(E)e\).

10.2.3 Theorem.
- A complete category \(E\) has an initial object if and only if it has a small weakly initial set.
- A cocomplete category \(E\) has a terminal object if and only if it has a small weakly terminal set.

Proof. If \(E\) has an initial object \(d\), then it has a small weakly initial set \(\{d\}\). Conversely, suppose that \(E\) has a small weakly initial set \(D\) and let \(D\) be the full subcategory of \(E\) generated by \(D\). Then \(D\) is left cofinal in \(E\) by Proposition 10.2.2. Since \(E\) is complete and \(D\) is small, the inclusion \(D \to E\) has a limit \(\mu : r \to D\), and this limit extends to a limit \(\mu : r \to E\) of the identity \(E \to E\) (see Remark 10.1.15(2)). Hence \(E\) has an initial object by Theorem 10.1.21.

10.2.4 Definition.
- Given a left module \(M : \ast \to A\), a weakly initial set in the comma category \([M]\) is called a solution set of \(M\); that is, a solution set of \(M\) is a set \(\{s_i : \ast \to a_i\}\) of \(M\)-arrows such that every \(M\)-arrow \(m : \ast \to a\) factors through some \(s_i\) along an \(A\)-arrow \(h : a_i \to a\) as shown in

\[
\begin{array}{ccc}
  \ast & \xrightarrow{s_i} & a_i \\
  \downarrow{m} & & \downarrow{h} \\
  \ast & \xrightarrow{\ast} & a
\end{array}
\]

- Given a right module \(M\), a weakly terminal set in the comma category \([M]\) is called a solution set of \(M\); that is, a solution set of \(M\) is a set \(\{s_i : x_i \to \ast\}\) of \(M\)-arrows such that every \(M\)-arrow \(m : x \to \ast\) factors through some \(s_i\) along an \(X\)-arrow \(h : x \to x_i\) as shown in

\[
\begin{array}{ccc}
  x_i & \xrightarrow{s_i} & \ast \\
  \downarrow{h} & & \downarrow{m} \\
  x & \xrightarrow{\ast} & \ast
\end{array}
\]

10.2.5 Theorem. (General Representability Theorem).
- A left module \(M : \ast \to A\) over a complete category \(A\) is representable if and only if \(M\) is continuous and has a small solution set.
- A right module \(M : X \to \ast\) over a cocomplete category \(X\) is representable if and only if \(M\) is cocontinuous and has a small solution set.

Proof. Suppose that \(M\) is representable. Then \(M\) is continuous by Corollary 8.7.10, and a unit \(u\) of \(M\) yields a small solution set \(\{u\}\). Conversely, suppose that \(M\) is continuous and has a small solution set. Then the comma category \([M]\) is complete by Corollary 8.8.3, and has an initial object by Theorem 10.2.3. Hence \(M\) is representable by Proposition 6.1.4.
10.3. Epimorphisms and Monomorphisms

In this section, we extend the notion of epimorphism and monomorphism to arrows of a module. In fact, the notion of epimorphism and monomorphism is ultimately associated with one-sided modules: an arrow \( m : x \to a \) in a category \( C \) is epic if it is so in the representable left module \( x(C) : * \to C \), and is monic if it is so in the representable right module \( (C)a : C \to * \).

10.3.1 Definition.

(1) An arrow \( m : * \to a \) in a left module \( M : * \to A \) is called epic (or an epimorphism) if for any parallel pair of \( A \)-arrows \( h, h' : a \to b \), \( m \circ h = m \circ h' \) implies that \( h = h' \).

(2) An arrow \( m : x \to * \) in a right module \( M : X \to * \) is called monic (or a monomorphism) if for any parallel pair of \( X \)-arrows \( h, h' : y \to x \), \( h \circ m = h' \circ m \) implies that \( h = h' \).

10.2.6 Corollary.

- A module \( M : X \to A \) with \( A \) complete is representable if and only if for every object \( x \in |X| \) the left module \( x(M) : * \to A \) is continuous and has a small solution set.

- A module \( M : X \to A \) with \( X \) cocomplete is corepresentable if and only if for every object \( a \in |A| \) the right module \( (M)a : X \to * \) is cocontinuous and has a small solution set.

Proof. Since a module \( M : X \to A \) is representable iff the left module \( x(M) : * \to A \) is representable for every \( x \in |X| \) (see Corollary 6.4.11), the assertion follows from Theorem 10.2.5.

10.2.7 Theorem. (General Adjoint Functor Theorem).

- A functor \( G : A \to X \) with \( A \) complete has a left adjoint if and only if the following conditions hold:
  (1) \( G \) is continuous;
  (2) for each object \( x \in |X| \), the left module \( x(X)G : * \to A \) has a small solution set, i.e. a small set \( \{ s_i : x \to G'a_i \} \) of \( X \)-arrows such that every \( X \)-arrow \( f : x \to G'a_i \) factors through some \( s_i \) along an \( A \)-arrow \( h : a_i \to a \) as shown in:

\[
\begin{array}{ccc}
x & \xrightarrow{s_i} & G'a_i \\
\downarrow{f} & & \downarrow{G'h} \\
G'a & \xrightarrow{h} & a
\end{array}
\]

- A functor \( F : X \to A \) with \( X \) cocomplete has a right adjoint if and only if the following conditions hold:
  (1) \( F \) is cocontinuous;
  (2) for each object \( a \in |A| \), the right module \( F(A)a : X \to * \) has a small solution set, i.e. a small set \( \{ s_i : x_i \to F'a \} \) of \( A \)-arrows such that every \( A \)-arrow \( f : x_i \to F'a \) factors through some \( s_i \) along an \( X \)-arrow \( h : x \to x_i \) as shown in:

\[
\begin{array}{ccc}
x_i & \xrightarrow{x_{i}} & F'a \\
\downarrow{h} & & \downarrow{h\circ F'} \\
x & \xrightarrow{F'a} & F'a
\end{array}
\]

Proof. By Remark 7.3.2(2), a functor \( G : A \to X \) has a left adjoint iff the module \( (X)G : X \to A \) is representable. By Corollary 10.2.6, this is the case iff for every \( x \in |X| \) the left module \( x(X)G : * \to A \) is continuous and has a small solution set. By Corollary 8.7.9, \( G \) is continuous iff \( x(X)G \) is continuous for every \( x \in |X| \). The assertion now follows.
10.3. Epimorphisms and Monomorphisms

(3) An arrow $m: x \to a$ in a category $C$ is called
  - epic (or an epimorphism) if for any parallel pair of $C$-arrows $h, h': a \to b$, $m \circ h = m \circ h'$ implies that $h = h'$.
  - monic (or a monomorphism) if for any parallel pair of $C$-arrows $h, h': y \to x$, $h \circ m = h' \circ m$ implies that $h = h'$.

10.3.2 Remark.

(1) An arrow $m: x \to a$ in a category $C$ is epic [op. monic] if and only if it is an epimorphism [op. monomorphism] in the hom-module $(C): C \to C$.

(2) An arrow $m: x \to a$ in a module $M: X \to A$ is
  - epic if and only if it is an epimorphism in the left module $x(M): * \to A$,
  - monic if and only if it is a monomorphism in the right module $(M) a: X \to *$,
  and an arrow $m: x \to a$ in a category $C$ is
  - epic if and only if it is an epimorphism in the left module $x(C): * \to C$.
  - monic if and only if it is a monomorphism in the right module $(C) a: C \to *$.

10.3.3 Proposition.

- Given a left module $M: * \to A$,
  1. an $M$-arrow $m: * \to a$ is epic if and only if it is an epimorphism in the collage category $[M]$.
  2. an $A$-arrow $f: a \to b$ is epic if and only if it is an epimorphism in the collage category $[M]$.
- Given a right module $M: X \to *$,
  1. an $M$-arrow $m: x \to *$ is monic if and only if it is a monomorphism in the collage category $[M]$.
  2. an $X$-arrow $f: y \to x$ is monic if and only if it is a monomorphism in the collage category $[M]$.

Proof. Obvious by the construction of a collage.

10.3.4 Remark. By Proposition 10.3.3, the properties that hold for epimorphisms [op. monomorphisms] in a category are also enjoyed by epimorphisms [op. monomorphisms] in a module.

10.3.5 Proposition.

- Let $M: * \to A$ be a left module. Given an $M$-arrow $m: * \to a$ and an $A$-arrow $f: a \to b$,
  1. if $m$ and $f$ are epic, so is $m \circ f$;
  2. if $m \circ f$ is epic, so is $f$.
- Let $M: X \to *$ be a right module. Given an $M$-arrow $m: x \to *$ and an $X$-arrow $f: y \to x$,
  1. if $m$ and $f$ are monic, so is $f \circ m$;
  2. if $f \circ m$ is monic, so is $f$.

Proof. See [AHS09] 7.34 and 7.41 (see also Remark 10.3.4).

10.3.6 Proposition. Let $M: X \to A$ be a module.

- If $u: x \to r$ is a direct universal $M$-arrow, then an $M$-arrow $m: x \to a$ is epic if and only if its adjunct $u \downarrow m: r \to a$ along $u$ (see Remark 6.2.2(3)) is epic.
- If $u: r \to a$ is an inverse universal $M$-arrow, then an $M$-arrow $m: x \to a$ is monic if and only if its adjunct $m / u: x \to r$ along $u$ (see Remark 6.2.2(3)) is monic.

Proof. $m: x \to a$ is epic if it is an epimorphism in the left module $x(M): * \to A$ and $u \downarrow m: r \to a$ is epic if it is an epimorphism in the left module $r(A): * \to A$. But the left module isomorphism $u(M \uparrow A): r(A) \to x(M): * \to A$ maps $u \downarrow m$ to $m$.

Note. We list below some properties of monomorphisms in a category.
10.3.7 Proposition.
(1) If a monomorphism \( m : x \to a \) has a section (i.e. if there exists an arrow \( h : a \to x \) such that \( h \circ m = 1 \)), then \( m \) is an isomorphism.
(2) Monomorphisms are pullback stable.
(3) A continuous function preserves monomorphisms.

Proof.
(1) See [AHS09] 7.36.
(2) See [AHS09] 11.18.
(3) See [AHS09] 13.5.

10.3.8 Definition.
- An arrow \( m : x \to a \) in a left module \( M : \ast \to A \) is called extremally epic (or an extremal epimorphism) if it does not factor through a proper monic \( A \)-arrow; that is, if it satisfies the following extremal condition: \( m = m' \circ u \) with \( u \) a monic \( A \)-arrow implies that \( u \) is an isomorphism.
- A monic arrow \( m : x \to \ast \) in a right module \( M : X \to \ast \) is called extremally monic (or an extremal monomorphism) if it does not factor through a proper epic \( X \)-arrow; that is, if it satisfies the following extremal condition: \( m = u \circ m' \) with \( u \) an epic \( X \)-arrow implies that \( u \) is an isomorphism.

10.3.9 Remark. An extremal epimorphism [op. monomorphism] in a two sided module and in a category is defined similarly.
- An arrow \( m : x \to a \) in a module \( M : X \to A \) is extremally epic if and only if it is an extremal epimorphism in the left module \( x(M) : \ast \to A \), and an arrow \( m : x \to a \) in a category \( C \) is extremally epic if and only if it is an extremal epimorphism in the left module \( x(C) : \ast \to C \).
- An arrow \( m : x \to a \) in a module \( M : X \to A \) is extremally monic if and only if it is an extremal monomorphism in the right module \( (M)a : X \to \ast \), and an arrow \( m : x \to a \) in a category \( C \) is extremally monic if and only if it is an extremal monomorphism in the right module \( (C)a : C \to \ast \).

10.3.10 Proposition. A unit of a left [op. right] module is extremally epic [op. monic].

Proof. Let \( u \) be a unit of a left module \( M : \ast \to A \). Clearly \( u \) is epic. Now consider a factorization \( u = v \circ k \) with \( k \) a monic \( A \)-arrow; we need to show that \( k \) is an isomorphism. If \( u \downarrow v \) is the adjunct of \( v \) along \( u \) as shown in

\[
\begin{array}{ccc}
\ast & \xrightarrow{u} & a \\
\downarrow{u} & \downarrow{k} & \downarrow{v} \\
\ast & \xrightarrow{k} & a \\
\end{array}
\]

, then since \( u \downarrow v \circ v \circ k = v \circ k = u \), we have \( u \downarrow v \circ k = 1 \) by the universality of \( u \). Hence, by Proposition 10.3.7(1), \( k \) is an isomorphism as required.

Note. The following is a restatement of Proposition 10.3.10 in terms of universal arrows.

10.3.11 Proposition. A direct [op. inverse] universal arrow is extremally epic [op. monic].

Proof. Since a direct universal arrow \( u : x \to r \) in a module \( M : X \to A \) is the same thing as a unit of the left module \( x(M) : \ast \to A \) (see Remark 6.2.2(2)), and \( u : x \to r \) is extremally epic in \( M \) if and only if it is so in \( x(M) \) (see Remark 10.3.9), the assertion follows from Proposition 10.3.10.

10.3.12 Proposition.
- Given a left module \( M : \ast \to A \), if \( A \) is complete and \( M \) is continuous, then any \( M \)-arrow satisfying the extremal condition is epic.
- Given a right module \( M : X \to \ast \), if \( X \) is cocomplete and \( M \) is cocontinuous, then any \( M \)-arrow satisfying the extremal condition is monic.
Proof. Suppose that an \( \mathcal{M} \)-arrow \( m \) satisfies the extremal condition. To show that \( m \) is epic, let \( h \) and \( h' \) be a parallel pair of \( A \)-arrows such that \( m \circ h = m \circ h' \). We need to show that \( h = h' \), and for this, it suffices to show that the pair \( (h, h') \) has an equalizer given by an isomorphism. Since \( A \) is complete, \( (h, h') \) has an equalizer \( u \), so it remains to show that \( u \) is iso. Since \( \mathcal{M} \) is continuous, \( u \) remains to be an equalizer of \( (h, h') \) in the collage category \( [\mathcal{M}] \) (see Remark 8.7.12). Hence there is a unique \( \mathcal{M} \)-arrow \( m/u \), the adjunct of \( m \) along \( u \), making the diagram

\[
\begin{array}{ccc}
\ast & \xrightarrow{u} & \ast \\
\downarrow{m/u} & & \downarrow{h} \\
\ast & \xrightarrow{h'} & \ast
\end{array}
\]

commute. Now that \( u \) is monic (because it is an equalizer) and \( m \) satisfies the extremal condition, we see that \( u \) is iso as required. \( \square \)

10.4 Subobjects

Just as we did for the notion of monomorphism and epimorphism, we extend the notion of subobjects and quotient objects to modules.

10.4.1 Definition.
- A subobject \( s \) in a right module \( \mathcal{M} : X \rightarrow * \) is a pair \( s = (s, s) \) consisting of an object \( s \in \|X\| \) and a monic \( \mathcal{M} \)-arrow \( s : s \rightarrow * \).
- A quotient object \( s \) in a left module \( \mathcal{M} : * \rightarrow A \) is a pair \( s = (s, s) \) consisting of an object \( s \in \|A\| \) and an epic \( \mathcal{M} \)-arrow \( s : * \rightarrow s \).

10.4.2 Remark.
(1) A subobject and its components are denoted using the same letter.
(2) We regard a subobject in a right module \( \mathcal{M} : X \rightarrow * \) as an object in the comma category \( [\mathcal{M}] \) and say that
   a) subobjects \( s \) and \( t \) are isomorphic when they are so in \( [\mathcal{M}] \); that is, when there is an iso \( X \)-arrow \( u : s \rightarrow t \) making the triangle

   \[
   \begin{array}{ccc}
   t & \xrightarrow{u} & * \\
   \downarrow{s} & & \downarrow{s} \\
   s & \xrightarrow{u} & *
   \end{array}
   \]

   commute;
   b) two sets \( S \) and \( T \) of subobjects are equivalent when any subobject in \( S \) is isomorphic to some subobject in \( T \), and vice versa;
   c) a set \( S \) of subobjects is essentially small when it is equivalent to a small set of subobjects.
(3) We write \( s \not\simeq t \) if subobjects \( s \) and \( t \) are isomorphic. Assuming the axiom of choice, the following conditions are equivalent for a set \( S \) of subobjects:
   a) \( S \) is essentially small;
   b) \( S \) has a small subset equivalent to \( S \);
   c) the quotient set \( S/\simeq \) is small.
(4) A subobject \( s \) in a right module \( \mathcal{M} : X \rightarrow * \) is called extremal if the \( \mathcal{M} \)-arrow \( s : s \rightarrow * \) is extremally monic. Dually, a quotient object \( s \) in a left module \( \mathcal{M} : * \rightarrow A \) is called extremal if the \( \mathcal{M} \)-arrow \( s : * \rightarrow s \) is extremally epic.

10.4.3 Definition. Let \( C \) be a category and \( c \) be an object in \( C \).
- A subobject of \( c \) is a subobject of the right module \( (C)c : C \rightarrow * \); that is, a subobject of \( c \) is a pair \( s = (s, s) \) consisting of an object \( s \in \|C\| \) and a monic \( C \)-arrow \( s : s \rightarrow c \).
- A quotient object of \( c \) is a quotient object of the left module \( c(C) : * \rightarrow C \); that is, a quotient object of \( c \) is a pair \( s = (s, s) \) consisting of an object \( s \in \|C\| \) and an epic \( C \)-arrow \( s : c \rightarrow s \).
10.4.4 Proposition.

- A continuous function $H: C \to B$ preserves subobjects; that is, $H$ sends each subobject of $c \in \|C\|$ to a subobject of $c: H \in \|B\|$.
- A cocontinuous function $H: C \to B$ preserves quotient objects; that is, $H$ sends each quotient object of $c \in \|C\|$ to a quotient object of $c: H \in \|B\|$.

Proof. Immediate from Proposition 10.3.7(3). \hfill \square

10.4.5 Definition.

- Let $S = \{s_i : s_i \twoheadrightarrow *\}$ be a set of subobjects in a right module $M: X \rightarrow *$. We say that an $M$-arrow $m: x \twoheadrightarrow *$ factors through $S$ if for each $s_i \in S$ there exists a (necessarily unique) $X$-arrow $\alpha_i: x \rightarrow s_i$ making the diagram

\[
\begin{array}{c}
\alpha_i \\
\downarrow \quad m \\
\downarrow \\
x \\
\end{array}
\]

commute.

- Let $S = \{s_i : * \twoheadrightarrow s_i\}$ be a set of quotient objects in a left module $M: * \rightarrow A$. We say that an $M$-arrow $m: * \twoheadrightarrow a$ factors through $S$ if for each $s_i \in S$ there exists a (necessarily unique) $A$-arrow $\alpha_i: s_i \rightarrow a$ making the diagram

\[
\begin{array}{c}
s_i \\
\downarrow \quad m \\
\downarrow \quad \alpha_i \\
s_i \\
\end{array}
\]

commute.

10.4.6 Remark. If an $M$-arrow $m: x \twoheadrightarrow *$ factors through $S = \{s_i : s_i \twoheadrightarrow *\}$, then it determines a unique discrete cone $(\alpha_i: x \rightarrow s_i)$ from $m$ to $S$ in the comma category $[M]$.

10.4.7 Definition.

- If $S$ is a set of subobjects in a right module $M: X \rightarrow *$, its intersection is an $M$-arrow $r: r \twoheadrightarrow *$ which is terminal among those that factors through $S$; that is;
  (1) $r$ factors through $S$;
  (2) if an $M$-arrow $m: x \twoheadrightarrow *$ factors through $S$, then it uniquely factors through $r: r \twoheadrightarrow *$.
- If $S$ is a set of quotient objects in a left module $M: * \rightarrow A$, its union is an $M$-arrow $r: * \twoheadrightarrow r$ which is initial among those that factors through $S$; that is;
  (1) $r$ factors through $S$;
  (2) if an $M$-arrow $m: * \twoheadrightarrow a$ factors through $S$, then it uniquely factors through $r: * \twoheadrightarrow r$.

10.4.8 Remark.

1. The unique factorization requirement in the definition implies that any intersection $r: r \twoheadrightarrow *$ of $S$ is monic, i.e. a subobject in $M$.
2. By Remark 10.4.6, an intersection of $S$ is identified with a product of $S$ in the comma category $[M]$. For example, an intersection $r$ of subobjects $s_0$ and $s_1$ is given by a product diagram

\[
\begin{array}{c}
S_0 \\
\downarrow \quad \mu_0 \\
r \\
\downarrow \quad \mu_1 \\
S_1 \\
\end{array}
\]

in $[M]$ ($\mu_0$ and $\mu_1$ are also monic by Proposition 10.3.5).
3. Note that, given a pair of subobjects $(s_0, s_1)$ of an object $c$ in a category $C$, a commutative diagram as in

\[
\begin{array}{c}
r \\
\downarrow \quad \mu_1 \\
S_1 \\
\downarrow \quad \mu_0 \\
S_0 \\
\end{array}
\]

\[
\begin{array}{c}
r \\
\downarrow \quad \mu_1 \\
S_1 \\
\downarrow \quad \mu_0 \\
S_0 \\
\end{array}
\]

\[
\begin{array}{c}
r \\
\downarrow \quad \mu_1 \\
S_1 \\
\downarrow \quad \mu_0 \\
S_0 \\
\end{array}
\]

is a pullback diagram of $s_0 \xrightarrow{s_0} c \xleftarrow{s_1} s_1$ in $C$ if and only if it is a product diagram of $s_0 \xrightarrow{s_0} c$ and $s_1 \xrightarrow{s_1} c$ in the comma category $[(C)c]$. Hence an intersection of $s_0$ and $s_1$ is identified with a pullback of $s_0 \xrightarrow{s_0} c \xleftarrow{s_1} s_1$. This is generalized for arbitrary set $S$ of subobjects of $c \in |C|$; an intersection of $S$ is identified with a limit, multiple pullback, of $S$.

10.4.9 Proposition.
- Two equivalent sets of subobjects have the same intersection.
- Two equivalent sets of quotient objects have the same union.

Proof. Let $S$ and $T$ be two sets of subobjects, and suppose that they are equivalent. Clearly an $M$-arrow $m : * \rightarrow a$ factors through $S$ if and only if it factors through $T$. Hence $S$ and $T$ have the same intersection.

10.4.10 Proposition.
- If a category $C$ is complete, then any small set of subobjects of an object $c \in |C|$ has an intersection. If a functor $H : C \rightarrow B$ is continuous, then it preserves subobjects and any intersection of a small set of subobjects.
- If a category $C$ is cocomplete, then any small set of quotient objects of an object $c \in |C|$ has a union. If a functor $H : C \rightarrow B$ is cocontinuous, then it preserves quotient objects and any union of a small set of quotient objects.

Proof. Immediate since an intersection is given as a limit (see Remark 10.4.8(3)). (We have already seen in Proposition 10.4.4 that a continuous functor preserves subobjects.)

10.4.11 Definition.
- A right module $M : X \rightarrow *$ is called well-powered (resp. extremally well-powered) if the set of subobjects (resp. extremal subobjects) in $M$ is essentially small.
- A left module $M : * \rightarrow A$ is called well-copowered (resp. extremally well-copowered) if the set of quotient objects (resp. extremal quotient objects) in $M$ is essentially small.

10.4.12 Remark. If $S$ denotes the set of subobjects (resp. extremal subobjects) of $M$, then, by Remark 10.4.2(3), $M$ is well-powered (resp. extremally well-powered) if and only if the quotient set $S/\sim$ is small.

10.4.13 Definition. A category $C$ is called
- well-powered (resp. extremally well-powered) if the set of subobjects (resp. extremal subobjects) of any $c \in |C|$ is essentially small.
- well-copowered (resp. extremally well-copowered) if the set of quotient objects (resp. extremal quotient objects) of any $c \in |C|$ is essentially small.

10.4.14 Remark. A category $C$ is
- well-powered (resp. extremally well-powered) if for every $c \in |C|$ the right module $(C)c : C \rightarrow *$ is well-powered (resp. extremally well-powered).
- well-copowered (resp. extremally well-copowered) if for every $c \in |C|$ the left module $c(C) : * \rightarrow C$ is well-copowered (resp. extremally well-copowered).

10.4.15 Proposition.
- If a category $C$ is complete and well-powered, then any set of subobjects of an object $c \in |C|$ has an intersection. If a functor $H : C \rightarrow B$ is continuous, then it preserves subobjects and intersections.
- If a category $C$ is cocomplete and well-copowered, then any set of quotient objects of an object $c \in |C|$ has a union. If a functor $H : C \rightarrow B$ is cocontinuous, then it preserves quotient objects and unions.

Proof. Since $C$ is well-powered, any set $S$ of subobjects of $c$ has a small set $S'$ of subobjects of $c$ equivalent to $S$, and, since any functor preserves isomorphisms, the images of $S$ and $S'$ under $H$ are also equivalent. Now, since equivalent sets of subobjects have the same intersection (see Proposition 10.4.9), the assertion is reduced to Proposition 10.4.10.
10.5 Epi-mono factorizations

Now that the notion of epimorphism and monomorphism has been extended to arrows of a module, we may extend the notion of epi-mono factorization to modules. Theorem 10.5.3 is the main result of this section and used in Section 10.7 to construct a solution set in the proof of the special adjoint functor theorem.

10.5.1 Definition.

- A left module \( M : \ast \rightarrow A \) is said to have \((\text{extremal-epi}, \text{mono})\)-factorizations when every \( M \)-arrow \( m \) factors as \( m = p \circ s \) with \( p \) an extremally epic \( M \)-arrow and \( s \) a monic \( A \)-arrow.
- A right module \( M : X \rightarrow \ast \) is said to have \((\text{epi}, \text{extremal-mono})\)-factorizations when every \( M \)-arrow \( m \) factors as \( m = s \circ p \) with \( p \) an extremally monic \( M \)-arrow and \( s \) an epic \( X \)-arrow.

10.5.2 Proposition.

- Let \( M : \ast \rightarrow A \) be a left module, and suppose that \( A \) is complete and \( M \) is continuous. Then \((\text{extremal-epi}, \text{mono})\)-factorizations are essentially unique; that is, if \( p_0 \circ s_0 = p_1 \circ s_1 \) are two \((\text{extremal-epi}, \text{mono})\)-factorizations of an \( M \)-arrow; then there exists a (necessarily unique) isomorphism \( u \) making the diagram

\[
\begin{array}{c}
p_0 \\
p_1 \\
u_0 \\
u_1 \\
\end{array} \quad \begin{array}{c}
s_0 \\
s_1 \\
\end{array} \quad \begin{array}{c}
m \end{array}
\]

commute.

- Let \( M : X \rightarrow \ast \) be a right module, and suppose that \( X \) is cocomplete and \( M \) is cocontinuous. Then \((\text{epi}, \text{extremal-mono})\)-factorizations are essentially unique; that is, if \( s_0 \circ p_0 = s_1 \circ p_1 \) are two \((\text{epi}, \text{extremal-mono})\)-factorizations of an \( M \)-arrow; then there exists a (necessarily unique) isomorphism \( u \) making the diagram

\[
\begin{array}{c}
u_0 \\
u_1 \\
\end{array} \quad \begin{array}{c}
\end{array} \quad \begin{array}{c}
p_0 \\
p_1 \\
\end{array} \quad \begin{array}{c}
s_0 \\
s_1 \\
\end{array}
\]

commute.

Proof. Let

\[
\begin{array}{c}
u_0 \\
u_1 \\
\end{array} \quad \begin{array}{c}
s_0 \\
s_1 \\
\end{array} \quad \begin{array}{c}
p_0 \\
p_1 \\
\end{array}
\]

be a pullback of \((s_0, s_1)\) in \( A \). Since \( M \) is continuous, this pullback remains to be a pullback in the collage category \([M]\) (see Remark 8.7.12). Hence there exists a unique \( M \)-arrow \( m \), the adjunct of \((p_0, p_1)\) along \((u_0, u_1)\), making the diagram

\[
\begin{array}{c}
p_0 \\
p_1 \\
u_0 \\
u_1 \\
\end{array} \quad \begin{array}{c}
s_0 \\
s_1 \\
\end{array} \quad \begin{array}{c}
m \end{array}
\]

commute. Since monomorphisms are pullback stable, \( u_0 \) and \( u_1 \) are monic, and hence they are isomorphisms by the extremal epicity of \( p_0 \) and \( p_1 \). Now \( u := u_0^{-1} \circ u_1 \) gives a desired isomorphism. \( \square \)

10.5.3 Theorem.

- Let \( M : \ast \rightarrow A \) be a left module, and suppose that \( A \) is complete and \( M \) is continuous. Under this condition, if \( A \) is well-powered, then \( M \) has essentially unique \((\text{extremal-epi}, \text{mono})\)-factorizations.
- Let \( M : X \rightarrow \ast \) be a right module, and suppose that \( X \) is cocomplete and \( M \) is cocontinuous. Under this condition, if \( X \) is well-copowered, then \( M \) has essentially unique \((\text{epi}, \text{extremal-mono})\)-factorizations.
Proof. The essential uniqueness of factorizations follows from Proposition 10.5.2. Now let \( m : * \to a \) be an \( M \)-arrow and consider all the possible factorizations \( m = m_k \circ s_k \) with \( s_k \) monic. By Proposition 10.4.15, there exists an intersection \( r : r \to a \) of all \( s_k \). Since \( M \) is continuous, by the second assertion of Proposition 10.4.15, it preserves this intersection. Hence \( m \) factors through \( r \) as \( m = n \circ r \) for some (necessarily unique) \( M \)-arrow \( n : * \to r \). The proof is complete if we show that \( n \) is extremally epic. For this, by Proposition 10.3.12, it suffices to show that \( n \) satisfies the extremal condition. Let \( n = n' \circ u \) be a factorization of \( n \) with \( u \) monic. We need to show that \( u \) is an isomorphism. Since \( m = n' \circ (u \circ r) \) and \( u \circ r \) is monic, there exists a unique \( A \)-arrow \( v \) making the diagram

\[
\begin{array}{c}
\ast \\
\downarrow n' \\
r \\
\downarrow v \\
a \\
\end{array}
\]

commute. Since \( r = v \circ u \circ r \) and \( r \) is monic, we have \( 1 = v \circ u \). Hence \( u \) is an isomorphism by Proposition 10.3.7(1) as required.

\[\Box\]

10.5.4 Definition. A category \( C \) is said to have

\( \bullet \) (extremal-epi,mono)-factorizations when every \( C \)-arrow \( m \) factors as \( m = p \circ s \) with \( p \) extremally epic and \( s \) monic.

\( \bullet \) (epi,extremal-mono)-factorizations when every \( C \)-arrow \( m \) factors as \( m = s \circ p \) with \( p \) extremally monic and \( s \) epic.

10.5.5 Remark. A category \( C \) has

\( \bullet \) (extremal-epi,mono)-factorizations if and only if for every \( c \in \| C \| \) the left module \( c(C) : * \to C \) has (extremal-epi,mono)-factorizations.

\( \bullet \) (epi,extremal-mono)-factorizations if and only if for every \( c \in \| C \| \) the right module \( (C)c : C \to * \) has (epi,extremal-mono)-factorizations.

10.5.6 Theorem.

\( \bullet \) A complete and well-powered category has essentially unique (extremal-epi,mono)-factorizations.

\( \bullet \) A cocomplete and well-copowered category has essentially unique (epi,extremal-mono)-factorizations.

Proof. Let \( C \) be a complete and well-powered category. By Remark 10.5.5, it suffices to show that for any \( c \in \| C \| \), the left module \( c(C) : * \to C \) has essentially unique (extremal-epi,mono)-factorizations. But this follows from Theorem 10.5.3. \[\Box\]

10.6 Generators

This section introduces the notion of generators\(^3\). The main result is Theorem 10.6.6, which is used to prove the special adjoint functor theorem in Section 10.7.

10.6.1 Definition. Let \( E \) be a category and \( M : X \to A \) be a module.

\( \bullet \) A cone \( \alpha : G \sim a : E* \sim M \) is said to be (jointly) epic if for any pair of parallel \( A \)-arrows \( h, h' : a \to b \), \( \alpha_a \circ h = \alpha_a \circ h' \) for every \( e \in \| E \| \) implies that \( h = h' \).

\( \bullet \) A cone \( \alpha : x \sim F : *E \sim M \) is said to be (jointly) monic if for any pair of parallel \( X \)-arrows \( h, h' : y \to x \), \( h \circ \alpha_e = h' \circ \alpha_e \) for every \( e \in \| E \| \) implies that \( h = h' \).

10.6.2 Remark. A cone \( \alpha : G \sim a : E* \sim M \) is epic if and only if \( \alpha \) is an epic \((E*, M)\)-arrow (see Definition 4.6.7). Dually, a cone \( \alpha : x \sim G : *E \sim M \) is monic if and only if \( \alpha \) is a monic \((*E, M)\)-arrow.

\[^3\text{Some authors use the term “separator” instead of “generator”.}\]
10.6.3 Proposition.
- If \( \mu : G \to r : E \to M \) is a universal cone, then a cone \( \alpha : G \to a : E \to M \) is epic if and only if its adjunct \( \alpha : r \to a \) along \( \mu \) is epic.
- If \( \mu : r \to F : *E \to M \) is a universal cone, then a cone \( \alpha : x \to F : *E \to M \) is monic if and only if its adjunct \( \alpha : x \to r \) along \( \mu \) is monic.

Proof. By Remark 10.6.2, this is an instance of Proposition 10.3.6 where \( M \) is given by \( (E^*, M) \).

10.6.4 Definition. A set \( \mathcal{E} \) of objects in a category \( C \) is said to
- generate \( C \) if for any parallel pair of \( C \)-arrows \( h, h' : c \to d \), \( h \neq h' \) implies that there is an \( x \in \mathcal{E} \) and a \( C \)-arrow \( f : x \to c \) with \( f \circ h \neq f \circ h' \).
- cogenerate \( C \) if for any parallel pair of \( C \)-arrows \( h, h' : d \to c \), \( h \neq h' \) implies that there is an \( x \in \mathcal{E} \) and a \( C \)-arrow \( f : c \to x \) with \( h \circ f \neq h' \circ f \).

10.6.5 Remark. Let \( C \) be a category. If \( c \) is an object in \( C \) and \( \mathcal{E} \) is a set of objects in \( C \), we denote by \( \{ \mathcal{E}, c \} \) the set of all \( C \)-arrows \( f : x \to c \) with \( x \in \mathcal{E} \), to be precise (recall that we do not require pairwise disjointness of hom-sets), the set of all pairs \( (x, f) \) with \( f \in x(C)c \) and \( x \in \mathcal{E} \). Indexed by itself, the set \( [\mathcal{E}, c] \) is seen as a discrete cone (see Remark 4.6.4(4)) from the family of objects \( \{ \text{dom}(f) \}_{f \in \mathcal{E}, c} \) to \( c \). Dually, we denote by \( \{ c, \mathcal{E} \} \) the set of all \( C \)-arrows \( f : c \to x \) with \( x \in \mathcal{E} \); \( [c, \mathcal{E}] \) is seen as a discrete cone from \( c \) to the family of objects \( \{ \text{cod}(f) \}_{f \in \mathcal{E}, c} \). With this notation and the terminology introduced in Definition 10.6.1, Definition 10.6.4 is stated more succinctly as:
- \( \mathcal{E} \) generate \( C \) if \( [\mathcal{E}, c] \) is epic for every \( c \in |C| \).
- \( \mathcal{E} \) cogenerate \( C \) if \( [c, \mathcal{E}] \) is monic for every \( c \in |C| \).

10.6.6 Theorem.
- Let \( M : * \to A \) be a left module, and suppose that \( A \) is complete and \( M \) is continuous. Under this condition, if \( A \) has a small cogenerating set, then \( M \) is extremally well-powered.
- Let \( M : X \to * \) be a right module, and suppose that \( X \) is cocomplete and \( M \) is cocontinuous. Under this generating condition, if \( X \) has a small generating set, then \( M \) is extremally well-powered.

Proof. We will use the notation in Remark 10.6.5. Let \( \mathcal{E} \) be a small cogenerating set of \( A \) and denote by \( \{ *, \mathcal{E} \} \) the set of all \( M \)-arrows \( m : * \to a \) with \( a \in \mathcal{E} \). Let \( \mathcal{P} \{ *, \mathcal{E} \} \) denote the set of all subsets of \( \{ *, \mathcal{E} \} \); since \( \mathcal{E} \) is small, so is \( \mathcal{P} \{ *, \mathcal{E} \} \). Given an extremal quotient object \( s \) of \( M \), define \( [s] \in \mathcal{P} \{ *, \mathcal{E} \} \) by \( [s] = \{ s \circ f \mid f \in [s, \mathcal{E}] \} \). By the smallness of \( \mathcal{P} \{ *, \mathcal{E} \} \) and noting Remark 10.4.12, the proof is complete if we show that, given two extremal quotient objects \( s \) and \( t \), \( [s] = [t] \) iff \( s \cong t \). If \( s \) and \( t \) are isomorphic, i.e. if there is an \( A \)-arrow \( u : t \to s \) such that \( s = s \circ u \), then, by the bijectivity of \( f \mapsto u \circ f : [s, \mathcal{E}] \to [t, \mathcal{E}] \), we have

\[
[s] = \{ s \circ f \mid f \in [s, \mathcal{E}] \} = \{ t \circ u \circ f \mid f \in [s, \mathcal{E}] \} = \{ t \circ g \mid g \in [t, \mathcal{E}] \} = [t].
\]

Now suppose that \( [s] = [t] \) and let \( \mathcal{I} = [s] = [t] \). Since \( s : * \to s \) and \( t : * \to t \) are epic, the assignments \( f \mapsto s \circ f : [s, \mathcal{E}] \to \mathcal{I} \) and \( f \mapsto t \circ f : [t, \mathcal{E}] \to \mathcal{I} \) are bijective. Hence the discrete cones \( [s, \mathcal{E}] \) and \( [t, \mathcal{E}] \) are written as \( \sigma = (\sigma_i : s \to a_i)_{i \in \mathcal{I}} \) and \( \tau = (\tau_i : t \to a_i)_{i \in \mathcal{I}} \) with \( s \circ \sigma_i = t \circ \tau_i \) for each \( i \in \mathcal{I} \). Let \( \mu = (\mu_i : r \to a_i)_{i \in \mathcal{I}} \) be a product diagram of the family of object \( (a_i)_{i \in \mathcal{I}} \) in \( A \). Since \( M \) is continuous, \( \mu \) remains to be a product diagram in the collage category \( [M] \) (see Remark 8.7.12). Hence there exists a unique \( M \)-arrow \( m : * \to r \), the adjunct of \( s \circ \sigma = t \circ \tau \) along \( \mu \), making the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma} & r \\
\downarrow{m} & & \downarrow{\mu} \\
\downarrow{t} & & \downarrow{\tau} \\
(a_i) & \xrightarrow{\sigma} & (a_i)
\end{array}
\]

commute. Since \( \sigma = [s, \mathcal{E}] \) and \( \tau = [t, \mathcal{E}] \) are monic by the definition of a cogenerating set, so are \( \sigma \circ \mu \) and \( \tau \circ \mu \) by Proposition 10.6.3. Hence \( s \circ \sigma \circ \mu \) and \( t \circ \tau \circ \mu \) are (extremal-epi,mono)-factorizations of \( m \), and by the essential uniqueness of such factorizations (see Proposition 10.5.2), \( s \) and \( t \) are isomorphic as required.

\( \square \)
10.6.7 Corollary.

- A complete category with a small cogenerating set is extremally well-copowered.
- A cocomplete category with a small generating set is extremally well-powered.

Proof. Let $C$ be a complete category with a small cogenerating set. By Remark 10.4.14, it suffices to show that for any $c \in |C|$, the left module $c(C) : \ast \to C$ is extremally well-copowered. But this follows from Theorem 10.6.6. 

10.7 Special adjoint functor theorem

We are now well equipped to prove SAFT (Special Adjoint Functor Theorem), which presents a sufficient condition for a functor to have a small solution set of GAFT (General Adjoint Functor Theorem). Just as we did for GAFT, we first prove the representability theorem (Theorem 10.7.1) and deduce SAFT from it.

10.7.1 Theorem. (Special Representability Theorem).

- If a category $A$ is complete, well-powered and with a small cogenerating set, then a left module $M : \ast \to A$ is representable if and only if $M$ is continuous.
- If a category $X$ is cocomplete, well-copowered and with a small generating set, then a right module $M : X \to \ast$ is representable if and only if $M$ is cocontinuous.

Proof. If $M$ is representable, then $M$ is continuous by Corollary 8.7.10. Conversely, suppose that $M$ is continuous. Then $M$ has (extremal-epi,mono)-factorizations by Theorem 10.5.3. This implies that the set of extremal quotient objects in $M$ forms a solution set of $M$, and this set is essentially small by Theorem 10.6.6. Hence $M$ is representable by Theorem 10.2.5. 

10.7.2 Corollary. Let $M : X \to A$ be a module.

- If a category $A$ is complete, well-powered and with a small cogenerating set, then $M$ is representable if and only if for every object $x \in |X|$ the left module $x(M) : \ast \to A$ is continuous.
- If a category $X$ is cocomplete, well-copowered and with a small generating set, then $M$ is representable if and only if for every object $a \in |A|$ the right module $(M)a : X \to \ast$ is cocontinuous.

Proof. Since a module $M : X \to A$ is representable iff the left module $x(M) : \ast \to A$ is representable for every $x \in |X|$ (see Corollary 6.4.11), the assertion follows from Theorem 10.7.1. 

10.7.3 Theorem. (Special Adjoint Functor Theorem).

- If a category $A$ is complete, well-powered and with a small cogenerating set, then a functor $G : A \to X$ has a left adjoint if and only if $G$ is continuous.
- If a category $X$ is cocomplete, well-copowered and with a small generating set, then a functor $F : X \to A$ has a right adjoint if and only if $F$ is cocontinuous.

Proof. By Remark 7.3.2(2), a functor $G : A \to X$ has a left adjoint iff the module $(X)G : X \to A$ is representable. By Corollary 10.7.2, this is the case iff for every object $x \in |X|$ the left module $x(X)G : \ast \to A$ is continuous. By Corollary 8.7.9, this is the case iff $G$ is continuous. 

10.7.4 Corollary.

- If a category $C$ is complete, well-powered and with a small cogenerating set, then $C$ is cocomplete as well.
- If a category $C$ is cocomplete, well-copowered and with a small generating set, then $C$ is complete as well.

Proof. Since $C$ is cocomplete iff for every small category $E$ the diagonal functor $[1_E, C]$ has a left adjoint (see Remark 8.1.14), and since $[1_E, C]$ is continuous by Theorem 8.4.15, the assertion follows from Theorem 10.7.3.
10.7.5 Corollary.
- If a category $\mathcal{C}$ is complete, well-powered and with a small cogenerating set, then $\mathcal{C}$ has an initial object.
- If a category $\mathcal{C}$ is cocomplete, well-copowered and with a small generating set, then $\mathcal{C}$ has a terminal object.

Proof. Immediate from Corollary 10.7.4 since any cocomplete category has an initial object. □

10.7.6 Remark. Conversely, Theorem 10.7.3 follows from Corollary 10.7.5. [ML98] takes this route to tackle the special adjoint functor theorem.
11 Collages and Commas (continued)

11.1 Module $\text{CYL} : \text{CAT} \to \text{MOD}$

In this section, we define and study the following modules:

$$\text{CYL} : \text{CAT} \to \text{MOD} \quad \uparrow\text{CYL} : \text{CAT} \to \text{CLG} \quad \downarrow\text{CYL} : \text{COM} \to \text{CAT}$$

The arrows of the module $\text{CYL}$ from a category $E$ to a module $M$ are all cylinders defined between $E$ and $M$. Towards the end of the section, we prove that the forgetful functor $\text{MOD} \to \text{CAT}$ defined in Section 3.4 is right adjoint to the embedding $\text{CAT} \to \text{MOD}$ by showing that $\text{CYL}$ is corepresented and represented by these functors.

11.1.1 Definition.

1. The module $\text{CYL} : \text{CAT} \to \text{MOD}$ is defined in the following way:
   
   a) A CYL-arrow from a category $E$ to a module $M : X \to A$, written $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$, is a cylinder

   $$\begin{array}{c}
   \text{X} \\
   \alpha \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{E} \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{G} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

   b) For a functor $K : D \to E$ and a CYL-arrow $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$ as in

   $$\begin{array}{c}
   \text{D} \\
   \downarrow \\
   \text{E} \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{X} \\
   \alpha \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{G} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

   , their composite is the CYL-arrow $K \circ \alpha : K \circ G \rightsquigarrow F \circ K : D \rightsquigarrow M$ (i.e. the cylinder

   $$\begin{array}{c}
   \text{X} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{D} \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{G} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

   ) given by the usual composition of a functor and a cylinder (see Definition 4.3.26).

c) For a CYL-arrow $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$ and a module cell $\psi : P \rightsquigarrow Q : M \to N$ as in

   $$\begin{array}{c}
   \text{X} \\
   \downarrow \\
   \text{M} \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{P} \\
   \psi \\
   \downarrow \\
   \text{N} \\
   \downarrow \\
   \text{B}
   \end{array}
   \quad
   \begin{array}{c}
   \text{E} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

   , their composite is the CYL-arrow $\alpha \circ \psi : G \circ P \rightsquigarrow Q \circ F : E \rightsquigarrow N$ (i.e the cylinder,

   $$\begin{array}{c}
   \text{Y} \\
   \downarrow \\
   \text{B}
   \end{array}
   \quad
   \begin{array}{c}
   \text{Y} \\
   \downarrow \\
   \text{N} \\
   \downarrow \\
   \text{B}
   \end{array}
   \quad
   \begin{array}{c}
   \text{E} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

   ) given by the usual composition of a cylinder and a cell (see Definition 4.3.15).

2. The module $\uparrow\text{CYL} : \text{CAT} \to \text{CLG}$ is defined in the following way:

   a) A $\uparrow\text{CYL}$-arrow from a category $E$ to a collage $M : X \to A$, written $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$, is a triple $(G, \alpha, F)$ consisting of a functor $G : E \to X$, a second functor $F : E \to A$, and a natural transformation

   $$\begin{array}{c}
   \text{X} \\
   \downarrow \\
   \text{M}_0 \\
   \downarrow \\
   \text{M}_1 \\
   \downarrow \\
   \text{A}
   \end{array}
   \quad
   \begin{array}{c}
   \text{E} \\
   \downarrow \\
   \text{F}
   \end{array}
   $$

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11.1. Module CYL : \textit{CAT} \to \textit{MOD}  

from $G \circ M_0$ to $M_1 \circ F$.

b) for a functor $K : D \to E$ and a $\uparrow$CYL-arrow $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$ as in

\[
\begin{array}{c}
D \\
\downarrow K \\
E \\
\alpha \\
| \\
A \\
\end{array}
\]

, their composite is the $\uparrow$CYL-arrow $K \circ \alpha : K \circ G \rightsquigarrow F \circ K : D \rightsquigarrow M$ (i.e. the natural transformation

\[
\begin{array}{c}
E \\
\alpha \\
| \\
A \\
\end{array}
\]

) given by the usual composition of a functor and a natural transformation.

c) for a $\uparrow$CYL-arrow $\alpha : G \rightsquigarrow F : E \rightsquigarrow M$ and a collage cell $\psi : P \rightsquigarrow Q : M \to N$ as in

\[
\begin{array}{c}
P \\
\downarrow \psi \\
Q \\
\end{array}
\]

, their composite is the $\uparrow$CYL-arrow $\alpha \circ \psi : G \circ P \rightsquigarrow Q \circ F : E \rightsquigarrow N$ (i.e. the natural transformation

\[
\begin{array}{c}
P \\
\alpha \\
\psi \\
\| \\
Q \\
\end{array}
\]

) defined by

$$\alpha \circ \psi = \alpha \circ [\psi]$$

, the usual composition of a natural transformation and a functor.

(3) The module $\downarrow$CYL : \textit{COM} \to \textit{CAT}$ is defined in the following way:

a) a $\downarrow$CYL-arrow from a comma \textit{K} : $X \to A$ to a category \textit{E}, written $\alpha : G \rightsquigarrow F : \textit{K} \rightsquigarrow \textit{E}$, is a triple $(G, \alpha, F)$ consisting of a functor $G : X \to E$, a second functor $F : A \to E$, and a natural transformation

\[
\begin{array}{c}
X \\
\textit{K} \\
\textit{G} \\
\alpha \\
\textit{E} \\
\downarrow F \\
A \\
\end{array}
\]

from $\textit{K}_0 \circ G$ to $F \circ \textit{K}_1$.

b) for a comma cell $\psi : P \rightsquigarrow Q : J \to K$ and a $\downarrow$CYL-arrow $\alpha : G \rightsquigarrow F : \textit{K} \rightsquigarrow \textit{E}$ as in

\[
\begin{array}{c}
P \\
\downarrow \psi \\
Q \\
\end{array}
\]

, their composite is the $\downarrow$CYL-arrow $\psi \circ \alpha : P \circ G \rightsquigarrow F \circ Q : J \rightsquigarrow E$ (i.e. the natural transformation

\[
\begin{array}{c}
P \\
\psi \circ \alpha \\
\| \\
E \\
\end{array}
\]

) defined by

$$\psi \circ \alpha = [\psi] \circ \alpha$$
11.1. Module CYL : CAT → MOD

, the usual composition of a functor and a natural transformation.

c) for a ↓CYL-arrow α : G → F : K → E and a functor K : E → D as in

\[
\begin{array}{c}
X \\ K_0 \\
\downarrow G \\
\alpha \\
E \\
\downarrow \alpha \circ K \\
\downarrow K \\
F \\
\downarrow K \circ F \\
D
\end{array}
\]

\[
\begin{array}{c}
X \\ \alpha \circ K \\
\downarrow G \circ K \\
E \\
\downarrow K \circ F \\
A
\end{array}
\]

, their composite is the ↓CYL-arrow α ◦ K : G ◦ K ↪ K ◦ F : K ↪ D (i.e. the natural transformation

\[
\begin{array}{c}
X \\ \alpha \circ K \\
\downarrow G \circ K \\
E \\
\downarrow K \circ F \\
A
\end{array}
\]

) given by the usual composition of a natural transformation and a functor.

Note. Recall from Theorem 3.1.14 that the functor MOD ↓ → CLG is an isomorphism.

11.1.2 Proposition. The identity

\[
\begin{array}{c}
\text{CAT} \xrightarrow{\text{CYL}} \text{MOD} \\
\downarrow 1 \downarrow 1 \\
\text{CAT} \xrightarrow{\uparrow \text{CYL}} \text{CLG}
\end{array}
\]

holds.

Proof. By Remark 4.3.2(2), a cylinder α : G ↪ F : E ↪ M is the same thing as a natural transformation α : G ◦ M₀ → M₁ ◦ F : E → [M]. □

11.1.3 Remark. The modules CYL : CAT → MOD and ↑CYL : CAT → CLG are thus the same thing under the identification MOD ≅ CLG.

11.1.4 Definition. (1) The unit cylinder of a module M : X → A is the cylinder

\[
\begin{array}{c}
\text{M}_0 \\
\text{M}_1 \\
\text{M} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{M}_0 \\
\text{M}_1 \\
\text{M} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

defined by

\[
[1^1_M]_m = m
\]

for m an arrow of M.

(2) The unit cylinder of a comma K : X → A is the cylinder

\[
\begin{array}{c}
\text{K}_0 \\
\text{K}_1 \\
\text{K} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{K}_0 \\
\text{K}_1 \\
\text{K} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

defined by

\[
[1^1_K]_k = k
\]

for k an object of [K].

11.1.5 Remark. By Remark 4.3.2(2), the unit cylinders 1^1_M and 1^1_k are also written as

\[
\begin{array}{c}
\text{M}_0 \\
\text{M}_1 \\
\text{M} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{K}_0 \\
\text{K}_1 \\
\text{K} \\
\text{X} \\
\downarrow \text{X} \\
\downarrow \text{A}
\end{array}
\]

and
using the collages of $\mathcal{M}$ and $\mathbb{K}^1$ (recall from Remark 3.4.29(2) that $[\mathbb{K}]$ denotes the collage category of $\mathbb{K}^1$).

**11.1.6 Proposition.** The unit cylinder $1^1_\mathcal{M}$ forms a CYL-arrow $1^1_\mathcal{M} : \mathcal{M}_0 \to \mathcal{M}_1 : [\mathcal{M}] \to \mathcal{M}$.

**Proof.** Immediate from the definitions. \qed

**11.1.7 Proposition.**

1. The unit cylinder $1^1_\mathcal{M}$ forms a ↑CYL-arrow $1^1_\mathcal{M} : \mathcal{M}_0 \to \mathcal{M}_1 : [\mathcal{M}] \to \mathcal{M}$.
2. The unit cylinder $1^1_\mathbb{K}$ forms a ↓CYL-arrow $1^1_\mathbb{K} : \mathbb{K}_0 \to \mathbb{K}_1 : \mathbb{K} \to [\mathbb{K}]$.

**Proof.** Immediate from Remark 11.1.5. \qed

**11.1.8 Proposition.**

1. Given a collage cell $\psi : \mathcal{P} \to \mathcal{Q} : \mathcal{M} \to \mathcal{N}$, the two compositions

\[
\begin{array}{ccc}
\mathcal{M}_0 & \downarrow [\mathcal{M}] & \mathcal{M}_1 \\
\mathcal{P} \rightarrow[\psi] & \downarrow 1^1_\mathcal{M} & \rightarrow \mathcal{A} \\
\mathcal{Y} \rightarrow[\psi] & \downarrow [\mathcal{N}] & \rightarrow \mathcal{B}
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{X} & \rightarrow[\mathcal{M}] & \mathcal{A} \\
\mathcal{P} \rightarrow[\psi] & \downarrow 1^1_\mathcal{M} & \rightarrow \mathcal{B} \\
\mathcal{Y} \rightarrow[\psi] & \downarrow [\mathcal{N}] & \rightarrow \mathcal{B}
\end{array}
\]

where $[\psi] : [\mathcal{M}] \to [\mathcal{N}]$ is the comma functor of $\psi$ (see Remark 3.4.24(3)), yield the same natural transformations $[\mathcal{M}] \to [\mathcal{N}]$; that is, the square

\[
\begin{array}{ccc}
[\mathcal{M}] & \rightarrow[\psi] & [\mathcal{M}] \\
\downarrow & & \downarrow \\
[\mathcal{N}] & \rightarrow[\psi] & [\mathcal{N}]
\end{array}
\]

commutes.

2. Given a comma cell $\psi : \mathcal{P} \to \mathcal{Q} : \mathcal{J} \to \mathbb{K}$, the two compositions

\[
\begin{array}{ccc}
\mathcal{Y} & \rightarrow[\mathcal{J}] & \mathcal{B} \\
\mathcal{P} \rightarrow[\psi] & \downarrow 1^1_\mathcal{J} & \rightarrow \mathcal{B} \\
\mathcal{X} & \rightarrow[\mathcal{K}] & \mathcal{A}
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{X} & \rightarrow[\mathcal{J}] & \mathcal{B} \\
\mathcal{P} \rightarrow[\psi] & \downarrow 1^1_\mathcal{J} & \rightarrow \mathcal{B} \\
\mathcal{Y} & \rightarrow[\mathcal{K}] & \mathcal{A}
\end{array}
\]

where $[\psi] : [\mathcal{J}] \to [\mathbb{K}]$ is the collage functor of $\psi$ (see Remark 3.4.29(2)), yield the same natural transformations $[\mathcal{J}] \to [\mathbb{K}]$; that is, the square

\[
\begin{array}{ccc}
[\mathcal{J}] & \rightarrow[\psi] & [\mathcal{J}] \\
\downarrow & & \downarrow \\
[\mathbb{K}] & \rightarrow[\psi] & [\mathbb{K}]
\end{array}
\]

commutes.

**Proof.** Immediate from the definitions of the unit cylinders $1^1_\mathcal{M}$ and $1^1_\mathbb{K}$, and the constructions of the comma functor $[\psi]$ and the collage functor $[\psi]$. \qed
11.1.9 Proposition.

(1) For a module \( \mathcal{M} : X \to A \), the unit cylinder \( 1^{i}_{\mathcal{M}} \) is given by the composition

\[
\begin{array}{cccc}
\mathcal{M}_0 & \mathcal{M}_{i} & \mathcal{M}_1 \\
X & [\mathcal{M}] & A \\
\mathcal{M}_0 & [\mathcal{M}] & \mathcal{M}_1
\end{array}
\]

; that is, the triangle

\[
\begin{array}{cccc}
[\mathcal{M}] & [\mathcal{M}_{i}] \\
1^{i} & [\epsilon_{\mathcal{M}}] & [\mathcal{M}]
\end{array}
\]

commutes, where \( 1^{i}_{\mathcal{M}} \) is the unit cylinder of the comma \( \mathcal{M}^{i} \) and \( \epsilon_{\mathcal{M}} \) is the isomorphism in Theorem 3.4.30.

(2) For a comma \( \mathcal{K} : X \to A \), the unit cylinder \( 1^{i}_{\mathcal{K}} \) is given by the composition

\[
\begin{array}{cccc}
\mathcal{K}_0 & \mathcal{K}_{i} & \mathcal{K}_1 \\
X & [\mathcal{K}] & A \\
\mathcal{K}_0 & [\mathcal{K}] & \mathcal{K}_1
\end{array}
\]

; that is, the triangle

\[
\begin{array}{cccc}
[\mathcal{K}_{i}] & [\mathcal{K}] \\
1^{i} & [\eta_{\mathcal{K}}] & [\mathcal{K}]
\end{array}
\]

commutes, where \( \eta_{\mathcal{K}} \) is the isomorphism in Theorem 3.4.30 and \( 1^{i}_{\mathcal{K}} \) is the unit cylinder of the module \( \mathcal{K}^{i} \).

Proof. Examining the constructions of \( \epsilon_{\mathcal{M}} \) and \( \eta_{\mathcal{K}} \), the verification is straightforward. \( \square \)

11.1.10 Definition.

(1) The comma adjunct of a \( \uparrow \)CYL-arrow \( \alpha : G \Rightarrow F : E \Rightarrow \mathcal{M} \) (i.e. a natural transformation

\[
\begin{array}{cccc}
G & E & F \\
X & [\mathcal{M}] & A \\
\mathcal{M}_0 & \mathcal{M}_{i} & \mathcal{M}_1
\end{array}
\]

) is the functor \([\alpha] : E \to [\mathcal{M}] \) defined by

\[
[\alpha] : e = \alpha_e \quad [\alpha] : h = (G \circ h, F \circ h)
\]

for \( e \) an object and \( h \) an arrow of \( E \).

(2) The collage adjunct of a \( \downarrow \)CYL-arrow \( \alpha : G \Rightarrow F : \mathcal{K} \Rightarrow E \) (i.e. a natural transformation

\[
\begin{array}{cccc}
G & E & F \\
X & [\mathcal{K}] & A \\
\mathcal{K}_0 & \mathcal{K}_{i} & \mathcal{K}_1
\end{array}
\]

) is the functor \([\alpha] : [\mathcal{K}] \to E \) given by the adjunct (see Theorem 3.1.16) of the cell

\[
\begin{array}{cccc}
X & \mathcal{K} & A \\
G \downarrow \alpha & \mathcal{K}_1 & \mathcal{K}_0 \\
E \downarrow \beta & E &=& E
\end{array}
\]
11.1. Module CYL : CAT → MOD

11.1.11 Remark. Proposition 11.1.12 below justifies the name “adjunct”.

11.1.12 Proposition.
(1) For any ↑CYL-arrow α : G ↣ F : E ↣ M, the comma adjunct [α] gives the unique functor E → [M] making the diagram

\[
\begin{array}{ccc}
[M] & \overset{1_M}{\cong} & M \\
\downarrow{[\alpha]} & \nearrow{\alpha} & \\
E & & \\
\end{array}
\]

commute; the unit cylinder \(1^\downarrow_M\) is therefore an inverse universal ↑CYL-arrow.

(2) For any ↓CYL-arrow α : G ↣ F : K ↣ E, the collage adjunct [α] gives the unique functor [K] → E making the diagram

\[
\begin{array}{ccc}
K & \overset{1_K}{\cong} & [K] \\
\downarrow{[\alpha]} & \nearrow{\alpha} & \\
E & & \\
\end{array}
\]

commute; the unit cylinder \(1^\uparrow_K\) is therefore a direct universal ↓CYL-arrow.

Proof. The identities

\[
[\alpha] \circ 1^\downarrow_M = \alpha \quad 1^\uparrow_K \circ [\alpha] = \alpha
\]

are verified easily. To prove the uniqueness of [α] and [α], it suffices to prove the identities

\[
[H \circ 1^\downarrow_M] = H \quad [1^\uparrow_K \circ H] = H
\]

for any functor H : E → [M] and any functor H : [K] → E. But these are also verified easily. □

11.1.13 Theorem.
(1) The functor [−] : CLG → CAT (see Remark 3.4.24(4)) and the family of unit cylinders \(1^\downarrow_M : [M] ↣ M\), one for each locally small module M, form a counit of the module ↑CYL : CAT → CLG.

(2) The functor [−] : COM → CAT (see Remark 3.4.29(3)) and the family of unit cylinders \(1^\uparrow_K : K ↣ [K]\), one for each locally small comma K, form a unit of the module ↓CYL : COM → CAT.

Proof. The family of unit cylinders \(1^\downarrow_M\) (resp. \(1^\uparrow_K\)) satisfies the naturality condition by Proposition 11.1.8, and each unit cylinder is universal as we have seen in Proposition 11.1.12. □

11.1.14 Theorem. The functor [−] : MOD → CAT (see Remark 3.4.24(1)) and the family of unit cylinders \(1^\downarrow_M : [M] ↣ M\), one for each locally small module M, form a counit of the module CYL : CAT → MOD.

Proof. Noting Proposition 11.1.2 and Remark 3.4.24(3), we see that the assertion translates to Theorem 11.1.13(1) along the isomorphism MOD \(\cong\) CLG. □

11.1.15 Definition. The unit cylinder of a category E is the cylinder

\[
E \xrightarrow{1} E \xrightarrow{1} E
\]

defined by

\[
[1_E]_e = 1_e
\]

for \(e \in [E]\), i.e. by the identity natural transformation \(1_E : 1_E \rightarrow 1_E\).
11.1.16 Proposition. For any functor $K : D \to E$, the square
\[
\begin{array}{ccc}
D & \sim & (D) \\
\uparrow_{[1(D)]} & & \downarrow_{\{K\}} \\
E & \sim & (E) \\
\end{array}
\]
commutes.

Proof. Easily verified. \qed

11.1.17 Proposition. For any locally small category $E$, the unit cylinder $[1(E)]$ forms a direct universal CYL-arrow.

Proof. The family of module isomorphisms $\Psi^E_M : \langle E, M \rangle \to \langle \{E\}, M \rangle$ in Corollary 5.5.3, one for each locally small module $M$, gives a representation of the left slice of CYL at $E$. The unit corresponding to this representation is given by the inverse image of identity cell $\langle E \rangle \to \langle E \rangle$ under $\Psi^E_{(E)}$, but it is immediately seen that this is nothing but the unit cylinder $[1(E)]$ of $E$. \qed

11.1.18 Theorem. The embedding $(-) : \text{CAT} \to \text{MOD}$ (see Theorem 1.2.32) and the family of unit cylinders $[1(E)] : E \to \langle E \rangle$, one for each locally small category $E$, form a unit of the module CYL : CAT $\to$ MOD.

Proof. Immediate from Proposition 11.1.16 and Proposition 11.1.17. \qed

11.1.19 Theorem. There is a canonical adjunction between the functor $[-] : \text{MOD} \to \text{CAT}$ and the embedding $(-) : \text{CAT} \to \text{MOD}$.

Proof. This follows by applying Theorem 7.3.15 to the counit and unit of the module CYL given in Theorem 11.1.14 and Theorem 11.1.18. \qed

11.1.20 Remark. The functor $[-] : \text{MOD} \to \text{CAT}$ is thus a right adjoint of the embedding $(-) : \text{CAT} \to \text{MOD}$ (cf. Remark 3.1.17).

11.2 Equivalence $[X \downarrow A] \simeq [X : A]$

The purpose of this section is to prove that the functors $[X \downarrow A] \to [X : A]$ and $[X : A] \to [X \downarrow A]$ constructed in Section 3.4 constitute an adjoint equivalence $[X \downarrow A] \simeq [X : A]$. To this end, we define the module $\langle X \uparrow A \rangle : [X \downarrow A] \to [X : A]$ (an $(X \downarrow A)$-arrow from a comma $K : X \rightarrow A$ to a module $M : X \rightarrow A$ is a cylinder $X \dashrightarrow \ldots M \rightarrow A$) and show that the module is represented by $[X \downarrow A] \rightarrow [X : A]$ and corepresented by $[X : A] \rightarrow [X \downarrow A]$; the adjunction resulting from these representation and corepresentation gives the desired equivalence. In fact, since they exhibit clear duality with commas, we work on collages instead of modules in most parts of the section, and establish an equivalence $[X \downarrow A] \simeq [X \uparrow A]$ (recall from Section 3.1 that $[X : A]$ and $[X \uparrow A]$ are isomorphic).

11.2.1 Definition. Let $X$ and $A$ be categories.

(1) The module $\langle X \uparrow A \rangle : [X \downarrow A] \to [X : A]$ is defined in the following way:

a) an $(X \downarrow A)$-arrow $\alpha : K \dashrightarrow M$ from a comma $K : X \rightarrow A$ to a module $M : X \rightarrow A$ is given by a cylinder $X \dashrightarrow \ldots M \rightarrow A$. 

b) for a comma morphism \( \psi : \mathcal{J} \to \mathcal{K} : X \to A \) and an \((X \downarrow A)\)-arrow \( \alpha : \mathcal{K} \to \mathcal{M} \) as in

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{K}] \ar[dr]_{\alpha} \ar[dl]^{K_0} & } & A \\
& \mathcal{M} & & \\
\end{array}
\]

, their composite is the \((X \downarrow A)\)-arrow \( \psi \circ \alpha : \mathcal{J} \to \mathcal{M} \) (i.e. the cylinder

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{J}] \ar[dr]_{\alpha} \ar[dl]^{J_0} & } & A \\
& \mathcal{M} & & \\
\end{array}
\]

) defined by

\[\psi \circ \alpha = [\psi] \circ \alpha\]

, the usual composition of a functor and a cylinder (see Definition 4.3.26).

c) for an \((X \downarrow A)\)-arrow \( \alpha : \mathcal{K} \to \mathcal{M} \) and a module morphism \( \psi : \mathcal{M} \to \mathcal{N} : X \to A \) as in

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{M}] \ar[dr]_{\psi} \ar[dl]^{\alpha} & } & A \\
& \mathcal{N} & & \\
\end{array}
\]

, their composite is the \((X \downarrow A)\)-arrow \( \alpha \circ \psi : \mathcal{K} \to \mathcal{N} \) (i.e. the cylinder

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{K}] \ar[dr]_{\psi} \ar[dl]^{K_0} & } & A \\
& \mathcal{N} & & \\
\end{array}
\]

) given by the usual composition of a cylinder and a module morphism (see Definition 4.3.11).

(2) The module \((X \downarrow A) : [X \downarrow A] \to [X \uparrow A]\) is defined in the following way:

a) an \((X \downarrow A)\)-arrow \( \alpha : \mathcal{K} \to \mathcal{M} \) from a comma \( \mathcal{K} : X \to A \) to a collage \( \mathcal{M} : X \to A \) is given by a natural transformation

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{K}] \ar[dr]_{\alpha} \ar[dl]^{K_0} & } & A \\
& \mathcal{M} & & \\
\end{array}
\]

from \(\mathcal{K}_0 \circ \mathcal{M}_0\) to \(\mathcal{M}_1 \circ \mathcal{K}_1\).

b) for a comma morphism \( \psi : \mathcal{J} \to \mathcal{K} : X \to A \) and an \((X \downarrow A)\)-arrow \( \alpha : \mathcal{K} \to \mathcal{M} \) as in

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{J}] \ar[dr]_{\alpha} \ar[dl]^{J_0} & } & A \\
& \mathcal{M} & & \\
\end{array}
\]

, their composite is the \((X \downarrow A)\)-arrow \( \psi \circ \alpha : \mathcal{J} \to \mathcal{M} \) (i.e. the natural transformation

\[
\begin{array}{ccc}
X & \xymatrix{ & [\mathcal{J}] \ar[dr]_{\alpha} \ar[dl]^{J_0} & } & A \\
& \mathcal{M} & & \\
\end{array}
\]

) defined by

\[\psi \circ \alpha = [\psi] \circ \alpha\]

, the usual composition of a functor and a natural transformation.
11.2. Equivalence \([X \downarrow A] \simeq [X : A]\)

\[c)\] for an \((X \downarrow A)\)-arrow \(\alpha : K \to M\) and a collage morphism \(\psi : M \to N : X \to A\) as in
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & M_0 \\
\downarrow \psi & & \downarrow \downarrow \psi \\
X & \xrightarrow{\alpha} & M_1 \\
\downarrow \psi & & \downarrow \downarrow \psi \\
N & & N_1
\end{array}
\]

, their composite is the \((X \downarrow A)\)-arrow \(\alpha \circ \psi : K \to N\) (i.e. the natural transformation
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha \circ \psi} & A \\
\downarrow \psi & & \downarrow \psi \\
N & & N_1
\end{array}
\]
) defined by
\[\alpha \circ \psi = \alpha \circ [\psi]\]

, the usual composition of a natural transformation and a functor.

**Note.** Recall from Theorem 3.1.14 that the functor \([X : A] \downarrow [X \uparrow A]\) is an isomorphism.

11.2.2 Proposition. **The identity**
\[
[X \downarrow A] \xrightarrow{\downarrow \downarrow} [X : A]
\]

holds.

**Proof.** By Remark 4.3.2(2), a cylinder \(\alpha : K_0 \to K_1 : [K] \to M\) is the same thing as a natural transformation \(\alpha : K_0 \circ M_0 \to M_1 \circ K_1 : [K] \to [M]\).

11.2.3 Remark. Because of this identity, we will deal with only the module \((X \downarrow A) : [X \downarrow A] \to [X \uparrow A]\) hereafter. However, any result in this section stated for \((X \uparrow A) : [X \downarrow A] \to [X \uparrow A]\) also holds with \([X \uparrow A]\) changed to \([X : A]\).

11.2.4 Proposition. **Each \((X \uparrow A)\)-arrow \(\alpha : K \to M\) gives**
- \(\uparrow \text{CYL} \text{-arrow} : \alpha : K_0 \to K_1 : [K] \to M\)
- \(\downarrow \text{CYL} \text{-arrow} : \alpha : M_0 \to M_1 : K \to [M]\)

(see Definition 11.1.1), defining the faithful cells:
\[
\begin{array}{ccc}
[X \downarrow A] & \xrightarrow{\downarrow \downarrow} & [X \uparrow A] \\
\downarrow \downarrow & & \downarrow \downarrow \\
\text{CAT} & \xrightarrow{\text{COM}} & \text{CLG}
\end{array}
\]

, where \(\rightarrow\) denotes the canonical embedding in Remark 3.1.6(3) and Remark 3.4.16(2).

**Proof.** Immediate from the definitions.

11.2.5 Proposition. **(1) The unit cylinder of a module \(M : X \to A\) in Definition 11.1.4(1) forms an \((X \downarrow A)\)-arrow \(1^M_M : M \downarrow \to M\).**

**Proof.** Immediate from Remark 11.1.5.
11.2.6 Definition. Given an \((X \uparrow A)\)-arrow \(\alpha : K \to M\) (i.e. a natural transformation)

\[
\begin{array}{c}
X \\
\alpha
\end{array}
\]

(1) the comma adjunct of \(\alpha\) is the comma morphism \(\alpha \downarrow : K \to M\) (i.e. a natural transformation \(\alpha : K_0 \to K_1 \to M\)) defined by the comma adjunct \(\alpha \downarrow : [K] \to [M]\) (see Definition 11.1.10(1)) of the \(\uparrow\)CYL-arrow \(\alpha : K_0 \to K_1 \to M\).

(2) the collage adjunct of \(\alpha\) is the collage morphism \(\alpha \uparrow : K \uparrow \to M\) (i.e. a natural transformation \(\alpha : M_0 \to M_1 \to K\)) defined by the collage adjunct \(\alpha \uparrow : [K] \to [M]\) (see Definition 11.1.10(2)) of the \(\downarrow\)CYL-arrow \(\alpha : M_0 \to M_1 \to K\).

11.2.7 Proposition. For any \((X \uparrow A)\)-arrow \(\alpha : K \to M\),

(1) the comma adjunct \(\alpha \downarrow\) gives the unique comma morphism \(\alpha \downarrow : K \to M\) making the diagram

\[
\begin{array}{c}
X \\
\alpha
\end{array}
\]

commute; the unit cylinder \(1_M\) is therefore an inverse universal \((X \uparrow A)\)-arrow.

(2) the collage adjunct \(\alpha \uparrow\) gives the unique collage morphism \(\alpha \uparrow : K \uparrow \to M\) making the diagram

\[
\begin{array}{c}
X \\
\alpha
\end{array}
\]

commute; the unit cylinder \(1_K\) is therefore a direct universal \((X \uparrow A)\)-arrow.

Proof. By the definitions of \(\alpha \downarrow\) and \(\alpha \uparrow\), this is reduced to Proposition 11.1.12.

Note. The following is the “reflection” of Theorem 11.1.13 along the faithful cells in Proposition 11.2.4.

11.2.8 Theorem.

(1) The functor \([X \downarrow A] \leftarrow [X \uparrow A]\) (see Remark 3.4.24(3)) and the family of unit cylinders \(1_M\) : \(M \to M\), one for each collage \(M : X \to A\), form a counit of the module \((X \downarrow A)\);

(2) The functor \([X \downarrow A] \rightarrow [X \uparrow A]\) (see Remark 3.4.29(2)) and the family of unit cylinders \(1_K\) : \(K \to K\), one for each comma \(K : X \to A\), form a unit of the module \((X \downarrow A)\).

Proof. We have seen in Proposition 11.2.7 the universality of each unit cylinder. It remains to show that the family of unit cylinders \(1_M\) (resp. \(1_K\)) satisfies the naturality condition. But this follows immediately from Proposition 11.1.8.
11.2.9 Remark. The counit and unit in Theorem 11.2.8 are depicted as

\[
\begin{array}{l}
\text{[X \downarrow A]} \xrightarrow{i^1_{\downarrow}} [X \uparrow A] \\
\text{(X\downarrow A)}
\end{array}
\quad \text{and} \quad
\begin{array}{l}
\text{[X \downarrow A]} \xleftarrow{i^1_{\uparrow}} [X \uparrow A] \\
\text{(X\uparrow A)}
\end{array}
\]

respectively.

11.2.10 Theorem. Given a pair of categories \(X\) and \(A\), there exists an adjoint equivalence

\[
[X \downarrow A] \xrightarrow{\eta} [X \uparrow A]
\]

with the unit \(\eta\) and the counit \(\epsilon\) given by the isomorphisms in Theorem 3.4.30.

Proof. This follows by applying Theorem 7.3.15 to the counit and unit of \((X \downarrow A)\) in Remark 11.2.9, and observing that the isomorphism \(\epsilon_M\) (resp. \(\eta_K\)) in Theorem 3.4.30 gives the adjunct of \(1^1_M\) along \(1^1_M\) (resp. \(1^1_K\) along \(1^1_K\)) by the commutativity of the triangle in Proposition 11.1.9. \(\square\)

11.2.11 Corollary. Given a pair of categories \(X\) and \(A\), the module \((X \downarrow A)\) is an equivalence and each unit cylinder is a two-way universal \((X \downarrow A)\)-arrow.

Proof. Since the functors \([X \downarrow A] \xrightarrow{\eta} [X \uparrow A]\) and \([X \downarrow A] \xleftarrow{\epsilon} [X \uparrow A]\) are equivalences as we have just seen in Theorem 11.2.10, the assertion follows by applying Corollary 7.11.10 to the counit and unit of \((X \downarrow A)\) in Remark 11.2.9. \(\square\)

11.3 Equivalence \(\text{COM} \simeq \text{MOD}\)

The purpose of this section is to prove that the functors \(\text{COM} \xrightarrow{\downarrow} \text{MOD}\) and \(\text{MOD} \xrightarrow{\downarrow} \text{COM}\) constructed in Section 3.4 constitute an adjoint equivalence \(\text{COM} \simeq \text{MOD}\). The section is completely analogous to Section 11.2. Again, we work on collages and establish an equivalence \(\text{COM} \simeq \text{CLG}\) instead of \(\text{COM} \simeq \text{MOD}\); the equivalence \([X \downarrow A] \simeq [X \uparrow A]\) established in the previous section is the “reflection” of \(\text{COM} \simeq \text{CLG}\) along the embedding \([X \downarrow A] \hookrightarrow \text{COM}\) and \([X \uparrow A] \hookrightarrow \text{CLG}\) (see Remark 11.3.7).

11.3.1 Definition.

1. The module \(\downarrow \text{CYL} : \text{COM} \to \text{MOD}\) is defined in the following way:
   a) A \(\downarrow \text{CYL}\)-arrow from a comma \(K : Y \to B\) to a module \(M : X \to A\), written \(\alpha : G \Rightarrow F : K \Rightarrow M\), is a triple \((G, \alpha, F)\) consisting of a functor \(G : Y \to X\), a second functor \(F : B \to A\), and a cylinder

   \[
   \begin{array}{c}
   Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
   X \xleftarrow{\alpha} \xrightarrow{F} A
   \end{array}
   \]

   from \(K_0 \circ \alpha\) to \(F \circ K_1\) along \(M\).

   b) For a comma cell \(\psi : P \Rightarrow Q : J \Rightarrow K\) and a \(\downarrow \text{CYL}\)-arrow \(\alpha : G \Rightarrow F : K \Rightarrow M\) as in

   \[
   \begin{array}{c}
   Z \xleftarrow{J_0} [J] \xrightarrow{J_1} C \\
   P \xleftarrow{\psi} \xrightarrow{\eta} \xleftarrow{\alpha} \xrightarrow{F} A
   \end{array}
   \]

   \[
   \begin{array}{c}
   Y \xleftarrow{K_0} [K] \xrightarrow{K_1} B \\
   \end{array}
   \]
(2) The module CYL-arrow \( \psi \circ \alpha : P \circ G \leadsto F \circ Q : \mathcal{J} \leadsto \mathcal{M} \) (i.e. the cylinder

\[
\begin{array}{c}
\mathcal{Y} \xrightarrow{J_0} \mathcal{J} \xrightarrow{J_1} \mathcal{B} \\
\xrightarrow{P \circ G} \psi \circ \alpha \xrightarrow{F \circ Q} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

) defined by

\[
\psi \circ \alpha = [\psi] \circ \alpha
\]

, the usual composition of a functor and a cylinder (see Definition 4.3.26).

(c) for a CYL-arrow \( \alpha : G \leadsto F : \mathcal{K} \leadsto \mathcal{M} \) and a module cell \( \psi : P \leadsto Q : \mathcal{M} \leadsto \mathcal{N} \) as in

\[
\begin{array}{c}
\mathcal{Y} \xleftarrow{K_0} \mathcal{K} \xrightarrow{K_1} \mathcal{B} \\
\xrightarrow{G \circ P} \alpha \xrightarrow{\psi} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

, their composite is the CYL-arrow \( \psi \circ \alpha : G \circ P \leadsto Q \circ F : \mathcal{K} \leadsto \mathcal{N} \) (i.e. the cylinder

\[
\begin{array}{c}
\mathcal{Y} \xleftarrow{K_0} \mathcal{K} \xrightarrow{K_1} \mathcal{B} \\
\xrightarrow{G \circ P} \alpha \xrightarrow{\psi} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

) given by the usual composition of a cylinder and a cell (see Definition 4.3.15).

(2) The module CYL : \text{COM} \rightarrow \text{CLG} is defined in the following way:

(a) a CYL-arrow from a comma \( \mathcal{K} : \mathcal{Y} \rightarrow \mathcal{B} \) to a collage \( \mathcal{M} : \mathcal{X} \rightarrow \mathcal{A} \), written \( \alpha : G \rightarrow F : \mathcal{K} \rightarrow \mathcal{M} \),

is a triple \((G, \alpha, F)\) consisting of a functor \( G : \mathcal{Y} \rightarrow \mathcal{X} \), a second functor \( F : \mathcal{B} \rightarrow \mathcal{A} \), and a natural transformation

\[
\begin{array}{c}
\mathcal{Y} \xleftarrow{K_0} \mathcal{K} \xrightarrow{K_1} \mathcal{B} \\
\xrightarrow{G} \alpha \xrightarrow{\psi} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

from \( K_0 \circ G \circ M_0 \) to \( M_1 \circ F \circ K_1 \).

(b) for a comma cell \( \psi : P \leadsto Q : \mathcal{J} \leadsto \mathcal{K} \) and a CYL-arrow \( \alpha : G \leadsto F : \mathcal{K} \leadsto \mathcal{M} \) as in

\[
\begin{array}{c}
\mathcal{Z} \xrightarrow{J_0} \mathcal{J} \xrightarrow{J_1} \mathcal{C} \\
\xrightarrow{P} \psi \xrightarrow{\psi} \mathcal{Q} \\
\mathcal{Y} \xleftarrow{K_0} \mathcal{K} \xrightarrow{K_1} \mathcal{B} \\
\xrightarrow{G} \alpha \xrightarrow{\psi} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

, their composite is the CYL-arrow \( \psi \circ \alpha : P \circ G \leadsto F \circ Q : \mathcal{J} \leadsto \mathcal{M} \) (i.e. the natural transformation

\[
\begin{array}{c}
\mathcal{Y} \xrightarrow{J_0} \mathcal{J} \xrightarrow{J_1} \mathcal{B} \\
\xrightarrow{P \circ G} \psi \circ \alpha \xrightarrow{F \circ Q} \mathcal{M} \\
\mathcal{X} \xrightarrow{M_0} \mathcal{A}
\end{array}
\]

) defined by

\[
\psi \circ \alpha = [\psi] \circ \alpha
\]

, the usual composition of a functor and a natural transformation.
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c) for a $\Uparrow\text{CYL}$-arrow $\alpha: G \to F: K \to M$ and a collage cell $\psi: P \to Q: M \to N$ as in

\[
\begin{array}{ccc}
Y & \xleftarrow{K_0} & [K] \xrightarrow{K_1} B \\
G & \xleftarrow{\alpha} & F \\
X & \xrightarrow{M_0} & [M] \xrightarrow{M_1} A \\
P & \xrightarrow{\psi} & Q \\
Z & \xrightarrow{N_0} & [N] \xrightarrow{N_1} C
\end{array}
\]

, their composite is the $\Uparrow\text{CYL}$-arrow $\alpha \circ \psi: G \circ P \to Q \circ F: K \to N$ (i.e. the natural transformation

\[
\begin{array}{ccc}
Y & \xleftarrow{K_0} & [K] \xrightarrow{K_1} B \\
G \circ P & \xleftarrow{\alpha \circ \psi} & Q \circ F \\
X & \xrightarrow{N_0} & [N] \xrightarrow{N_1} A
\end{array}
\]

) defined by

\[
\alpha \circ \psi = \alpha \circ [\psi]
\]

, the usual composition of a natural transformation and a functor.

\textbf{Note.} Recall from Theorem 3.1.14 that the functor $\text{MOD} \xrightarrow{\downarrow} \text{CLG}$ is an isomorphism.

\textbf{11.3.2 Proposition.} The identity

\[
\begin{array}{ccc}
\text{COM} & \xrightarrow{\Uparrow\text{CYL}} & \text{MOD} \\
\downarrow & & \downarrow 1 \\
\text{COM} & \xrightarrow{\Uparrow\text{CYL}} & \text{CLG}
\end{array}
\]

holds.

\textbf{Proof.} By Remark 4.3.2(2), a cylinder $\alpha : K_0 \circ G \to F \circ K_1 : [K] \to M$ is the same thing as a natural transformation $\alpha : K_0 \circ G \circ M_0 \to M_1 \circ F \circ K_1 : [K] \to [M]$.

\textbf{11.3.3 Remark.} Because of this identity, we will deal with only the module $\Uparrow\text{CYL} : \text{COM} \to \text{CLG}$ hereafter. However, any result in this section stated for $\Uparrow\text{CYL} : \text{COM} \to \text{CLG}$ also holds with $\text{CLG}$ changed to $\text{MOD}$.

\textbf{11.3.4 Proposition.} Each $\Uparrow\text{CYL}$-arrow $\alpha: G \to F: K \to M$ gives

- $\Uparrow\text{CYL}$-arrow $\alpha : K_0 \circ G \to F \circ K_1 : [K] \to M$

- $\downarrow\text{CYL}$-arrow $\alpha : G \circ M_0 \to M_1 \circ F : K \to [M]$

(see Definition 11.1.1), defining the faithful cells:

\[
\begin{array}{ccc}
\text{COM} & \xrightarrow{\Uparrow\text{CYL}} & \text{CLG} \\
\text{CAT} & \xrightarrow{\Uparrow\text{CYL}} & \text{CLG}
\end{array}
\]

\[
\begin{array}{ccc}
\text{COM} & \xrightarrow{\downarrow\text{CYL}} & \text{CLG} \\
\text{CAT} & \xrightarrow{\downarrow\text{CYL}} & \text{CLG}
\end{array}
\]

\textbf{Proof.} Immediate from the definitions.

\textbf{11.3.5 Proposition.}

1. The unit cylinder of a module $M : X \to A$ in Definition 11.1.4(1) forms a $\Uparrow\text{CYL}$-arrow $1^1_M : 1_X \to 1_A : M \to M$.

2. The unit cylinder of a comma $K : X \to A$ in Definition 11.1.4(2) forms a $\Uparrow\text{CYL}$-arrow $1^1_K : 1_X \to 1_A : K \to K$.

\textbf{Proof.} Immediate from Remark 11.1.5.
11.3.6 Proposition. Each \((X \uparrow A)\)-arrow \(\alpha : K \to M\) (see Definition 11.2.1) gives the \(\uparrow\text{CYL}\)-arrow \(\alpha : 1_X \sim 1_A : K \to M\), defining the faithful cell

\[
\begin{array}{c}
\text{COM} \\
\dashv
\end{array}
\begin{array}{c}
\downarrow \\
\dashv
\end{array}
\begin{array}{c}
\text{CLG}
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
A
\end{array}
\]

\[
\begin{array}{c}
K \\
\downarrow
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
M
\end{array}
\]

, where \(\to\) denotes the canonical embedding in Remark 3.1.6(3) and Remark 3.4.16(2).

Proof. Immediate from the definitions. \(\square\)

11.3.7 Remark. All definitions and results given for the module \(X \uparrow A\) in the previous section are the “reflections” of those given for the module \(\downarrow\text{CYL}\) in this section along the faithful cell above.

11.3.8 Definition. Given a \(\downarrow\text{CYL}\)-arrow \(\alpha : G \sim F : K \to M\) (i.e. a natural transformation

\[
\begin{array}{c}
Y \\
\downarrow
\end{array}
\begin{array}{c}
\to
\end{array}
\begin{array}{c}
B
\end{array}
\]

\[
\begin{array}{c}
G
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
F
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow
\end{array}
\begin{array}{c}
\sim
\end{array}
\begin{array}{c}
M
\end{array}
\]

\[
\begin{array}{c}
M_0 \\
\downarrow
\end{array}
\begin{array}{c}
\sim
\end{array}
\begin{array}{c}
M_1
\end{array}
\]

),

1. the comma adjunct of \(\alpha\) is the comma cell \(\alpha^\downarrow : G \sim F : K \to M^\downarrow\) ( \(\text{Y} \leftarrow \begin{array}{c} K_0 \\
\downarrow \\
\alpha \end{array} \begin{array}{c} K_1 \\
\downarrow \\
F \end{array} \begin{array}{c} B \\
\end{array} \)

\(\text{X} \sim \begin{array}{c} M_0 \\
\downarrow \\
\alpha \end{array} \begin{array}{c} M_1 \\
\downarrow \\
A \end{array}\) ) defined by the comma adjunct \([\alpha] : [K] \to [M]\) (see Definition 11.1.10(1)) of the \(\uparrow\text{CYL}\)-arrow \(\alpha : K_0 \circ G \sim F \circ K_1 : [K] \sim [M]\).

2. the collage adjunct of \(\alpha\) is the collage cell \(\alpha^\uparrow : G \sim F : K^\uparrow \to M\) ( \(\text{Y} \leftarrow \begin{array}{c} K_0 \\
\downarrow \\
\alpha \end{array} \begin{array}{c} K_1 \\
\downarrow \\
F \end{array} \begin{array}{c} B \\
\end{array} \)

\(\text{X} \sim \begin{array}{c} M_0 \\
\downarrow \\
\alpha \end{array} \begin{array}{c} M_1 \\
\downarrow \\
A \end{array}\) ) defined by the collage adjunct \([\alpha] : [K] \to [M]\) (see Definition 11.1.10(2)) of the \(\downarrow\text{CYL}\)-arrow \(\alpha : G \circ M_0 \sim M_1 \circ F : K \sim [M]\).

11.3.9 Proposition. For any \(\uparrow\text{CYL}\)-arrow \(\alpha : G \sim F : K \to M\),

1. the comma adjunct \(\alpha^\downarrow\) gives the unique comma cell \(K \to M^\downarrow\) making the diagram

\[
\begin{array}{c}
M^\downarrow \\
\sim
\end{array}
\begin{array}{c}
\alpha^\downarrow
\end{array}
\begin{array}{c}
K
\end{array}
\]

commute; the unit cylinder \(1^\downarrow_M\) is therefore an inverse universal \(\uparrow\text{CYL}\)-arrow.

2. the collage adjunct \(\alpha^\uparrow\) gives the unique collage cell \(K^\uparrow \to M\) making the diagram

\[
\begin{array}{c}
K \\
\sim
\end{array}
\begin{array}{c}
\alpha^\uparrow
\end{array}
\begin{array}{c}
K^\uparrow
\end{array}
\]

\[
\begin{array}{c}
\sim
\end{array}
\begin{array}{c}
\alpha
\end{array}
\begin{array}{c}
M
\end{array}
\]

commute; the unit cylinder \(1^\uparrow_K\) is therefore a direct universal \(\uparrow\text{CYL}\)-arrow.

Proof. By the definitions of \(\alpha^\downarrow\) and \(\alpha^\uparrow\), this is reduced to Proposition 11.1.12. \(\square\)
11.3.10 Theorem.  
(1) The functor $\text{COM} \xrightarrow{\downarrow} \text{CLG}$ (see Remark 3.4.24(3)) and the family of unit cylinders $1^\downarrow_M : M \to \text{COM}$, one for each comma $M$, form a counit of the module $\downarrow\text{CYL}$.

(2) The functor $\text{COM} \xleftarrow{\uparrow} \text{CLG}$ (see Remark 3.4.29(2)) and the family of unit cylinders $1^\uparrow_K : K \to K'$, one for each comma $K$, form a unit of the module $\uparrow\text{CYL}$.

Proof. We have seen in Proposition 11.3.9 the universality of each unit cylinder. It remains to show that the family of unit cylinders $1^\downarrow_M$ (resp. $1^\uparrow_K$) satisfies the naturality condition. But this follows immediately from Proposition 11.1.8.

11.3.11 Remark. The counit and unit in Theorem 11.3.10 are depicted as

$$
\begin{array}{cc}
\text{COM} & \xrightarrow{\downarrow} & \text{CLG} \\
\xleftarrow{\uparrow} & \downarrow_{\text{CYL}} & \xleftarrow{\uparrow} \\
\end{array}
$$

respectively.

11.3.12 Theorem. There exists an adjoint equivalence

$$
\text{COM} \xleftarrow{\downarrow} \text{CLG}
$$

with the unit $\eta$ and the counit $\epsilon$ given by the isomorphisms in Theorem 3.4.30.

Proof. This follows by applying Theorem 7.3.15 to the counit and unit of $\downarrow\text{CYL}$ in Remark 11.3.11, and observing that the isomorphism $\epsilon_M$ (resp. $\eta_K$) in Theorem 3.4.30 gives the adjunct of $1^\downarrow_M$ along $1^\downarrow_M$ (resp. $1^\uparrow_K$ along $1^\uparrow_K$) by the commutativity of the triangle in Proposition 11.1.9.

11.3.13 Corollary. The module $\downarrow\text{CYL}$ is an equivalence and each unit cylinder is a two-way universal $\downarrow\text{CYL}$-arrow.

Proof. Since the functors $\text{COM} \xrightarrow{\downarrow} \text{CLG}$ and $\text{COM} \xleftarrow{\uparrow} \text{CLG}$ are equivalences as we have just seen in Theorem 11.3.12, the assertion follows by applying Corollary 7.11.10 to the counit and unit of $\downarrow\text{CYL}$ in Remark 11.3.11.

11.4 Equivalences $[X \downarrow] \simeq [X:]$ and $[\downarrow A] \simeq [: A]$  

The purpose of this section is to prove that the functors $[X \downarrow] \xrightarrow{\downarrow} [X:]$ and $[X:] \xrightarrow{\uparrow} [X \downarrow]$ constructed in Section 3.3 constitute an adjoint equivalence $[X \downarrow] \simeq [X:]$. To this end, we define the module $(X \downarrow) : [X \downarrow] \to [X:]$ (an $(X \downarrow)$-arrow from a right comma $K : X \to *$ to a right module $M : X \to *$ is a cone $[X \downarrow] \xrightarrow{\alpha} \text{COM} \xrightarrow{\uparrow} [X:]$) and show that the module is represented by $[X \downarrow] \xrightarrow{\downarrow} [X:]$ and corepresented by $[X:] \xrightarrow{\uparrow} [X \downarrow]$; the adjunction resulting from these representation and corepresentation gives the desired equivalence. The equivalence module $(X \uparrow) : [X \downarrow] \to [X:]$ is used later in Section 12.1 to establish a bijective correspondence between conic cells and cones. The unit of $(X \uparrow)$ consists of a family of unit cones (see Definition 11.4.1); we use them in Section 12.8 to give a characterization of the denseness of a module.

This section is quite analogous to Section 11.2; in fact, the equivalence $[X \downarrow] \simeq [X:]$ is regarded as special cases of the equivalence $[X \downarrow A] \simeq [X : A]$ where $A$ is the terminal category (see Remark 11.4.5).

Note. The following definition is regarded as a special case of Definition 11.1.4 where $A$ [op. $X$] is the terminal category.
11.4.1 Definition.

(1) The unit cone of a right module \( \mathcal{M} : X \to * \) [op. left module \( \mathcal{M} : * \to A \)] is the cone

\[
\begin{array}{c}
\xymatrix{
X \ar@{->}[rr]^\alpha \ar@{-}[dd]_M & & * \\
M \ar@{->}[rr]^\psi & & A \\
\end{array}
\]

defined by

\[ [1^\psi_M]_m = m \]

for \( m \) an arrow of \( \mathcal{M} \).

(2) The unit cone of a right comma \( \mathcal{K} : X \to * \) [op. left comma \( \mathcal{K} : * \to A \)] is the cone

\[
\begin{array}{c}
\xymatrix{
X \ar@{->}[rr]^\alpha \ar@{-}[dd]_K & & * \\
K \ar@{->}[rr]^\psi & & A \\
\end{array}
\]

defined by

\[ [1^\psi_K]_k = k \]

for \( k \) an object of \( \mathcal{K} \).

Note. The following definition is regarded as a special case of Definition 11.2.1 where \( A \) [op. \( X \)] is the terminal category.

11.4.2 Definition. Given a category \( X \) [op. \( A \)], the module

\( \langle X \downarrow \rangle : [X \downarrow] \to [X:] \) [op. \( \langle \uparrow A \rangle : [\uparrow A] \to [:A] \)]

is defined in the following way:

(1) an arrow \( \alpha : \mathcal{K} \to \mathcal{M} \) of \( \langle X \downarrow \rangle \) [op. \( \langle \uparrow A \rangle \)] from a right comma \( \mathcal{K} : X \to * \) [op. left comma \( \mathcal{K} : * \to A \)] to a right module \( \mathcal{M} : X \to * \) [op. left module \( \mathcal{M} : * \to A \)] is given by a cone

\[
\begin{array}{c}
\xymatrix{
X \ar@{->}[rr]^\alpha \ar@{-}[dd]_K & & * \\
\mathcal{K} \ar@{->}[rr]^\psi & & A \\
\end{array}
\]

(2) the composition of \( \alpha : \mathcal{K} \to \mathcal{M} \) with a right [op. left] comma morphism \( \psi : \mathcal{J} \to \mathcal{K} \) as in

\[
\begin{array}{c}
\xymatrix{
X \ar@{->}[rr]^\alpha \ar@{-}[dd]_K & & * \\
\mathcal{K} \ar@{->}[rr]^\psi \ar@{->}[urr] & & \mathcal{J} \\
\end{array}
\]

is given by \([\psi] \circ \alpha\), the usual composition of a functor and a cone (see Definition 4.6.27).

(3) the composition of \( \alpha : \mathcal{K} \to \mathcal{M} \) with a right [op. left] module morphism \( \psi : \mathcal{M} \to \mathcal{N} \) as in

\[
\begin{array}{c}
\xymatrix{
X \ar@{->}[rr]^\alpha \ar@{-}[dd]_M & & * \\
\mathcal{K} \ar@{->}[rr]^\psi \ar@{->}[urr] & & \mathcal{M} \\
\end{array}
\]

is given by \( \alpha \circ \psi \), the usual composition of a cone and a module morphism (see Definition 4.6.11).

11.4.3 Remark.

(1) The unit cone of a right module \( \mathcal{M} : X \to * \) forms an \( \langle X \downarrow \rangle \)-arrow \( 1^\mathcal{M}_{\downarrow} : \mathcal{M} \downarrow \to \mathcal{M} \). Dually, the unit cone of a left module \( \mathcal{M} : * \to A \) forms an \( \langle \uparrow A \rangle \)-arrow \( 1^\mathcal{M}_{\uparrow} : \mathcal{M} \uparrow \to \mathcal{M} \).

(2) The unit cone of a right comma \( \mathcal{K} : X \to * \) forms an \( \langle X \downarrow \rangle \)-arrow \( 1^\mathcal{K}_{\downarrow} : \mathcal{K} \downarrow \to \mathcal{K} \). Dually, the unit cone of a left comma \( \mathcal{K} : * \to A \) forms an \( \langle \uparrow A \rangle \)-arrow \( 1^\mathcal{K}_{\uparrow} : \mathcal{K} \uparrow \to \mathcal{K} \).

Note. We saw the isomorphisms \([X:] \cong [X:*]\) and \([X \downarrow] \cong [X \downarrow *]\) in Remark 1.1.14(4) and Remark 3.4.14.
11.4.4 Proposition. The identity
\[ [X \downarrow] \cong \frac{X}{\Xi} [X : : ] \quad \text{op.} \quad [\downarrow A] \cong \frac{A}{\Lambda} [\cdot : A] \]
holds, giving canonical isomorphism
\[ \langle X \downarrow \rangle \cong \langle X \downarrow * \rangle \quad \text{op.} \quad \langle \downarrow A \rangle \cong \langle * \downarrow A \rangle. \]

Proof. Immediate from the definitions.

11.4.5 Remark. By the isomorphism above, the results obtained in Section 11.2 for the module
\((X \downarrow A)\) are carried over, as a special case where \(A [\text{op. } X]\) is the terminal category, to the module
\((X \downarrow X) [\text{op. } \langle \downarrow A \rangle]\).

Note. Recall from Remark 3.4.24(2) and Remark 3.4.29(1) that the functors \([X : ] \downarrow [X \downarrow]\) and
\([X \downarrow] \downarrow [X : ]\) are special instances of the functors \([X : A] \downarrow [X \downarrow A]\) and \([X \downarrow A] \downarrow [X : A]\) where
\(A\) is the terminal category.

11.4.6 Theorem.

(1) **The functor** \( [X \downarrow] \downarrow [X : ]\) **and the family of unit cones** \(1^1_{\Xi} : \Xi^1 \to \Xi\), **one for each right module** \(\Xi : X \to *, \) **form a unit of the module** \((X \downarrow)\);
- **The functor** \( [\downarrow A] \downarrow [\cdot : A]\) **and the family of unit cones** \(1^1_{\Lambda} : \Lambda^1 \to \Lambda\), **one for each left module** \(\Lambda : * \to A\), **form a unit of the module** \((\downarrow A)\);

(2) **The functor** \( [X \downarrow] \uparrow [X : ]\) **and the family of unit cones** \(1^1_{\Xi} : \Xi \to \Xi^1\), **one for each right comma** \(\Xi : X \to *, \) **form a unit of the module** \((X \uparrow)\);
- **The functor** \( [\downarrow A] \uparrow [\cdot : A]\) **and the family of unit cones** \(1^1_{\Lambda} : \Lambda \to \Lambda^1\), **one for each left comma** \(\Lambda : * \to A\), **form a unit of the module** \((\uparrow A)\).

Proof. This is a special case of Theorem 11.2.8 where \(A [\text{op. } X]\) is the terminal category.

11.4.7 Remark. The counit and unit in Theorem 11.4.6 are depicted as
\[ [X \downarrow] \cong \frac{X}{\Xi} [X : : ] \quad \text{op.} \quad [\downarrow A] \cong \frac{A}{\Lambda} [\cdot : A] \]
and
\[ [X \downarrow] \cong \frac{X}{\Xi} [X : : ] \quad \text{op.} \quad [\downarrow A] \cong \frac{A}{\Lambda} [\cdot : A] \]
respectively.

11.4.8 Theorem. **Given a category** \(X [\text{op. } A]\), **there exists an adjoint equivalence**
\[ [X \downarrow] \cong \frac{(\eta, \epsilon)}{\Xi} [X : : ] \quad \text{op.} \quad [\downarrow A] \cong \frac{(\eta, \epsilon)}{\Lambda} [: A] \]
with the unit \(\eta\) and the counit \(\epsilon\) given by the isomorphisms in Theorem 3.3.22.

Proof. This is a special case of Theorem 11.2.10 where \(A [\text{op. } X]\) is the terminal category.

11.4.9 Corollary.
- **Given a category** \(X\), **the module** \((X \downarrow) : [X \downarrow] \to [X : ]\) **is an equivalence and each unit cone is a two-way universal** \((X \downarrow)\)-arrow.
- **Given a category** \(A\), **the module** \((\downarrow A) : [\downarrow A] \to [: A]\) **is an equivalence and each unit cone is a two-way universal** \((\downarrow A)\)-arrow.
Proof. This is a special case of Corollary 11.2.11 where $A$ [op. $X$] is the terminal category. 

Note. Since the module $(X \downarrow)$ is an equivalence, Theorem 7.13.16 allows the following definition.

11.4.10 Definition. Let $X$ and $A$ be categories.

1. The equivalence cells

\[
\begin{array}{c}
\begin{array}{c}
X \downarrow \xrightarrow{\text{[1]}} [X :] \\
\uparrow 1 \\
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
X \downarrow \xrightarrow{\text{[1]}} [X :] \\
\downarrow 1 \\
\end{array}
\end{array}
\]

are quasi-inverse to each other are defined by

\[
\langle 1 \rangle = \left( [X \downarrow]^{[1]} \right)^{-1}
\quad \text{and} \quad
\langle 1 \rangle = [X \downarrow]^{[1]}
\]

where $[X \downarrow]^{[1]}$ and $[X \downarrow]^{[1]}$ are the module morphisms generated by $[X \downarrow]$ direct along the counit and unit of $(X \downarrow)$ (see Remark 11.4.7).

2. The equivalence cells

\[
\begin{array}{c}
\begin{array}{c}
[\downarrow A] \xrightarrow{\text{[(1)]}} [: A] \\
\uparrow 1 \\
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
[\downarrow A] \xrightarrow{\text{[(1)]}} [: A] \\
\downarrow 1 \\
\end{array}
\end{array}
\]

are quasi-inverse to each other are defined by

\[
\langle 1 \rangle = \left( [\downarrow A]^{[1]} \right)^{-1}
\quad \text{and} \quad
\langle 1 \rangle = [\downarrow A]^{[1]}
\]

where $[\downarrow A]^{[1]}$ and $[\downarrow A]^{[1]}$ are the module morphisms generated by $[\downarrow A]$ direct along the counit and unit of $(\downarrow A)$ (see Remark 11.4.7).

(2) The cell $(X \uparrow) \xrightarrow{i} (X \downarrow)$ [op. $(\uparrow A) \xrightarrow{i} (\downarrow A)$] sends each cone

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X \xleftarrow{\alpha} [K] \\
\downarrow \alpha \\
\end{array}
\end{array}
\end{array}
\quad \text{op.} \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[X :] \\
\uparrow \alpha \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

11.4.11 Remark.

(1) The cell $(X \uparrow) \xrightarrow{i} (X \downarrow)$ [op. $(\uparrow A) \xrightarrow{i} (\downarrow A)$] sends each cone

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X \xleftarrow{\alpha} [K] \\
\downarrow \alpha \\
\end{array}
\end{array}
\end{array}
\quad \text{op.} \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[X :] \\
\uparrow \alpha \\
\end{array}
\end{array}
\end{array}
\end{array}
\]
11.4. Equivalences $[X \downarrow] \simeq [X:]$ and $[\downarrow A] \simeq [:A]$

to the right [op. left] comma morphism

$$\alpha^! : K \to M^! : X \to *$$

with the comma functor

$$X \xleftarrow{\alpha^!} \frac{K}{M} \xrightarrow{[\alpha]} \frac{M}{A}$$

such that

$$[\alpha] \cdot k = \alpha_k \quad [\alpha] \cdot h = K \cdot h$$

for each object $k$ and each arrow $h$ of $[K]$, given as the adjunct of $\alpha$ along the unit cone of $M$ as indicated in

$$\xymatrix{ K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \
\ar@{|->}[u]_{\alpha^!} }$$

(cf. Definition 11.2.6(1)).

(2) The cell $\langle X \downarrow \rangle \xrightarrow{\psi} \langle X \downarrow \rangle$ [op. $\langle \downarrow A \rangle \xrightarrow{\psi} \langle \downarrow A \rangle$] sends each right [op. left] comma morphism

$$\psi : J \to K : X \to *$$

to the right [op. left] cone

$$\xymatrix{ J \ar@{|->}[r]^\psi \ar@{|->}[d]_{i^J} & K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \ar@{|->}[u]_{i^M} \
A \ar@{|->}[u]_{\psi^!} & \ar@{|->}[u]_{\psi^!} & \ar@{|->}[u]_{\psi^!} }$$

, "the collage transpose of $\psi$", such that

$$[\psi^!]_J = [\psi] \cdot j$$

for each object $j$ of $[J]$, given by postcomposition with the unit cone of $K$ as indicated in

$$\xymatrix{ J \ar@{|->}[r]^\psi \ar@{|->}[d]_{i_J} & K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \ar@{|->}[u]_{i^K} \
A \ar@{|->}[u]_{\psi^!} & \ar@{|->}[u]_{\psi^!} & \ar@{|->}[u]_{\psi^!} }$$

(3) The cell $\langle X \downarrow \rangle \xrightarrow{\psi} \langle X \downarrow \rangle$ [op. $\langle \downarrow A \rangle \xrightarrow{\psi} \langle \downarrow A \rangle$] sends each cone

$$\xymatrix{ K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & A \
\ar@{|->}[u]_{\alpha^!} & \ar@{|->}[u]_{\alpha^!} & \ar@{|->}[u]_{\alpha^!} }$$

to the right [op. left] module morphism

$$\alpha^! : K^! \to M : X \to *$$

such that

$$\alpha^! \cdot k = \alpha_k$$

for each object $k$ of $[K]$, given as the adjunct of $\alpha$ along the unit cone of $K$ as indicated in

$$\xymatrix{ K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \ar@{|->}[d]_{i^K} \
A \ar@{|->}[u]_{\alpha^!} & \ar@{|->}[u]_{\alpha^!} }$$

(cf. Definition 11.2.6(2)).

(4) The cell $\langle X \downarrow \rangle \xrightarrow{\psi} \langle X \downarrow \rangle$ [op. $\langle \downarrow A \rangle \xrightarrow{\psi} \langle \downarrow A \rangle$] sends each right [op. left] module morphism

$$\psi : M \to N : X \to *$$

such that

$$\psi \cdot k = \psi_k$$

for each object $k$ of $[K]$, given as the adjunct of $\psi$ along the unit cone of $K$ as indicated in

$$\xymatrix{ K \ar@{|->}[r]^\alpha \ar@{|->}[d]_{i^K} & M \ar@{|->}[d]_{i^K} \
A \ar@{|->}[u]_{\psi^!} & \ar@{|->}[u]_{\psi^!} }$$
to the cone

\[
\begin{array}{c}
X \xrightarrow{\psi} M \xrightarrow{\downarrow} A
\end{array}
\]

, “the comma transpose of \( \psi \)”, such that

\[
\left[ \psi^i \right]_m = \psi \cdot m
\]

for each \( \mathcal{M} \)-arrow \( m \), given by precomposition with the unit cone of \( \mathcal{M} \) as indicated in

\[
\mathcal{M}^i \xrightarrow{\psi^i} \mathcal{M} \xrightarrow{\downarrow} \mathcal{N}
\]

### 11.5 coYoneda embedding

The equivalence \([X:] \simeq [X \downarrow] \) we established in Section 11.4 turns the Yoneda embedding into the coYoneda embedding.

#### 11.5.1 Definition. Given a module \( \mathcal{M} : X \to A \), the functor

\[
[\mathcal{M}^\varphi] : A \to [X \downarrow] \quad \text{op.} \quad [\varphi \mathcal{M}] : X \to [\downarrow A]^\varphi
\]

is defined by the composition

\[
A \xrightarrow{\mathcal{M}^\varphi} [X:] \xrightarrow{\downarrow} [X \downarrow] \quad \text{op.} \quad X \xrightarrow{\mathcal{M}^\varphi} [: A]^\varphi \xrightarrow{\downarrow} [\downarrow A]^\varphi
\]

of the right [op. left] exponential transpose of \( \mathcal{M} \) and the right adjoint in Theorem 11.4.8, so that \( \mathcal{M}^\varphi \) [op. \( \mathcal{M}^\varphi \)] sends each object

\[
a \in \| A \| \quad \text{op.} \quad x \in \| X \|
\]

to the comma

\[
\langle \mathcal{M} a \rangle^\dagger : X \to *[\mathcal{M} a] \quad \text{op.} \quad \langle x \mathcal{M} \rangle^\dagger : * \to A
\]

of the right [op. left] module

\[
\mathcal{M} a : X \to *[\mathcal{M} a] \quad \text{op.} \quad x \mathcal{M} : * \to A
\]

(the right [op. left] slice of \( \mathcal{M} \) at \( a \in \| A \| \) [op. \( x \in \| X \| \) ).

#### 11.5.2 Remark. By Theorem 11.4.8, the diagram

\[
\begin{array}{c}
X \xrightarrow{\mathcal{M}^\varphi} [X:] \xrightarrow{\downarrow} [X \downarrow] \quad \text{op.} \quad X \xrightarrow{\mathcal{M}^\varphi} [: A]^\varphi \xrightarrow{\downarrow} [\downarrow A]^\varphi
\end{array}
\]

commutes up to isomorphism.

**Note.** The following definition is a special case of Definition 11.5.1 where \( \mathcal{M} \) is given by the homomodule of a category.

#### 11.5.3 Definition. Given a category \( X \) [op. \( A \)], the right [op. left] coYoneda functor

\[
[X^\varphi] : X \to [X \downarrow] \quad \text{op.} \quad [\varphi A] : A \to [\downarrow A]^\varphi
\]

is defined by the composition

\[
X \xrightarrow{X^\varphi} [X:] \xrightarrow{\downarrow} [X \downarrow] \quad \text{op.} \quad A \xrightarrow{\varphi A} [: A]^\varphi \xrightarrow{\downarrow} [\downarrow A]^\varphi
\]

of the right [op. left] Yoneda functor and the equivalence functor \([X:] \xrightarrow{\downarrow} [X \downarrow] \) [op. \( [: A] \xrightarrow{\downarrow} [\downarrow A] \)], so that \( X^\varphi \) [op. \( \varphi A \)] sends each object

\[
x \in \| X \| \quad \text{op.} \quad a \in \| A \|
\]
11.6 Slice category over a module

The purpose of this section is to prove Theorem 11.6.8, which asserts the equivalence \([X:] / \mathcal{M} \simeq [\mathcal{M}] : \) for a right module \(\mathcal{M} : X \to *\) (see Remark 11.6.9 for its implication); we first present the isomorphism \([X \downarrow] / K \simeq [K] \downarrow\) for a right comma \(K : X \to *\), and show that the asserted equivalence translates into this isomorphism via the equivalence \([X \downarrow] \simeq [X :]\) in Section 11.4.

Note. Recall from Remark 3.3.12(1) that the category \([X \downarrow]\) of right commas over a category \(X\) is the same thing as the slice category of CFR over \(X\).

11.6.1 Definition. Given a fibre-small right comma fibration \(P : X \to Y\), the postcomposition functor \(\text{CFR}/P : \text{CFR}/X \to \text{CFR}/Y\) (cf. Preliminary 0.0.8) is denoted by \([P \downarrow] : [X \downarrow] \to [Y \downarrow]\).

Note. Proposition 3.3.3 allows the following assertion.

11.6.2 Proposition. Given a right comma \(K : X \to *\), there is the canonical isomorphism

\[\Sigma_K : [X \downarrow] / K \to [[K] \downarrow]\]
between the slice category over \( K \) and the category of right commas over the comma category \([K]\) such that the diagram

\[
\begin{array}{ccc}
[X \downarrow]_{/K} & \xrightarrow{\Sigma X/K} & [[K]\downarrow] \\
\Sigma_k & \searrow & [K]\downarrow \\
X \downarrow & \swarrow & \end{array}
\]

commutes, where \([K]\downarrow\) denotes the postcomposition with the right comma fibration \( K : [K] \to X \).

**Proof.** Since the category \([K]\downarrow\) is the same thing as the slice category \( CFR/X \) and the functor \([K]\downarrow : [[K]\downarrow] \to [X \downarrow]\) is the same thing as the postcomposition functor \( CFR/K : CFR/[[K] \to CFR/X\), the assertion is just an instance of Proposition 3.3.10. \(\square\)

**Note.** Since the category \( CFR \) is locally cartesian as we saw in Theorem 9.1.29, it admits pullback functors (see Definition 9.1.17).

**11.6.3 Definition.** The pullback functor along a fibre-small right comma fibration \( P : X \to Y \) is denoted by \( P^* : [Y \downarrow] \to [X \downarrow] \).

**11.6.4 Remark.** By Proposition 9.1.19, the pullback functor \( P^* : [Y \downarrow] \to [X \downarrow] \) is right adjoint to the postcomposition functor \( [P \downarrow] : [X \downarrow] \to [Y \downarrow] \) defined in Definition 11.6.1.

**11.6.5 Proposition.** Given a fibre-small right comma fibration \( P : X \to Y \), the diagram

\[
\begin{array}{ccc}
[X \downarrow] & \xrightarrow{p} & [X :] \\
\downarrow\downarrow P^* & \searrow & \downarrow\downarrow [P] \\
[Y \downarrow] & \xrightarrow{p} & [Y :] \\
\end{array}
\]

commutes; that is, the pullback functor \( P^* \) is conjugate (see Definition 7.12.3) to the precomposition functor \( [P :] \) along the equivalences \([X \downarrow] \simeq [X :]\) and \([Y \downarrow] \simeq [Y :]\) (see Theorem 11.4.8).

**Proof.** This follows from Theorem 3.3.18. \(\square\)

**11.6.6 Definition.** Given a fibre-small right comma fibration \( P : X \to Y \), the functor \( P_! : [X :] \to [Y :] \) is defined by the conjugate (see Definition 7.12.2) of the postcomposition functor \( [P \downarrow] : [Y \downarrow] \to [X \downarrow] \) along the equivalences \([X \downarrow] \simeq [X :]\) and \([Y \downarrow] \simeq [Y :]\) (see Theorem 11.4.8) such that the diagram

\[
\begin{array}{ccc}
[X \downarrow] & \xleftarrow{p} & [X :] \\
\downarrow\downarrow [P_!] & \searrow & \downarrow\downarrow [P] \\
[Y \downarrow] & \xrightarrow{p} & [Y :] \\
\end{array}
\]

commutes.

**11.6.7 Proposition.** Given a fibre-small right comma fibration \( P : X \to Y \), the functor \( P_! : [X :] \to [Y :] \) is left adjoint to the precomposition functor \( [P :] : [Y :] \to [X :] \) so that the diagram

\[
\begin{array}{ccc}
[X \downarrow] & \xrightarrow{p} & [X :] \\
\downarrow\downarrow [P_i] & \searrow & \downarrow\downarrow [P] \\
[Y \downarrow] & \xrightarrow{p} & [Y :] \\
\end{array}
\]

commutes up to isomorphism, where the adjunctions on the top and bottom are the equivalences in Theorem 11.4.8 and the adjunction on the left is that noted in Remark 11.6.4.

**Proof.** Since (recall Definition 11.6.6 and Proposition 11.6.5) \( P_i \) and \( [P :] \) are respectively the conjugates of \([P \downarrow]\) and \( P^* \) along \([X \downarrow] \simeq [X :]\) and \([Y \downarrow] \simeq [Y :]\), the assertion follows from Theorem 7.12.6. \(\square\)
11.6.8 Theorem. Given a right module $\mathcal{M}: X \to \ast$, there is the canonical equivalence

$[X:] / \mathcal{M} \simeq [[\mathcal{M}] :]$

between the slice category over $\mathcal{M}$ and the category of right [op. left] modules over the comma category $[\mathcal{M}]$ such that the diagram

$\begin{array}{c}
[X:] / \mathcal{M} \\
\downarrow \Sigma_{\mathcal{M}}
\end{array} \Rightarrow
\begin{array}{c}
[[\mathcal{M}] :] \\
\downarrow [\mathcal{M}]_{i}
\end{array}$

commute, where $[\mathcal{M}^i :]$ denotes the left adjoint (see Proposition 11.6.7) of the precomposition functor $[\mathcal{M}^i :] : [X :] \to [[\mathcal{M}] :]$ (precomposition with the right comma fibration $\mathcal{M}^i : [\mathcal{M}] \to X$).

Proof. By Proposition 11.6.2, there is the canonical isomorphism $\Sigma_{X / \mathcal{M}^i} : [X \downarrow] / \mathcal{M}^i \to [[\mathcal{M}] \downarrow]$ making the diagram

$\begin{array}{c}
[X \downarrow] / \mathcal{M}^i \\
\downarrow \Sigma_{\mathcal{M}^i}
\end{array} \Rightarrow
\begin{array}{c}
[[\mathcal{M}] \downarrow] \\
\downarrow [\mathcal{M}]_{i}
\end{array}$

commute. Now the required equivalence is given by the conjugate of the isomorphism $\Sigma_{X / \mathcal{M}^i}$ as shown in the diagram

$\begin{array}{c}
[X \downarrow] / \mathcal{M}^i \\
\downarrow \Sigma_{\mathcal{M}^i}
\end{array} \Rightarrow
\begin{array}{c}
[[\mathcal{M}] \downarrow] \\
\downarrow [\mathcal{M}]_{i}
\end{array}$

, where $\downarrow / \mathcal{M}$ is the slice functor (see Preliminary 0.0.8) of the equivalence $[X :] \downarrow [X \downarrow]$ (see Theorem 11.4.8) over $\mathcal{M}$ (note that a slice of an equivalence is again an equivalence). Since all squares in the prism commute up to isomorphism, the commutativity (up to isomorphism) of the right-hand triangle follows from the commutativity of the left-hand triangle.

11.6.9 Remark. Theorem 11.6.8 implies that a “non-evil” property of the category $[X :]$ that holds for arbitrary $X$ is stable under slicing.

Note. The following is an example of Remark 11.6.9.

11.6.10 Theorem. For any small category $X$, the category $[X :]$ of right modules over $X$ is locally cartesian closed.

Proof. We saw in Theorem 9.5.3 that $[X :]$ is cartesian closed for any small category $X$. Now let $\mathcal{M}: X \to \ast$ be a small right module. Then $[X :] / \mathcal{M} \simeq [[\mathcal{M}] :]$ by Theorem 11.6.8. Since the smallness of $\mathcal{M}$ implies the smallness of $[\mathcal{M}]$, the category $[[\mathcal{M}] :]$ is again cartesian closed, and hence so is $[X :] / \mathcal{M}$ by Proposition 9.3.6.
12 Extensions

12.1 Degenerate cells

In this section, we study the corepresentable module \( (M \triangleright) : [X :] \rightarrow A \) of the functor \( \mathcal{M} : A \rightarrow [X :] \) introduced in Section 2.1, and the corepresentable module \( (M \triangleright E) : [X : E] \rightarrow [E, A] \) of the functor \( \mathcal{M} \triangleright E : [E, A] \rightarrow [X : E] \) introduced in Section 2.2. Interesting facts about these corepresentable modules are that the arrows of \( M \triangleright \) are (degenerate) conic cells and the arrows of \( M \triangleright E \) are (degenerate) two-sided module cells. In Section 12.3, a limit is defined by a universal arrow of the module \( M \triangleright \), and in Section 12.4, an extension is defined by a universal arrow of the module \( M \triangleright E \).

12.1.1 Definition. Let \( M : \mathcal{X} \rightarrow A \) be a module.

- Given a right module \( J : \mathcal{X} \rightarrow \ast \) and an object \( a \in A \), a (degenerate) conic cell \( \theta \) from \( J \) to \( a \) along \( M \), written as \( \theta : J \rightharpoonup a : M \triangleright A \), or diagrammatically as

\[
\begin{array}{ccc}
X & \rightharpoonup & A \\
\downarrow & & \downarrow \\
J & \rightharpoonup & a
\end{array}
\]

is defined by a right module morphism \( \theta : J \rightarrow (M)a : \mathcal{X} \rightarrow \ast \).

- Given a left module \( J : \ast \rightarrow A \) and an object \( x \in \mathcal{X} \), a (degenerate) conic cell \( \theta \) from \( x \) to \( J \) along \( M \), written as \( \theta : x \rightharpoonup J : \mathcal{X} \rightarrow M \), or diagrammatically as

\[
\begin{array}{ccc}
X & \rightharpoonup & A \\
\downarrow & & \downarrow \\
x & \rightharpoonup & J
\end{array}
\]

is defined by a left module morphism \( \theta : J \rightarrow x(M) : \ast \rightarrow A \).

12.1.2 Remark.

1. A conic cell defined in Definition 1.4.1 is called an ordinary conic cell to distinguish it from a degenerate cell defined in Definition 12.1.1. However, the adjectives “ordinary” and “degenerate” are often omitted when the context makes it clear which type of cell is being talked about.

2. In fact, a degenerate conic cell

\[
\begin{array}{ccc}
J & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
X & \rightharpoonup & A
\end{array}
\quad \text{op.}
\begin{array}{ccc}
E & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
X & \rightharpoonup & A
\end{array}
\]

is depicted as an ordinary right conic cell

\[
\begin{array}{ccc}
X & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
J & \rightharpoonup & a
\end{array}
\quad \text{op.}
\begin{array}{ccc}
* & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
x & \rightharpoonup & J
\end{array}
\]

defined by the right [op. left] module morphism

\[
\theta : J \rightarrow (M)a : \mathcal{X} \rightarrow \ast \quad \text{op.}
\theta : J \rightarrow x(M) : \ast \rightarrow A.
\]

3. Conversely, an ordinary right conic cell

\[
\begin{array}{ccc}
E & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
X & \rightharpoonup & A
\end{array}
\quad \text{op.}
\begin{array}{ccc}
X & \rightharpoonup & \ast \\
\downarrow & & \downarrow \\
E & \rightharpoonup & F
\end{array}
\]

is being defined by a right module morphism

\[
\theta : J \rightarrow G(M)a : E \rightarrow \ast \quad \text{op.}
\theta : J \rightarrow x(M)F : \ast \rightarrow A.
\]
12.1.4 Remark. A conic cell defined in Definition 12.1.1 is in fact given as an arrow of the representable module defined below.

12.1.3 Definition. Let \( \mathcal{M} \colon X \to A \) be a module.

- The corepresentable module of the right exponential transpose of \( \mathcal{M} \) is denoted by \( \mathcal{M}^\gamma \); that is, the module
  \[
  \langle \mathcal{M}^\gamma \rangle : [X:] \to A
  \]
  is defined by the composition
  \[
  [X:] \xrightarrow{\langle X^\gamma \rangle} [X:] \xrightarrow{\mathcal{M}^\gamma} A.
  \]

- The representable module of the left exponential transpose of \( \mathcal{M} \) is denoted by \( \& \mathcal{M} \); that is, the module
  \[
  \langle \& \mathcal{M} \rangle : X \to [A]^\gamma
  \]
  is defined by the composition
  \[
  X \xrightarrow{\& \mathcal{M}} [A]^\gamma = [A]^{\langle A \rangle^\gamma}.
  \]

12.1.4 Remark.

1. For a right module \( \mathcal{J} : X \to * \) and an object \( a \in \| A \| \), the set
   \[
   (\mathcal{J} \langle \mathcal{M}^\gamma \rangle)(a) = (\mathcal{J} \langle X^\gamma \rangle)((\mathcal{M}^\gamma) a)
   \]
   consists of all right module morphisms \( \mathcal{J} \to \langle \mathcal{M}^\gamma \rangle a : X \to * \), i.e. all degenerate conic cells \( \mathcal{J} \to \mathcal{M}^\gamma \) defined in Definition 12.1.1, and for a left module \( \mathcal{J} : * \to A \) and an object \( x \in \| X \| \), the set
   \[
   (x) \langle \& \mathcal{M} \rangle (\mathcal{J}) = (\mathcal{J} \langle A \rangle) (x \langle \mathcal{M} \rangle)
   \]
   consists of all left module morphisms \( \mathcal{J} \to x \langle \mathcal{M} \rangle : * \to A \), i.e. all degenerate conic cells \( x \to \mathcal{J} : \& \mathcal{M} \).

2. With the notation introduced above, the right [op. left] Yoneda morphism for \( \mathcal{M} \) is depicted as
   \[
   X \xrightarrow{\mathcal{M}^\gamma} A
   \]
   so that it sends each \( \mathcal{M} \)-arrows \( m : x \sim a \) to the conic cell
   \[
   X \xrightarrow{\mathcal{M}^\gamma} A
   \]
   ; with this depiction of the Yoneda morphism, Theorem 5.2.10 is restated as follows:

- for any pair of objects \( x \in \| X \| \) and \( a \in \| A \| \), the assignment \( m \mapsto X \downarrow m \) yields a bijection
  \[
  x \langle \mathcal{M} \rangle a \cong (x \langle X \rangle) \langle \mathcal{M}^\gamma \rangle (a)
  \]
  from the set of \( \mathcal{M} \)-arrows \( x \sim a \) to the set of conic cells \( \langle X \rangle \sim a \) along \( \mathcal{M} \); moreover, the bijection is natural in \( x \) and \( a \).

- for any pair of objects \( x \in \| X \| \) and \( a \in \| A \| \), the assignment \( m \mapsto m \downarrow A \) yields a bijection
  \[
  x \langle \& \mathcal{M} \rangle a \cong (x \langle \& \mathcal{M} \rangle (a \langle A \rangle))
  \]
  from the set of \( \mathcal{M} \)-arrows \( x \sim a \) to the set of conic cells \( x \sim a \langle A \rangle \) along \( \mathcal{M} \); moreover, the bijection is natural in \( x \) and \( a \).
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(3) For any module $\mathcal{M}$,
$$\langle\mathcal{M} \mathcal{M}\rangle = \langle \mathcal{M} \mathcal{M}\rangle.$$

(4) The assignment $\mathcal{M} \mapsto \mathcal{M} \mathcal{M}$ [op. $\mathcal{M} \mapsto \mathcal{M} \mathcal{M}$] extends to a functor; indeed,

- the functor
$$[X : A] \rightarrow [[X : ] : A]$$

from the category of modules $X \rightarrow A$ to the category of modules $[[X : ] : A]$ is defined by the composition

of the right exponential transposition and the right generalized Yoneda functor for the functor category $[A, [X : ]]$; the object function of the functor $[X : A] \rightarrow [[X : ] : A]$ sends each module $\mathcal{M} : X \rightarrow A$ to the module $\langle\mathcal{M} \mathcal{M}\rangle : [X : ] \rightarrow A$.

- the functor
$$[X : A] \rightarrow [X : [A ]^-]$$

from the category of modules $X \rightarrow A$ to the category of modules $X \rightarrow [A ]^-$ is defined by the composition
$$[X : A] \rightarrow [X, [A ]^-] \rightarrow [X : [A ]^-]$$

of the left exponential transposition and the left generalized Yoneda functor for the functor category $[X, [A ]^-]$; the object function of the functor $[X : A] \rightarrow [X : [A ]^-]$ sends each module $\mathcal{M} : X \rightarrow A$ to the module $\langle\mathcal{M} \mathcal{M}\rangle : X \rightarrow [A ]^-$.

Note. The following definition is a special case of Definition 12.1.1 where $\mathcal{M}$ is given by the homomodule of a category.

12.1.5 Definition.

- Given a right module $\mathcal{M} : X \rightarrow *$ and an object $x \in (X)$, a (degenerate) conic cell $\theta$ from $\mathcal{M}$ to $x$, written as $\theta : \mathcal{M} \rightarrow x : X \mathcal{M}$, or diagrammatically as $\xymatrix{\mathcal{M} \ar@{->}[r]^-{\theta} \ar@{->}[d]_{X} & x \ar@{->}[d]_{X} \ar@{->}[l]^{X} \ar@{->}[r]_{X} & X}$, is defined by a right module morphism $\theta : \mathcal{M} \rightarrow (X) x : X \rightarrow *$.

- Given a left module $\mathcal{M} : * \rightarrow A$ and an object $a \in \mathcal{M} A$, a (degenerate) conic cell $\theta$ from $a$ to $\mathcal{M}$, written as $\theta : a \rightarrow \mathcal{A} : \mathcal{A}$, or diagrammatically as $\xymatrix{a \ar@{->}[r]_{\theta} \ar@{->}[d]_{a} & \mathcal{A} \ar@{->}[d]_{\mathcal{A}} \ar@{->}[l]^{\mathcal{A}} \ar@{->}[r]_{\mathcal{A}} & \mathcal{A}}$, is defined by a left module morphism $\theta : \mathcal{M} \rightarrow a(\mathcal{A}) : * \rightarrow A$.

Note. The following is a special case of Definition 12.1.3 where $\mathcal{M}$ is given by the homomodule of a category; a conic cell defined in Definition 12.1.5 is given as an arrow of the representable module defined below.

12.1.6 Definition.

- The corenrepresentable module of the right Yoneda functor for a category $X$ is denoted by $X \mathcal{M}$; that is, the module
$$\langle X \mathcal{M}\rangle : [X : ] \rightarrow X$$

is given by the composition
$$[X : ] \rightarrow [X, [X : ]] \rightarrow X$$

- The representable module of the left Yoneda functor for a category $\mathcal{A}$ is denoted by $\mathcal{A} \mathcal{A}$; that is, the module
$$\langle \mathcal{A} \mathcal{A}\rangle : A \rightarrow [A ]^-$$

is given by the composition
12.1.7 Remark.
(1) The identity
\[
\langle X \varnothing \rangle = \langle (X) \varnothing \rangle \quad \text{op.} \quad \langle \varnothing A \rangle = \langle \varnothing \langle A \rangle \rangle
\]
follows from the identity
\[
[X \varnothing] = [(X) \varnothing] \quad \text{op.} \quad [\varnothing A] = [\varnothing \langle A \rangle]
\]
in Definition 2.3.1. Hence the module \( X \varnothing \) [op. \( \varnothing A \)] is a special instance of a module \( M \varnothing \) [op. \( \varnothing M \)] defined in Definition 12.1.3 where \( M \) is given by the hom-module of \( X \) [op. \( A \)]

(2) For a right module \( M : X \rightarrow A \) and an object \( x \in \| X \| \), the set
\[
(X \varnothing)(x) = (M)(X : (x) a)
\]
consists of all right module morphisms \( M \rightarrow (X) x : X \rightarrow * \), i.e. all degenerate conic cells \( M \rightarrow x : X \varnothing \) defined in Definition 12.1.5, and for a left module \( M : * \rightarrow A \) and an object \( a \in \| A \| \), the set
\[
a(\varnothing A)(M) = (M)(A) (a(A))
\]
consists of all left module morphisms \( M \rightarrow a(A) : * \rightarrow \), i.e. all degenerate conic cells \( a \rightarrow M : \varnothing A \)

12.1.8 Definition. Let \( E \) be a category and \( M : X \rightarrow A \) be a module.
- Given a module \( J : X \rightarrow E \) and a functor \( F : E \rightarrow A \), a (degenerate) cell \( \theta \) from \( J \) to \( F \) along \( M \), written as \( \theta : J \rightarrow F : M \varnothing E \), or diagrammatically as
\[
\begin{array}{cccc}
X & \xrightarrow{\theta} & \rightarrow M \rightarrow & A \\
F & & & \\
\end{array}
\]

- Given a module \( J : E \rightarrow A \) and a functor \( G : E \rightarrow X \), a (degenerate) cell \( \theta \) from \( G \) to \( J \) along \( M \), written as \( \theta : G \rightarrow J : E \varnothing M \), or diagrammatically as
\[
\begin{array}{cccc}
X & \xleftarrow{\theta} & \triangleright M \triangleleft & A \\
\rightarrow & & & \\
\end{array}
\]

12.1.9 Remark. Remarks similar to those in Remark 12.1.2 also apply to two-sided module cells.
(1) Where ambiguity is possible, the adjective “degenerate” is used to distinguish a cell defined in Definition 12.1.8 from an “ordinary” cell defined in Definition 12.1.1.

(2) A degenerate cell
\[
\begin{array}{cc}
J & \xrightarrow{\theta} \rightarrow E \\
X & \xleftarrow{\varnothing} & \triangleright M \triangleleft & A \\
\end{array}
\]

is in fact depicted as an ordinary cell
\[
\begin{array}{cc}
X & \xleftarrow{\varnothing} & \triangleright E \triangleleft \\
1 & \triangleright F & \\
\end{array}
\]

\[
\begin{array}{cc}
X & \xrightarrow{\varnothing} & \triangleright M \triangleleft \\
\end{array}
\]

defined by the module morphism
\[
\theta : J \rightarrow (M) F : X \rightarrow E \quad \text{op.} \quad \theta : J \rightarrow G(M) : E \rightarrow A.
\]

(3) Conversely, an ordinary cell
\[
\begin{array}{cc}
D & \xrightarrow{\varnothing} \rightarrow E \\
G & \triangleright F & \\
X & \xleftarrow{\varnothing} & \triangleright M \triangleleft \\
\end{array}
\]

, being defined by a module morphism
\[
\theta : J \rightarrow G(M) F : D \rightarrow E
\]
12.1. Degenerate cells

, is depicted as a degenerate cell

\[ \xymatrix{ D \ar[r]^G \ar[d]_\theta & A \ar[d]_\theta \ar[r]^{\langle M \rangle F} & X \ar[r]^{\theta F} \ar[d]_\theta & \langle M \rangle \ar[d]_\theta & E \ar[r]^F & E \ar[ld]^G } \]

along the composite module \( G \langle M \rangle [\text{op. } (M) F] \).

(4) A conic cell \[ \xymatrix{ \mathcal{J} \ar[r]^a & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^a & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^a & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^a & \mathcal{J} } \]
in Definition 12.1.1 is identified with a special instance of a cell in Definition 12.1.8 where \( E \) is the terminal category.

Note. A cell defined in Definition 12.1.8 is in fact given as an arrow of the representable module defined below.

12.1.10 Definition. Let \( E \) be a category and \( M : X \to A \) be a module.

- The corepresentable module of the right action of \( M \) on \([E, A]\) is denoted by \( M \not\rightharpoonup E \); that is, the module \( \langle M \not\rightharpoonup E \rangle : [X : E] \to [E, A] \)
is defined by the composition \( [X : E] \xrightarrow{(X, E)} [X : E] \xrightarrow{M \not\rightharpoonup E} [E, A] \).

- The representable module of the left action of \( M \) on \([E, X]\) is denoted by \( E \not\rightharpoonup M \); that is, the module \( \langle E \not\rightharpoonup M \rangle : [E, X] \to [E : A]^\rightharpoonup \)
is defined by the composition \( [E, X] \xrightarrow{E \not\rightharpoonup M} [E : A]^\rightharpoonup \xrightarrow{(E : A)^\rightharpoonup} [E : A]^\rightharpoonup \).

12.1.11 Remark.

(1) For a module \( \mathcal{J} : X \to E \) and a functor \( F : E \to A \), the set \( (\mathcal{J}) \langle M \not\rightharpoonup E \rangle (F) = (\mathcal{J}) (X : E) (\langle M \rangle F) \)
consists of all module morphisms \( \mathcal{J} \to \langle M \rangle F : X \to E \), i.e. all degenerate cells \( \mathcal{J} \rightharpoonup F : M \not\rightharpoonup \)
defined in Definition 12.1.8, and for a module \( \mathcal{J} : E \rightharpoonup A \) and a functor \( G : E \rightharpoonup X \), the set \( (G) \langle E \not\rightharpoonup M \rangle (\mathcal{J}) = (\mathcal{J}) (E : A) (G \langle M \rangle) \)
consists of all module morphisms \( \mathcal{J} \to G \langle M \rangle : E \rightharpoonup A \), i.e. all degenerate cells \( G \rightharpoonup \mathcal{J} : E \not\rightharpoonup M \).

(2) The right [op. left] exponential transpose of a cell

\[ \xymatrix{ \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} } \]

, i.e. a module morphism \( \theta : \mathcal{J} \to \langle M \rangle F : X \to E \)

is a natural transformation

\[ \xymatrix{ [X : ] \ar[r]^{\theta \rightharpoonup} & A \ar[d]_\theta \ar[r]^{\langle M \rangle F} & X \ar[d]_\theta \ar[r]^{E \rightharpoonup} & [ : A]^\rightharpoonup } \]

(see Proposition 2.1.6), i.e. a cylinder

\[ \xymatrix{ \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} \ar[d]_\theta & \mathcal{J} \ar[r]^E & \mathcal{J} } \]

\[ \xymatrix{ [X : ] \ar[r]^{\theta \not\rightharpoonup} & A \ar[d]_\theta \ar[r]^{\langle M \rangle \rightharpoonup} & X \ar[d]_\theta \ar[r]^{E \not\rightharpoonup} & [ : A]^\rightharpoonup } \]
12.1. Degenerate cells

(4) The module for any category

12.1.12 Definition. module of a category.

Note.

12.1.13 Definition. module defined below. module of a category; a cell defined in Definition 12.1.12 is given as an arrow of the representable

Note. The following definition is a special case of Definition 12.1.10 where \( \mathcal{M} \) is identified with \( (\mathcal{M}, \mathcal{M}) \).

(3) For any category \( \mathcal{E} \) and any module \( \mathcal{M} \),

\[
(\mathcal{M} \otimes \mathcal{E}) \cong (\mathcal{E}, (\mathcal{M} \otimes \mathcal{M})).
\]

(4) The module

in Definition 12.1.3 is identified with

\[
(\mathcal{M} \otimes \ast) : [\mathcal{X} : \ast] \to [\ast, \mathcal{A}]
\]

; that is, the module \( (\mathcal{M} \otimes \ast) \) is regarded as a special instance of a more general \( (\mathcal{M} \otimes \mathcal{E}) \) where \( \mathcal{E} \) is the terminal category (cf. Remark 12.1.9(4)).

Note. The following definition is a special case of Definition 12.1.8 where \( \mathcal{M} \) is given by the hom-module of a category.

12.1.12 Definition.

- Given a module \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) and a functor \( \mathcal{G} : \mathcal{A} \to \mathcal{X} \), a (degenerate) cell \( \theta \) from \( \mathcal{M} \) to \( \mathcal{G} \), written as \( \theta : \mathcal{M} \to \mathcal{G} \), or diagrammatically as \( \mathcal{M} \otimes \mathcal{G} \), is defined by a module morphism

\[
\theta : \mathcal{M} \to (\mathcal{X} \otimes \mathcal{G}).
\]

- Given a module \( \mathcal{M} : \mathcal{X} \to \mathcal{A} \) and a functor \( \mathcal{F} : \mathcal{X} \to \mathcal{A} \), a (degenerate) cell \( \theta \) from \( \mathcal{F} \) to \( \mathcal{M} \), written as \( \theta : \mathcal{F} \to \mathcal{M} \), or diagrammatically as \( \mathcal{F} \otimes \mathcal{M} \), is defined by a module morphism

\[
\theta : \mathcal{F} \to (\mathcal{F} \otimes \mathcal{A}).
\]

Note. The following definition is a special case of Definition 12.1.10 where \( \mathcal{M} \) is given by the hom-module of a category; a cell defined in Definition 12.1.12 is given as an arrow of the representable module defined below.

12.1.13 Definition. Let \( \mathcal{X} \) and \( \mathcal{A} \) be categories.

- The corepresentable module of the right generalized Yoneda functor for \( [\mathcal{A}, \mathcal{X}] \) is denoted by \( \mathcal{X} \otimes \mathcal{A} \); that is, the module

\[
(\mathcal{X} \otimes \mathcal{A}) : [\mathcal{X} : \mathcal{A}] \to [\mathcal{A}, \mathcal{X}]
\]

is given by the composition

\[
[\mathcal{X} : \mathcal{A}] \xrightarrow{(\mathcal{X} \mathcal{A})} [\mathcal{A}, \mathcal{X}] \xrightarrow{\mathcal{X} \mathcal{A}} [\mathcal{A}, \mathcal{X}].
\]
12.2. coYoneda lemma

We define the module \( \langle X, A \rangle : [X, A] \to [X : A] \) for a category \( X \) by composing the right Yoneda functor \( \langle X, A \rangle : X \to [X :] \) and the equivalence module \( \langle X \downarrow \rangle : [X \downarrow] \to [X :] \) defined in Section 11.1.4 so that the arrows of \( \langle X, A \rangle \) consist of cones from a right comma over \( X \) to an object of \( X \). The equivalence between the module \( \langle X, A \rangle : [X \downarrow] \to X \) so defined and the module \( \langle X \downarrow, A \rangle : [X :] \to X \) defined in Section 12.1 then follows from the equivalence \( [X \downarrow] \to [X :] \) and yields a bijective correspondence between cones and cocones (the “coYoneda lemma”).

Note. Recall from Corollary 11.1.9 that the module \( \langle X \downarrow \rangle : [X \downarrow] \to [X :] \) is an equivalence.

12.2.1 Definition. Given a module \( M : X \to A \), the module

\[
\langle M \rangle : [X] \to [X : A]
\]

is defined by the composition

\[
[X] \xrightarrow{\langle X \rangle} [X : A] \xrightarrow{\langle M \rangle} [X : A]
\]

of the right [op. left] exponential transpose of \( M \) and the module in Definition 11.4.2.

12.2.2 Remark.

1. Given a right comma \( K : X \to * \) and an object \( a \in [A] \), an \( \langle M \rangle \)-arrow \( \alpha : K \to a \) is a cone from the right comma fibration \( K : [K] \to X \) to \( a \) along \( M \).

2. Given a left comma \( K : * \to A \) and an object \( x \in [X] \), an \( \langle M \rangle \)-arrow \( \alpha : x \to K \) is a cone from \( x \) to the left comma fibration \( K : [K] \to A \) along \( M \).
(2) The assignment $\mathcal{M} \mapsto \mathcal{M}^{\mathcal{C}}$ [op. $\mathcal{M} \mapsto \mathcal{C}^{\mathcal{M}}$] extends to a functor; indeed,

- the functor

$$[X : A] \xrightarrow{\sim} [[X \downarrow] : A]$$

from the category of modules $X \rightarrow A$ to the category of modules $[X \downarrow] \rightarrow A$ is defined by the composition

$$[X : A] \xrightarrow{\sim} [A, [X :]] \xrightarrow{(X \downarrow)^{-A}} [[X \downarrow] : A]$$

of the right exponential transposition and the right action of the module $\langle X \downarrow \rangle : [X \downarrow] \rightarrow [X :]$ on the functor category $[A, [X :]]$; the object function of the functor $[X : A] \xrightarrow{\sim} [[X \downarrow] : A]$ sends each module $\mathcal{M} : X \rightarrow A$ to the module $\langle \mathcal{M}^{\mathcal{C}} \rangle : [X \downarrow] \rightarrow A$.

- the functor

$$[X : A] \xrightarrow{\sim} [X : [\downarrow A]^{-}]$$

from the category of modules $X \rightarrow A$ to the category of modules $X \rightarrow [\downarrow A]^{-}$ is defined by the composition

$$[X : A] \xrightarrow{\sim} [X, [: A]^{-}] \xrightarrow{([A]^{-} \times X)} [X : [\downarrow A]^{-}]$$

of the left exponential transposition and the left action of the module $\langle [A]^{-} \rangle : [: A]^{-} \rightarrow [\downarrow A]^{-}$ on the functor category $[X, [: A]^{-}]$; the object function of the functor $[X : A] \xrightarrow{\sim} [X : [\downarrow A]^{-}]$ sends each module $\mathcal{M} : X \rightarrow A$ to the module $\langle \mathcal{M}^{\mathcal{C}} \rangle : X \rightarrow [\downarrow A]^{-}$.

(3) Since the module $\langle X \downarrow \rangle : [X \downarrow] \rightarrow [X :]$ is corepresented by the equivalence functor $[X \downarrow] \xrightarrow{\sim} [X :]$ (see Definition 11.4.10(1)), by Proposition 2.3.16, the composition

$$[X \downarrow] \xrightarrow{\sim} [X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A$$

yields a corepresentation of the module $\langle \mathcal{M}^{\mathcal{C}} \rangle : [X \downarrow] \rightarrow A$ by the functor $\langle \mathcal{M}^{\mathcal{C}} \rangle : A \rightarrow [X \downarrow]$ in Definition 11.5.1, and dually the composition

$$X \xrightarrow{\mathcal{M}^{\mathcal{C}}} [: A]^{-} \xrightarrow{\sim} [\downarrow A]^{-}$$

yields a representation of the module $\langle \mathcal{M} \rangle : X \rightarrow [\downarrow A]^{-}$ by the functor $\langle \mathcal{M} \rangle : X \rightarrow [\downarrow A]^{-}$.

Note. We will see below that the modules defined in Definition 12.1.3 and Definition 12.2.1 are equivalent.

12.2.3 Theorem.

- The modules $\langle \mathcal{M}^{\mathcal{C}} \rangle : [X :] \rightarrow A$ and $\langle \mathcal{M}^{\mathcal{C}} \rangle : [X \downarrow] \rightarrow A$ are equivalent. Indeed, a pair of equivalence cells

$$\begin{array}{c}
[X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\downarrow \downarrow 1 \\
[X \downarrow] \xrightarrow{\sim \mathcal{M}^{\mathcal{C}}} A \\
\end{array} \quad \text{and} \quad \begin{array}{c}
[X \downarrow] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\downarrow \downarrow 1 \\
[X :] \xrightarrow{\sim \mathcal{M}^{\mathcal{C}}} A \\
\end{array}$$

quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 11.4.10(2) by the pasting compositions

$$\begin{array}{c}
[X :] \xrightarrow{(X \downarrow)} [X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\downarrow \downarrow 1 \\
[X \downarrow] \xrightarrow{\sim (X \downarrow)} [X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\end{array} \quad \text{and} \quad \begin{array}{c}
[X \downarrow] \xrightarrow{(X \downarrow)} [X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\downarrow \downarrow 1 \\
[X :] \xrightarrow{\sim (X \downarrow)} [X :] \xrightarrow{\mathcal{M}^{\mathcal{C}}} A \\
\end{array}$$

- The modules $\langle \mathcal{M} \rangle : X \rightarrow [: A]^{-}$ and $\langle \mathcal{M} \rangle : X \rightarrow [\downarrow A]^{-}$ are equivalent. Indeed, a pair of equiv-
alence cells

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^-{\simeq} & \mathcal{A} \\
\downarrow^{1} & \downarrow^{1} \\
X \ar[r]^-{\simeq} & \mathcal{A} 
}
\text{ and }
\begin{array}{c}
\xymatrix{
X \ar[r]^-{\simeq} & \mathcal{A} \\
\downarrow^{1} & \downarrow^{1} \\
X \ar[r]^-{\simeq} & \mathcal{A} 

\end{array}
\end{array}
\]

quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 11.4.10(2) by the pasting compositions

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^-{\Psi} & \mathcal{N} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
\mathcal{M} \ar[r]^-{\Psi} & \mathcal{N} 
}
\end{array}
\text{ and }
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^-{\Psi} & \mathcal{N} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
\mathcal{M} \ar[r]^-{\Psi} & \mathcal{N} 

\end{array}
\]

Proof. Self explanatory. □

12.2.4 Remark.

(1) By the construction, the equivalence \( (\mathcal{M} \varnothing) \simeq (\mathcal{M}^{\varnothing}) \) [op. \( (\mathcal{M}, \mathcal{M}) \simeq (\mathcal{M}, \mathcal{M}) \)] is natural in \( \mathcal{M} \) (see Remark 12.1.4(4) and Remark 12.2.2(2)); that is, the square

\[
\begin{array}{c}
\xymatrix{
\mathcal{M} \ar[r]^-{\downarrow} & \mathcal{N} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
\mathcal{M} \ar[r]^-{\downarrow} & \mathcal{N} 

\end{array}
\]

commutes for any module morphism \( \varphi : \mathcal{M} \to \mathcal{N} \).

(2) The equivalence cell \( \downarrow_{\mathcal{M}} : (\mathcal{M} \varnothing) \to (\mathcal{M}^{\varnothing}) \) [op. \( \downarrow_{\mathcal{M}} : (\mathcal{M}, \mathcal{M}) \simeq (\mathcal{M}, \mathcal{M}) \)] sends each conic cell

\[
\begin{array}{c}
\xymatrix{
X \ar[r]^-{a} & \mathcal{A} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
X \ar[r]^-{a} & \mathcal{A} 

\end{array}
\text{ to its comma transpose, i.e. to the cone }
\begin{array}{c}
\xymatrix{
X \ar[r]^-{a} & \mathcal{A} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
X \ar[r]^-{a} & \mathcal{A} 

\end{array}
\]

(cf. Remark 11.4.11(4)).

(3) As a special case (see Remark 12.1.2(3)), the equivalence cell

\[
\begin{array}{c}
\xymatrix{
[E] \ar[r]^-{\varphi} & \mathcal{A} \\
\downarrow^{i} & \downarrow^{i} \\
[E] \ar[r]^-{\varphi} & \mathcal{A} 

\end{array}
\text{ for the composite module } G(\mathcal{M}) [\text{ op. } (\mathcal{M}, F) \text{ sends each conic cell }
\begin{array}{c}
\xymatrix{
E \ar[r]^-{a} & \mathcal{A} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
E \ar[r]^-{a} & \mathcal{A} 

\end{array}
\text{ to the cone }
\begin{array}{c}
\xymatrix{
E \ar[r]^-{a} & \mathcal{A} \\
\downarrow^{\varphi} & \downarrow^{\varphi} \\
E \ar[r]^-{a} & \mathcal{A} 

\end{array}
\]

\]
12.2. coYoneda lemma

(4) The composition

\[
\begin{array}{ccc}
\mathbf{X} \ar[r]^-\mathcal{M} & \mathbf{A} \quad \text{op.} \\
\mathbf{X} \ar[u]^-\mathcal{M} \ar[r] & \mathcal{M} \ar[u]^-\mathcal{M} \ar[r] & \mathbf{A} \quad \text{op.}
\end{array}
\]

of the Yoneda morphism in Remark 12.1.4(2) and the equivalence cell \(\downarrow_M\) yields a fully faithful cell

\[
\begin{array}{ccc}
\mathbf{X} \ar[r]^-\mathcal{M} & \mathbf{A} \quad \text{op.} \\
\mathbf{X} \ar[u]^-\mathcal{M} \ar[r] & \mathcal{M} \ar[u]^-\mathcal{M} \ar[r] & \mathbf{A} \quad \text{op.}
\end{array}
\]

\((\mathbf{X} \triangleright op. \triangleright \mathbf{A})\) is the coYoneda functor in Definition 11.5.3) making the diagram

\[
\begin{array}{ccc}
\mathcal{M} \ar[r]^-\mathcal{M} \ar[d]_-\downarrow M & \mathcal{M} \ar[r]^-\mathcal{M} \ar[d]_-\downarrow M & \mathcal{M} \\
\mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \\
\mathbf{A} \ar[r]^-\mathbf{A} & \mathbf{A} & \mathbf{A}
\end{array}
\]

commute up to isomorphism. The cell \((\mathbf{X} \triangleright \mathcal{M} \triangleright \mathbf{A})\) establishes a bijection between the set of \(\mathcal{M}\)-arrows \(x \sim a\) and the set of cones from the forgetful functor \(\Sigma_x : \mathbf{X}/x \to \mathbf{X}\) to \(a \in \mathbf{A} \) [op. from \(x \in \mathbf{X}\)] to the forgetful functor \(\Sigma^a : \mathbf{A} \to \mathbf{A}\) (see Remark 11.5.4(2)) along \(\mathcal{M}\).

Note. The following definition is a special case of Definition 12.2.1 where \(\mathcal{M}\) is given by the hom-module of a category.

12.2.5 Definition. Given a category \(\mathbf{X} \) [op. \(\mathbf{A}\)], the right [op. left] coYoneda module

\[(\mathbf{X} \triangleright) : \mathbf{X} \to \mathbf{X} \quad \text{op.} \quad (\triangleright \mathbf{A}) : \mathbf{A} \to [\downarrow \mathbf{A}]^\sim\]

is defined by the composition

\[
\begin{array}{ccc}
\mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \\
\mathbf{A} \ar[r]^-\mathbf{A} & \mathbf{A} & [\downarrow \mathbf{A}]^\sim
\end{array}
\]

of the right [op. left] Yoneda functor and the module in Definition 11.5.3.

12.2.6 Remark.

(1) • Given a right comma \(\mathbb{K} : \mathbf{X} \to \ast\) and an object \(x \in \mathbf{X}\), an \((\mathbf{X} \triangleright)\)-arrow \(\alpha : \mathbb{K} \to x\) is a cone

\[
\begin{array}{c}
\mathbb{K} \ar[d]^-\alpha \\
x \ar[d]^-\alpha
\end{array}
\]

from the right comma fibration \(\mathbb{K} : [\mathbb{K}] \to \mathbf{X}\) to \(x\).

• Given a left comma \(\mathbb{K} : \ast \to \mathbf{A}\) and an object \(a \in \mathbf{A}\), an \(\triangleright \mathbf{A}\)-arrow \(\alpha : a \to \mathbb{K}\) is a cone

\[
\begin{array}{c}
\mathbb{K} \ar[d]^-\alpha \\
a \ar[r]^-\alpha
\end{array}
\]

from \(a\) to the left comma fibration \(\mathbb{K} : [\mathbb{K}] \to \mathbf{A}\).

(2) Since the module \((\mathbf{X} \triangleright) : \mathbf{X} \to \mathbf{X}\) is corepresented by the equivalence functor \([\mathbf{X} \downarrow] \xrightarrow{\downarrow} [\mathbf{X}]\) (see Definition 11.4.10(1)), by Proposition 2.3.16, the composition

\[
\begin{array}{ccc}
\mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \ar[r]^-\mathbf{X} \ar[d]_-\downarrow M & \mathbf{X} \\
[\mathbf{X} \downarrow] \ar[r]^-\downarrow M & [\mathbf{X}] \ar[r]^-\downarrow M & \mathbf{X}
\end{array}
\]

yields a corepresentation of the right coYoneda module \((\mathbf{X} \triangleright) : \mathbf{X} \to \mathbf{X}\) by the right coYoneda functor \([\mathbf{X} \triangleright] : \mathbf{X} \to [\mathbf{X} \downarrow]\) in Definition 11.5.3, and dually the composition

\[
\begin{array}{ccc}
\mathbf{A} \ar[r]^-\mathbf{A} \ar[d]_-\downarrow M & \mathbf{A} \ar[r]^-\mathbf{A} \ar[d]_-\downarrow M & \mathbf{A} \\
[\downarrow \mathbf{A}]^\sim \ar[r]^-\downarrow M & [\downarrow \mathbf{A}]^\sim \ar[r]^-\downarrow M & [\downarrow \mathbf{A}]^\sim
\end{array}
\]
yields a representation of the left coYoneda module \(\langle \cdot, \mathcal{A} \rangle : A \to [\downarrow A]^\circ\) by the left coYoneda functor \([\cdot \mathcal{A}] : A \to [\downarrow A]^\circ\).

Note. The following is a special case of Theorem 12.2.3 where \(\mathcal{M}\) is given by the hom-module of a category; we see that the modules defined in Definition 12.1.6 and Definition 12.2.5 are equivalent.

12.2.7 Theorem.

- The modules \((X, \mathcal{A}) : A \to [A]^{-}\) and \((X, \mathcal{A}^\prime) : [A]^{-} \to X\) are equivalent. Indeed, a pair of equivalence cells
  \[
  [X] : \xymatrix{X \ar[r]^-{X, \mathcal{A}} & X} \quad \text{and} \quad [X] : \xymatrix{X \ar[r]^-{X, \mathcal{A}^\prime} & X}
  \]
  quasi-inverse to each other is obtained from the pair of equivalence cells in Definition 11.4.10(2)
  by the pasting compositions
  \[
  [X] : \xymatrix{X \ar[r]^-{(X)} & X} \quad \text{and} \quad [X] : \xymatrix{X \ar[r]^-{(X)} & X}.
  \]

- The modules \((\mathcal{A}, \mathcal{A}^\prime) : [A]^{-} \to A\) is given by the hom-module of \(M\).

12.2.8 Remark.

(1) The equivalence cell \(\downarrow\mathcal{A} : (X, \mathcal{A}) \to (X, \mathcal{A}^\prime)\) [op. \(\downarrow\mathcal{A} : (\mathcal{A}, \mathcal{A}^\prime) \simeq (\mathcal{A}, \mathcal{A})\)] sends each conic cell

\[
\xymatrix{X \ar[r]^-{\mathcal{A}} & A \ar[l]^-{(\mathcal{A})}} \quad \text{op.} \quad \xymatrix{A \ar[r]^-{\mathcal{A}} & A \ar[l]^-{(\mathcal{A})}}
\]

to its comma transpose, i.e. to the cone

\[
\xymatrix{M \ar[r]^-{\mathcal{A}} & A \ar[l]^-{(\mathcal{A})}} \quad \text{op.} \quad \xymatrix{A \ar[r]^-{\mathcal{A}} & A \ar[l]^-{(\mathcal{A})}}
\]

(cf. Remark 11.4.11(4)).

(2) The equivalence \((X, \mathcal{A}) \simeq (X, \mathcal{A}^\prime)\) is called the coYoneda lemma in [ML98]. It may be more appropriate to call the fully faithful cell \((X, \mathcal{A})^\prime \mathcal{M} \mathcal{A}^\prime\) in Remark 12.2.4(4) “coYoneda” instead; in any case, however, it is the equivalence \((\mathcal{M}, \mathcal{A}) \simeq (\mathcal{M}, \mathcal{A}^\prime)\) that plays a critical role in the sequel, providing a way to transform a cellular limit to a conical limit.
12.3 Cellular limits

Definition 12.3.1 and Definition 12.3.5 give two definitions of a cellular limit—a universal conic cell; Definition 12.3.1 defines a limit by a universal arrow of the corepresentable module \( M \mapsto \) introduced in Section 12.1, while Definition 12.3.5 defines a limit (weighted limit) by a universal arrow of the module \( \langle J, M \rangle \) introduced in Section 1.4. The two definitions are shown to be equivalent in Theorem 12.3.7. We will see that a cellular limit transforms into a conical limit and vice versa, and see that a cell preserves cellular limits precisely when it preserves conical limits.

Note. In the following, we consider a conic cell as an arrow of the module \( M \mapsto \) defined in Definition 12.1.3.

12.3.1 Definition. A conic cell \( \xymatrix{ J \ar[r]^* & r \ar[l]_{\omega} \ar[r]_{\mathbb{M}} & A } \)

right module \( J : X \rightarrow *, \) a universal conic cell \( \omega : J \rightarrow r : M \mapsto \) or the pair \( (r, \omega) \), or the object \( r \) itself, is called a (cellular) colimit of \( J \) along \( M \).

A conic cell \( \xymatrix{ r \ar[r]^* & J \ar[r]_{\omega} \ar[l]_{\mathbb{M}} & A } \)

left module \( J : * \rightarrow \), a universal conic cell \( \omega : r \rightarrow J : \mathbb{M} \) or the pair \( (r, \omega) \), or the object \( r \) itself, is called a (cellular) limit of \( J \) along \( M \).

12.3.2 Remark.

1. A conic cell \( \omega : J \rightarrow r : M \mapsto \) is universal if and only if to every conic cell \( \theta : J \rightarrow a : M \mapsto \) there is a unique \( A \)-arrow \( \omega \theta : r \rightarrow a \) (the adjunct of \( \theta \) along \( \omega \)) such that \( \theta = \omega \theta \theta \). Dually, a conic cell \( \omega : r \rightarrow J : \mathbb{M} \) is universal if and only if to every conic cell \( \theta : x \rightarrow J : \mathbb{M} \) there is a unique \( X \)-arrow \( \theta \omega : x \rightarrow r \) such that \( \theta = \theta \omega \omega \).

2. Limits and colimits are unique up to isomorphism by Corollary 6.2.8.

3. A colimit [op. limit] defined in Definition 12.3.1 is referred to as a cellular colimit [op. limit] to distinguish it from a (conical) colimit [op. limit] defined in Section 8.1.

Note. Recall from Remark 12.2.4(2) that a conic cell is transposed to a cone.

12.3.3 Proposition. A conic cell

\[
\xymatrix{ J \ar[r]^* & r \ar[l]_{\omega} \ar[r]_{\mathbb{M}} & A } \quad \text{op.} \quad \xymatrix{ J \ar[r]^* & r \ar[l]_{\omega} \ar[r]_{\mathbb{M}} & A }
\]

is universal if and only if its comma transpose

\[
\xymatrix{ J_i \ar[r]^* & J \ar[l]_{\omega} \ar[r]_{\mathbb{M}} & A } \quad \text{op.} \quad \xymatrix{ J_i \ar[r]^* & J \ar[l]_{\omega} \ar[r]_{\mathbb{M}} & A }
\]

is a universal cone.

Proof. By Theorem 7.13.13, the equivalence cell \( \downarrow M : \langle M \mapsto \rangle \rightarrow \langle M \mapsto \mapsto \rangle \) in Theorem 12.2.3 preserves and reflects direct universal arrows.

Note. Recall from Remark 12.1.4(2) the Yoneda morphism transforms a module arrow into a conic cell.

12.3.4 Theorem. Let \( M : X \rightarrow A \) be a module.

\[
\xymatrix{ \ast \ar[r]^* & x \ar[l]_{\mathbb{M}} \ar[r]_{\theta} & A }
\]

An \( M \)-arrow \( u : x \rightarrow r \) is direct universal if and only if the conic cell \( \xymatrix{ (X) \ar[r]^* & X \ar[l]_{\mathbb{M}} \ar[r]_{\theta} & A } \)

is universal.
12.3.5 Definition. 
- A conic cell \( \mathbf{E} \xrightarrow{\mathbf{J}} \mathbf{A} \) is called universal if it is a direct universal \( \langle \mathbf{J}, \mathcal{M} \rangle \)-arrow. Given a functor \( \mathbf{G} : \mathbf{E} \to \mathbf{X} \), a universal conic cell \( \mathbf{ω} : \mathbf{G} \xrightarrow{\mathbf{r}} \mathbf{J} \to \mathcal{M} \) or the pair \( (\mathbf{r}, \mathbf{ω}) \), or the object \( \mathbf{r} \) itself, is called a colimit of \( \mathbf{G} \) weighted by \( \mathbf{J} \) (or \( \mathbf{J} \)-weighted colimit of \( \mathbf{G} \)) along \( \mathcal{M} \), with the object \( \mathbf{r} \) denoted by \( \prod \mathbf{J} \mathbf{G} \).
- A conic cell \( \mathbf{E} \xleftarrow{\mathbf{r}} \mathbf{A} \) is called universal if it is an inverse universal \( \langle \mathbf{J}, \mathcal{M} \rangle \)-arrow. Given a functor \( \mathbf{F} : \mathbf{E} \to \mathbf{A} \), a universal conic cell \( \mathbf{ω} : \mathbf{r} \xrightarrow{} \mathbf{F} : \mathbf{J} \to \mathcal{M} \) or the pair \( (\mathbf{r}, \mathbf{ω}) \), or the object \( \mathbf{r} \) itself, is called a limit of \( \mathbf{F} \) weighted by \( \mathbf{J} \) (or \( \mathbf{J} \)-weighted limit of \( \mathbf{F} \)) along \( \mathcal{M} \), with the object \( \mathbf{r} \) denoted by \( \prod \mathbf{J} \mathbf{F} \).

12.3.6 Remark. 
(1) A conic cell \( \mathbf{ω} : \mathbf{G} \xrightarrow{\mathbf{r}} \mathbf{J} \to \mathcal{M} \) is universal if and only if to every conic cell \( \mathbf{θ} : \mathbf{G} \xrightarrow{\mathbf{a}} \mathbf{J} \to \mathcal{M} \) there is a unique \( \mathbf{A} \)-arrow \( \mathbf{ω} \triangleleft \mathbf{θ} : \mathbf{r} \to \mathbf{a} \) (the adjunct of \( \mathbf{θ} \) along \( \mathbf{ω} \)) such that \( \mathbf{θ} = \mathbf{ω} \circ \mathbf{θ} \). Dually, a conic cell \( \mathbf{ω} : \mathbf{r} \xrightarrow{} \mathbf{F} : \mathbf{J} \to \mathcal{M} \) is universal if and only if to every conic cell \( \mathbf{θ} : \mathbf{x} \xrightarrow{} \mathbf{F} : \mathbf{J} \to \mathcal{M} \) there is a unique \( \mathbf{X} \)-arrow \( \mathbf{θ} \circ \mathbf{ω} : \mathbf{x} \to \mathbf{r} \) such that \( \mathbf{θ} = \mathbf{θ} \circ \mathbf{ω} \).
(2) As a special case where \( \mathcal{M} \) is given by the hom-module of a category \( \mathbf{C} \),
- a universal conic cell \( \mathbf{E} \xleftarrow{\mathbf{J}} \mathbf{C} \) or the pair \( (\mathbf{r}, \mathbf{ω}) \), or the object \( \mathbf{r} \) itself, is called a \( \mathbf{J} \)-weighted colimit of \( \mathbf{L} \) in \( \mathbf{C} \).
- a universal conic cell \( \mathbf{E} \xrightarrow{\mathbf{J}} \mathbf{C} \) or the pair \( (\mathbf{r}, \mathbf{ω}) \), or the object \( \mathbf{r} \) itself, is called a \( \mathbf{J} \)-weighted limit of \( \mathbf{L} \) in \( \mathbf{C} \).
(3) Weighted limits and colimits are unique up to isomorphism by Corollary 6.2.8.

Note. Recall from Remark 12.1.2(2) and (3) that a degenerate conic cell can be depicted as an ordinary conic cell, and vice versa.

12.3.7 Theorem. Definition 12.3.1 subsumes Definition 12.3.5 in the following sense:
- a conic cell \( \mathbf{E} \xleftarrow{\mathbf{J}} \mathbf{A} \) is universal in the sense of 12.3.5 if and only if it, depicted as a degenerate conic cell \( \mathbf{E} \xrightarrow{\mathbf{J}} \mathbf{A} \), is universal in the sense of 12.3.1; that is, a \( \mathbf{J} \)-weighted colimit of \( \mathbf{G} \) along \( \mathcal{M} \) is the same thing as a colimit of \( \mathbf{J} \) along \( \mathbf{G} \).
• a conic cell \( * \xrightarrow{J} E \) is universal in the sense of 12.3.5 if and only if it, depicted as a degenerate
\[
\begin{array}{c}
X \\ (-M)^\rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
is universal in the sense of 12.3.1; that is, a \( J \)-weighted limit of \( F \) along \( M \)
is the same thing as a limit of \( J \) along \( F \).

Conversely, Definition 12.3.5 subsumes Definition 12.3.1 in the following sense:

• a conic cell \( \xrightarrow{J} \ast \) is universal in the sense of 12.3.1 if and only if it, depicted as an ordinary conic cell
\[
\begin{array}{c}
X \\ (-M) \rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
is universal in the sense of 12.3.5; that is, a colimit of \( J \) along \( M \) is the same thing as a \( J \)-weighted colimit of the identity \( X \rightarrow X \) along \( M \).

• a conic cell \( \xrightarrow{r} \ast \) is universal in the sense of 12.3.1 if and only if it, depicted as an ordinary conic cell
\[
\begin{array}{c}
X \\ (-M) \rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
is universal in the sense of 12.3.5; that is, a limit of \( J \) along \( M \) is the same thing as a \( J \)-weighted limit of the identity \( A \rightarrow A \) along \( M \).

Proof. The left slice of the module \( (J,M) \) at \( G \) and the left slice of the module \( (G(M)) \) at \( J \) are the same left module over \( A \) given by
\[
a \mapsto (J)(E): (G(M)a)
\]
hence \( \omega : G \rightleftharpoons r \) is a direct universal \( (J,M) \)-arrow iff \( \omega : J \rightleftharpoons r \) is a direct universal \( (G(M)) \)-arrow.

The left slice of the module \( M \) at \( J \) and the left slice of the module \( (J,M) \) at \( 1_X \) are the same left module over \( A \) given by
\[
a \mapsto (J)(X): (M) \)
hence \( \omega : J \rightleftharpoons r \) is a direct universal \( (M) \)-arrow iff \( \omega : 1_X \rightleftharpoons r \) is a direct universal \( (J,M) \)-arrow.

Note. Recall from Remark 12.2.4(3) that a conic cell is transposed to a cone.

12.3.8 Theorem. A conic cell
\[
\begin{array}{c}
E \\ (-M) \rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
is universal if and only if its comma transpose
\[
\begin{array}{c}
E \\ (\Delta M) \rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
is a universal cone.

Proof. Since the conic cell \( E \xrightarrow{J} \ast \) is universal iff the conic cell \( J \xrightarrow{r} \ast \) along the composite
\[
\begin{array}{c}
E \\ G(M) \rightarrow \downarrow \omega \\
\downarrow r \\
A
\end{array}
\]
module $G(M)$ is universal (see Theorem 12.3.7), and since the cone $E \xleftarrow{\mathcal{J}^i} \mathcal{J}$ is universal if $\frac{G}{\mathcal{M}} \xrightarrow{\omega^i} \frac{E}{\mathcal{M}} \xrightarrow{\Delta_r} \mathcal{M} \rightarrow A$, the cone $\xleftarrow{\mathcal{J}^i} \mathcal{J} \xrightarrow{\omega^i} \frac{E}{\mathcal{M}} \xrightarrow{\Delta_r} \mathcal{M} \rightarrow A$ along the composite module $G(M)$ is universal (see Theorem 8.1.4), the assertion is reduced to Proposition 12.3.3.

12.3.9 Remark. By Theorem 12.3.8 and by the bijectivity of comma transposition, we see that

- a $\mathcal{J}$-weighted colimit of $G$ along $M$ is the same thing as a colimit of $\mathcal{J}^i \circ G$ along $M$.
- a $\mathcal{J}$-weighted limit of $F$ along $M$ is the same thing as a limit of $F \circ \mathcal{J}^i$ along $M$.

The notion of a limit thus subsumes that of a weighted limit. We will see that the converse is also the case in Remark 12.3.12.

12.3.10 Corollary. Let $\mathcal{J}$ and $M$ be modules as in Theorem 12.3.8.
- If $M$ is cocomplete and $\mathcal{J}$ is small, then a functor $G : E \rightarrow X$ has a $\mathcal{J}$-weighted colimit along $M$.
- If $M$ is complete and $\mathcal{J}$ is small, then a functor $F : E \rightarrow A$ has a $\mathcal{J}$-weighted limit along $M$.

Proof. By Remark 12.3.9, it suffices to show that $\mathcal{J}^i \circ G : [\mathcal{J}] \rightarrow X$ has a colimit along $M$. But since the smallness of $\mathcal{J}$ guarantees the smallness of the comma category $[\mathcal{J}]$, this holds by the cocompleteness of $M$.

Note. The bijective correspondence between cones and conic cells stated in Corollary 5.5.5 gives the following result.

12.3.11 Theorem. A cone

\[
\begin{array}{ccc}
E & \xleftarrow{\mathcal{J}^i} & \ast \\
\xrightarrow{G} \mathcal{M} & \xrightarrow{\mu} & \xrightarrow{r} \mathcal{M} \\
X & \xrightarrow{\Delta_r} & A
\end{array}
\quad
\begin{array}{ccc}
\ast & \xrightarrow{\mathcal{J}^i} & E \\
\xrightarrow{r} \mathcal{M} & \xrightarrow{\mu} & \xrightarrow{\Delta_r} \mathcal{M} \\
X & \xrightarrow{\Delta_r} & A
\end{array}
\]

is universal if and only if the conic cell

\[
\begin{array}{ccc}
E & \xleftarrow{\Delta_E} & \ast \\
\xrightarrow{G} \mathcal{M} & \xrightarrow{(\mu)} & \xrightarrow{r} \mathcal{M} \\
X & \xrightarrow{\Delta_r} & A
\end{array}
\quad
\begin{array}{ccc}
\ast & \xrightarrow{\Delta_E} & E \\
\xrightarrow{r} \mathcal{M} & \xrightarrow{(\mu)} & \xrightarrow{\Delta_r} \mathcal{M} \\
X & \xrightarrow{\Delta_r} & A
\end{array}
\]

is universal.

Proof. Immediate from the isomorphism in Corollary 5.5.5.

12.3.12 Remark. By Theorem 12.3.11 and by the bijectivity of the assignment $\mu \mapsto \{\mu\}$, we see that given a module $M : X \rightarrow A$,

- a colimit of a functor $G : E \rightarrow X$ along $M$ is the same thing as an $\langle \Delta_E \ast \rangle$-weighted colimit of $G$ along $M$.
- a limit of a functor $F : E \rightarrow A$ along $M$ is the same thing as an $\langle \ast \Delta_E \rangle$-weighted limit of $F$ along $M$.

Now recalling Remark 12.3.9, we see that the notion of a weighted limit and that of a conical limit subsume each other.

Note. Since a weighted limit is defined by a universal $\langle \mathcal{J}, M \rangle$-arrow, we can describe the preservation of weighted limits using the postcomposition cell $\langle \mathcal{J}, \psi \rangle$ defined in Definition 1.4.12.
12.3.13 **Definition.** Let \( J \) be a right \([\text{op. left}]\) module over a category \( E \). Then a cell
\[
\begin{array}{ccc}
X \xrightarrow{M} A & \xrightarrow{\psi} & Y \xrightarrow{N} B
\end{array}
\]
is said to preserve (resp. reflect, create) \( J \)-weighted colimits \([\text{op. limits}]\) if the postcomposition cell
\[
\begin{array}{c}
[E, X] \xrightarrow{(J, M)} A \quad \text{op.} \quad X \xrightarrow{(J, M)} [E, A]
\end{array}
\]
\[
\begin{array}{c}
[E, P] \xrightarrow{(J, \psi)} Q \quad \text{op.} \quad P \xrightarrow{(J, \psi)} [E, Q]
\end{array}
\]
\[
\begin{array}{c}
[E, Y] \xrightarrow{(J, N)} B \quad \text{op.} \quad Y \xrightarrow{(J, N)} [E, B]
\end{array}
\]
preserves (resp. reflects, creates) direct \([\text{op. inverse}]\) universal arrows.

12.3.14 **Remark.**

(1) Recalling the definition of the postcomposition cell, Definition 12.3.13 can be stated in elementary terms as follows: \( \psi \) is said to

a) preserve

- \( J \)-weighted colimits if each universal conic cell \( \omega : G \rightrightarrows J \to M \) yields by composition with \( \psi \) a universal conic cell \( \omega \circ \psi : G \circ P \rightrightarrows Q \circ r : J \to N \).
- \( J \)-weighted limits if each universal conic cell \( \omega : r \rightrightarrows F : J \to M \) yields by composition with \( \psi \) a universal conic cell \( \omega \circ \psi : r : P \rightrightarrows Q \circ F : J \to N \).

b) reflect

- \( J \)-weighted colimits if a conic cell \( \omega : G \rightrightarrows J \to M \) is universal whenever the conic cell \( \omega \circ \psi : G \circ P \rightrightarrows Q \circ r : J \to N \) is universal.
- \( J \)-weighted limits if a conic cell \( \omega : r \rightrightarrows F : J \to M \) is universal whenever the conic cell \( \omega \circ \psi : r : P \rightrightarrows Q \circ F : J \to N \) is universal.

c) create

- \( J \)-weighted colimits if for every functor \( G : E \to X \) and for every universal conic cell \( \theta : G \circ P \rightrightarrows s : J \to N \) there is exactly one conic cell \( \omega : G \rightrightarrows r : J \to M \) with \( \omega \circ \psi = \theta \), and if this \( \omega \) is universal.
- \( J \)-weighted limits if for every functor \( F : E \to A \) and for every universal conic cell \( \theta : s \rightrightarrows Q \circ F : J \to N \) there is exactly one conic cell \( \omega : r \rightrightarrows F : J \to M \) with \( \omega \circ \psi = \theta \), and if this \( \omega \) is universal.

(2) A cell \( \psi \) is said to preserve (resp. reflect, create) weighted colimits \([\text{op. limits}]\) if it preserves (resp. reflects, creates) \( J \)-weighted colimits \([\text{op. limits}]\) for any right \([\text{op. left}]\) module \( J \).

12.3.15 **Theorem.** If a cell has a right \([\text{op. left}]\) adjoint (see Definition 7.10.1), then it preserves weighted colimits \([\text{op. limits}]\).

**Proof.** By Definition 12.3.13, the assertion is equivalent to saying that if a cell \( \psi \) has a right adjoint, then for any right module \( J \), the postcomposition cell \( (J, \psi) \) preserves direct universal arrows. But this follows from Theorem 7.10.6 because if \( \psi \) has a right adjoint, so does \( (J, \psi) \) by Corollary 7.10.4.

12.3.16 **Remark.** In fact, by virtue of Theorem 12.3.17, Theorem 12.3.15 follows from Theorem 8.11.1 (and vice versa).

12.3.17 **Theorem.** A cell preserves (resp. reflects, creates) weighted colimits \([\text{op. limits}]\) if and only if it preserves (resp. reflects, creates) colimits \([\text{op. limits}]\).

**Proof.** \( (\Rightarrow) \) By Theorem 12.3.11 and since the isomorphism in Corollary 5.5.5 is natural in \( M \), for any category \( E \), if a cell \( \psi : M \to N \) preserves (resp. reflects, creates) colimits weighted by \( \Delta E^\ast \), then it preserves (resp. reflects, creates) colimits over \( E \).

\( (\Leftarrow) \) By Theorem 12.3.8 and since the equivalence \( (M, \mathcal{P}) \simeq (M, \mathcal{K}) \) is natural in \( M \) (see Remark 12.2.4(1)), for any right module \( J \), if a cell \( \psi : M \to N \) preserves (resp. reflects, creates)
colimits over the comma category \( [\mathcal{J}] \), then it preserves (resp. reflects, creates) colimits weighted by \( \mathcal{J} \).

### 12.4 Extensions

Definition 12.4.1 and Definition 12.4.10 give two definitions of an extension (i.e. a universal cell); Definition 12.4.1 defines an extension by a universal arrow of the corepresentable module \( M \not\rightarrow E \) introduced in Section 12.1, while Definition 12.4.10 defines it by a universal arrow of the module \( (\mathcal{J}, M) \) introduced in Section 1.2. The two definitions are shown to be equivalent in Theorem 12.4.14.

A pointwise (or strong) extension is defined in Definition 12.4.3 and Definition 12.4.12: a cell is a pointwise extension if each slice of it is universal; that is, if it consists of a family of cellular limits. We will see that a pointwise extension transforms reversibly into a pointwise lift in this section, and see the converse later in Section 12.6.

**Note.** In the following, we consider a cell as an arrow of the module \( (M \not\rightarrow E) \) defined in Definition 12.1.10.

#### 12.4.1 Definition.

- A cell \( \mathcal{J} \leftarrow E \xleftarrow{\omega} R \xrightarrow{M} A \) along \( M \) is called universal if it is a direct universal \( (M \not\rightarrow E) \)-arrow. Given a module \( \mathcal{J} : X \rightarrow E \), a universal cell \( \omega : \mathcal{J} \sim R : M \not\rightarrow E \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called an extension of \( \mathcal{J} \) along \( M \).

- A cell \( R \xrightarrow{E} \mathcal{J} \xleftarrow{\omega} M \xrightarrow{A} \) along \( M \) is called universal if it is an inverse universal \( (E \not\leftarrow M) \)-arrow. Given a module \( \mathcal{J} : E \rightarrow A \), a universal cell \( \omega : R \sim \mathcal{J} : E \not\leftarrow M \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a coextension of \( \mathcal{J} \) along \( M \).

#### 12.4.2 Remark.

1. A cell \( \omega : \mathcal{J} \sim R : M \not\rightarrow E \) is universal if and only if to every cell \( \theta : \mathcal{J} \sim F : M \not\rightarrow E \) there is a unique natural transformation \( \omega \theta : R \rightarrow F \) (the adjunct of \( \theta \) along \( \omega \) ) such that \( \theta = \omega \circ \omega \theta \).

   Dually, a cell \( \omega : R \sim \mathcal{J} : E \not\leftarrow M \) is universal if and only if to every cell \( \theta : G \sim \mathcal{J} : E \not\leftarrow M \) there is a unique natural transformation \( \theta \omega : G \rightarrow R \) such that \( \theta = \theta \circ \omega \).

2. By Remark 12.1.11(4), a universal conic cell \( \mathcal{J} \leftarrow * \xleftarrow{\omega} R \xrightarrow{M} A \) in Definition 12.3.1 is identified with a special instance of a universal cell in Definition 12.4.1 where \( E \) is the terminal category.

3. Extensions are unique up to isomorphism by Corollary 6.2.8.

**Note.** Recall from Remark 12.1.11(2) that a cell is sliced into pieces of conic cells. In the following, we consider the case where these conic cells are universal (cf. Remark 12.4.2(2)).

#### 12.4.3 Definition.

- A cell \( \mathcal{J} \leftarrow E \xleftarrow{\omega} R \xrightarrow{M} A \) along \( M \) is called pointwise universal if each slice \( \mathcal{J} \leftarrow \omega \xleftarrow{R} \xrightarrow{M} A \) is a universal conic cell, i.e. a direct universal \( (M \not\rightarrow \omega) \)-arrow; in this case, the cell \( \omega \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a pointwise extension of \( \mathcal{J} \) along \( M \).

- A cell \( R \xrightarrow{E} \mathcal{J} \xleftarrow{\omega} M \xrightarrow{A} \) along \( M \) is called pointwise universal if each slice \( \omega \xrightarrow{R} \xleftarrow{M} \xrightarrow{A} A \) is a universal conic cell, i.e. an inverse universal \( (\not\leftarrow M \omega) \)-arrow; in this case, the cell \( \omega \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a pointwise coextension of \( \mathcal{J} \) along \( M \).
12.4.4 Remark. We will see in Theorem 12.4.7 that a pointwise extension is an extension in the sense of Definition 12.4.1.

**Note.** The isomorphism in Remark 12.1.11(2) yields the following.

12.4.5 Theorem.

- A cell

\[
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{E} & \mathcal{R} \\
\mathcal{X} & \xrightarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is an extension (resp. pointwise extension) of \( \mathcal{J} \) along \( \mathcal{M} \) if and only if its right exponential transpose

\[
\begin{array}{ccc}
\mathcal{J} & \xleftarrow{E} & \mathcal{R} \\
\mathcal{X} & \xleftarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is a left Kan lift (resp. pointwise left Kan lift) of \( \mathcal{J} \) along \( \mathcal{M} \), i.e. if and only if the cylinder

\[
\begin{array}{ccc}
\mathcal{J} & \xrightarrow{E} & \mathcal{R} \\
\mathcal{X} & \xrightarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is a colift (resp. pointwise colift) of \( \mathcal{J} \) along \( \mathcal{M} \).

- A cell

\[
\begin{array}{ccc}
\mathcal{J} & \xleftarrow{E} & \mathcal{R} \\
\mathcal{X} & \xleftarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is a coextension (resp. pointwise coextension) of \( \mathcal{J} \) along \( \mathcal{M} \) if and only if its left exponential transpose

\[
\begin{array}{ccc}
\mathcal{J} & \xleftarrow{E} & \mathcal{R} \\
\mathcal{X} & \xleftarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is a right Kan lift (resp. pointwise right Kan lift) of \( \mathcal{J} \) along \( \mathcal{M} \), i.e. if and only if the cylinder

\[
\begin{array}{ccc}
\mathcal{J} & \xleftarrow{E} & \mathcal{R} \\
\mathcal{X} & \xleftarrow{\omega} & \mathcal{M} \\
\end{array}
\]

is a lift (resp. pointwise lift) of \( \mathcal{J} \) along \( \mathcal{M} \).

**Proof.** By the isomorphism in Remark 12.1.11(2), \( \omega \) is universal iff the cylinder \( \omega \xrightarrow{\varphi} \) is direct universal. Since the slice of \( \omega \) at each \( e \in \| E \| \) is given by the component of \( \omega \xrightarrow{\varphi} \) at \( e \), \( \omega \) is pointwise universal iff the cylinder \( \omega \xrightarrow{\varphi} \) is pointwise direct universal. □

12.4.6 Remark. By Theorem 12.4.5 and by the bijectiveness of exponential transposition, we see that

- an extension (resp. pointwise extension) of \( \mathcal{J} \) direct along \( \mathcal{M} \) is the same thing as a left Kan lift (resp. pointwise left Kan lift) of \( \mathcal{J} \) along \( \mathcal{M} \), i.e. a colift (resp. pointwise colift) of \( \mathcal{J} \) along \( \mathcal{M} \).

- an extension (resp. pointwise extension) of \( \mathcal{J} \) inverse along \( \mathcal{M} \) is the same thing as a right Kan lift (resp. pointwise right Kan lift) of \( \mathcal{J} \) along \( \mathcal{M} \), i.e. a lift (resp. pointwise lift) of \( \mathcal{J} \) along \( \mathcal{M} \).

Hence the notion of lift (resp. pointwise lift), in fact that of Kan lift (resp. pointwise Kan lift), subsume the notion of extension (resp. pointwise extension). We will see in Remark 12.6.13 that the converse is also the case.
12.4.7 Theorem. A pointwise universal cell in Definition 12.4.3 is universal in the sense of Definition 12.4.1.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of conditions in Theorem 12.4.5. □

12.4.8 Theorem. Given a pair of modules $E \xrightarrow{J} X \xrightarrow{M} A$,

- if there is a family of universal conic cells $(J)e_j^*, r_e$, one for each object $e \in \|E\|$, then there is a unique functor $R : E \to A$ with $e : R = r_e$ such that $\omega := (x(\omega_e))_{e \in \|E\|}$ forms a cell

\[
\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{\omega} & A \\
\downarrow_{J} & & \downarrow_{R} \\
X & \xrightarrow{M} & A
\end{array}
\end{array}
\]

and $\omega$ is pointwise universal.

- if there is a family of universal conic cells $(J)e_j^*, r_e$, one for each object $e \in \|E\|$, then there is a unique functor $R : E \to X$ with $e : R = r_e$ such that $\omega := (x(\omega_e))_{e \in \|E\|}$ forms a cell

\[
\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{\omega} & X \\
\downarrow_{J} & & \downarrow_{R} \\
X & \xrightarrow{M} & A
\end{array}
\end{array}
\]

and $\omega$ is pointwise universal.

Proof. Since by Definition 12.3.1, a universal conic cell $(J)e_j^*, r_e$ is a direct universal $(M \not\rightarrow)$-arrow $\omega_e : (J)e \rightarrow r_e$, the problem translates by Theorem 12.4.5 into getting the unique colift of $J \not\rightarrow$ along the module $M \not\rightarrow$, i.e. into an instance of Theorem 6.5.14. □

12.4.9 Theorem. Suppose that $D$ is an isomorphism-dense subcategory of a category $E$. Then a cell $\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{\omega} & A \\
\downarrow_{J} & & \downarrow_{R} \\
X & \xrightarrow{M} & A
\end{array}
\end{array}$ is pointwise universal if and only if its right [op. left] slice at each $d \in \|D\|$ is universal.

Proof. Since the slice of $\omega$ at each $e \in \|E\|$ is given by the component of $\omega \not\rightarrow$ at $e$ (see Remark 12.1.11(2)), by Theorem 12.4.5, the assertion is reduced to Theorem 6.5.24. □

Note. In the following, we consider a cell $J \to M$ as an arrow of the module $(J, M)$ defined in Definition 12.1.8.

12.4.10 Definition.

- A cell $D \xrightarrow{J} E$ is called direct universal if it is a direct universal $(J, M)$-arrow. Given a functor $G : D \to X$, a direct universal cell $\omega : G \not\rightarrow R : J \to M$ or the pair $(R, \omega)$, or the functor $R$ itself, is called an extension of $G$ along $J$ and $M$.

- A cell $E \xleftarrow{F} D$ is called inverse universal if it is an inverse universal $(J, M)$-arrow. Given a functor $F : D \to A$, an inverse universal cell $\omega : R \not\rightarrow F : J \to M$ or the pair $(R, \omega)$, or the functor $R$ itself, is called a coextension of $F$ along $J$ and $M$.

12.4.11 Remark.

1. A cell $\theta : G \not\rightarrow R : J \to M$ is direct universal if and only if to every cell $\theta : G \not\rightarrow F : J \to M$ there is a unique natural transformation $\omega \theta : R \not\rightarrow F$ (the adjunct of $\theta$ along $\omega$) such that $\theta = \omega \theta \omega \theta$. Dually, a cell $\theta : R \not\rightarrow F : J \to M$ is inverse universal if and only if to every cell $\theta : G \not\rightarrow F : J \to M$ there is a unique natural transformation $\theta \omega : G \not\rightarrow R$ such that $\theta = \theta \omega \omega \theta$.
(2) As a special case where \( \mathcal{M} \) is given by the hom-module of a category \( \mathcal{C} \),
- a direct universal cell \( \xymatrix{ D \ar[r]^-{f} & E } \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called an extension of \( L \) along \( \mathcal{J} \).
- an inverse universal cell \( \xymatrix{ E \ar[r]^-{g} & D } \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a coextension of \( L \) along \( \mathcal{J} \).

(3) A weighted colimit \( \xymatrix{ E \ar[r]^-{g} & \ast } \) [op. limit \( \ast \)] in Definition 12.3.5 is a special instance
\[
\begin{array}{ccc}
X & \xrightarrow{\omega} & A \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{\mathcal{M}} & A
\end{array}
\]

of an extension [op. coextension] in Definition 12.4.10 where \( D \) is the terminal category.

(4) Extensions are unique up to isomorphism by Corollary 6.2.8.

Note. Recall from Definition 2.1.8 that a cell is sliced into pieces of conic cells. In the following, we consider the case where these conic cells are universal (cf. Remark 12.4.11(3)).

12.4.12 Definition.
- A cell \( \xymatrix{ D \ar[r]^-{f} & E } \) is called pointwise direct universal if each right slice \( \xymatrix{ G \ar[r]^-{\omega} & R } \) is a universal conic cell, i.e. a direct universal \( \langle (\mathcal{J}) \omega, \mathcal{M} \rangle \)-arrow; in this case, the cell \( \omega \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a pointwise extension of \( G \) along \( \mathcal{J} \) and \( \mathcal{M} \).
- A cell \( \xymatrix{ E \ar[r]^-{g} & D } \) is called pointwise inverse universal if each left slice \( \xymatrix{ R \ar[r]^-{\omega} & F } \) is a universal conic cell, i.e. an inverse universal \( \langle e(\mathcal{J}), \mathcal{M} \rangle \)-arrow; in this case, the cell \( \omega \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a pointwise coextension of \( F \) along \( \mathcal{J} \) and \( \mathcal{M} \).

12.4.13 Remark. We will see in Theorem 12.4.15 that a pointwise extension is an extension in the sense of Definition 12.4.10.

Note. Recall from Remark 12.1.9(2) and (3) that a degenerate cell can be depicted as an ordinary cell, and vice versa.

12.4.14 Theorem. Definition 12.4.1 and Definition 12.4.3 subsumes Definition 12.4.10 and Definition 12.4.12 respectively in the following sense:
- a cell \( \xymatrix{ D \ar[r]^-{f} & E } \) is direct universal (resp. pointwise direct universal) in the sense of 12.4.10
  \[
  \begin{array}{ccc}
  X & \xrightarrow{\omega} & A \\
  \downarrow & & \downarrow \\
  \mathcal{M} & \xrightarrow{\mathcal{M}} & A
  \end{array}
  \]
(resp. 12.4.12) if and only if it, depicted as a degenerate cell \( \xymatrix{ D \ar[r]^-{f} & E } \), is universal (resp. pointwise universal) in the sense of 12.4.1 (resp. 12.4.3); that is, an extension (resp. pointwise extension) of \( G \) along \( \mathcal{J} \) and \( \mathcal{M} \) is the same thing as an extension (resp. pointwise extension) of \( \mathcal{J} \) along \( G(\mathcal{M}) \).
- a cell \( \xymatrix{ E \ar[r]^-{g} & D } \) is inverse universal (resp. pointwise inverse universal) in the sense of 12.4.10
  \[
  \begin{array}{ccc}
  X & \xleftarrow{\omega} & A \\
  \downarrow & & \downarrow \\
  \mathcal{M} & \xleftarrow{\mathcal{M}} & A
  \end{array}
  \]
(resp. 12.4.12) if and only if it, depicted as a degenerate cell \( \xymatrix{ E \ar[r]^-{g} & D } \), is universal (resp. pointwise universal) in the sense of 12.4.1 (resp. 12.4.3).
pointwise universal) in the sense of 12.4.1 (resp. 12.4.3); that is, a coextension (resp. pointwise coextension) of $\mathcal{F}$ along $\mathcal{J}$ and $\mathcal{M}$ is the same thing as a coextension (resp. pointwise coextension) of $\mathcal{J}$ along $\langle \mathcal{M} \rangle \mathcal{F}$.

Conversely, Definition 12.4.10 and Definition 12.4.12 subsumes Definition 12.4.1 and Definition 12.4.3 respectively in the following sense:

- a cell $\begin{array}{c} \mathcal{J} \xrightarrow{\omega} \mathbb{E} \end{array}$ is universal (resp. pointwise universal) in the sense of 12.4.1 (resp. 12.4.3) if and only if it, depicted as an ordinary cell $\begin{array}{c} X \xrightarrow{\omega} \mathbb{E} \end{array}$, is direct universal (resp. pointwise direct universal) in the sense of 12.4.10 (resp. 12.4.12); that is, a coextension (resp. pointwise extension) of the identity $X \to X$ along $\mathcal{J}$ and $\mathcal{M}$.

- a cell $\begin{array}{c} \mathbb{E} \xrightarrow{\omega} \mathcal{J} \end{array}$ is universal (resp. pointwise universal) in the sense of 12.4.1 (resp. 12.4.3) if and only if it, depicted as an ordinary cell $\begin{array}{c} \mathbb{E} \xrightarrow{\omega} \mathcal{J} \end{array}$, is inverse universal (resp. pointwise inverse universal) in the sense of 12.4.10 (resp. 12.4.12); that is, an extension (resp. pointwise extension) of the identity $\mathbb{E} \to \mathbb{E}$ along $\mathcal{J}$ and $\mathcal{M}$.

Proof. The left slice of the module $\langle \mathcal{J}, \mathcal{M} \rangle$ at $\mathcal{G}$ and the left slice of the module $\langle \mathcal{G}(\mathcal{M}) \rangle \mathcal{E}$ at $\mathcal{J}$ are the same left module over $[\mathbb{E}, \mathbb{A}]$ given by

$$R \mapsto (\mathcal{J})(\mathcal{D} : \mathbb{E})(\mathcal{G}(\mathcal{M}) \mathcal{R})$$

; hence the cell $\begin{array}{c} \mathcal{D} \xrightarrow{\mathcal{J}} \mathbb{E} \end{array}$ is direct universal iff the cell $\begin{array}{c} \mathcal{E} \xrightarrow{\mathcal{J}} \mathbb{D} \end{array}$ is universal. Since by Theorem 12.3.7, the slice $\begin{array}{c} \mathcal{J}(\mathcal{G}(\mathcal{M}) \mathcal{R}) \end{array}$ of the cell $\begin{array}{c} \mathcal{D} \xrightarrow{\mathcal{J}} \mathbb{E} \end{array}$ at $e \in [\mathbb{E}]$ is universal iff the slice $\begin{array}{c} \mathcal{D} \xrightarrow{\mathcal{J}} \mathbb{A} \end{array}$ is universal, the pointwise version follows.

The left slice of the module $\mathcal{M} \not\supseteq \mathbb{E}$ at $\mathcal{J}$ and the left slice of the module $\langle \mathcal{J}, \mathcal{M} \rangle$ at $1_\mathbb{X}$ are the same left module over $[\mathbb{E}, \mathbb{A}]$ given by

$$R \mapsto (\mathcal{J})(\mathcal{X} : \mathbb{E})((\mathcal{M}) \mathcal{R})$$

; hence the cell $\begin{array}{c} \mathcal{J} \xrightarrow{\mathcal{E}} \mathbb{X} \end{array}$ is universal iff the cell $\begin{array}{c} \mathbb{X} \xrightarrow{\mathcal{J}} \mathbb{E} \end{array}$ is direct universal. Since by Theorem 12.3.7, the slice $\begin{array}{c} \mathcal{J}(\mathcal{G}(\mathcal{M}) \mathcal{R}) \end{array}$ of the cell $\begin{array}{c} \mathbb{X} \xrightarrow{\mathcal{E}} \mathbb{A} \end{array}$ at $e \in [\mathbb{E}]$ is universal iff the slice $\begin{array}{c} \mathbb{X} \xrightarrow{\mathcal{E}} \mathbb{A} \end{array}$ is universal, the pointwise version follows. \qed
12.4.15 Theorem. A pointwise universal cell in Definition 12.4.12 is universal in the sense of Definition 12.4.10.

Proof. By the equivalence in Theorem 12.4.14, this is reduced to Theorem 12.4.7.

Note. Theorem 12.4.16 and Theorem 12.4.8 are reduced to each other by the equivalence of conditions in Theorem 12.4.14 and that in Theorem 12.3.7.

12.4.16 Theorem. Let $\mathcal{J} : D \to E$ [op. $\mathcal{J} : E \to D$] and $\mathcal{M} : X \to A$ be modules. 

- Given a functor $G : D \to X$, if there is a family of universal conic cells $D \xrightarrow{(\mathcal{J})^*} E$ and $X \xrightarrow{\mathcal{M}} A$, for each object $e \in |E|$, then there is a unique functor $R : E \to A$ with $e : R = r_e$ such that $\omega := (d(\omega_e))_{d \in |D|, e \in |E|}$ forms a cell $D \xrightarrow{\mathcal{J}} E$ and $X \xrightarrow{\mathcal{M}} A$.

- Given a functor $F : D \to A$, if there is a family of universal conic cells $\mathcal{J}^* D$ and $X \xrightarrow{\mathcal{M}} A$, for each object $e \in |E|$, then there is a unique functor $R : E \to X$ with $e : R = r_e$ such that $\omega := (\langle \omega_e \rangle d)_{e \in |E|, d \in |D|}$ forms a cell $E \xrightarrow{\mathcal{J}} D$ and $X \xrightarrow{\mathcal{M}} A$.

Note. The axiom of choice is used in the proof of the following.

12.4.17 Corollary. Let $\mathcal{J} : D \to E$ [op. $\mathcal{J} : E \to D$] and $\mathcal{M} : X \to A$ be modules. 

- If $\mathcal{M}$ is cocomplete and $D$ is small, then a functor $G : D \to X$ has a pointwise extension $D \xrightarrow{\mathcal{J}} E$ along $\mathcal{J}$ and $\mathcal{M}$.

- If $\mathcal{M}$ is complete and $D$ is small, then a functor $F : D \to A$ has a pointwise coextension $E \xrightarrow{\mathcal{J}} D$ along $\mathcal{J}$ and $\mathcal{M}$.

Proof. By Corollary 12.3.10, $G$ has a colimit along $\mathcal{M}$ weighted by $\langle \mathcal{J} \rangle e$ for any $e \in |E|$, and by Theorem 12.4.16, a family of universal conic cells $\omega_e : G \rightharpoonup r_e : \langle \mathcal{J} \rangle e \rightharpoonup \mathcal{M}$, one chosen for each $e \in |E|$, extends to a pointwise direct universal cell $\omega : G \rightharpoonup R : \mathcal{J} \rightharpoonup \mathcal{M}$.

12.4.18 Definition. A cell $X \xrightarrow{\mathcal{J}} A$ is said to

- preserve extensions along a module $\mathcal{J} : D \to E$ if any direct universal cell $D \xrightarrow{\mathcal{J}} E$ yields by $G \blacktriangleright \psi \blacktriangleleft Q \rightharpoonup R \psi$ compositions with $\psi$ a direct universal cell $D \xrightarrow{\mathcal{J}} E$. 

$\square$
12.5. Extensions and representations

- preserve coextensions along a module \( \mathcal{J} : E \to D \) if any inverse universal cell \( \xymatrix{ E \ar[r]^-J & D } \) yields
  \( \xymatrix{ E \ar[r]^-J & D } \)
  by composition with \( \psi \) an inverse universal cell

\[
\xymatrix{ E \ar[r]^-J & D \\
X \ar[r]_-M & A }
\]

12.4.19 Remark.
(1) Since a direct universal cell \( \omega : J \to M \) is the same thing as a direct universal \( \langle J, M \rangle \)-arrow, to say that a cell \( \psi : M \to N \) preserves extensions along \( J \) is to say that the postcomposition cell \( \langle J, \psi \rangle \) (see Definition 1.2.25) preserves direct universal arrows. Dually, a cell \( \psi : M \to N \) preserves coextensions along \( J \) precisely when the postcomposition cell \( \langle J, \psi \rangle \) preserves inverse universal arrows.

(2) A cell is said to preserve extensions if it preserves extensions along any modules.

(3) When we say that a cell \( \psi \) preserves pointwise extensions [op. coextensions], we require that \( \psi \) preserves pointwise universality.

12.4.20 Proposition. If a cell preserves weighted colimits [op. limits], or equivalently (see Theorem 12.3.17), preserves colimits [op. limits], then it preserves pointwise extensions [op. coextensions].

Proof. Immediate because the right slice of the composite cell \( \omega \circ \psi \) at \( e \in \|E\| \) is given by \( (\omega \circ \psi) e = (\omega) e \circ \psi \).

12.4.21 Remark. Since a weighted colimits [op. limits] is a special instance of a pointwise extension [op. coextension] (see Remark 12.4.11(3)), the converse also holds.

12.4.22 Theorem. If a cell has a right [op. left] adjoint (see Definition 7.10.1), then it preserves extensions [op. coextensions] and pointwise extensions [op. coextensions].

Proof. By Remark 12.4.19(1), the assertion is equivalent to saying that if a cell \( \psi \) has a right adjoint, then the postcomposition cell \( \langle J, \psi \rangle \) preserves direct universal arrows for any module \( J \). But this follows from Theorem 7.10.6 because if \( \psi \) has a right adjoint, so does \( \langle J, \psi \rangle \) by Theorem 7.10.3. The pointwise version is reduced to Theorem 12.3.15 by virtue of Proposition 12.4.20.

12.5 Extensions and representations

In this section, we consider a special case of a cellular limit and an extension defined in Definition 12.3.1 and Definition 12.4.1 where \( M \) is given by the hom-module of a category, and study the relation with the representations of a module. It is shown that a universal conic cell

\[
\xymatrix{ M \ar[r]^-* & X \ar[l]_\omega \ar[r]_r & X }
\]
gives a representation \( \omega : M \cong (X)r \) of a right module \( M : X \to \ast \) precisely when \( M \) preserves colimits. The result is then generalized for two-sided modules.

Note. The following definition is a special case of Definition 12.3.1 where \( M \) is given by the hom-module of a category; we consider a conic cell as an arrow of the module \( X \mathcal{G} \) defined in Definition 12.1.6.

12.5.1 Definition.
- A conic cell \( \xymatrix{ M \ar[r]^-* & X } \) is called universal if it is a direct universal \( (X \mathcal{G}) \)-arrow. Given a right module \( M : X \to \ast \), a universal conic cell \( \omega : M \to r : X \mathcal{G} \) or the pair \( (r, \omega) \), or the object \( r \) itself, is called a colimit of \( M \) in \( X \).
12.5. Extensions and representations

A conic cell $\xymatrix{ \mathcal{M} \ar[r]^-{\omega} & \mathcal{A} & \mathcal{M} \ar@{-->}[l]^-{\mathcal{M}} \ar[r]^-{\omega} & \mathcal{A} }$ is called universal if it is an inverse universal $(\mathcal{A})$-arrow. Given a left module $\mathcal{M} : \ast \to \mathcal{A}$, a universal conic cell $\omega : \mathcal{M} \to \mathcal{A}$ or the pair $(\mathcal{M}, \omega)$, or the object $\mathcal{M}$ itself, is called a limit of $\mathcal{M}$ in $\mathcal{A}$.

12.5.2 Remark. A colimit of a right module $\mathcal{M} : \mathcal{X} \to \ast$ in $\mathcal{X}$ is the same thing as a colimit of $\mathcal{M}$ along the hom-module $(\mathcal{X})$. Dually, a limit of a left module $\mathcal{M} : \ast \to \mathcal{A}$ in $\mathcal{A}$ is the same thing as a limit of $\mathcal{M}$ along the hom-module $(\mathcal{A})$.

Note. Recall from Remark 12.2.8(1) that a conic cell is transposed to a cone.

12.5.3 Proposition. A conic cell

$\xymatrix{ \mathcal{X} \ar[r]^-{\omega} & \mathcal{A} & \mathcal{X} \ar@{-->}[l]^-{\mathcal{X}} \ar[r]^-{\omega} & \mathcal{A} }$ is universal if and only if its comma transpose

$\xymatrix{ \mathcal{X} \ar[r]^-{\omega} & \mathcal{A} & \mathcal{X} \ar@{-->}[l]^-{\mathcal{X}} \ar[r]^-{\omega} & \mathcal{A} }$ is a universal cone.

Proof. By Theorem 7.13.13, the equivalence cell $\downarrow_{\mathcal{X}}: (\mathcal{X}) \to (\mathcal{X})$ in Theorem 12.2.7 preserves and reflects direct universal arrows.

Note. Recall from Remark 2.3.4(2) that a representation of a right module is expressed by a conic cell.

12.5.4 Proposition.

- A representation $(\mathcal{M}, \omega)$ of a right module $\mathcal{M} : \mathcal{X} \to \ast$ gives a colimit $\xymatrix{ \mathcal{X} \ar[r]^-{\omega} & \mathcal{A} & \mathcal{X} \ar@{-->}[l]^-{\mathcal{X}} \ar[r]^-{\omega} & \mathcal{A} }$ of $\mathcal{M}$ in $\mathcal{X}$.

- A representation $(\mathcal{M}, \omega)$ of a left module $\mathcal{M} : \ast \to \mathcal{A}$ gives a limit $\xymatrix{ \mathcal{X} \ar[r]^-{\omega} & \mathcal{A} & \mathcal{X} \ar@{-->}[l]^-{\mathcal{X}} \ar[r]^-{\omega} & \mathcal{A} }$ of $\mathcal{M}$ in $\mathcal{A}$.

Proof. Since $\omega : \mathcal{M} \to (\mathcal{X}) \ar[r]^-{\mathcal{M}} & \ast$ is an isomorphism and the Yoneda functor $\mathcal{X} \to (\mathcal{X})$ is fully faithful, $\omega$ is a direct universal $(\mathcal{X})$-arrow by Corollary 6.2.12.

12.5.5 Remark. The converse does not hold in general: a colimits does not necessarily give a representation. For example, consider the conic cell
given by the unique function $!$ from the set $\{0,1\}$ to the set $\{0\}$ (i.e. the hom-module of the terminal category). The cell, being the only cell $\{0,1\} \to \ast$, forms a colimit of $\{0,1\}$ but not a representation.

Note. Despite Remark 12.5.5, we have the following; if $\mathcal{M}$ preserves colimits [op. limits], the converse of Proposition 12.5.4 holds.
For any right module \( M : X \to * \), the following conditions are equivalent:
1. \( M \) is representable;
2. \( M \) preserves colimits and \( M \) has a colimit (in the sense of Definition 12.5.1) in \( X \);
3. \( M \) preserves colimits and the right comma fibration \( M^1 : [M] \to X \) has a colimit in \( X \).

When these conditions hold, a conic cell \( \xymatrix{ \omega : M \ar[r] & (X) \ar@{..>}[r] & X } \) is universal if and only if the right module morphism \( \omega : M \to (X) \) is iso; that is, if and only if the pair \((r, \omega)\) forms a representation of \( M \).

For any left module \( M : * \to A \), the following conditions are equivalent:
1. \( M \) is representable;
2. \( M \) preserves limits and \( M \) has a limit (in the sense of Definition 12.5.1) in \( A \);
3. \( M \) preserves limits and the left comma fibration \( M^1 : [M] \to A \) has a limit in \( A \).

When these conditions hold, a conic cell \( \xymatrix{ \omega : M \ar[r] & A \ar@{..>}[r] & X } \) is universal if and only if the left module morphism \( \omega : M \to A \) is iso; that is, if and only if the pair \((r, \omega)\) forms a representation of \( M \).

**Proof.** (2)\(\Rightarrow\) (3) By Proposition 12.3.3.

(1)\(\Rightarrow\)(3) \( M \) preserves colimits by Corollary 8.7.10. Since the comma category \( [M] \) has a terminal object by Proposition 6.1.4, \( M^1 : [M] \to X \) has a colimit by Corollary 10.1.20.

(3)\(\Rightarrow\)(1) \( M^1 : [M] \to X \) creates a colimit of the identity \( [M] \to [M] \) from its own colimit. Hence the comma category \( [M] \) has a terminal object by Theorem 10.1.21, and \( M \) is representable by Proposition 6.1.4.

The “if” part of the second assertion is already seen in Proposition 12.5.4. Since a universal conic cell \( \omega : M \rightarrow (X) \) is the same thing as a direct universal \( (X \mathcal{F}) \)-arrow, the “only if” part follows by applying Theorem 6.2.14 to the right Yoneda functor \( X \mathcal{F} \).

**Note.** The following definition is a special case of Definition 12.4.1 where \( M \) is given by the hom-module of a category; we consider a cell as an arrow of the module \( (X \mathcal{F} A) \) defined in Definition 12.1.13.

**12.5.7 Definition.**

- A cell \( \xymatrix{ \omega : M \ar[r] & A \ar@{..>}[r] & X } \) is called universal if it is a direct universal \( (X \mathcal{F} A) \)-arrow. Given a module \( M : X \to A \), a universal cell \( \omega : J \rightarrow R : M \mathcal{F} E \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called an extension of \( M \) in \( X \).

- A cell \( \xymatrix{ \omega : A \ar[r] & M \ar@{..>}[r] & A } \) is called universal if it is an inverse universal \( (X \mathcal{G} A) \)-arrow. Given a module \( M : X \to A \), a universal cell \( \omega : R \rightarrow M : X \mathcal{G} A \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a coextension of \( M \) in \( A \).

**12.5.8 Remark.**

1. A extension of a module \( M : X \to A \) is the same thing as an extension of \( M \) along the hom-module \( (X) \), and dually a coextension of a module \( M : X \to A \) is the same thing as a coextension of \( M \) along the hom-module \( (A) \).

2. A universal conic cell \( \xymatrix{ \omega : M \ar[r] & X \ar@{..>}[r] & (X) } \) in Definition 12.5.1 is identified with a special instance of a universal cell in Definition 12.5.7 where \( A \mathcal{G} [op. \ X] \) is the terminal category.
Note. The following definition is a special case of Definition 12.4.3 where \( \mathcal{M} \) is given by the hom-module of a category; we consider the case where each slice of a cell is a universal conic cell defined in Definition 12.5.1.

12.5.9 Definition. A cell \( \mathcal{M} : X \to A \) is called pointwise universal if each slice \( \mathcal{M} \) of \( \mathcal{M} \) is a universal conic cell, i.e. a direct universal \( (X \rightarrow) \)-arrow; in this case, the cell \( \omega \) or the pair \((R, \omega)\), or the functor \( R \) itself, is called a pointwise extension of \( \mathcal{M} \) in \( X \).

12.5.10 Remark. (1) A pointwise extension of a module \( \mathcal{M} : X \to A \) is the same thing as a pointwise extension of \( \mathcal{M} \) along the hom-module \( (X) \), and dually a pointwise coextension of a module \( \mathcal{M} : X \to A \) is the same thing as a pointwise coextension of \( \mathcal{M} \) along the hom-module \( (A) \).

(2) By Theorem 12.4.7, a pointwise universal cell is universal in the sense of Definition 12.5.7.

12.5.11 Proposition. Let \( \mathcal{M} : X \to A \) be a module.

- A corepresentation \( (R, \omega) \) of \( \mathcal{M} \) gives a pointwise extension

\[
\mathcal{M} : X \to A
\]

\[
\xymatrix{ X \ar[r]^-R & A \ar@{..>}[l]_\omega }
\]

of \( \mathcal{M} \).

- A representation \( (R, \omega) \) of \( \mathcal{M} \) gives a pointwise coextension

\[
\mathcal{M} : X \to A
\]

\[
\xymatrix{ A \ar[r]^-R & X \ar@{..>}[l]^\omega }
\]

of \( \mathcal{M} \).

Proof. By definition, \( \omega \) is a pointwise extension of \( \mathcal{M} \) iff the slice \( (\mathcal{M})_a \) is a universal conic cell for each \( a \in \| A \| \), and by Proposition 2.3.15, \( (R, \omega) \) is a corepresentation of \( \mathcal{M} \) iff for each \( a \in \| A \| \), \((R \cdot a, (\omega) a)\) is a representation of \( (\mathcal{M})_a \). The assertion is thus reduced to Proposition 12.5.4.

12.5.12 Remark. The converse does not hold as we saw in Remark 12.5.5; note that Proposition 12.5.4 is a special case of Proposition 12.5.11 where \( A \) [op. \( X \)] is the terminal category.

12.5.13 Theorem. Given a module \( \mathcal{M} : X \to A \),

- if there is a family of universal conic cells \( (\mathcal{M})_a \), one for each object \( a \in \| A \| \), then there is a unique functor \( \mathcal{R} : A \to X \) with \( \mathcal{R} a = r_a \) such that \( \omega := (x(a)a)_{a \in \| A \|} \) forms a cell.

\[
\xymatrix{ X \ar[r]^-R & A \ar@{..>}[l]_\omega }
\]

- if there is a family of universal conic cells \( (\mathcal{M})_a \), one for each object \( x \in \| X \| \), then there is a unique functor \( X \to A \) with \( x : \mathcal{R} = r_x \) such that \( \omega := (x(a)a)_{a \in \| A \|} \) forms a cell.

\[
\xymatrix{ X \ar[r]^-R & A \ar@{..>}[l]^\omega }
\]
Proof. Recalling Remark 12.5.10(1) and Remark 12.5.2, we see that this is a special case of Theorem 12.4.8 where $\mathcal{M}$ is given by the hom-module of a category.

Note. The following generalizes Theorem 12.5.6 for two-sided modules.

12.5.14 Theorem. Let $\mathcal{M} : X \to A$ be a module.

- The following conditions are equivalent:
  1. $\mathcal{M}$ is representable;
  2. each right slice $\langle \mathcal{M} \rangle a : X \to *$ preserves colimits and $\mathcal{M}$ has a pointwise extension.

When these conditions hold, a cell

\[
\begin{array}{ccc}
X & \xrightarrow{\omega} & A \\
\downarrow \mathcal{M} & & \downarrow R \\
\langle X \rangle & \xrightarrow{\omega} & \langle A \rangle
\end{array}
\]

is pointwise universal if and only if the module morphism $\omega : M \to \langle X \rangle R$ is iso; that is, if and only if the pair $(R, \omega)$ forms a corepresentation of $\mathcal{M}$.

- The following conditions are equivalent:
  1. $\mathcal{M}$ is representable;
  2. each left slice $\langle \mathcal{M} \rangle : * \to A$ preserves limits and $\mathcal{M}$ has a pointwise coextension.

When these conditions hold, a cell

\[
\begin{array}{ccc}
A & \xleftarrow{\omega} & X \\
\downarrow \mathcal{M} & & \downarrow \mu \\
\langle A \rangle & \xleftarrow{\omega} & \langle X \rangle
\end{array}
\]

is pointwise universal if and only if the module morphism $\omega : M \to R(A)$ is iso; that is, if and only if the pair $(R, \omega)$ forms a representation of $\mathcal{M}$.

Proof. By Corollary 6.4.11, $\mathcal{M}$ is corepresentable iff each right slice $\langle \mathcal{M} \rangle a : X \to *$ is representable. By Theorem 12.5.13, $\mathcal{M}$ has a pointwise extension iff each right slice $\langle \mathcal{M} \rangle a : X \to *$ has a colimit. The first assertion is thus reduced to Theorem 12.5.6. The “if” part of the second assertion is already seen in Proposition 12.5.11. Since a pointwise universal cell is universal (see Remark 12.5.10(2)) and a universal cell $\omega : M \to R : X \not\to A$ is the same thing as a direct universal $\langle X \not\to A \rangle$-arrow, the “only if” part follows by applying Theorem 6.2.14 to the right generalized Yoneda functor $X \not\to A$. 

### 12.6 Cylindrical extensions

A cylindrical extension is defined as a universal weighted cylinder; however, it transforms reversibly into an instance of a (cellular) extension defined in Section 12.4. A cylindrical extension is introduced here to reconcile the notion of extensions (universal cells) and lifts (universal cylinders); it is shown that a pointwise lift transforms reversibly into a pointwise extension via a pointwise cylindrical extension (together with the converse we saw in Section 12.4, we now see that the notion of pointwise lift and that of pointwise extension subsume each other). Cylindrical extensions also bridge the gap between cellular extensions and Kan extensions to be studied in Section 12.7.

Note. In the following, we consider a left (op. right) $K$-weighted cylinder along $\mathcal{M}$ as an arrow of the module $(K \triangleright \mathcal{M})$ (op. $(K \triangleleft \mathcal{M})$) defined in Definition 4.5.3.

#### 12.6.1 Definition.

- A left $K$-weighted cylinder $\begin{array}{c} E \xrightarrow{K} D \\ \downarrow \mu \downarrow R \\ X \xrightarrow{\mathcal{M}} A \end{array}$ is called universal if it is a direct universal $(K \triangleright \mathcal{M})$-arrow. Given a functor $G : E \to X$, a universal left $K$-weighted cylinder $\mu : G \rightleftharpoons R \circ K : E \rightleftharpoons \mathcal{M}$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a (cylindrical) extension of $G$ along $K$ and $\mathcal{M}$.

- A right $K$-weighted cylinder $\begin{array}{c} D \xleftarrow{K} E \\ \downarrow \mu \downarrow F \\ X \xleftarrow{\mathcal{M}} A \end{array}$ is called universal if it is an inverse universal $(K \triangleleft \mathcal{M})$-arrow. Given a functor $F : E \to A$, a universal right $K$-weighted cylinder $\mu : K \triangleleft R \rightleftharpoons F : E \rightleftharpoons \mathcal{M}$.
along \(\mathcal{M}\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a (cylindrical) coextension of \(F\) along \(K\) and \(\mathcal{M}\).

**12.6.2 Remark.**

(1) A left \(K\)-weighted cylinder \(\mu: G \Rightarrow R \circ K: E \Rightarrow \mathcal{M}\) is universal if and only if to every left \(K\)-weighted cylinder \(\alpha: G \Rightarrow F \circ K: E \Rightarrow \mathcal{M}\) there is a unique natural transformation \(\mu|\alpha: R \Rightarrow F\) (the adjunct of \(\alpha\) along \(\mu\)) such that \(\alpha = \mu \circ [K \circ \mu|\alpha]\) (see Definition 4.5.3(2)). Dually, a right \(K\)-weighted cylinder \(\mu: K \circ R \Rightarrow F: E \Rightarrow \mathcal{M}\) is universal if and only if to every right \(K\)-weighted cylinder \(\alpha: K \circ G \Rightarrow F: E \Rightarrow \mathcal{M}\) there is a unique natural transformation \(\alpha/\mu: G \Rightarrow R\) such that \(\alpha = [K \circ \alpha/\mu] \circ \mu\).

(2) Since \((E^*, \mathcal{M}) = (\{E \rightharpoonup \mathcal{M}\})\) by Theorem 4.6.22, a colimit \[
E \xrightarrow{\mu} \star
\]

is direct [op. inverse] universal.

**12.6.3 Proposition.** A left [op. right] \(K\)-weighted cylinder

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{K} & \text{D} \\
\text{G} & \mu & \downarrow R \\
\text{X} & \xrightarrow{\mu|\mathcal{M}} & \text{A}
\end{array}
\]

is universal if and only if the cell

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{K(D)} & \text{D} \\
\text{G} & \mu|\mathcal{D} & \downarrow R \\
\text{X} & \xrightarrow{\mu|\mathcal{M}} & \text{A}
\end{array}
\]

is direct [op. inverse] universal.

**Proof.** By the isomorphism in Theorem 5.5.1, the cylinder \(\mu\) is universal iff the cell \(\mu|D\) is universal.

\(\square\)

**12.6.4 Remark.** A cylindrical extension is thus regarded as a special instance of an extension in Definition 12.4.10 where \(\mathcal{J}\) is representable.

**Note.** The pointwise universality of a weighted cylinder is defined below indirectly using the pointwise universality of a cell defined in Definition 12.4.12 so that Proposition 12.6.3 holds with “universal” replaced by “pointwise universal”.

**12.6.5 Definition.**

- A left \(K\)-weighted cylinder \(\xrightarrow{\mu} D\) is called pointwise universal if the cell \(\xrightarrow{K(D)} D\) is pointwise universal; in this case, the weighted cylinder \(\mu\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a pointwise (cylindrical) extension of \(G\) along \(K\) and \(\mathcal{M}\).
A right $K$-weighted cylinder $\underbrace{D \xrightarrow{K} E}_{\text{R} \uparrow \mu \downarrow F}$ is called pointwise universal if the cell $\underbrace{D \xrightarrow{(D)K} E}_{\text{R} \uparrow D \uparrow \mu \downarrow F}$ is pointwise universal; in this case, the weighted cylinder $\mu$ or the pair $(R, \mu)$, or the functor $R$ itself, is called a pointwise (cylindrical) coextension of $F$ along $K$ and $M$.

12.6.6 Remark.

(1) We will see in Theorem 12.6.9 that a pointwise extension is an extension in the sense of Definition 12.6.1.

(2) A pointwise extension defined in Definition 12.6.5 is referred to as a pointwise cylindrical extension to distinguish it from a pointwise (cellular) extension defined in Definition 12.4.12. A pointwise cylindrical extension is regarded as a special instance of a pointwise extension in Definition 12.4.12 where $J$ is representable.

12.6.7 Theorem. Let $K : E \to D$ be a functor and $M : X \to A$ be module.

- Given a functor $G : E \to X$, a functor $R : D \to A$ is an extension (resp. pointwise extension) $\underbrace{E \xrightarrow{K} D}_{\text{G} \uparrow \mu \downarrow R}$ of $G$ along $K$ and $M$ if and only if it is an extension (resp. pointwise extension) $\underbrace{E \xrightarrow{K(D)} D}_{\text{G} \uparrow \omega \downarrow R}$ of $G$ along the representable module $K(D)$ and $M$.

- Given a functor $F : E \to A$, a functor $R : D \to X$ is a coextension (resp. pointwise coextension) $\underbrace{D \xrightarrow{K} E}_{\text{R} \uparrow \mu \downarrow F}$ of $F$ along $K$ and $M$ if and only if it is a coextension (resp. pointwise coextension) $\underbrace{D \xrightarrow{(D)K} E}_{\text{R} \uparrow \omega \downarrow F}$ of $F$ along the corepresentable module $(D)K$ and $M$.

Proof. This is immediate from Proposition 12.6.3, Definition 12.6.5, and the bijectivity of the generalized Yoneda morphism $\mu \mapsto \mu \downarrow D$: if a cell $\underbrace{E \xrightarrow{K(D)} D}_{\text{G} \uparrow \omega \downarrow R}$ is inverse universal (resp. pointwise inverse universal), then the weighted cylinder $\underbrace{E \xrightarrow{K} D}_{\text{G} \uparrow \mu \downarrow R}$ (see Theorem 5.5.1) is universal (resp. pointwise universal). 

12.6.8 Remark. The notion of (cellular) extension thus subsumes that of cylindrical extension. We will see in Remark 12.6.13 that the converse is also the case.

12.6.9 Theorem. A pointwise universal weighted cylinder in Definition 12.6.5 is universal in the sense of Definition 12.6.1.

Proof. This is reduced to Theorem 12.4.15 by the equivalence of conditions in Theorem 12.6.7.

12.6.10 Theorem. Let $K : E \to D$ be a functor and $M : X \to A$ be module.

- If $M$ is cocomplete and $E$ is small, then a functor $G : E \to X$ has a pointwise extension $\underbrace{E \xrightarrow{K} D}_{\text{G} \uparrow \mu \downarrow R}$ along $K$ and $M$.
If \( \mathcal{M} \) is complete and \( \mathcal{E} \) is small, then a functor \( F: \mathcal{E} \to \mathcal{A} \) has a pointwise coextension

\[
\begin{array}{c}
D \xrightarrow{K} \mathcal{E} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mu} \xrightarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

along \( K \) and \( \mathcal{M} \).

**Proof.** By the equivalence of conditions in Theorem 12.6.7, this is reduced to a special case of Corollary 12.4.17 where \( J \) is given by the representable module \( K(D) \).

**Note.** Since a left \( K \)-weighted cylinder

\[
\begin{array}{c}
\mathcal{G} \xrightarrow{\mu} \mathcal{E} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is depicted as an ordinary two-sided cylinder

\[
\begin{array}{c}
\mathcal{G} \xrightarrow{\mu} \mathcal{E} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

we may ask if an extension transforms into a colift. The answer is negative in general; however, we have the following.

**12.6.11 Theorem.** Given a fully faithful functor \( K: \mathcal{E} \to \mathcal{D} \),

- if a left \( K \)-weighted cylinder

\[
\begin{array}{c}
\mathcal{G} \xrightarrow{\mu} \mathcal{E} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is a pointwise extension of \( \mathcal{G} \), then the cylinder

\[
\begin{array}{c}
\mathcal{G} \xrightarrow{\mu} \mathcal{E} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is a pointwise colift of \( \mathcal{G} \).

- if a right \( K \)-weighted cylinder

\[
\begin{array}{c}
\mathcal{D} \xleftarrow{K} \mathcal{E} \\
\xrightarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is a pointwise coextension of \( \mathcal{F} \), then the cylinder

\[
\begin{array}{c}
\mathcal{D} \xleftarrow{K} \mathcal{E} \\
\xrightarrow{\mathcal{R}} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is a pointwise lift of \( \mathcal{F} \).

The converse holds if \( K \) is fully faithful and essentially surjective, i.e. if \( K \) is an equivalence.

**Proof.** This follows as a special case of the lemma below where \( \mathcal{M} \) is given by the composite module \( G(M) \). (Conversely, the lemma is a special case of the theorem where \( G \) is an identity.)

**Lemma.** Given a fully faithful functor \( K: \mathcal{X} \to \mathcal{D} \), if a left \( K \)-weighted cylinder

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{K} \mathcal{D} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mu} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is pointwise universal, i.e. if the cell

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{K} \mathcal{D} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mu} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

is pointwise direct universal. The converse holds if \( K \) is fully faithful and essentially surjective.

**Proof.** The first assertion follows immediately from the claim below. The second assertion follows from the claim and Theorem 12.4.9.

**Claim.** For any \( x \in \mathcal{X} \), the right slice

\[
\begin{array}{c}
\mathcal{X} \xrightarrow{K(D)(x)} \mathcal{D} \\
\xleftarrow{\mathcal{R}} \xleftarrow{\mu} \xleftarrow{\mathcal{M}} \mathcal{A}
\end{array}
\]

of \( \mu \uparrow \mathcal{D} \) at \( K \cdot x \) is a universal conic cell iff the component \( \mu_x: x \to R \cdot K \cdot x \) of \( \mu \) at \( x \) is a direct universal \( \mathcal{M} \)-arrow.
Proof. Depict $\mu$ as $X \xrightarrow{\mu} \mathcal{M} \xrightarrow{\psi} \mathcal{Q}$ and apply Proposition 5.3.6 and Corollary 5.3.12 to these cylinders respectively. Then we have a commutative diagram

\begin{align*}
\xymatrix{
\langle X \rangle \ar[r]^{(K)x} & K(D)(K'x) \\
x_{\langle \mu \rangle x} \ar[u] & \downarrow_{(\mu)(D)(K'x)} \\
\langle \mathcal{M} \rangle (R ; K'x) & \downarrow_{(\mathcal{M} R)(K'x)}
}
\end{align*}

(the upper right triangle commutes by Proposition 5.3.6, and the lower left triangle commutes by Corollary 5.3.12). Since $K$ is fully faithful, $(K)x$ is iso. Hence $(\mu \uparrow D)(K'x) : K(D)(K'x) \rightarrow R ; K'x$ is a direct universal $(\mathcal{M} \varphi)$-arrow if so is $X \uparrow \mu x : \langle X \rangle x \sim R ; K'x$, and by Theorem 12.3.4, this is the case iff $\mu x : x \sim R ; K'x$ is a direct universal $\mathcal{M}$-arrow.

12.6.12 Corollary. Let $\mathcal{M} : X \rightarrow A$ be a module.

> Given a pair of functors $X \xrightarrow{G} E \xrightarrow{\mu} A$, the following conditions are equivalent:

1. $R$ is a colift (resp. pointwise colift) $\langle X \rangle \xrightarrow{E \rightarrow E} \mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{G}$ along $\mathcal{M}$;

2. $R$ is an extension (resp. pointwise extension) $\mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{G}$ along the identity functor $E \rightarrow E$ and $\mathcal{M}$;

3. $R$ is an extension (resp. pointwise extension) $\mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{G}$ along the hom-module $(E)$ and $\mathcal{M}$.

> Given a pair of functors $X \xrightarrow{R} E \xrightarrow{\mu} A$, the following conditions are equivalent:

1. $R$ is a lift (resp. pointwise lift) $\langle X \rangle \xrightarrow{E \rightarrow E} \mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{F}$ along $\mathcal{M}$;

2. $R$ is a coextension (resp. pointwise coextension) $\mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{F}$ along the identity functor $E \rightarrow E$ and $\mathcal{M}$;

3. $R$ is a coextension (resp. pointwise coextension) $\mathcal{E} \xrightarrow{\mu} A$ of $\mathcal{F}$ along the hom-module $(E)$ and $\mathcal{M}$.

Proof. The equivalences (1)$\Leftrightarrow$(2) and (2)$\Leftrightarrow$(3) are an instance of Theorem 12.6.11 and an instance of Theorem 12.6.7, respectively, where $K$ is given by the identity $E \rightarrow E$. 

12.6.13 Remark. Now recalling Remark 12.4.6 and Remark 12.6.8, we see that the notion of extension (cellular or cylindrical) and that of lift (or Kan lift) subsume each other.

12.6.14 Definition. A cell $X \xrightarrow{\mathcal{M}} A$ is said to

\[ Y \xrightarrow{\psi} B \]

> preserve cylindrical extensions along a functor $K : E \rightarrow D$ if any universal left $K$-weighted cylinder $E \xrightarrow{\mathcal{K}} D$ yields by compositions with $\psi$ a universal left $K$-weighted cylinder $E \xrightarrow{\mathcal{K}} D$. 

\[ G \xrightarrow{\mu} \mathcal{N} \xrightarrow{\mathcal{Q}} \mathcal{R} \]

\[ X \xrightarrow{\mathcal{M}} A \]

\[ Y \xrightarrow{\mathcal{N}} B \]
- preserve cylindrical coextensions along a functor \( K : E \to D \) if any universal right \( K \)-weighted cylinder
  \[
  \begin{array}{c}
  \mu \downarrow \\
  \xymatrix@C=20pt{D \ar[r]^-K & E \\
  R \otimes F \ar@{|->}[u] & \ar@{|->}[u] \times M \ar[r]^-A & Y \ar[r]^-B \\
  \end{array}
  \]
  yields by composition with \( \psi \) a universal right \( K \)-weighted cylinder
  \[
  \begin{array}{c}
  \mu \downarrow \\
  \xymatrix@C=20pt{D \ar[r]^-K & E \\
  R \otimes F \ar@{|->}[u] & \ar@{|->}[u] \times M \ar[r]^-A & Y \ar[r]^-B \\
  \end{array}
  \]

12.6.15 Remark.

1. Since a universal left \( K \)-weighted cylinder along \( M \) is the same thing as a direct universal \( \langle K \rhd M \rangle \)-arrow, to say that a cell \( \psi : M \to N \) preserves cylindrical extensions along \( K \) is to say that the postcomposition cell \( \langle K \rhd \psi \rangle \) (see Definition 4.5.7) preserves direct universal arrows. Dually, a cell \( \psi : M \to N \) preserves cylindrical coextensions along \( K \) precisely when the postcomposition cell \( \langle K \triangleleft \psi \rangle \) preserves inverse universal arrows.

2. By the naturality square

\[
\begin{array}{ccc}
\langle K \rhd M \rangle & \psi_{\langle K \rhd M \rangle}^N \downarrow & (K \rhd M) \\
\langle K \rhd N \rangle & \psi_{\langle K \rhd N \rangle}^N \downarrow & (K \rhd N)
\end{array}
\]

in Theorem 5.5.1, the notion of preservation of cylindrical extensions \([\text{op. coextensions}]\) translates faithfully along the natural isomorphism \( \Psi_{K^r}^\Psi \) \([\text{op. } \Psi_{K^r}^\Psi] \) to the notion of preservation of extensions \([\text{op. coextensions}]\) defined in Definition 12.4.18.

3. A cell is said to preserve cylindrical extensions if it preserves cylindrical extensions along any functors.

4. When we say that a cell \( \psi \) preserves pointwise cylindrical extensions \([\text{op. coextensions}]\), we require that \( \psi \) preserves pointwise universality.

12.6.16 Proposition. If a cell preserves colimits \([\text{op. limits}]\), then it preserves pointwise cylindrical extensions \([\text{op. coextensions}]\).

Proof. By Remark 12.6.15(2), the assertion faithfully translates to Proposition 12.4.20.

12.6.17 Theorem. If a cell has a right \([\text{op. left}]\) adjoint, then it preserves cylindrical extensions \([\text{op. coextensions}]\) and pointwise cylindrical extensions \([\text{op. coextensions}]\).

Proof. By Remark 12.6.15(2), the assertion faithfully translates to Theorem 12.4.22.

### 12.7 Kan extensions

A Kan extension is a special case of a cylindrical extension defined in Definition 12.6.1 where \( M \) is given by the hom-module of a category. All results in Section 12.6 thus also apply to Kan extensions. A Kan extension is presented in the form of a weighted natural transformation introduced in Section 4.5; however, it transforms reversibly into a special instance of a (cellular) extension defined in Section 12.4. In fact, a pointwise Kan extension is defined indirectly via this transformation. Just like we did for a limit in Section 8.7, we characterize a pointwise Kan extension in terms of Kan extensions in \( \text{Set} \).

Note. In the following, we consider a left \([\text{op. right}]\) weighted natural transformation as an arrow of the module \( \langle K \rhd C \rangle \) \([\text{op. } \langle K \triangleleft C \rangle] \) defined in Definition 4.5.17.

12.7.1 Definition. Given a pair of functors \( C \xleftarrow{\uparrow} E \xrightarrow{\downarrow} D \),
12.7. Kan extensions

12.7.2 Remark.
(1) The definition is a special case of Definition 12.6.1 where \(M\) is given by the hom-module of a category \(C\).

(2) A left \(K\)-weighted natural transformation \(\mu : L \rightarrow R \circ K\) forms a left Kan extension if and only if to every left \(K\)-weighted natural transformation \(\alpha : L \rightarrow J \circ K\) there is a unique natural transformation \(\mu \alpha : R \rightarrow J\) (the adjunct of \(\alpha\) along \(\mu\)) such that \(\alpha = \mu \circ [K \circ \mu \alpha]\) (see Remark 4.5.18(2)). Dually, a right \(K\)-weighted natural transformation \(\mu : J \circ R \rightarrow L\) forms a right Kan extension if and only if to every right \(K\)-weighted natural transformation \(\alpha : K \circ J \rightarrow L\) there is a unique natural transformation \(\alpha / \mu : J \rightarrow R\) such that \(\alpha = [K \circ \alpha / \mu] \circ \mu\).

(3) Since \((E^* , C) = \langle !_E \circ C \rangle\) [op. \((E^* , C) = \langle !_E \circ C \rangle\)] by Remark 4.9.4(2), a colimit \(E \xrightarrow{\text{lim}} \ast\) [op. \(E \xrightarrow{\text{lim}} \ast\)] in Definition 8.1.9 is a special instance of a left [op. right] Kan extension in Definition 12.7.1 where \(D\) is the terminal category.

Note. Pointwise Kan extensions are defined indirectly via the isomorphism in Theorem 5.5.1:

12.7.3 Definition. Given a pair of functors \(C \leftarrow E \rightarrow D\),

- a left \(K\)-weighted natural transformation \(E \xrightarrow{\text{lim}} D\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a pointwise left Kan extension of \(L\) along \(K\) if the cell \(E \xrightarrow{\text{lim}} D\) generated by \(D\) inverse along \(\mu\) is pointwise direct universal.

- a right \(K\)-weighted natural transformation \(D \xleftarrow{\text{lim}} E\) or the pair \((R, \mu)\), or the functor \(R\) itself, is called a pointwise right Kan extension of \(L\) along \(K\) if the cell \(D \xleftarrow{\text{lim}} E\) generated by \(D\) direct along \(\mu\) is pointwise inverse universal.

12.7.4 Remark. The definition is a special case of Definition 12.6.5 where \(M\) is given by the hom-module of a category \(C\). By Theorem 12.6.9, a pointwise Kan extension is a Kan extension in the sense of Definition 12.7.1.

12.7.5 Theorem. Given functors as in Definition 12.7.3,

- the following conditions are equivalent:
(1) $R$ is a left Kan extension (resp. pointwise left Kan extension) of $L$ along $K$;
\[ \begin{array}{ccc}
E & \xrightarrow{K} & D \\
\downarrow & \mu & \downarrow R \\
C & \rightarrow & \omega C \\
\end{array} \]

(2) $R$ is an extension (resp. pointwise extension) of $L$ along the representable module $K(D)$;
\[ \begin{array}{ccc}
D & \xrightarrow{(D)K} & E \\
\downarrow & \omega & \downarrow L \\
C & \rightarrow & \omega C \\
\end{array} \]

(3) $R$ is an extension (resp. pointwise extension) of the representable module $K(D)$ along the representable module $L(C)$.

The following conditions are equivalent:
(1) $R$ is a right Kan extension (resp. pointwise right Kan extension) of $L$ along $K$;
(2) $R$ is a coextension (resp. pointwise coextension) of $L$ along the corepresentable module $(D)K$;
(3) $R$ is a coextension (resp. pointwise coextension) of the corepresentable module $(D)K$ along the representable module $L(C)$.

Proof. (1)$\iff$(2) and (2)$\iff$(3) are an instance of Theorem 12.6.7 and an instance of Theorem 12.4.14, respectively, where $\mathcal{M}$ is given by the hom-module of a category $\mathcal{C}$.

12.7.6 Remark. A Kan extension is thus regarded as a special instance of an extension in Remark 12.4.11(2) where $\mathcal{J}$ is representable, or as a special instance of an extension defined in Definition 12.4.1 and Definition 12.4.3 where both $\mathcal{J}$ and $\mathcal{M}$ are representable.

12.7.7 Theorem. If $E$ is small and $\mathcal{C}$ is cocomplete [op. complete], then any functor $L : E \to \mathcal{C}$ has a pointwise left [op. right] Kan extension along any functor $K : E \to \mathcal{D}$.

Proof. Since the completeness of a category is the same thing as the completeness of its hom-module (see Remark 8.1.16), this is a special case of Theorem 12.6.10 where $\mathcal{M}$ is given by the hom-module $\mathcal{C}$.

12.7.8 Theorem. If $K : E \to \mathcal{D}$ is a fully faithful functor, then

- a pointwise left Kan extension of a functor $L : E \to \mathcal{C}$ along $K$ is a natural isomorphism $\mu : L \to R \circ K$.
- a pointwise right Kan extension of a functor $L : E \to \mathcal{C}$ along $K$ is a natural isomorphism $\mu : K \circ R \to L$.

Proof. This is a special case of Theorem 12.6.11 where $\mathcal{M}$ is given by the hom-module of $\mathcal{C}$, noting that a natural isomorphism $L \to R \circ K$ in $\mathcal{C}$ is the same thing as a pointwise universal cylinder $L \sim R \circ K$ along the hom-module $\langle \mathcal{C} \rangle$ (see Theorem 6.5.13).
12.7.9 Theorem.

- For a functor $F : X \to A$ the following conditions are equivalent:
  1. $F$ has a right adjoint;
  2. $F$ preserves colimits and the identity $X \to X$ has a pointwise left Kan extension along $F$.

When these conditions hold, a natural transformation $\eta : 1_X \to G \circ F$ forms a pointwise left Kan extension $X \xrightarrow{F} A$ of the identity $X \to X$ along $F$ if and only if $\eta$ is the unit of an adjunction $G \dashv F$.

- For a functor $G : A \to X$ the following conditions are equivalent:
  1. $G$ has a left adjoint;
  2. $G$ preserves limits and the identity $A \to A$ has a pointwise right Kan extension along $G$.

When these conditions hold, a natural transformation $\epsilon : G \circ F \to 1_A$ forms a pointwise right Kan extension $X \leftarrow G A$ of the identity $A \to A$ along $G$ if and only if $\epsilon$ is the counit of an adjunction $G \dashv F$.

Proof. We see that this faithfully translates into an instance of Theorem 12.5.14 where $\mathcal{M}$ is given by the representable module $F(A)$, noting that

1. a right adjoint of $F$ is the same thing as a corepresentation of the module $F(A)$ (see Remark 7.3.2(2));
2. $F$ preserves colimits iff for each $a \in \| A \|$ the right module $F(A) a : X \to \ast$ preserves colimits (see Corollary 8.7.9);
3. a pointwise left Kan extension of the identity $X \to X$ along $F$ is the same thing as a pointwise extension of the module $F(A)$ (see Theorem 12.7.5);
4. a natural transformation $\eta : 1_X \to G \circ F$ is the unit of an adjunction $G \dashv F$ iff the module morphism $\eta A : F(A) \to (X) G$ is iso (see Proposition 7.3.9), and by definition, $\eta$ forms a pointwise left Kan extension iff the cell $\begin{array}{ccc} F(A) & \xrightarrow{F(A)} & A \\ \downarrow{G} & \downarrow{\eta A} & \downarrow{G} \\ X & \xrightarrow{\eta} & X \end{array}$ is pointwise universal.

\[\square\]

12.7.10 Definition. A functor $H : C \to B$ is said to

- preserve left Kan extensions along a functor $K : E \to D$ if any left Kan extension $\begin{array}{ccc} E & \xrightarrow{K} & D \\ \downarrow{L} & \downarrow{\mu H} & \downarrow{L H \circ R} \\ C & \xrightarrow{1_C} & C \end{array}$ yields by composition with $H$ a left Kan extension $\begin{array}{ccc} E & \xrightarrow{H \circ K} & D \\ \downarrow{L \circ H} & \downarrow{\mu H} & \downarrow{L H \circ R} \\ B & \xrightarrow{1_B} & B \end{array}$.

- preserve right Kan extensions along a functor $K : E \to D$ if any right Kan extension $\begin{array}{ccc} D & \xrightarrow{K} & E \\ \downarrow{R} & \downarrow{\mu H H \circ L} & \downarrow{R H \circ L} \\ C & \xrightarrow{1_C} & C \end{array}$ yields by composition with $H$ a right Kan extension $\begin{array}{ccc} D & \xrightarrow{H \circ K} & E \\ \downarrow{R \circ H} & \downarrow{\mu H} & \downarrow{R H \circ L} \\ B & \xrightarrow{1_B} & B \end{array}$.

12.7.11 Remark.

1. Since a left Kan extension of $L : E \to C$ along a left Kan extension of $K : E \to D$ is the same thing as a direct universal $(K \circ C)$-arrow from $L$, to say that a functor $H$ preserves left Kan extensions along a functor $K$
is to say that the postcomposition cell \( (K \ast H) \) (see Definition 4.5.19) preserves direct universal arrows. Dually, a functor \( H \) preserves right Kan extensions along a functor \( K \) precisely when the postcomposition cell \( (K \circ H) \) preserves inverse universal arrows.

(2) Since \( (K \circ H) = (K H) \) and \( (K \ast H) \) by Remark 4.5.20(2), the definition is a special case of Definition 12.6.14 where \( \varphi \) is given by the hom-cell of \( H \).

(3) A functor is said to preserve left [op. right] Kan extensions if it preserves left [op. right] Kan extensions along any functor.

(4) When we say that a functor \( H \) preserves pointwise Kan extensions, we require that \( H \) preserves pointwise universality (cf. Remark 12.4.19(3)).

12.7.12 Proposition. If a functor preserves colimits [op. limits], then it preserves pointwise left [op. right] Kan extensions.

Proof. By Remark 12.7.11(2) and noting Remark 8.2.12(2), we see that this is a special case of Proposition 12.6.16.

12.7.13 Theorem. If a functor has a right [op. left] adjoint, then it preserves left [op. right] Kan extensions and pointwise left [op. right] Kan extensions.

Proof. By Remark 12.7.11(2) and noting Remark 7.10.2(2), we see that this is a special case of Theorem 12.6.17.

Note. The following roughly says that Yoneda functors preserve and reflect pointwise Kan extensions (cf. Corollary 8.7.8).

12.7.14 Theorem. Given a pair of functors \( C \xleftarrow{L} E \xrightarrow{K} D \), a left [op.right] \( K \)-weighted natural transformation

\[
\begin{array}{ccc}
C & \xrightarrow{\mu} & D \\
\mu & \downarrow & \downarrow \text{R} \\
\text{op.} & C & \xrightarrow{\ast} & \text{Set} \\
\mu & \downarrow & \downarrow \text{L} \\
\text{op.} & C & \xrightarrow{\ast} & \text{Set}
\end{array}
\]

is a pointwise left [op. right] Kan extension if and only if, for every \( c \in \|C\| \), its composition with the representable right [op. left] module

\[
\langle c \rangle \circ : C \to \text{Set}
\]

yields a right Kan extension

\[
\begin{array}{ccc}
\text{Set} & \xleftarrow{K} & E \\
\text{op.} & \text{Set} & \xrightarrow{K} \text{Set} \\
\text{left} & \text{right} & \text{right}
\end{array}
\]

Proof. (\( \Rightarrow \)) Since a representable right module preserves colimits (Corollary 8.7.10), the “only if” part follows from Proposition 12.7.12.

(\( \Leftarrow \)) We need to show that the right slice

\[
\begin{array}{ccc}
C & \xrightarrow{\text{op.}} & \ast \\
\mu & \downarrow & \downarrow \text{R} \circ \text{d} \\
\text{op.} & C & \xrightarrow{\ast} \text{Set}
\end{array}
\]

of the cell \( \mu \text{d} \) at each \( \text{d} \in \|D\| \) is a universal conic cell. For this, it suffices (see Remark 12.3.6(1)) to show that for any \( c \in \|C\| \), the assignment

\[
f \mapsto \langle \mu \text{d} \rangle \circ f : \langle \text{d} \circ \text{R} \rangle \langle C \rangle \circ c \to \langle K \langle \text{d} \rangle \rangle \langle E \rangle \langle L \circ c \rangle
\]

is bijective. Since \( \mu \langle C \rangle \circ c \) is a left Kan extension by assumption, the assignment

\[
\psi \mapsto K \langle \psi \rangle \circ \mu \langle C \rangle \circ c : \langle \langle \text{d} \rangle \rangle \langle \text{d} \circ \text{R} \rangle \langle C \rangle \circ c \to \langle K \langle \text{d} \rangle \rangle \langle E \rangle \langle L \circ c \rangle
\]
is bijective (see Remark 12.7.2(2)). Composing this bijection with “Yoneda”

\[ f \mapsto D \downarrow f : (d \downarrow R)(C) \mapsto ((D)(d) \downarrow (R(C)c)) \]

(see Corollary 5.2.13), we have a bijection

\[ f \mapsto K(D)(f) \circ \mu(C) : (d \downarrow R)(C) \mapsto (K(D)(d) \downarrow (E \downarrow (L(C)c)). \]

The proof is thus complete if we show that

\[ (\mu \downarrow D)(f) \circ \mu(C) = K(D)(f) \circ \mu(C). \]

Noting Example 5.3.5(3), we see that the right module morphism \((\mu \downarrow D)(f) \circ \mu(C): E \to *\) maps \(h : e : K \to d\) to the composite \(e : L \overset{\rho_C}{\to} e : K \overset{\rho_R}{\to} d : R \overset{f}{\to} c\). On the other hand, the right module morphism \(K(D \uparrow f) : K(D)(d) \to K(R(C)c)) : E \to *\) maps \(h : e : K \to d\) to the composite \(e : K \overset{h}{\to} R \overset{\rho_R}{\to} d : R \overset{f}{\to} c\) (cf. Example 5.2.7(3)), and the right module morphism \(\mu(C) : K(R(C)c) \to L(C)c) : E \to *\) then maps this arrow to the composite \(e : L \overset{\rho_C}{\to} e : K \overset{\rho_R}{\to} d : R \overset{f}{\to} c\). We have thus shown that two right module morphisms \((\mu \downarrow D)(f) \circ \mu(C)\) map each \((K(D)(d))-arrow \circ\) h : e : K → d to the same \((L(C)c))-arrow e : L → e : K : R → d : R → c\).

\[ \square \]

12.7.15 Remark.

(1) [ML98] and [Rie16] define pointwise Kan extensions using this characterization.

(2) Since a limit is a special instance of a Kan extension (see Remark 12.7.2(3)), Theorem 12.7.14 for Kan extensions may be seen as a generalization of Theorem 8.7.6 and Corollary 8.7.7.

**Problem.** In Remark 12.4.6, we saw that the notion of pointwise Kan lift subsumes the notion of pointwise extension, hence in particular that of pointwise Kan extension. Does the converse hold? Does the notion of pointwise Kan extension subsume that of pointwise Kan lift?

### 12.8 Density

In this section, we generalize the notion of denseness to modules. Theorem 12.8.7 characterizes the denseness of a representable module using the notion of a unit cone introduced in Section 11.4. If applied to the Yoneda representation (see Section 5.2), the theorem yields the “density theorem”, which states that every right module is a colimit of representable right modules.

#### 12.8.1 Definition.

(1) A module \(M: X \to A\) is called

- dense if its right exponential transpose \([M \to A] : A \to [X : ]\) is fully faithful.
- codense if its left exponential transpose \([\times, M] : X \to [\cdot, A]^-\) is fully faithful.

(2) A functor \(K : D \to E\) is called

- dense if its representable module \(K(E) : D \to E\) is dense.
- codense if its corepresentable module \((E)K : E \to D\) is codense.

(3) A subcategory \(D\) of a category \(E\) is called

- dense in \(E\) if the inclusion \(D \to E\) is dense.
- codense in \(E\) if the inclusion \(D \to E\) is codense.

#### 12.8.2 Proposition.** Given a module \(M: X \to A\), the following conditions are equivalent:

(1) \(M\) is dense \([op. codense]\);

(2) the identity \(M \to M\) forms a pointwise universal cell

\[
\begin{array}{ccc}
M & \xrightarrow{1_M} & A \\
\downarrow & & \downarrow 1 \\
X & \xrightarrow{M} & A
\end{array}
\]

**op.

\[
\begin{array}{ccc}
X & \xrightarrow{M} & A \\
\downarrow & & \downarrow 1_M \\
X & \xrightarrow{1_M} & A
\end{array}
\]

**
(3) for every \(a \in \|A\|\) \([\text{op. } x \in \|X\|]\), the identity \((M) a \rightarrow (M) a\) \([\text{op. } x(M) \rightarrow x(M)]\) forms a universal conic cell

\[
\begin{array}{c}
\xymatrix{
(M)a \ar[r]^{\star} & a \\
X \ar[r]_{\Delta a} & A \\
M \ar[ur]^{1_{(M)a}} & 
}
\end{array}
\]  \quad \text{op.}

\[
\begin{array}{c}
\xymatrix{
X \ar[r]_{\Delta x} & x(M) \\
M \ar[ur]^{1_{x(M)}} & 
}
\end{array}
\]

(4) for every \(a \in \|A\|\) \([\text{op. } x \in \|X\|]\), the unit cone (see Definition 11.4.1)

\[
\begin{array}{c}
\xymatrix{
\xymatrix{X \ar[r]_{\Delta x} & x(M) \ar[ur]^{1_{x(M)}(x(M))} & A \\
M \ar[ur]^{1_{x(M)}} & 
}
\end{array}
\]

of \((M) a\) \([\text{op. } x(M)]\) is universal.

**Proof.** (1)⇔(2) By Theorem 12.4.5, the cell

\[
\xymatrix{M \ar[r]^{1_{M}} & A \\
X \ar[ur]^{1_{x(M)}} & 
}
\]

is pointwise universal iff its right exponential transpose

\[
\xymatrix{M \ar[r]^{1_{M}} & A \\
X \ar[ur]^{1_{x(M)}} & 
}
\]

forms a pointwise left Kan lift of \(M \rightarrow A\). By Theorem 6.6.3, this is the case iff \(M \rightarrow A\) is fully faithful.

(2)⇔(3) Evident by the definition of pointwise universality.

(3)⇔(4) Since the unit cone \(1_{(M)a}\) is the comma transpose of the identity \(1_{(M)a}\) (see Remark 11.4.11(4)), this is immediate from Proposition 12.3.3. \(\square\)

**12.8.3 Theorem.** The right \([\text{op. left}]\) Yoneda module is dense \([\text{op. codense}]\).

**Proof.** Immediate from Proposition 5.1.3. \(\square\)

**12.8.4 Corollary.** The right \([\text{op. left}]\) Yoneda functor is dense \([\text{op. codense}]\).

**Proof.** Since the Yoneda module is represented by the Yoneda functor (see Theorem 5.2.15), the assertion follows from Theorem 12.8.3. \(\square\)

**Note.** If \(K\) in Corollary 6.2.12 is dense \([\text{op. codense}]\), then we have the following.

**12.8.5 Theorem.** Let \(K : D \rightarrow E\) be a functor.

- If \(K\) is fully faithful and dense, then an \(E\)-arrow \(u : r : K \rightarrow e\) is iso if and only if it is universal from \(K\) to \(e\), i.e. the \(K(E)\)-arrow \(u : r \rightarrow e\) is inverse universal.
- If \(K\) is fully faithful and codense, then an \(E\)-arrow \(u : e \rightarrow K \cdot r\) is iso if and only if it is universal from \(e\) to \(K\), i.e. the \(E\)-arrow \(u : e \rightarrow r\) is direct universal.

**Proof.** Since \((K(E)) \cdot \rightarrow\) is fully faithful by the denseness of \(K\), \(u\) is iso iff \(K(E)u\) is iso. The assertion thus follows from Theorem 6.2.11. \(\square\)

**Note.** Further to Theorem 7.3.14, we have yet another equivalent condition.

**12.8.6 Theorem.**

- For an adjunction \(\Phi : G \vdash F : X \rightarrow A\), the following conditions are equivalent:
  1. the functor \(G : A \rightarrow X\) is fully faithful;
  2. the functor \(F : X \rightarrow A\) is dense.
- For an adjunction \(\Phi : G \vdash F : X \rightarrow A\), the following conditions are equivalent:
  1. the functor \(F : X \rightarrow A\) is fully faithful;
  2. the functor \(G : A \rightarrow X\) is codense.
Proof. The module isomorphism
\[ \phi : (X) G \to F(A) : X \to A \]
yields the natural isomorphism
\[ [(X) G : X \to A] \cong [F(A) : A \to [X : ] \]
; but by Proposition 2.3.9,
\[ [(X) G : X \to A] = [X : ] \delta G \]
; so we have the isomorphism
\[ [X : ] \delta G \cong [F(A) : X \to A] \]
; hence, noting that the Yoneda functor \( X \cdot \) is fully faithful, we see that \( G \) is fully faithful iff \( F(A) \cdot \) is fully faithful, i.e. \( \text{iff } F \) is dense. \( \square \)

12.8.7 Theorem. Suppose that a module \( M : X \to A \) has a representation \( \text{[op. corepresentation]}\) \((R, \phi)\). Then \( M \) (or \( R \) equivalently) is dense \( \text{[op. codense]} \) if and only if, for any \( a \in \|A\| \text{[op.} x \in \|X\|\text{]}, the composition
\[ [(M) a]^\dagger \delta R \]
yields a universal cone
\[ [(M) a]^\dagger \delta R \sim a \]
in \( A \) \( \text{[op.} X\text{]} \).

Proof. By Proposition 12.8.2, \( M \) is dense iff the unit cone \( 1^\dagger_{(M)a} \) is universal for any \( a \in \|A\| \). By Theorem 8.2.8, \( 1^\dagger_{(M)a} \) is universal iff so is its composite with \( \phi \). \( \square \)

12.8.8 Corollary. \( \text{(Density Theorem).} \) Any right \( \text{[op. left]} \) module is a colimit of representables.
Specifically, given a right module \( M : X \to \ast \) \( \text{[op. left]} \) module \( M : \ast \to A \), the composition
\[ [\delta (X \cdot) \sim \delta : \] yields a universal cone
\[ [\delta (X \cdot) \sim \delta : \]
in \( [X : ] \) \( \text{[op.} [\ast : A]\text{]} \).

Proof. Since the Yoneda module \( X \cdot \) is dense (see Theorem 12.8.3), and since \( (X \cdot) (M) = M \) (see Proposition 5.1.3), the assertion follows by applying Theorem 12.8.7 to the Yoneda representation. \( \square \)

12.8.9 Remark. Since \( 1^\dagger_{\|M\|} = m \) for any \( M \)-arrow \( m \) (see Definition 11.4.1), recalling the definition of the Yoneda representation, we see that the component of the universal cone
\[ (1^\dagger_{\|M\|} \delta (1)) : [M^\dagger \delta [X \cdot ] \sim M : [M] \to [X : ] \text{ op.} \]
at an \( M \)-arrow
\[ m : x \sim \ast \text{ op.} \]
\[ m : \ast \sim a \]
is the module morphism
\[ X \mid m : (X) x \to M : X \to \ast \op. \quad m \mid A : a(A) \to M : \ast \to A. \]

### 12.9 Yoneda extensions

In this section, we study Kan extensions along a Yoneda functor.

#### 12.9.1 Theorem.
- Given functors \( L : E \to C \) and \( R : [E] \to C \), the following conditions are equivalent:
  1. \( R \) is a pointwise left Kan extension
     \[
     \begin{array}{c}
     E \xrightarrow{E^n} [E] \\
     \downarrow_{\mu} \downarrow_{\{R\}} \\
     C \xrightarrow{(C)^n} C
     \end{array}
     \]
     of \( L \) along the right Yoneda functor \( E^n \);
  2. \( R \) is a pointwise extension
     \[
     \begin{array}{c}
     E \xrightarrow{E^n} [E] \\
     \downarrow_{\omega} \downarrow_{\{R\}} \\
     C \xrightarrow{(C)^n} C
     \end{array}
     \]
     of \( L \) along the right Yoneda module \( E^n \);
  3. \( R \) is a pointwise extension
     \[
     \begin{array}{c}
     E \xleftarrow{E^n} [E] \\
     \downarrow_{\omega} \downarrow_{\{R\}} \\
     C \xleftarrow{(C)^n} C
     \end{array}
     \]
     of the right Yoneda module \( E^n \) along the representable module \( L(C) : E \to C \);
  4. \( R \) is a left adjoint
     \[
     \begin{array}{c}
     [E] \xrightarrow{(L(C))^n} C \\
     \downarrow_{\{R\}} \\
     C \xrightarrow{(C)^n} C
     \end{array}
     \]
     of the right exponential transpose of the representable module \( L(C) : E \to C \).
- Given functors \( L : E \to C \) and \( R : [E] \to C \), the following conditions are equivalent:
  1. \( R \) is a pointwise right Kan extension
     \[
     \begin{array}{c}
     [E] \xleftarrow{[E]} E \\
     \downarrow_{\mu} \downarrow_{L} \\
     C \xleftarrow{(C)^n} C
     \end{array}
     \]
     of \( L \) along the left Yoneda functor \( \ast, E \);
  2. \( R \) is a pointwise coextension
     \[
     \begin{array}{c}
     [E] \xleftarrow{[E]} E \\
     \downarrow_{\omega} \downarrow_{L} \\
     C \xleftarrow{(C)^n} C
     \end{array}
     \]
     of \( L \) along the left Yoneda module \( \ast, E \);
  3. \( R \) is a pointwise coextension
     \[
     \begin{array}{c}
     C \xleftarrow{[E]} E \\
     \downarrow_{\omega} \downarrow_{L} \\
     C \xleftarrow{(C)^n} C
     \end{array}
     \]
     of the left Yoneda module \( \ast, E \) along the corepresentable module \( (C)L : C \to E \);
Proof. (1)⇔(2)⇔(3) Since \((\mathbf{E}, \star) \cong [\mathbf{E}, \star](\mathbf{E})\) (see Theorem 5.2.15), this follows from Theorem 12.7.5.

(3)⇔(4) By Remark 12.4.6, and since \((\mathbf{E}, \star) \dashv 1\) (see Proposition 5.1.3), \(\mathbf{R}\) is a pointwise extension of \(\mathbf{E}, \star\) along \(\mathbf{L}(\mathbf{C})\) iff \(\mathbf{R}\) is a pointwise left Kan extension of the identity \([\mathbf{E}]: \to [\mathbf{E}]:\)

\[
\begin{array}{c}
\mathbf{E} \\
\downarrow \mathbf{L} \downarrow \mathbf{R} \downarrow \mathbf{J} \\
\mathbf{C} \leftarrow \mathbf{(C)} \leftarrow \mathbf{C}
\end{array}
\]

along \((\mathbf{L}(\mathbf{C})), \star\), and by the equivalence of (2) and (5) in Proposition 7.3.9, this is the case iff \(\eta\) is the unit of an adjunction \([\mathbf{(L(C))}, \star] \rightharpoonup \mathbf{R}; [\mathbf{E}]: \to \mathbf{C}\). \(\square\)

12.9.2 Corollary.

- If a functor \(\mathbf{L}: \mathbf{E} \to \mathbf{C}\) has for each right module \(\mathbf{J}: \mathbf{E} \to \star\) an \(\mathbf{J}\)-weighted colimit

\[
\begin{array}{c}
\mathbf{E} \xrightarrow{\mathbf{J}} \star \\
\downarrow \mathbf{L} \downarrow \mathbf{\omega} \downarrow \mathbf{R} \downarrow \mathbf{J} \\
\mathbf{C} \xrightarrow{\mathbf{(C)}} \mathbf{\star} \mathbf{C}
\end{array}
\]

, then \(\mathbf{L}\) has a pointwise left Kan extension

\[
\begin{array}{c}
\mathbf{E} \xrightarrow{\mathbf{E}, \star} [\mathbf{E}]: \\
\downarrow \mathbf{L} \downarrow \mathbf{\mu} \downarrow \mathbf{R} \\
\mathbf{C} \xrightarrow{\mathbf{(C)}} \mathbf{\star} \mathbf{C}
\end{array}
\]

along the right Yoneda functor \([\mathbf{E}, \star]: \mathbf{E} \to [\mathbf{E}]:\) with \(\mathbf{J}: \mathbf{R} = \mathbf{R} \mathbf{J}\).

- If a functor \(\mathbf{L}: \mathbf{E} \to \mathbf{C}\) has for each left module \(\mathbf{J}: \star \to \mathbf{E}\) an \(\mathbf{J}\)-weighted limit

\[
\begin{array}{c}
\star \xrightarrow{\mathbf{J}} \mathbf{E} \\
\downarrow \mathbf{R} \downarrow \mathbf{\omega} \downarrow \mathbf{L} \\
\mathbf{C} \xrightarrow{\mathbf{(C)}} \mathbf{\star} \mathbf{C}
\end{array}
\]

, then \(\mathbf{L}\) has a pointwise right Kan extension

\[
\begin{array}{c}
[\mathbf{E}]: \xleftarrow{\mathbf{\sqrt{E}}} \mathbf{E} \\
\downarrow \mathbf{R} \downarrow \mathbf{\mu} \downarrow \mathbf{L} \\
\mathbf{C} \xrightarrow{\mathbf{(C)}} \mathbf{\star} \mathbf{C}
\end{array}
\]

along the left Yoneda functor \(\mathbf{\sqrt{E}}: \mathbf{E} \to [\mathbf{E}]:\) with \(\mathbf{J}: \mathbf{R} = \mathbf{R} \mathbf{J}\).

Proof. By the equivalence of (1) and (2) in Theorem 12.9.1, the problem translates into showing that \(\mathbf{L}\) has a pointwise extension \(\mathbf{R}: [\mathbf{E}]: \to \mathbf{C}\) along the right Yoneda module \((\mathbf{E}, \star): \mathbf{E} \to [\mathbf{E}]:\) with \(\mathbf{J}: \mathbf{R} = \mathbf{R} \mathbf{J}\). But since \((\mathbf{E}, \star)(\mathbf{J}) = \mathbf{J}\) for any right module \(\mathbf{J}: \mathbf{E} \to \star\) (see Proposition 5.1.3), this follows from Theorem 12.4.16. \(\square\)

12.9.3 Theorem.

- If \(\mathbf{L}: \mathbf{E} \to \mathbf{C}\) is a functor with \(\mathbf{E}\) small and \(\mathbf{C}\) cocomplete, then \(\mathbf{L}\) has a pointwise left Kan extension

\[
\begin{array}{c}
\mathbf{E} \xrightarrow{\mathbf{E}, \star} [\mathbf{E}]: \\
\downarrow \mathbf{L} \downarrow \mathbf{\mu} \downarrow \mathbf{R} \\
\mathbf{C} \xrightarrow{\mathbf{(C)}} \mathbf{\star} \mathbf{C}
\end{array}
\]

along the right Yoneda functor \(\mathbf{E}, \star\), such that \(\mu\) is an isomorphism and \(\mathbf{R}\) preserves colimits. Conversely, if a functor \(\mathbf{S}: [\mathbf{E}]: \to \mathbf{C}\) is cocontinuous and \(\mathbf{L} \cong \mathbf{S}\) \(\mathbf{[E, \star]}\), then \(\mathbf{S} \cong \mathbf{R}\).
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If \( \mathbf{L} : \mathbf{E} \to \mathbf{C} \) is a functor with \( \mathbf{E} \) small and \( \mathbf{C} \) complete, then \( \mathbf{L} \) has a pointwise right Kan extension

\[
\begin{array}{c}
\lceil : [\mathbf{E}] \to \mathbf{E} \\
\mathbf{R} \downarrow \mu \\
\mathbf{C} \to \mathbf{C}
\end{array}
\]

along the left Yoneda functor \( \mathbf{\Psi} \), such that \( \mu \) is an isomorphism and \( \mathbf{R} \) preserves limits. Conversely, if a functor \( \mathbf{S} : [\lceil : \mathbf{E}] \to \mathbf{C} \) is cocontinuous and \( [\mathbf{\Psi}] \circ \mathbf{R} \approx \mathbf{L} \), then \( \mathbf{S} \approx \mathbf{R} \).

Proof. By Theorem 12.7.7, a pointwise left Kan extension \( (\mathbf{R}, \mu) \) exists. Since the Yoneda functor is fully faithful, \( \mu \) is a natural isomorphism by Theorem 12.7.8. Since \( \mathbf{R} \) is given by a left adjoint of the functor \( \left[ (\mathbf{L}(\mathbf{C})) \mathbf{S} \right] : \mathbf{C} \to [\mathbf{E}] \) by Theorem 12.9.1, \( \mathbf{R} \) preserves colimits by Theorem 8.11.2.

Now suppose that a functor \( \mathbf{S} : [\mathbf{E}] \to \mathbf{C} \) is cocontinuous and has a natural isomorphism \( \nu : \mathbf{L} \to \mathbf{S} \circ [\mathbf{\Psi}] \). We will show that the adjunct of \( \nu \) along \( \mu \), i.e. the unique natural transformation \( \mu \circ \nu : \mathbf{R} \to \mathbf{S} \) with \( \nu = \mu \circ [\mathbf{\Psi}] \circ \mu \circ \nu \) (see Remark 12.7.2(2)), is a natural isomorphism. For this, it suffices to show that the component of \( \mu \circ \nu \) at each right module \( \mathcal{J} : \mathbf{E} \to \ast \) is an isomorphism. By Corollary 12.8.8, there is a universal cone \( \{1^* \circ (\mathbf{E})\} : [\mathcal{J} \circ [\mathbf{\Psi}]] \to \mathcal{J} \). Now consider the commutative square

\[
\begin{array}{ccc}
\mathcal{J} \circ [\mathbf{\Psi}] & \circ \mathbf{R} & \to \mathcal{J} \circ \mathbf{S} \\
\mathcal{J} \circ [\mathbf{\Psi}] \circ \mu \circ \nu & \sim & \mathcal{J} \circ \mathbf{S} \\
\mathcal{J} \circ [\mathbf{\Psi}] \circ \nu & \to & \mathcal{J} \circ \mathbf{S}
\end{array}
\]

giving the composite cone \( \{1^* \circ (\mathbf{E})\} \circ \mu \circ \nu : [\mathcal{J} \circ [\mathbf{\Psi}] \circ \mathbf{R}] \to \mathcal{J} \circ \mathbf{S} \). Since \( \mathbf{R} \) and \( \mathbf{S} \) are cocontinuous, \( 1^* \circ (\mathbf{E}) \circ \mathbf{R} \) and \( 1^* \circ (\mathbf{E}) \circ \mathbf{S} \) are universal cones, and since \( \mu \) and \( \nu \) are natural isomorphisms, so is \( \mathcal{J} \circ [\mathbf{\Psi}] \circ \mu \circ \nu = \mathcal{J} \circ [\mathbf{\Psi}] \circ [\mu^{-1} \circ \nu] \); hence \( \mathcal{J} \circ \mu \circ \nu \) is an isomorphism by Proposition 6.3.3(2).

12.9.4 Corollary. If \( \mathbf{K} : \mathbf{D} \to \mathbf{E} \) is a functor between small categories, then the precomposition functor \( [\mathbf{K}] : [\mathbf{E}] \to [\mathbf{D}] \) has both left and right adjoints.

Proof. Consider the Yoneda functors as in

\[
\begin{array}{ccc}
\mathbf{D} & \to & [\mathbf{D}] \\
\mathbf{K} \downarrow & & \downarrow \mathbf{\Psi} \\
\mathbf{E} & \to & [\mathbf{E}]
\end{array}
\]

; since \( \mathbf{D} \) is small and \( [\mathbf{E}] \) is cocomplete (see Corollary 8.6.7), \( \mathbf{K} \circ [\mathbf{\Psi}] \) has a pointwise left Kan extension

\[
\begin{array}{ccc}
\mathbf{D} & \to & [\mathbf{D}] \\
\mathbf{K} \downarrow \mu & & \downarrow \mathbf{R} \\
\mathbf{E} & \to & [\mathbf{E}]
\end{array}
\]

along \( \mathbf{D} \) by Theorem 12.9.3, and \( \mathbf{R} \) is a left adjoint of the functor \( \{[\mathbf{K} \circ [\mathbf{\Psi}]] (\mathbf{E}) \} \mathbf{\Psi} \) by Theorem 12.9.1; but

\[
\begin{align*}
\{[\mathbf{K} \circ [\mathbf{\Psi}]] (\mathbf{E}) \} \mathbf{\Psi} & = \{\mathbf{K} \circ [\mathbf{\Psi}] \} (\mathbf{E}) \mathbf{\Psi} \\
& \approx \{\mathbf{K} \circ [\mathbf{\Psi}] \} \mathbf{\Psi} \\
& = [\mathbf{K}] \circ [\mathbf{\Psi}] \\
& = [\mathbf{K}]
\end{align*}
\]

\((^1 \text{ by Theorem } 5.2.15; ^2 \text{ by Proposition } 2.1.7; ^3 \text{ by Proposition } 5.1.3)\). Hence \( [\mathbf{K}] \) has a left
adjoint, namely $\text{R}$. Now consider the composition

$$
\begin{array}{c}
\text{E} \\
[ \text{E} \circ \text{r} \downarrow \text{[K]} ] \\
\downarrow \text{[K]} \\
\text{D}
\end{array}
$$

; since $[\text{K} : ]$ is cocontinuous (see Corollary 8.6.8), $[\text{K} : ]$ is a pointwise left Kan extension of $[\text{E} \circ \text{r} ] \downarrow [\text{K} : ]$ along $\text{E} \circ \text{r}$ by Theorem 12.9.3, and thus has a right adjoint by Theorem 12.9.1.

\section{12.10 Tiny-projective modules}

We will show that if a category $\text{C}$ is idempotent complete, it is recovered (up to equivalence) from the category $[\text{C} : ]$.

\begin{proposition}
An object $\text{x}$ of a category $\text{C}$ is said to be tiny-projective if the representable left module $\text{x}(\text{C})$ preserves colimits.
\end{proposition}

\begin{remark}
(1) Since an epimorphism $f : y \rightarrow z$ is regarded as a colimit (pushout) of the diagram $y \leftarrow y \rightarrow y$, if $\text{x}$ is tiny-projective, $\text{x}(\text{C})$ preserves epimorphisms. Hence a tiny-projective object is projective.

(2) In the category $\text{Set}$, the tiny-projective objects are precisely the singleton sets. To see this, first let $\mathcal{S}$ be a singleton set. Since $[\mathcal{S}, -] : \text{Set} \rightarrow \text{Set}$ is an equivalence, $[\mathcal{S}, -]$ preserves colimits. Now suppose a set $\mathcal{T}$ is a tiny-projective object in $\text{Set}$. Then, in particular, $[\mathcal{T}, 2] = \mathcal{T} \uplus 1 \equiv [\mathcal{T}, 1] \uplus [\mathcal{T}, 1]$ and $[\mathcal{T}, 0] = [\mathcal{T}, 0 \uplus 0] \equiv [\mathcal{T}, 0] \uplus [\mathcal{T}, 0]$; $[\mathcal{T}, 2] \equiv [\mathcal{T}, 1] \uplus [\mathcal{T}, 1]$ holds when $\mathcal{T} = 0$ or $\mathcal{T} \equiv 1$, and $[\mathcal{T}, 0] \equiv [\mathcal{T}, 0] \uplus [\mathcal{T}, 0]$ holds when $\mathcal{T} \neq 0$.
\end{remark}

\begin{proposition}
A retract of a tiny-projective object is tiny-projective.
\end{proposition}

\begin{proof}
Suppose that $\text{x} \in [\text{C}]$ is tiny-projective and $\text{x} \xrightarrow{\sigma \circ \rho} \text{r}$ is a retract of $\text{x}$. The left Yoneda functor sends $\text{x} \xrightarrow{\sigma \circ \rho} \text{r}$ to a retract $\text{x}(\text{C}) \xrightarrow{\sigma \circ \rho} \text{r}(\text{C})$ of $\text{x}(\text{C})$. So $\sigma(\text{C}) : \text{x}(\text{C}) \rightarrow \text{r}(\text{C})$ is a coequalizer (see Definition 8.9.11) of the idempotent $\rho \circ (\sigma(\text{C})) : \text{x}(\text{C}) \rightarrow \text{x}(\text{C})$. Hence, by Theorem 8.4.19, if $\text{x}(\text{C})$ preserves colimits, so does $\text{r}(\text{C})$.
\end{proof}

\begin{remark}
In particular, any object isomorphic to a tiny-projective object is tiny-projective.
\end{remark}

\begin{proposition}
Any equivalence functor sends a tiny-projective object to a tiny-projective object.
\end{proposition}

\begin{proof}
Let $\text{F} : \text{X} \rightarrow \text{A}$ be an equivalence. We need to show that if $\text{x} \in [\text{X}]$ is tiny-projective, then so is $\text{x} \xrightarrow{\text{F}} \text{F} \in [\text{A}]$. But since $\text{x}(\text{X})$ and $(\text{x} \circ \text{F})(\text{A})$ are conjugate to each other as we saw in Proposition 7.12.5, this follows from Corollary 8.2.18.
\end{proof}

\begin{definition}
A right module $\mathcal{M} : \text{X} \rightarrow \ast$ is called tiny-projective if it is a tiny-projective object of the category $[\text{X} : ]$. Given a category $\text{X}$, Tiny$[\text{X} : ]$ denotes the full subcategory of $[\text{X} : ]$ whose objects are tiny-projective right modules over $\text{X}$.
\end{definition}

\begin{proposition}
For any object $\text{x} \in [\text{X}]$, the representable right module $(\text{X}) \text{x}$ is tiny-projective.
\end{proposition}

\begin{proof}
By the Yoneda representation (see Definition 5.2.16), $(\text{x}(\text{X}) \circ (\text{X}) : \ast \rightarrow [\text{X} : ]$ is isomorphic to $\text{x}(\text{X}) \circ [\text{X} : ]$. Hence, by Proposition 8.2.14, the problem translates into showing that the left module $\text{x}(\text{X})$—evaluation at $\text{x}$ (see Definition 5.1.1)—preserves colimits. But this is the case because a colimit in the category $[\text{X} : ]$ is given pointwise (see Corollary 8.6.7).
\end{proof}
12.10.8 Theorem. A small right module $\mathcal{M} : X \to \ast$ is tiny-projective if and only if it is a retract of the representable right module $(X) x$ for some $x \in \|X\|$; that is,

$$\text{Tiny}[X :] = \text{Ret}_{[X]} \mathcal{M}.$$

Proof. ($\Leftarrow$) By Proposition 12.10.7 and Proposition 12.10.3.

($\Rightarrow$) First recall from Corollary 12.8.8 that $\mathcal{M}$ is presented as a colimit with a universal cone given by the composite

$$
\begin{array}{ccc}
[\mathcal{M}] & \xrightarrow{i} & * \\
\mathcal{M} & \xrightarrow{1_{\mathcal{M}}} & \mathcal{M} \\
X & \xrightarrow{\cdot(x \cdot)} & [X :] \\
x & \xrightarrow{1} & [X :] \\
[\mathcal{X}] & \xrightarrow{\cdot(x \cdot)} & [X :]
\end{array}
$$

; if $\mathcal{M}$ is tiny-projective, then by definition $(\mathcal{M}) (X :)$ preserves colimit, and we have a universal cone

$$
(\mathcal{M}) (X :)[1_{\mathcal{M}} \cdot (X :)] : (\mathcal{M}) (X :)[\mathcal{M} \cdot (X :)] \sim (\mathcal{M}) (X :)(\mathcal{M})
$$

; since a colimit of a small left module $\mathcal{M}$ is given by the set of orbits of $\mathcal{M}$ (Theorem 8.6.2), the universal cone $(\mathcal{M}) (X :)[1_{\mathcal{M}} \cdot (X :)]$ induces a bijection between the set of orbits of $(\mathcal{M}) (X :)[\mathcal{M} \cdot (X :)]$ and the set $(\mathcal{M}) (X :)(\mathcal{M})$ of right module morphisms $\mathcal{M} \to \mathcal{M} : X \to \ast$. Hence, noting Remark 12.8.9, we see that there exists an $\mathcal{M}$-arrow $m : x \to \ast$ and a right module morphism $\theta : \mathcal{M} \to (X :)$ such that

$$
\theta \cdot 1_{\mathcal{M}} = \theta \cdot \mathcal{X} m = 1_{\mathcal{M}}.
$$

\(\Box\)

12.10.9 Theorem. If a small category $\mathcal{D}$ is idempotent complete, then $\mathcal{D}$ is equivalent to $\text{Tiny}[\mathcal{D} :]$.

Proof. By Theorem 7.11.28, Theorem 8.9.37, and Theorem 12.10.8, we have

$$
\mathcal{D} \simeq \mathcal{M} = \text{Ret}_{[\mathcal{D}]} \mathcal{M} = \text{Tiny}[\mathcal{D} :].
$$

\(\Box\)

12.10.10 Corollary. Given two small categories $\mathcal{C}$ and $\mathcal{C}'$ and idempotent completions $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C}' \to \mathcal{D}'$, the following conditions are equivalent:

1. $\mathcal{C} :$ and $\mathcal{C}' :$ are equivalent;
2. $\mathcal{D}$ and $\mathcal{D}'$ are equivalent.

Proof. By Theorem 8.9.36, we may assume that $\mathcal{D}$ and $\mathcal{D}'$ are small. The assertion now follows from the following lemma.

Lemma. For idempotent completions $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C}' \to \mathcal{D}'$ with $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C}'$, and $\mathcal{D}'$ small, the following conditions are equivalent:

1. $\mathcal{C} : \simeq \mathcal{C}' :$;
2. $\mathcal{D} : \simeq \mathcal{D}' :$;
3. $\text{Tiny}[\mathcal{D} :] \simeq \text{Tiny}[\mathcal{D}' :]$;
4. $\mathcal{D} \simeq \mathcal{D}'$.

Proof. (1)$\iff$ (2) Because $\mathcal{C} : \simeq \mathcal{D} :$ and $\mathcal{C}' : \simeq \mathcal{D}' :$ by Theorem 8.9.39.

(2)$\Rightarrow$ (3) By Proposition 12.10.5, the equivalence $\mathcal{D} : \simeq \mathcal{D}' :$ restricts to the equivalence $\text{Tiny}[\mathcal{D} :] \simeq \text{Tiny}[\mathcal{D}' :]$.

(3)$\iff$ (4) Because $\text{Tiny}[\mathcal{D} :] \simeq \mathcal{D}$ and $\text{Tiny}[\mathcal{D}' :] \simeq \mathcal{D}'$ by Theorem 12.10.9.

(4)$\Rightarrow$ (2) This is Theorem 7.11.27.
13 Ends

13.1 Ends

An end is defined as a universal extraordinary cylinder; specifically, an end of a bifunctor \( E^* \times E \to A \) along a module \( M : X \to A \) is defined by a universal arrow of the module \( (E,M) \) introduced in Section 4.4. A cell \( \psi : M \to N \) is then said to preserve ends of bifunctors \( E^* \times E \to A \) along \( M \) when the postcomposition cell \( (E,\psi) : (E,M) \to (E,N) \) preserves universal arrows.

A conical limit is seen as a special instance of an end. We will see on the other hand that an end transforms reversibly into a weighted limit (and hence into a conical limit as shown in Section 12.3), and see that a cell preserves ends precisely when it preserves limits.

**Note.** In the following, we consider an extraordinary cylinder \( E \rightharpoonup M \) as an arrow of the module \( (E,M) \) defined in Definition 4.4.5.

13.1.1 Definition. Let \( E \) be a category and \( M : X \to A \) be a module.

\( \triangleright \) An extraordinary cylinder \( E \rightharpoonup E \) is called universal if and only if the conic cell

\[
\begin{array}{c}
\text{E} \\
\downarrow \mu \\
\text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{X} \\
\downarrow r
\end{array}
\quad \text{E} \\
\downarrow \psi
\]

there is a unique \( \text{A} \rightarrow \text{X} \rightarrow \text{A} \) such that \( \alpha = \alpha/\mu \circ \mu \). Dually, a cylinder \( \mu : G \rightharpoonup E \rightharpoonup M \) is universal if and only if to every cylinder \( \alpha : G \rightharpoonup a : E \rightharpoonup M \) there is a unique \( G \rightarrow A \rightarrow a \) such that \( \alpha = \mu \circ \mu \alpha \).

13.1.2 Remark.

(1) A cylinder \( \mu : r \rightharpoonup F : E \rightharpoonup M \) is universal if only if to every cylinder \( \alpha : X \rightharpoonup F : E \rightharpoonup M \) there is a unique \( X \rightarrow \text{r} \rightarrow \text{F} \rightarrow \mu \) (the adjunct of \( \alpha \) along \( \mu \)) such that \( \alpha = \alpha/\mu \circ \mu \). Dually, a cylinder \( \mu : G \rightharpoonup r : E \rightharpoonup M \) is universal if only if to every cylinder \( \alpha : G \rightharpoonup a : E \rightharpoonup M \) there is a unique \( a \rightarrow \text{A} \rightarrow \text{X} \rightarrow \mu \alpha \).

(2) An end of a bifunctor \( L : E^* \times E \to C \) is defined as a special case of an end defined in Definition 13.1.1 where \( M \) is the hom-module of a category \( C \); and an end of \( L \) is given by a universal extraordinary natural transformation \( \mu : r \rightharpoonup L : E \rightharpoonup C \), i.e. by an inverse universal \( (E,C) \)-arrow (see Remark 4.4.6(4)), and dually a coend of \( L \) is given by a universal extraordinary natural transformation \( \mu : L \rightharpoonup r : E^* \rightharpoonup C \) i.e. by a direct universal \( (E,C) \)-arrow.

(3) Ends and coends are unique up to isomorphism by Corollary 6.2.8.

**Note.** The bijective correspondence between cylinders and conic cells stated in Theorem 5.5.8 gives the following result.

13.1.3 Theorem. Given a category \( E \) and a module \( M : X \to A \), a cylinder

\[
\begin{array}{c}
\text{E} \\
\downarrow \mu \\
\text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{X} \\
\downarrow r
\end{array}
\]

is universal if and only if the conic cell

\[
\begin{array}{c}
\text{E} \\
\downarrow \mu \\
\text{A}
\end{array}
\]

\[
\begin{array}{c}
\text{X} \\
\downarrow r
\end{array}
\]

is universal.
Proof. Immediate from the isomorphism in Theorem 5.5.8.

13.1.4 Remark. By Theorem 13.1.3 and by the bijectivity of the assignment $\mu \mapsto (\mu)$, we see that given a module $M : X \to A$,
- an end of a bifunctor $F : E^* \times E \to A$ along $M$ is the same thing as an $(E)$-weighted limit of $F$ along $M$.
- a coend of a bifunctor $G : E^* \times E \to X$ along $M$ is the same thing as an $(E^*)$-weighted colimit of $G$ along $M$.

The notion of a weighted limit thus subsumes that of an end.

Note. In Remark 4.6.4(2), we noted that a cone is regarded as a special instance of an extraordinary cylinder. As expected, we have the following.

13.1.5 Theorem. Let $E$ be a category and $M : X \to A$ be a module.
- A cone $\mu : r \Rightarrow F : *E \Rightarrow M$ is universal if and only if the cylinder $\mu : r \Rightarrow F \circ [!_E \times E] : E \Rightarrow M$ is universal.
- A cone $\mu : G \Rightarrow r : E^* \Rightarrow M$ is universal if and only if the cylinder $[!_E \times E] \circ G : r : E^* \Rightarrow M$ is universal.

Proof. Since a cone $\mu : r \Rightarrow F : *E \Rightarrow M$ is universal iff it is an inverse universal $(*E,M)$-arrow, and the cylinder $\mu : r \Rightarrow F \circ [!_E \times E] : E \Rightarrow M$ is universal iff it is an inverse universal $(E,M)$-arrow, the assertion follows by applying Theorem 6.2.19 to the composite in Theorem 4.6.34.

13.1.6 Remark. Theorem 13.1.5 says that given a module $M : X \to A$,
- a limit of a functor $F : E \to A$ along $M$ is the same thing as an end of the bifunctor $F \circ [!_E \times E] : E^* \times E \to A$ along $M$.
- a colimit of a functor $G : E \to X$ along $M$ is the same thing as a coend of the bifunctor $[!_E \times E] \circ G : E^* \times E \to X$ along $M$.

A limit is thus regarded as an end with a dummy variable. Now recalling Remark 13.1.4 and Remark 12.3.9, we see that each of the notions “end”, “conical limit”, and “weighted limit” subsumes the other two.

Note. Since an end is defined by a universal $(E,M)$-arrow, we can describe the preservation of ends using the postcomposition cell $(E, \psi)$ defined in Definition 4.4.15.

13.1.7 Definition. A cell $X \xrightarrow{M} A$ is said to preserve (resp. reflect, create) ends [op. coends] $\overrightarrow{\psi}$ $\overrightarrow{\psi}$ $\overrightarrow{\psi}$ $\overrightarrow{\psi}$ $\overrightarrow{\psi}$ $\overrightarrow{\psi}$ over a category $E$ if the postcomposition cell

$$\begin{array}{c}
X \xrightarrow{E,M} \xrightarrow{E \times E,A} \xrightarrow{E \times E,Y} B
\end{array}$$

preserves (resp. reflects, creates) inverse [op. direct] universal arrows.

13.1.8 Remark.
(1) Recalling the definition of the postcomposition cell, Definition 13.1.7 can be stated in elementary terms as follows: $\psi$ is said to preserve

- ends over $E$ if each universal cylinder $\mu : r \Rightarrow F : E \Rightarrow M$ yields by composition with $\psi$ a universal cylinder $\mu \circ \psi : r' : P \Rightarrow Q \circ F : E \Rightarrow N$.
- coends over $E$ if each universal cylinder $\mu : G \Rightarrow r : E^* \Rightarrow M$ yields by composition with $\psi$ a universal cylinder $\mu \circ \psi : G \circ P \Rightarrow Q \circ r : E^* \Rightarrow N$. 
13.2. Ends with parameters

A parameterized end is defined as a pointwise universal bicylinder or any of its transposes (see Section 4.7). Left exponential transposition transforms a pointwise universal bicylinder into a pointwise lift along the module of extraordinary cylinders, allowing us to apply the results in Section 6.5 to parameterized ends.

Note. In the following, we consider a complex bicylinder \(\textbf{E} \times \textbf{D} \rightrightarrows \text{M} \) (see Definition 4.7.1) as an arrow of the module \((\textbf{E} \times \textbf{D}, \text{M})\) in Remark 4.7.2(1).

13.2.1 Definition. Given a module \(\text{M} : \text{X} \rightrightarrows \text{A} \) and categories \textbf{E} and \textbf{D}, a complex bicylinder

\[
\begin{array}{c}
\text{E} \xrightarrow{\mu} \text{E} \times \text{D}^{-} \xrightarrow{\text{E} \times \text{D}^{-} \xrightarrow{\text{D}} \text{E}} \\
\text{R} \xrightarrow{\mu} \text{E} \times \text{D}^{-} \xrightarrow{\text{D}} \text{E} \xrightarrow{\text{E} \times \text{D}^{-} \xrightarrow{\text{D}} \text{E}} \\
\text{X} \xrightarrow{\text{M}} \text{A} \xrightarrow{\text{M}} \text{A}
\end{array}
\]

is called universal if it is an inverse universal \((\text{E} \times \text{D}, \text{M})\)-arrow [op. direct universal \((\text{E} \times \text{D}^{-}, \text{M})\)-arrow].

- Given a bifunctor \(\text{F} : \text{E} \times \text{D}^{-} \times \text{D} \rightarrow \text{A} \), a universal complex bicylinder \(\mu : \text{R} \rightrightarrows \text{F} : \text{E} \times \text{D} \rightrightarrows \text{M} \) or the pair \((\text{R}, \mu)\), or the functor \text{R} itself, is called an end of \text{F} along \text{M}, with the functor \text{R} denoted by \(\prod_{\text{D}} \text{F} \).
- Given a bifunctor \(\text{G} : \text{E} \times \text{D}^{-} \times \text{D} \rightarrow \text{X} \), a universal complex bicylinder \(\mu : \text{G} \rightrightarrows \text{R} : \text{E} \times \text{D} \rightrightarrows \text{M} \) or the pair \((\text{R}, \mu)\), or the functor \text{R} itself, is called a coend of \text{G} along \text{M}, with the functor \text{R} denoted by \(\prod_{\text{D}} \text{G} \).

Note. Recall from Definition 4.7.9 that a complex bicylinder is sliced into pieces of extraordinary cylinders. In the following, we consider the case where these extraordinary cylinders are universal (see Definition 13.1.1).
13.2.2 Definition. Given a module $\mathcal{M} : X \to A$ and categories $E$ and $D$, a complex bicylinder

$$\begin{array}{ccc}
E & E \times D^\perp \times D & \text{op.} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
X - \mathcal{M} - \to A & X - \mathcal{M} - \to A
\end{array}$$

is called pointwise universal if each left slice

$$\begin{array}{ccc}
\ast & D^\perp \times D & \text{op.} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
e^R & e^R & e^R \\
\downarrow & \downarrow & \downarrow \\
X - \mathcal{M} - \to A & X - \mathcal{M} - \to A
\end{array}$$

is a universal extraordinary cylinder, i.e. an inverse universal $(D,\mathcal{M})$-arrow [op. direct universal $(D^\perp,\mathcal{M})$-arrow].

- Given a bifunctor $F : E \times D^\perp \times D \to A$, a pointwise universal complex bicylinder $\mu : R \Rightarrow F : E \times D \Rightarrow \mathcal{M}$ or the pair $(R,\mu)$, or the functor $R$ itself, is called an $E$-parameterized end of $F$ along $\mathcal{M}$, with the functor $R$ denoted by $\prod^E_D F$ or just by $\prod^E_D F$.

- Given a bifunctor $G : E \times D^\perp \times D \Rightarrow X$, a pointwise universal complex bicylinder $\mu : G \Rightarrow R : E \times D \Rightarrow \mathcal{M}$ or the pair $(R,\mu)$, or the functor $R$ itself, is called an $E$-parameterized coend of $G$ along $\mathcal{M}$, with the functor $R$ denoted by $\prod^E_D G$ or just by $\prod^E_D G$.

Note. Since a complex bicylinder $E \times D \Rightarrow \mathcal{M}$ is transposed into an extraordinary cylinder $D \Rightarrow (E,\mathcal{M})$ (see Definition 4.7.9), Definition 13.2.2 has the following variation.

13.2.3 Definition. Given a module $\mathcal{M} : X \to A$ and categories $E$ and $D$, an extraordinary cylinder

$$\begin{array}{ccc}
\ast & D^\perp \times D & \text{op.} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
\left[ E \times X \right]_{(E,M)} - \mathcal{M} - \to \left[ E \times A \right]_{(E,M)} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
\left[ e \times X \right]_{(e,M)} - \mathcal{M} - \to \left[ e \times A \right]_{(e,M)}
\end{array}$$

along the module $(E,\mathcal{M})$ is called pointwise universal if each slice

$$\begin{array}{ccc}
\ast & D^\perp \times D & \text{op.} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
\left[ E \times X \right]_{(E,M)} - \mathcal{M} - \to \left[ E \times A \right]_{(E,M)} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
\left[ e \times X \right]_{(e,M)} - \mathcal{M} - \to \left[ e \times A \right]_{(e,M)}
\end{array}$$

(see Remark 4.7.12(2)) is a universal extraordinary cylinder, i.e. an inverse universal $(D,\mathcal{M})$-arrow [op. direct universal $(D^\perp,\mathcal{M})$-arrow].

- Given a bifunctor $F : D^\perp \times D \Rightarrow [E, A]$, a pointwise universal extraordinary cylinder $\mu : R \Rightarrow F : D \Rightarrow (E,\mathcal{M})$ or the pair $(R,\mu)$, or the functor $R$ itself, is called a pointwise end of $F$ along $(E,\mathcal{M})$, with the functor $R$ denoted by $\prod^E_D F$ or just by $\prod^E_D F$.

- Given a bifunctor $G : D^\perp \times D \Rightarrow [E, X]$, a pointwise universal extraordinary cylinder $\mu : G \Rightarrow R : D \Rightarrow (E,\mathcal{M})$ or the pair $(R,\mu)$, or the functor $R$ itself, is called a pointwise coend of $G$ along $(E,\mathcal{M})$, with the functor $R$ denoted by $\prod^E_D G$ or just by $\prod^E_D G$.

13.2.4 Proposition. The exponential transposition in Definition 4.7.9 preserves and reflects universality and pointwise universality; that is, the following conditions are equivalent;

(1) a complex bicylinder

$$\begin{array}{ccc}
E & E \times D^\perp \times D & \text{op.} \\
\mu & \mu & \mu \\
\downarrow & \downarrow & \downarrow \\
X - \mathcal{M} - \to A & X - \mathcal{M} - \to A
\end{array}$$

is universal (resp. pointwise universal);
(2) its right exponential transpose
\[
\begin{array}{c|c|c}
R & \mu^\tau & [E, X] \\
\mu & F & [E, A] \\
\hline \end{array}
\]

is a universal (resp. pointwise universal) extraordinary cylinder;

(3) its left exponential transpose
\[
\begin{array}{c|c|c}
R & \mu & [E, X] \\
\hline \mu & F & [D^\leftarrow \times D, A] \\
\end{array}
\]

is a universal (resp. pointwise universal) ordinary cylinder.

Proof. By the isomorphisms in Remark 4.7.10, \(\mu\) is universal iff \(\mu^\tau\) is universal iff \(\mu^\tau\) is universal. The pointwise version is immediate from Remark 4.7.12(3).

13.2.5 Remark. By Proposition 13.2.4 and noting by the bijectivity of exponential transposition, we see that given a module \(M : X \to A\),

- the following mean the same thing:
  1. an end (resp. \(E\)-parameterized end) of \(F : E \times D^\leftarrow \times D \to A\) along \(M\);
  2. an end (resp. pointwise-parameterized end) of \([F^\tau] : D^\leftarrow \times D \to [E, A]\) along the module \((E, M)\);
  3. a lift (resp. pointwise lift) of \([\mu^\tau] : E \to [D^\leftarrow \times D, A]\) along the module \((D, M)\).

- the following mean the same thing:
  1. a coend (resp. \(E\)-parameterized coend) of \(G : E \times D^\leftarrow \times D \to X\) along \(M\);
  2. a coend (resp. pointwise coend) of \([G^\tau] : D^\leftarrow \times D \to [E, X]\) along the module \((E, M)\);
  3. a colift (resp. pointwise colift) of \([\mu^\tau] : E \to [D^\leftarrow \times D, X]\) along the module \((D, M)\).

13.2.6 Proposition. A pointwise universal complex bicylinder in Definition 13.2.2 is universal in the sense of Definition 13.2.1.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of (1) and (3) in Proposition 13.2.4.

13.2.7 Proposition. The twist transposition in Definition 4.8.7 preserves and reflects universality and pointwise universality; that is, the following conditions are equivalent;

(1) an extraordinary cylinder
\[
\begin{array}{c|c|c}
R & \mu & [E, X] \\
\mu & F & [E, A] \\
\hline \end{array}
\]

is universal (resp. pointwise universal);

(2) its twist transpose
\[
\begin{array}{c|c|c}
R & \mu^\tau & [E, X] \\
\mu & F & [D^\leftarrow \times D, A] \\
\hline \mu & F & [D^\leftarrow \times D, A] \\
\end{array}
\]

is a universal (resp. pointwise universal) ordinary cylinder.

Proof. By the isomorphism in Remark 4.7.12(1), \(\mu\) is universal iff \(\mu^\tau\) is universal. Since the slice of \(\mu\) at each \(e \in [E]\) is given by the component of \(\mu^\tau\) at \(e\), \(\mu\) is pointwise universal iff \(\mu^\tau\) is pointwise universal.

13.2.8 Remark. By Proposition 13.2.7 and noting the bijectivity of twist transposition, we see that given a module \(M : X \to A\),
13.2.9 Proposition. A pointwise universal extraordinary cylinder in Definition 13.2.3 is universal in the sense of Definition 13.1.1.

Proof. This is reduced to Proposition 6.5.10 by the equivalence of (1) and (2) in Proposition 13.2.7.

13.2.10 Theorem. (Parameter Theorem for ends). Let $E$ and $D$ be categories and $M : X \to A$ be a module as in Definition 13.2.2.

- Given a bifunctor $F : E \times D^\rightarrow \to A$, suppose that for each object $e \in |E|$, the bifunctor $[e \times F] : D^\rightarrow \to A$ has an end along $M$ and a universal extraordinary cylinder

$$ r_e \leftarrow M \rightarrow A $$

is chosen. Then there is a unique functor $R : E \to X$ with $e : R = r_e$ such that $\mu := ([\mu_e]_{d,e})_{(e,d) \in |E \times D|}$ forms a complex bicylinder

$$ E \xrightarrow{\mu} E \times D^\rightarrow \to D^\rightarrow \to A $$

- Given a bifunctor $G : E \times D^\rightarrow \to X$, suppose that for each object $e \in |E|$, the bifunctor $[e \times G] : D^\rightarrow \to X$ has a coend along $M$ and a universal extraordinary cylinder

$$ e \leftarrow M \rightarrow A $$

is chosen. Then there is a unique functor $R : E \to A$ with $e : R = r_e$ such that $\mu := ([\mu_e]_{d,e})_{(e,d) \in |E \times D|}$ forms a complex bicylinder

$$ E \times D^\rightarrow \to D \xrightarrow{\mu} E \xrightarrow{\mu} \leftarrow M \rightarrow A $$

Proof. By the equivalence of (1) and (3) in Proposition 13.2.4, this is reduced to an instance of Theorem 6.5.14 where $M$ is given by the module $(D,M)$.

13.2.11 Theorem. Let $M : X \to A$ be a module.

- Suppose that a bifunctor $F : E^\rightarrow \to E \times D^\rightarrow \to D \to A$ has an $[E^\rightarrow \to E]$-parameterized end

$$ E^\rightarrow \to E \xrightarrow{\mu} E \times D^\rightarrow \to D \xrightarrow{\mu} \leftarrow M \rightarrow A $$

along $M$. Then an end of $F$ (regarded as a bifunctor $[E \times D]^\rightarrow \times E \times D \to A$) along $M$ exists if and only if an end of $R$ in $X$ exists; specifically, if $R$ has an end $\nu : r \sim R$, then the composite $\nu \circ \mu : r \sim F$ (see Proposition 4.7.3) gives an end of $F$, and conversely if $F$ has an end $\kappa : r \sim F$, then there is a unique cylinder $\nu : r \sim R$ such that $\kappa = \nu \circ \mu$, and $\nu$ gives an end of $R$. 

Suppose that a bifunctor $G : E^\times E \times D^\times D \to X$ has an $[E^\times E]$-parameterized coend
\[G \parallel \mu \downarrow \Rightarrow \parallel A^\times A \times A \parallel\]
along $M$. Then a coend of $G$ (regarded as a bifunctor $[E^\times D] \times [E^\times D] \to X$) along $M$ exists if and only if a coend of $R$ in $A$ exists; specifically, if $R$ has a coend $\nu : R \to r$, then the composite $\mu \circ \nu : G \to r$ (see Proposition 4.7.3) gives a coend of $G$, and conversely if $G$ has a coend $\kappa : G \to r$, then there is a unique cylinder $\nu : R \to r$ such that $\kappa = \mu \circ \nu$, and $\nu$ gives a coend of $R$.

Proof. This follows from the lemma below.

Lemma. If the complex bicylinder $\mu : R \to F$ in Proposition 4.7.3 is pointwise universal, then the right module morphism in Remark 4.7.4 is an isomorphism; that is, for any extraordinary bicylinder $\kappa : x \to F$, there is a unique extraordinary natural transformation $\nu : x \to R$ such that $\kappa = \mu \circ \nu$.

Proof. Since $\mu$ is pointwise universal, each left slice $[(e,e)\land \mu] : R(e,e) \to [(e,e)\land F]$ of $\mu$ at $(e,e) \in [E^\times E]$ is a universal extraordinary cylinder; hence for the left slice $[e\land \kappa] : x \to [(e,e)\land F]$ of $\kappa$ at $e \in E$ (see Definition 4.7.7), there is a unique $X$-arrow $\nu_e : x \to R(e,e)$ such that $\nu_e \circ [(e,e)\land \mu] = [e\land \kappa]$. The proof is thus complete if we show that $\nu := (\nu_e)_{e \in [E]}$ forms extraordinary natural transformation $c : R$. For this, let $h : e \to e'$ be an $E$-arrow and consider the diagram

\[
\begin{array}{ccc}
x & \xrightarrow{\nu_e} & R(e,e) \\
\downarrow{\nu_{e'}} & & \downarrow{R(e,h)} \\
R(e',e') & \xrightarrow{R(h,e')} & R(e,e') \\
\end{array}
\]

; we need to show that the top left square commutes. Since the right and bottom trapezoids commute by the naturality of $\land \mu$, and the outer square commutes by the naturality of $\land \kappa$, simple diagram chasing shows that
\[\nu_e \circ R(e,h) \circ [(e,e')\land \mu] = \nu_e \circ R(h,e') \circ [(e,e')\land \mu]\]

; hence
\[\nu_e \circ R(e,h) = \nu_e \circ R(h,e')\]

by the universality of $[(e,e')\land \mu]$.

\[\Box\]

13.2.12 Corollary. Let $M : X \to A$ be a module.

- Suppose that a bifunctor $F : E^\times E \times D^\times D \to A$ has an $[E^\times E]$-parameterized end and a $[D^\times D]$-parameterized end along $M$ as shown below

\[\begin{array}{ccc}
E^\times E & \xrightarrow{\mu} & E^\times E \\
\downarrow{R} & & \downarrow{F} \\
X & \xrightarrow{M} & A \\
\end{array}
\]

\[\begin{array}{ccc}
D^\times D & \xrightarrow{\mu'} & E^\times E \\
\downarrow{R'} & & \downarrow{F'} \\
X & \xrightarrow{M} & A \\
\end{array}
\]

; then an end of $R$ in $X$ exists if and only if an end of $R'$ in $X$ exists. If this is the case, then there exist universal cylinders $\nu : r \to R$ and $\nu' : r \to R'$ in $X$ such that the diagram

\[
\begin{array}{ccc}
r & \xrightarrow{\nu_d} & R'(d,d) \\
\downarrow{\nu_e} & & \downarrow{\nu'_d,d,d} \\
R(e,e) & \xrightarrow{\mu'_e,d,d} & F(e,e,d,d) \\
\end{array}
\]

commutes for every $(e,d) \in [E^\times D]$.\[\Box\]
• Suppose that a bifunctor $G : E^r \times E \times D^r \times D \to X$ has an $[E^r \times E]$-parameterized coend and a $[D^r \times D]$-parameterized coend along $M$ as shown below

$$
\begin{array}{c}
E^r \times E \times D^r \times D & E^r \times E & D^r \times D \\
\mu & \mu' \\
X \to \to \to \to \to A & X \to \to \to \to \to A
\end{array}
$$

Then a coend of $R$ in $A$ exists if and only if a coend of $R'$ in $A$ exists. If this is the case, then there exist universal cylinders $\nu : R \to r$ and $\nu' : R' \to r$ in $A$ such that the diagram

$$
\begin{array}{ccc}
G(e,e,d,d) & \xrightarrow{\nu_{e,d,d}} & R(e,e) \\
\nu'_{e,d,d} & \downarrow & \downarrow \nu_e \\
R'(d,d) & \xrightarrow{\nu'_d} & r
\end{array}
$$

commutes for every $(e,d) \in [E \times D]$.

Proof. Immediate from Theorem 13.2.11.

\[ \Box \]

13.2.13 Remark.

(1) The results in Theorem 13.2.11 and Corollary 13.2.12 are expressed by

$$
\prod_{e \in E} \prod_{d \in D} F(e,e,d,d) \cong \prod_{e \in E} \prod_{d \in D} F(e,e,d,d)
$$

or, more informatively,

$$
\prod_{e \in E} \prod_{d \in D} F(e,e,d,d) \cong \prod_{(e,d) \in E \times D} F(e,e,d,d)
$$

op.

$$
\prod_{e \in E} \prod_{d \in D} G(e,e,d,d) \cong \prod_{(e,d) \in E \times D} G(e,e,d,d)
$$

and called the Fubini theorem for ends [op. coends].

(2) Since a limit is regarded as an end with a dummy variable, the result in (1) above includes the corresponding facts for limits: we have the end-limit interchange property

$$
\prod_{e \in E} \prod_{d \in D} F(e,d,d) \cong \prod_{(e,d) \in E \times D} F(e,d,d)
$$

by regarding $F(e,d,d)$ as $F(e,e,d,d)$ dummy in the first variable $e$, and then the limits interchange property

$$
\prod_{e \in E} \prod_{d \in D} F(e,d) \cong \prod_{(e,d) \in E \times D} F(e,d)
$$

by regarding $F(e,d)$ as $F(e,e,d,d)$ dummy in the first variable $d$. Dually, we have the coend-colimit interchange property and the colimits interchange property.

13.3 Ends of modules

In this section, we study ends in the category $\text{Set}$; that is, ends of endomodules. Just like the frame-set of a small left module $M : * \to E$ gives a limit of $M$, the frame-set of a small endomodule $M : E \to E$ gives an end of $M$.

13.3.1 Theorem. For a small endomodule $M : E \to E$, i.e. a bifunctor $M : E^r \times E \to \text{Set}$ with $E$ small, an end of $M$ is given by an equalizer $\pi$ as in

$$
\begin{array}{ccc}
\Pi_{e \in [E]} e(M) e & \xrightarrow{\pi} & \Pi_{e \in [E]} (\Gamma_{e,e'}) e(M) e' \\
\Pi_{e' \in [E]} e'(M) e' & \xrightarrow{\mu_{e' \in [E]} (\Gamma_{e',e'})} & \Pi_{e \in [E]} \prod_{e' \in [E]} (\Gamma_{e,e'}) e(M) e'
\end{array}
$$
, where

\[ \Gamma_{e, e'} : e(\mathcal{M}) e \to [e(\mathcal{E}) e, e(\mathcal{M}) e'] : m \mapsto [h \mapsto m \circ h] \]

and

\[ \Gamma_{e, e'}^\prime : e'(\mathcal{M}) e' \to [e(\mathcal{E}) e', e(\mathcal{M}) e'] : m \mapsto [h \mapsto h \circ m] \]

are the functions given by the exponential transpose of the composition between \( M \)-arrows and \( E \)-arrows.

**Proof.** By the claim below, an end of \( \mathcal{M} \) is given by a universal fork on \( \prod_{e \in \mathcal{E}} (\Gamma_{e, e'})_{e' \in \mathcal{E}} \) and \( \prod_{e' \in \mathcal{E}} (\Gamma_{e, e'})_{e \in \mathcal{E}} \), i.e. by an equalizer of them.

**Claim.** Given a small set \( S \), a family of functions \( \xi_e : S \to e(\mathcal{E}) e \), one for each object \( e \in \mathcal{E} \), forms a cylinder \( S \to \mathcal{M} \) if and only if the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{(\xi_e)_{e \in \mathcal{E}}} & \prod_{e \in \mathcal{E}} e(\mathcal{M}) e \\
\downarrow (\xi_{e'})_{e' \in \mathcal{E}} & & \downarrow [\prod_{e' \in \mathcal{E}} (\Gamma_{e, e'})_{e \in \mathcal{E}}] \\
\prod_{e' \in \mathcal{E}} e'(\mathcal{M}) e' & \xrightarrow{\prod_{e' \in \mathcal{E}} (\Gamma_{e, e'})_{e \in \mathcal{E}}} & \prod_{e' \in \mathcal{E}} e(\mathcal{E}) e' e(\mathcal{M}) e'
\end{array}
\]

commutes, i.e. if and only if the function \( (\xi_e)_{e \in \mathcal{E}} : S \to \prod_{e \in \mathcal{E}} e(\mathcal{M}) e \) forms a fork on the functions \( \prod_{e \in \mathcal{E}} (\Gamma_{e, e'})_{e' \in \mathcal{E}} \) and \( \prod_{e' \in \mathcal{E}} (\Gamma_{e, e'})_{e \in \mathcal{E}} \).

**Proof.** The diagram in the claim commutes iff the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\xi_e} & e(\mathcal{M}) e \\
\downarrow \xi_{e'} & & \downarrow (\Gamma_{e, e'})_{e' \in \mathcal{E}} \\
e'(\mathcal{M}) e' & \xrightarrow{\Gamma_{e, e'}} & e(\mathcal{E}) e' e(\mathcal{M}) e'
\end{array}
\]

commutes for each \( e \in \mathcal{E} \), and this diagram commutes iff the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\xi_e} & e(\mathcal{M}) e \\
\downarrow \xi_{e'} & & \downarrow \Gamma_{e, e'} \\
e'(\mathcal{M}) e' & \xrightarrow{\Gamma_{e, e'}} & [e(\mathcal{E}) e', e(\mathcal{M}) e']
\end{array}
\]

commutes for each \( e' \in \mathcal{E} \), and this diagram commutes iff the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\xi_e} & e(\mathcal{M}) e \\
\downarrow \xi_{e'} & & \downarrow \Gamma_{e, e'} \\
e'(\mathcal{M}) e' & \xrightarrow{\Gamma_{e, e'}} & e(\mathcal{M}) e'
\end{array}
\]

commutes for each \( E \)-arrow \( h : e \to e' \) (because for any \( s \in S \),

\[
s : [\xi_e \circ e(\mathcal{M}) h] = (s : \xi_e) \circ h = h : [s : (\xi_e) : \Gamma_{e, e'}] = h : [s : \xi_e \circ \Gamma_{e, e'}]
\]

and

\[
s : [\xi_{e'} \circ e(\mathcal{M}) h] = h : [s : (\xi_{e'}) : \Gamma_{e, e'}] = h : [s : \xi_{e'} \circ \Gamma_{e, e'}]
\]

by the definitions of \( \Gamma_{e, e'} \) and \( \Gamma_{e, e'}^\prime \). \( \square \)

**13.3.2 Theorem.** For a small endomodule \( \mathcal{M} : E \to E \), the frame-set (see Definition 4.1.1) of \( \mathcal{M} \) gives an end of \( \mathcal{M} \) with the universal cylinder

\[
\tau : \prod E \mathcal{M} \rightrightarrows \mathcal{M} : E \to E
\]

defined such that its component

\[
\tau_e : \prod E \mathcal{M} \to e(\mathcal{M}) e
\]
at \( e \in \| E \| \) maps each frame \( \alpha \) of \( \mathcal{M} \) to its component at \( e \); that is,

\[ \alpha : \pi_e := \alpha_e. \]

**Proof.** This follows from the observation that an element \( \alpha \) of the product \( \prod_{e \in \| E \|} e(\mathcal{M}) \) is a frame of \( \mathcal{M} \) iff it lies in the equalizer in Theorem 13.3.1. \( \square \)

13.3.3 **Remark.** For a small endomodule \( \mathcal{M} : \mathcal{E} \to \mathcal{E} \), the notation \( \prod_{\mathcal{E}} \mathcal{M} \) thus denotes both the frame-set of \( \mathcal{M} \) and an end of \( \mathcal{M} \).

13.3.4 **Definition.** Given a pair of left modules \( \mathcal{M}, \mathcal{N} : \ast \to \mathcal{E} \), the hom-join of \( \mathcal{M} \) and \( \mathcal{N} \) is the endomodule

\[ \langle \mathcal{M} \circ \mathcal{N} \rangle : \mathcal{E} \to \mathcal{E} \]

defined by the composition

\[ \mathcal{E} \xrightarrow{\mathcal{M}} \text{Set} \xrightarrow{\langle \text{Set} \rangle} \text{Set} \xrightarrow{\mathcal{N}} \mathcal{E} \]

, where \( \langle \text{Set} \rangle \) denotes the hom-module of the category \( \text{Set} \); that is, for \( e, d \in \mathcal{E} \),

\[ e \langle \mathcal{M} \circ \mathcal{N} \rangle d := ((\mathcal{M})_e)(\langle \mathcal{N} \rangle d) = ([\mathcal{M}]_e, [\mathcal{N}]_d) \]

(for \( S, T \in \text{Set} \), we write \( [S, T] \) instead of \( S \langle \text{Set} \rangle T \) (see Remark 1.1.22(5))).

13.3.5 **Remark.** A frame \( \psi \) of the hom-join \( \langle \mathcal{M} \circ \mathcal{N} \rangle : \mathcal{E} \to \mathcal{E} \) is the same thing as a natural transformation \( \psi : \mathcal{M} \to \mathcal{N} : \ast \to \text{Set} \), i.e., a left module morphism \( \psi : \mathcal{M} \to \mathcal{N} \).

13.3.6 **Proposition.** Let \( \mathcal{M}, \mathcal{N} : \ast \to \mathcal{E} \) be a pair of left modules. Given a small set \( S \), there is a canonical bijection between the set of cylinders

\[ \ast \xrightarrow{\mathcal{M}} \mathcal{E} \times \mathcal{E} \xrightarrow{\langle \text{Set} \rangle} \text{Set} \xrightarrow{\mathcal{N}} \mathcal{E} \]

and the set of conic cells

\[ \ast \xrightarrow{\mathcal{M}} \mathcal{E} \xrightarrow{\langle \text{Set} \rangle} \text{Set} \]

, with each component

\[ (\xi)_e : \langle \mathcal{M} \rangle e \to [S, \langle \mathcal{N} \rangle e] \]

of a cell \( (\xi) \) given by the twist transpose of the component

\[ [\xi]_e : S \to ([\mathcal{M}]_e, [\mathcal{N}]_e) \]

of the corresponding frame \( [\xi] \). Moreover, the bijection is natural in \( S \).

**Proof.** Immediate since the twist transposition

\[ [\mathcal{A}, [S, \mathcal{B}]] \xrightarrow{\langle \text{Set} \rangle} [S, [\mathcal{A}, \mathcal{B}]] \]

is bijective and natural in all three variables. \( \square \)

13.3.7 **Remark.** Hence an \( \mathcal{M} \)-weighted limit of \( \mathcal{N} \) in \( \text{Set} \) is the same thing as an end of \( \langle \mathcal{M} \circ \mathcal{N} \rangle \) in \( \text{Set} \):

\[ \prod^{\mathcal{M}} \mathcal{N} \cong \prod_{\mathcal{E}} \langle \mathcal{M} \circ \mathcal{N} \rangle. \]

13.3.8 **Proposition.** For a pair of small left modules \( \mathcal{M}, \mathcal{N} : \ast \to \mathcal{E} \), an end of the hom-join \( \langle \mathcal{M} \circ \mathcal{N} \rangle : \mathcal{E} \to \mathcal{E} \) exists and is given by the set of module morphisms \( \mathcal{M} \to \mathcal{N} : \ast \to \mathcal{E} \):

\[ \prod_{\mathcal{E}} \langle \mathcal{M} \circ \mathcal{N} \rangle \cong \mathcal{M} : \mathcal{E} \to \mathcal{N}. \]
To apply Proposition 13.3.8, the universe may have to be enlarged temporarily so that $E$. Hence the frame-set of $(\mathcal{M} \circ \mathcal{N})$ gives an end of $(\mathcal{M} \circ \mathcal{N})$ by Theorem 13.3.2. But the frame-set of $(\mathcal{M} \circ \mathcal{N})$ is the same thing as the set of left module morphisms $\mathcal{M} \to \mathcal{N}$ (see Remark 13.3.5).

13.3.9 Theorem. For any left module $\mathcal{M}: \ast \to E$ and any object $e \in \mathcal{E}$, there is a bijection $(\mathcal{M}) e \cong \prod_{E} (e(E) \circ \mathcal{M})$.

Proof. By Theorem 5.2.8 and Proposition 13.3.8, we have $(\mathcal{M}) e \cong (e(E)) (\mathcal{M}) \cong \prod_{E} (e(E) \circ \mathcal{M})$.

13.3.10 Remark.

1. To apply Proposition 13.3.8, the universe may have to be enlarged temporarily so that $E$ becomes small. However, the result holds in the original universe, in which $\mathcal{M}$ is locally small.

2. Some authors call the bijection above the end form of the Yoneda lemma.

13.4 Coends of modules

In this section, we study coends in the category $\text{Set}$; that is, coends of endomodules. Just like the set of orbits of a small left module $\mathcal{M}: \ast \to E$ gives a colimit of $\mathcal{M}$, the set of orbits of a small endomodule $\mathcal{M}: E \to E$ gives a coend of $\mathcal{M}$.

13.4.1 Theorem. For a small endomodule $\mathcal{M}: E \to E$, i.e. a bifunctor $\mathcal{M}: E^\ast \times E \to \text{Set}$ with $E$ small, a coend of $\mathcal{M}$ is given by a coequalizer $\pi$ as in

\[
\begin{array}{ccc}
\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e \times e(E) e' & \xrightarrow{\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} (e'_e)^{\text{set}[\mathcal{E}]}} & \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e' \\
\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e \times e(E) e' & \xrightarrow{\pi} & \coprod_{E} E \mathcal{M}
\end{array}
\]

, where $\Gamma_{e,e'}: e'(\mathcal{M}) e \times e(E) e' \to e(\mathcal{M}) e; (m,h) \mapsto h \circ m$ and $\Gamma_{e,e'}': e'(\mathcal{M}) e \times e(E) e' \to e'(\mathcal{M}) e'; (m,h) \mapsto m \circ h$ are the functions given by the composition between $\mathcal{M}$-arrows and $E$-arrows.

Proof. By the claim below, a coend of $\mathcal{M}$ is given by a universal fork on $\coprod_{e \in \mathcal{E}} (\Gamma_{e,e'})_{e \in \mathcal{E}}$ and $\coprod_{e' \in \mathcal{E}} (\Gamma_{e,e'})_{e \in \mathcal{E}}$, i.e. by a coequalizer of them.

Claim. Given a small set $S$, a family of functions $\xi_{e}: e(\mathcal{M}) e \to S$, for each object $e \in \mathcal{E}$, forms a cylinder $\mathcal{M} \to S$ if and only if the diagram

\[
\begin{array}{ccc}
\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e \times e(E) e' & \xrightarrow{\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} (\xi_{e'})_{e \in \mathcal{E}}} & \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e' \\
\coprod_{e \in \mathcal{E}} \coprod_{e' \in \mathcal{E}} e'(\mathcal{M}) e \times e(E) e' & \xrightarrow{(\xi_{e})_{e \in \mathcal{E}}} & S
\end{array}
\]

commutes, i.e. if and only if the function $(\xi_{e})_{e \in \mathcal{E}}: \coprod_{e \in \mathcal{E}} e(\mathcal{M}) e \to S$ forms a fork on the functions $\coprod_{e \in \mathcal{E}} (\Gamma_{e,e'})_{e \in \mathcal{E}}$ and $\coprod_{e' \in \mathcal{E}} (\Gamma_{e,e'})_{e \in \mathcal{E}}$. 


Proof. The diagram in the claim commutes iff the diagram
\[
\coprod_{e' \in \|E\|} e' \langle \mathcal{M} \rangle e \times e(\mathcal{E}) e' \cong \coprod_{e' \in \|E\|} e' \langle \mathcal{M} \rangle e' \\
\bigg/ (\mathcal{R}_{e,e'})_{e' \in \|E\|} \bigg| \\
e(\mathcal{M}) e \xrightarrow{\iota_e} S 
\]
commutes for each \(e \in \|E\|\), and this diagram commutes iff the diagram
\[
e'(\mathcal{M}) e \times e(\mathcal{E}) e' \cong e'(\mathcal{M}) e' \\
\bigg/ (\mathcal{R}_{e,e'}) \bigg| \\
e(\mathcal{M}) e \xrightarrow{\iota_e} S 
\]
commutes for each \(e' \in \|E\|\), and this diagram commutes iff the diagram
\[
e'(\mathcal{M}) e \cong e'(\mathcal{M}) e' \\
h(\mathcal{M}) e \bigg/ (\mathcal{R}_{e,e'}) \bigg| \\
e(\mathcal{M}) e \xrightarrow{\iota_e} S 
\]
commutes for each \(E\)-arrow \(h : e \rightarrow e'\) (because for any \(\mathcal{M}\)-arrow \(m : e' \rightarrow e\),
\[
m : [h(\mathcal{M}) e \circ \xi_e] = (h \circ m) : \xi_e = (m, h) : \mathcal{R}_{e,e'} : \xi_e = (m, h) : \mathcal{R}_{e,e'} : \xi_e 
\]
and
\[
m : [e(\mathcal{M}) h \circ \xi_{e'}] = (m \circ h) : \xi_{e'} = (m, h) : \mathcal{R}_{e,e'} : \xi_{e'} = (m, h) : \mathcal{R}_{e,e'} : \xi_{e'} 
\]
by the definitions of \(\mathcal{R}_{e,e'}\) and \(\mathcal{R}_{e,e'}\).
\[\square\]

13.4.2 Theorem. For a small endomodule \(\mathcal{M} : E \rightarrow E\), the set of orbits (see Definition 4.11.6) of \(\mathcal{M}\) gives a coend of \(\mathcal{M}\) with the universal cylinder
\[
\pi : \mathcal{M} \rightarrow \coprod_E \mathcal{M} : E \rightarrow E 
\]
defined by the natural projection such that
\[
m : \pi = m^n 
\]
for each \(\mathcal{M}\)-arrow \(m\).

Proof. The coequalizer in Theorem 13.4.1 is given by the quotient of \(\coprod_{e \in E} e(\mathcal{M}) e\) by the equivalence relation generated by the pairs \((h \circ m, m \circ h)\) for every \(\mathcal{M}\)-arrow \(m : e' \rightarrow e\) and every \(E\)-arrow \(h : e \rightarrow e'\); that is (see Remark 4.11.4), by the equivalence relation defined in Definition 4.11.1.
\[\square\]

13.4.3 Remark. For a small endomodule \(\mathcal{M} : E \rightarrow E\), the notation \(\coprod_E \mathcal{M}\) thus denotes both the set of orbits of \(\mathcal{M}\) and a coend of \(\mathcal{M}\).

13.4.4 Definition. The product-join of a right module \(\mathcal{M} : E \rightarrow *\) and a left module \(\mathcal{N} : * \rightarrow E\) is the endomodule
\[
\langle \mathcal{M} \times \mathcal{N} \rangle : E \rightarrow E 
\]
defined by the composition
\[
E \times E \xrightarrow{\mathcal{M} \times \mathcal{N}} Set \times Set \xrightarrow{\times} Set 
\]
, where \(Set \times Set \xrightarrow{\times} Set\) is the set theoretic product; that is
\[
e(\mathcal{M} \times \mathcal{N}) d := e(\mathcal{M}) \times e(\mathcal{N}) d 
\]
for \(e, d \in E\).

13.4.5 Proposition. Let \(\mathcal{M} : E \rightarrow *\) be a right module and \(\mathcal{N} : * \rightarrow E\) be a left module. Given a
small set $S$, there is a canonical bijection between the set of cylinders

$$E^* \times E \xrightarrow{e} \ast$$

$$\mathcal{M} \times \mathcal{N} \xrightarrow{[\xi]} S$$

$$\text{Set} \xrightarrow{\pi} \text{Set}$$

and the set of conic cells

$$E \xrightarrow{\mathcal{M}} \ast$$

$$\mathcal{N} \xrightarrow{[\xi]} S$$

$$\text{Set} \xrightarrow{\pi} \text{Set}$$

, with each component

$$e(\xi) : e(\mathcal{M}) \to [(\mathcal{N}) e, S]$$

of a cell $[\xi]$ given by the exponential transpose of the component

$$[\xi]_e : e(\mathcal{M}) \times (\mathcal{N}) e \to S$$

of the corresponding frame $[\xi]$. Moreover, the bijection is natural in $S$.

Proof. Immediate since the exponential transposition

$$[A \times B, S] \xrightarrow{\sim} [A, [B, S]]$$

is bijective and natural in all three variables. 

13.4.6 Remark. Hence an $\mathcal{M}$-weighted colimit of $\mathcal{N}$ in $\text{Set}$ is the same thing as a coend of $(\mathcal{M} \times \mathcal{N})$ in $\text{Set}$:

$$\int^{\mathcal{M}\mathcal{N}} \cong \int^{\mathcal{M}} \{\mathcal{M} \times \mathcal{N}\}$$

13.4.7 Theorem.

- Given a right module $\mathcal{M} : E \to \ast$ and an object $e \in \|E\|$, there is a canonical universal cylinder

$$\pi : (\mathcal{M} \times e(\mathcal{E})) \to e(\mathcal{M})$$

with each component $\pi_w : e'(\mathcal{M}) \times e(\mathcal{E}) e' \to e(\mathcal{M})$ given by the assignment $(m, h) \mapsto h \circ m$; for a set $S$ and a cylinder $\xi : (\mathcal{M} \times e(\mathcal{E})) \to S$, the adjunct $\pi \xi : e(\mathcal{M}) \to S$ is given by the partial evaluation of $\xi_e : e(\mathcal{M}) \times e(\mathcal{E}) e \to S$ at $1_e$, i.e. by the assignment $m \mapsto \xi_e(m, 1_e)$.

- Given a left module $\mathcal{M} : \ast \to E$ and an object $e \in \|E\|$, there is a canonical universal cylinder

$$\pi : (\mathcal{M} \times e(\mathcal{E})) \to (\mathcal{M}) e$$

with each component $\pi_w : e'(\mathcal{E}) e \times (\mathcal{M}) e' \to (\mathcal{M}) e$ given by the assignment $(h, m) \mapsto m \circ h$; for a set $S$ and a cylinder $\xi : (\mathcal{M} \times e(\mathcal{E})) \to S$, the adjunct $\pi \xi : (\mathcal{M}) e \to S$ is given by the partial evaluation of $\xi_e : e(\mathcal{E}) e \times (\mathcal{M}) e \to S$ at $1_e$, i.e. by the assignment $m \mapsto \xi_e(1_e, m)$.

Proof. The naturality of $\pi$ follows immediately from the functoriality of $\mathcal{M}$. To prove the universality of $\pi$, it suffices to show that $\pi \xi$, defined as above gives a unique factorization $\xi = \pi \circ \pi \xi$.

Let $h : e \to e'$ be an E-arrow and consider the diagram

$$\begin{array}{ccc}
E'(\mathcal{M}) \times e(\mathcal{E}) & \xrightarrow{1 \times e(\mathcal{E}) h} & E'(\mathcal{M}) \\
/_{h(\mathcal{M}) \times 1} & & \\
\pi_e \downarrow & & \pi_w \downarrow \\
e(\mathcal{M}) \times e(\mathcal{E}) & \xrightarrow{\pi_w} & e(\mathcal{M}) \\
\xi_e \downarrow & & \pi \xi \\
S & &
\end{array}$$

; the inner commutative square maps $(m, 1_e) \in e'(\mathcal{M}) \times e(\mathcal{E}) e$ as follows:

$$\begin{array}{ccc}
(m, 1_e) & \xrightarrow{1 \times e(\mathcal{E}) h} & (m, h) \\
/_{h(\mathcal{M}) \times 1} & & \\
\pi_w \downarrow & & \pi_w \downarrow \\
(h \circ m, 1_e) & \xrightarrow{\pi_w} & h \circ m
\end{array}$$

\[\square\]
For a small right module $\xi$, we have
\[
[\pi \backslash \xi](h \circ m) = \xi_e(h \circ m, 1_e) = \xi_e(m, h)
\]

; hence, by the commutativity of the outer diagram and by the definition of $\pi \backslash \xi$, we have

\[
\text{the diagram}
\]

\[
\begin{array}{ccc}
\mathcal{E}(M) & \xrightarrow{\pi_m} & S \\
\downarrow{\pi_e} & & \downarrow{\pi_e} \\
\mathcal{E}(M) & \xrightarrow{\xi_e} & S
\end{array}
\]

thus commutes for arbitrary $\xi \in \mathcal{E}$. Hence $\xi = \pi \pi \backslash \xi$.

Since $\pi_e$ maps $(m, 1_e)$ to $1_e \circ m = m$, in order for the diagram

\[
\begin{array}{ccc}
\mathcal{E}(M) \times \mathcal{E}(E) & \xrightarrow{\pi_m \mathcal{E}} & S \\
\downarrow{\xi_e} & & \downarrow{\xi_e} \\
\mathcal{E}(M) & \xrightarrow{\xi_e} & S
\end{array}
\]

\[\text{to commute, } \pi \backslash \xi \text{ must map } m \text{ to } \xi_e(m, 1_e). \text{ The uniqueness of } \pi \backslash \xi \text{ follows.}\]

13.4.8 Corollary.

- For any right module $\mathcal{M} : \mathcal{E} \to *$ and any object $e \in \mathcal{E}$, there is a bijection
  \[\mathcal{E}(\mathcal{M}) \cong \bigcup_{(\mathcal{M} \star e)} (\mathcal{M} \star e(\mathcal{E})).\]

- For any left module $\mathcal{M} : * \to \mathcal{E}$ and any object $e \in \mathcal{E}$, there is a bijection
  \[\mathcal{E}(\mathcal{M}) e \cong \bigcup_{(\mathcal{E} e)} (\mathcal{E} e \star \mathcal{M}).\]

Proof. We have shown in Theorem 13.4.7 that there is a canonical universal cylinder $(\mathcal{M} \star e(\mathcal{E})) \sim e(\mathcal{M})$.

13.4.9 Remark. Some authors call the bijections above the coend form of the Yoneda lemma (cf. Remark 13.3.10(2)). [Lor19] contains a slick proof for the bijection using the Yoneda lemma without constructing a universal cylinder.

13.4.10 Definition. Given a small right module $\mathcal{M} : \mathcal{E} \to *$ and a small left module $\mathcal{N} : * \to \mathcal{E}$, their tensor product, written $\mathcal{M} \otimes \mathcal{E} \mathcal{N}$ or just $\mathcal{M} \otimes \mathcal{N}$, is given by the set $\bigcup_{\mathcal{E}} (\mathcal{M} \star \mathcal{N})$ of orbits of the product-join $(\mathcal{M} \star \mathcal{N}) : \mathcal{E} \to \mathcal{E}$; specifically, the tensor product $\mathcal{M} \otimes \mathcal{E} \mathcal{N}$ is given by the quotient set of $\bigcup_{\mathcal{E}} e(\mathcal{M} \star \mathcal{N}) e$ by the equivalence relation generated by all pairs $(m, n) \approx (m', n')$ such that the diagram

\[
\begin{array}{ccc}
\star & \xrightarrow{n} & \star \\
\downarrow{h} & & \downarrow{m'} \\
\star & \xrightarrow{m} & \star
\end{array}
\]

commutes for some $\mathcal{E}$-arrow $h$ (cf. Remark 4.11.4).

13.4.11 Remark. An amalgamation of a right module $\mathcal{M} : \mathcal{E} \to *$ and a left module $\mathcal{N} : * \to \mathcal{E}$ is a pushout

\[
\begin{array}{ccc}
\mathcal{E} \times [\mathcal{N}] & \xrightarrow{\mathcal{M}} & [\mathcal{N}] \\
\downarrow{[\mathcal{M}]} & & \downarrow{[\mathcal{N}]} \\
[\mathcal{M}] \star [\mathcal{N}] & \xrightarrow{+_{\mathcal{E}}} & [\mathcal{N}]
\end{array}
\]

in CAT. For small $\mathcal{M}$ and $\mathcal{N}$, the small category $[\mathcal{M}] +_{\mathcal{E}} [\mathcal{N}]$ is constructed by “pasting” the collage categories $[\mathcal{M}]$ and $[\mathcal{N}]$ together in the obvious way and defining the hom-set between the two endpoints by the tensor product $\mathcal{M} \otimes \mathcal{E} \mathcal{N}$.

13.4.12 Proposition. For a small right module $\mathcal{M} : \mathcal{E} \to *$ and a small left module $\mathcal{N} : * \to \mathcal{E}$, a coend of the product-join $(\mathcal{M} \star \mathcal{N}) : \mathcal{E} \to \mathcal{E}$ exists and is given by the tensor product of $\mathcal{M}$ and $\mathcal{N}$:

\[\bigcup_{\mathcal{E}} (\mathcal{M} \star \mathcal{N}) \cong \mathcal{M} \otimes \mathcal{E} \mathcal{N}\]

Proof. Immediate from Definition 13.4.10 and Theorem 13.4.2.
13.4.13 Theorem.\footnote{Remark 13.4.6} For any small right module $\mathcal{M} : E \rightarrow \ast$ and any object $e \in \|E\|$, there is a bijection
\[ e(\mathcal{M}) \cong \mathcal{M} \otimes_E e(E). \]

For any small left module $\mathcal{M} : \ast \rightarrow E$ and any object $e \in \|E\|$, there is a bijection
\[ (\mathcal{M}) e \cong (E) e \otimes_E \mathcal{M}. \]

Proof. By Corollary 13.4.8 and Proposition 13.4.12, we have
\[ e(\mathcal{M}) \cong \bigsqcup_E (\mathcal{M} \otimes e(E)) \cong \mathcal{M} \otimes_E e(E). \]

13.4.14 Remark. Some authors call the bijections above the coYoneda lemma (cf. Remark 12.2.8(2)).

13.4.15 Theorem. A left small module $\mathcal{N} : \ast \rightarrow E$ has a pointwise left Kan extension along the right Yoneda functor $E \rightarrow$, given by the tensor product as shown in
\[
\begin{array}{ccc}
E & \overset{\mathcal{E}}{\longrightarrow} & [E:] \\
\downarrow \mathcal{N} & \pi \downarrow \mathcal{N} & \\
\text{Set} & \overset{\otimes \mathcal{N}}{\longrightarrow} & \text{Set}
\end{array}
\]

; moreover, $\pi$ is a natural isomorphism.

Proof. By Corollary 12.9.2, it suffices to show that for any right module $\mathcal{M} : E \rightarrow \ast$, $\mathcal{M} \otimes \mathcal{N}$ gives an $\mathcal{M}$-weighted colimit of $\mathcal{N}$. But since a $\mathcal{M}$-weighted colimit of $\mathcal{N}$ is the same thing as a coend of $(\mathcal{M} \otimes \mathcal{N})$ (see Remark 13.4.6), this follows from Proposition 13.4.12. Since the Yoneda functor $E \rightarrow$ is fully faithful, the second assertion follows from Theorem 12.7.8. \qed
References


[MM92] Mac Lane, S., Moerdijk, I., Sheaves in geometry and logic: a first introduction to topos theory. Springer-Verlag, 1992.


Pare, R., On absolute colimits. J. Algebra 19,80-95, 1971.


### List of Notation

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