Further Insights into Thermal Relativity Theory and Black Hole Thermodynamics

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Abstract

We continue to explore the consequences of Thermal Relativity Theory to the physics of black holes. The thermal analog of Lorentz transformations in the *tangent* space of the thermodynamic manifold are studied in connection to the Hawking evaporation of Schwarzschild black holes and one finds that there is *no* bound to the thermal analog of proper accelerations despite the maximal bound on the thermal analog of velocity given by the Planck temperature. The proper entropic infinitesimal interval corresponding to the Kerr-Newman black hole involves a 3×3 non-Hessian metric with diagonal and off-diagonal terms of the form $(d\mathbf{s})^2 = g_{ab}(M, Q, J) dZ^a dZ^b$, where $Z^a = M, Q, J$ are the mass, charge and angular momentum, respectively. Black holes in asymptotically Anti de Sitter (de Sitter) spacetimes are more subtle to study since the mass turns out to be related to the *enthalpy* rather that the internal energy. We finalize with some remarks about the thermal-relativistic analog of proper force, the need to extend our analysis of Gibbs-Boltzmann entropy to the case of Reny and Tsallis entropies, and to complexify spacetime.

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The deep origins of the connection between Black Holes and Thermodynamics is still a mystery (to our knowledge). The idea of describing classical thermodynamics using geometric approaches has a long history. The thermodynamic length is a metric distance between equilibrium thermodynamic states. Among various treatments, Weinhold [2] used the Hessian of internal energy $g^W_{ij} = \partial_i \partial_j U(S, \mathcal{E}^k)$ to define a metric for thermodynamic fluctuations. U is the internal energy of the system and $\mathcal{E}^k$ refers to the extensive parameters of the system, $i, j, k = 1, 2, \cdots, n$.

Ruppeiner [3] used the negative Hessian of the entropy for the same purpose to define the metric $g^R_{ij} = -\partial_i \partial_j S(U, \mathcal{E}^k)$. Since the net entropy of the system is extremal (maximized) at equilibrium the second derivatives are negative so that $g^R_{ij} > 0$. From the first law of thermodynamics $dU = TdS + I_a d\mathcal{E}^a$ (given in terms of the intensive and extensive variables $I_a, \mathcal{E}^a$) one arrives at the relations $I_a = \frac{\partial U}{\partial \mathcal{E}^a} = -T \frac{\partial S}{\partial \mathcal{E}^a}$ and one finds that the Ruppeiner metric is conformally related to the Weinhold metric $(ds)^2_R = g^R_{ij} d\mathcal{E}^i d\mathcal{E}^j = T^{-1}(ds)^2_W = T^{-1} g^W_{lk} d\mathcal{E}^l d\mathcal{E}^k$.

It has long been observed that the Ruppeiner metric is flat for systems with noninteracting underlying statistical mechanics such as the ideal gas [4]. Curvature singularities signal critical behaviors. In addition, it has been applied to a number of statistical systems including Van de Waals gas. Recently the anyon gas has been studied using this approach. In the past years, this geometry has been applied to black hole thermodynamics, with some physically relevant results. The most physically significant case is for the Kerr black hole in higher dimensions, where the curvature singularity signals thermodynamic instability, as found earlier by conventional methods [4]. There are also many important applications of Finsler geometry to the field of Thermodynamics, contact geometry and many other topics [8].

Quevedo [5] introduced a formalism called Geometrothermodynamics (GTD) which also introduces metric structures on the configuration space $\mathcal{E}$ of the thermodynamic equilibrium states spanned by all the extensive variables. In GTD, to study the geometric properties of the equilibrium space three thermodynamic metrics have been proposed so far. These metrics were obtained by using the condition of Legendre invariance and can be computed explicitly once a thermodynamic potential is specified as fundamental equation.

The three classes of Legendre invariant metrics are of the form $ds^2 = (d\Phi - I_a d\mathcal{E}^a)^2 + h_{ab}(I_a, \mathcal{E}^a) d\mathcal{E}^a dI^b$ where Φ is the thermodynamic potential. The remaining diffeomorphism invariance in the phase and equilibrium spaces can be used to show that the components of the GTD-metrics can be interpreted as the second moment of the fluctuation of a new thermodynamic potential. This result establishes a direct connection between GTD and fluctuation theory. In this way, the diffeomorphism invariance of GTD allows us to introduce *new* thermodynamic coordinates and *new* thermodynamic potentials, which are *not* related by means of Legendre transformations to the fundamental thermodynamic potentials [6].

Zhao [7] was able to outline the essential principles of Thermal Relativity; i.e. invariance under the group $\mathcal{G}$ of general coordinate transformations on the thermodynamic configuration space, and *introduced* a metric with a *Lorentzian* signature on the space. The line element was identified as the square of the *proper* entropy, and which was also *invariant* under the action of the group $\mathcal{G}$. Thus the first and second law of thermodynamics admitted an invariant formulation under general coordinate transformations, which justified the foundations for the principle of Thermal Relativity (frame independence and thermal causality). It is important to emphasize that one must *not* confuse Thermal Relativity with Relativistic Thermodynamics, nor with (GTD) Geometrothermodynamics, nor with Thermodynamic (Information) Geometry.

Recently we derived the *exact* thermal relativistic *corrections* to the Schwarzschild black hole entropy and provided a detailed analysis of the many *novel* applications and consequences of the Thermal Relativity principle to the physics of black holes, quantum gravity, minimal area, minimal mass, Yang-Mills mass gap, information paradox, arrow of time, dark matter, and dark energy [1].

We shall briefly review our results in [1] which involved the thermal analog of Lorentz boosts transformations and then extend our analysis to the thermal analog of proper accelerations in connection to the physics of Hawking's evaporation of black holes.

One may implement Zhao's formulation [7] of Thermal Relativity in the flat analog of Minkowski space as

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ \leftrightarrow \ (d\tau)^2 \ = \ (cdt)^2 \ - \ (dx)^2 \qquad (1)$$

The maximal Planck temperature T_P plays the role of the speed of light, and $\mathbf{s}$ is the so-called *proper entropy* which is invariant under the thermodynamical version of Lorentz transformations [7]. Note the $\mathbf{s} \leftrightarrow \tau$ correspondence. Thus the flow of the proper entropy $\mathbf{s}$ is consistent with the arrow of time.

The left hand side of (1) yields, after recurring to the first law of Thermodynamics $TdS = dM \Rightarrow T = \frac{dM}{dS}$,

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ \left(1 \ - \ \frac{T^2}{T_P^2} \right) \ \Rightarrow \ (d\mathbf{s}) \ = \ (T_P \ dS) \ \sqrt{\left(1 \ - \ \frac{T^2}{T_P^2} \right)} \ =$$

$$T_P \ (\frac{dM}{T}) \ \sqrt{\left(1 \ - \ \frac{T^2}{T_P^2} \right)} \ \Rightarrow \ dM \ = \ \frac{T}{T_P} \ \frac{1}{\sqrt{1 - \frac{T^2}{T_P^2}}} \ d\mathbf{s} \qquad (2)$$

Given the thermal dilation factor one can always define an "effective" temperature by

$$T_{eff} \ = \ \frac{T}{\sqrt{1 - \frac{T^2}{T_P^2}}} \qquad (3a)$$

such that $dM = \gamma(T) T (d\mathbf{s}/T_P)$ becomes then the thermal relativistic analog of the Energy-Momentum relations $E = m_o c^2 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $\vec{p} = m_o \vec{v} (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ in Special Relativity, in terms of the rest mass m_o, velocity v, and maximal speed of light c.

Setting T equal to the Hawking temperature of a Schwarzschild black hole $T = T_H = (8\pi G M)^{-1}$, the effective temperature (3a) can be written in terms of the mass as

$$T_{eff} \ = \ \frac{T}{\sqrt{1 - \frac{T^2}{T_P^2}}} \ = \ \frac{1}{\sqrt{(8\pi G M)^2 - \frac{1}{T_P^2}}} \qquad (3b)$$

The effective $T_{eff} = \infty$ blows up when one reaches the maximal temperature $T = T_H = T_P$ at the reduced Planck mass $M = \frac{1}{8\pi G T_P} = \frac{M_P}{8\pi}$. The mass value of $\frac{M_P}{8\pi}$ corresponds to the *minimal* mass at the point when the black hole evaporation process *stops* since the maximal Planck T_P cannot be surpassed beyond that point.

Eq-(1) allowed to *derive* the thermal relativistic corrections to the Black Hole Entropy [1] as follows. After renaming $\tilde{S} \equiv (\mathbf{s}/T_P)$, in terms of the *proper* entropy $\mathbf{s}$, the first law of black hole thermal-relativity dynamics $dM = \gamma(T_H) T_H d\tilde{S}$ yields the *corrected* entropy

$$\int_{\tilde{S}_o}^{\tilde{S}} d\tilde{S} \;=\; \tilde{S} - \tilde{S}_o \;=\; \int_{M_o}^{M} \frac{dM}{\gamma(T_H) T_H} \;=\; \int_{M_o}^{M} dM \; \frac{\sqrt{1 - (T_H^2/T_P^2)}}{T_H} \tag{4}$$

inserting the expression for the Hawking temperature $T_H(M) = (8\pi G M)^{-1}$ into eq-(4), and after setting $(T_P)^{-2} = (M_P)^{-2} = L_P^2 = G$, yields the following integral

$$\tilde{S} - \tilde{S}_o \;=\; \int_{M_o}^{M} dM \; (8\pi G M) \sqrt{1 - \frac{G}{(8\pi G M)^2}} \;=\; \int_{M_o}^{M} dM \; \sqrt{(8\pi G M)^2 \; - \; G} \tag{5}$$

where $\tilde{S}_o \equiv \tilde{S}(M_o)$. The indefinite integral

$$\int dx \; \sqrt{a^2 x^2 \; - \; b} \;=\; \frac{ax \; \sqrt{a^2 x^2 \; - \; b}}{2a} \; - \; \frac{b}{2a} \; ln \Big(\; a \; [\sqrt{a^2 x^2 \; - \; b} \; + \; ax] \; \Big) \tag{6}$$

permits to evaluate the definite integral in the right hand side of (5) between the *upper* limit M, and a *lower* limit M_o defined by $(8\pi G M_o)^2 - G = 0$, giving

$$\tilde{S} \;=\; \frac{A}{4G} \sqrt{1 - \frac{1}{16\pi} (\frac{A}{4G})^{-1}} - \frac{1}{16\pi} \; ln \left(\; 4\sqrt{\pi} \; (\frac{A}{4G})^{\frac{1}{2}} \; [\; 1 \; + \; \sqrt{1 - \frac{1}{16\pi} (\frac{A}{4G})^{-1}} \;] \right) \tag{7}$$

after using the relation for the ordinary entropy in the Schwarzschild black hole (adopting the units $\hbar = c = k_B = 1$)

$$S = \frac{A}{4G} \;=\; 4\pi G M^2 \;\Rightarrow\; M \;=\; (\frac{A}{16\pi G^2})^{\frac{1}{2}} \tag{8}$$

and $(8\pi G M_o)^2 = G \Rightarrow 8\pi G M_o = \sqrt{G} \Rightarrow M_o = \frac{M_P}{8\pi}$. The lower limit M_o of integration is required in eq-(5) to ensure the terms inside the square root are positive definite and the integral is real-valued. This lower mass M_o is associated to a black hole remnant which has reached the maximal Planck temperature T_P (the thermal dilation factor diverges) and where black hole evaporation stops. One of the most salient features of eq-(7) is that the *modified* entropy

$(\tilde{S})$, corresponding to the minimal area $A_o = \frac{G}{4\pi} = \frac{L_P^2}{4\pi}$, $\tilde{S}_o = \tilde{S}(A_o) = 0$ is *zero*. The fact that the third law of thermodynamics (zero entropy at zero absolute temperature) is violated here should not be bothersome because it is well known that the entropy of *extremal* Kerr, Kerr-Newman black holes is *not* zero despite having a *zero* Hawking temperature. It also violates the third law of thermodynamics

Given the correspondence between an interval in thermal configuration space and the interval in spacetime

$$(d\mathbf{s})^2 \;=\; (T_P dS)^2 \left(1 \;-\; \frac{T^2}{T_P^2} \right) \;\Leftrightarrow\; (d\tau)^2 \;=\; (cdt)^2 \left(1 \;-\; \frac{v^2}{c^2} \right) \tag{9}$$

one will also have the thermal "velocity" boost transformations analog of the Lorentz boost transformation that leave invariant the interval

$$(d\mathbf{s})^2 \;=\; (T_P dS)^2 \;-\; (dM)^2 \;=\; (T_P dS')^2 \;-\; (dM')^2 \tag{10}$$

These thermal boosts transformations which leave invariant eq-(10) are given by

$$S' \;=\; \frac{S - \frac{TM}{T_P^2}}{\sqrt{1 - (\frac{T}{T_P})^2}}, \quad M' \;=\; \frac{M - TS}{\sqrt{1 - (\frac{T}{T_P})^2}} \tag{11}$$

Based on this thermal relativistic analogy with the Lorentz velocity-boosts transformations, and on the correspondence $S \leftrightarrow t; M \leftrightarrow x$, one learns that the analog of length-contraction in Special Relativity is going to be M-contraction under thermal boosts transformations. Since the black hole Hawking radiation leads to a mass loss of black holes one could then postulate that a continuous succession of thermal boosts transformations, whose respective thermal boost rapidity parameters are continuously increasing, may mimick the black hole evaporation process. As the black hole evaporates it gets hotter and hotter so that the corresponding thermal boost rapidity parameters $\xi = arctanh(\frac{T}{T_P})$ increase as the temperature increases.

Similarly, the analog of time-dilation in Special Relativity is going to be S-dilation under thermal boosts transformations, meaning that a continuous succession of thermal boosts transformations continuously increases the entropy S of the emitted Hawking thermal radiation. A thermal gas of photons at a temperature of T has an entropy proportional to T^3, so an increase in T leads to an increase in entropy S. Therefore, the thermal boosts transformations can reflect the loss of the black hole mass and the increase in entropy of the emitted radiation. Because the black hole entropy decreases due to the loss of mass and decrease of its horizon area, there is no S-dilation for the black hole, however, the *net* entropy of the black hole plus the emitted radiation (the outside region) must always increase consistent with the second law of thermodynamics : $\Delta S_{BH} + \Delta S_{rad} \geq 0$.

Since a continuous succession of thermal boosts transformations can be modeled by the analog of thermal accelerations, let us introduce the following dot derivatives

$$\dot{S} \equiv \frac{dS}{d\tilde{S}}, \quad \dot{M} \equiv \frac{dM}{d\tilde{S}} \tag{12}$$

defined in terms of $\tilde{S} \equiv \frac{\mathbf{s}}{T_P}$, a dimensionless quantity that has the same units as the entropy. One may generalize the notion of a tangent spacetime interval to the notion of a *thermal tangent*-space infinitesimal interval given by

$$(d\sigma)^2 = (dS)^2 - T_P^{-2} (dM)^2 + T_P^{-2} [(T_P d\dot{S})^2 - (d\dot{M})^2] \tag{13}$$

After factoring out $(d\tilde{S})^2 \equiv (dS)^2 - T_P^{-2} (dM)^2$ in eq-(13) gives

$$(d\sigma)^2 = (d\tilde{S})^2 \left(1 - \frac{\mathcal{T}^2(\tilde{S})}{T_P^2} \right) \tag{14}$$

where the expression (whose physical units are those of a temperature-squared)

$$- \mathcal{T}^2(\tilde{S}) \equiv (T_P \ddot{S})^2 - (\ddot{M})^2 = T_P^2 (\frac{d^2 S}{d\tilde{S}^2})^2 - (\frac{d^2 M}{d\tilde{S}^2})^2 \leq 0 \tag{15}$$

defines the Thermal Relativity analog of the spacelike proper acceleration-squared $(\ddot{t})^2 - (\ddot{x})^2 \leq 0$. When the velocity is timelike $v_\mu v^\mu = 1 > 0$, the acceleration is spacelike $a_\mu a^\mu < 0$ as a result of differentiating $\frac{d}{d\tau}(v_\mu v^\mu) = \frac{d}{d\tau} 1 = 2 \frac{dv_\mu}{d\tau} v^\mu = 2 a_\mu v^\mu = 0$. From which one learns that the spacelike a^μ is orthogonal to the timelike velocity v^μ.

Eq-(14) can be rewritten as

$$(d\sigma)^2 = (d\tilde{S})^2 \left(1 - \frac{\mathcal{T}^2(\tilde{S})}{T_P^2} \right) = (dM)^2 (\frac{d\tilde{S}}{dM})^2 \left(1 - \frac{\mathcal{T}^2(\tilde{S}(M))}{T_P^2} \right) \tag{16a}$$

By using the identity $(dy/dx) = (dx/dy)^{-1}$ one has $(\frac{d\tilde{S}}{dM})^2 = (\frac{dM}{d\tilde{S}})^{-2}$ so that eq-(16a) becomes

$$(d\sigma)^2 = (dM)^2 (\frac{dM}{d\tilde{S}})^{-2} \left(1 - \frac{\mathcal{T}^2(\tilde{S}(M))}{T_P^2} \right) \tag{16b}$$

To evaluate $\frac{d^2 S}{d\tilde{S}^2}$ and $\frac{d^2 M}{d\tilde{S}^2}$ in eq-(15) is highly nontrivial if one recurs to the complicated expression (7) found for the thermal relativistic corrections to the black hole entropy and given by $\tilde{S} = \tilde{S}(M)$, after expressing the horizon area $A = 4\pi(2GM)^2$ in terms of the mass. The reason being that one must invert (if possible) the expression $\tilde{S}(M)$ leading to $M = M(\tilde{S})$, and to the Bekenstein-Hawking entropy $S(\tilde{S}) = \frac{A}{4G} = 4\pi G M^2(\tilde{S})$. And, in doing so, one may then evaluate the first $\dot{M}, \dot{S}$, and the second derivatives $\ddot{M}, \ddot{S}$ with respect to $\tilde{S}$.

Because this is extremely cumbersome it is far simpler to use the relation

$$\dot{M} = \frac{dM}{d\tilde{S}} = T_{eff} \equiv T\gamma(T) = \left((8\pi GM)^2 - \frac{1}{T_P^2} \right)^{-1/2} \Rightarrow$$

$$\ddot{M} = \frac{d^2M}{d\tilde{S}^2} = -(8\pi G)^2 M \left((8\pi GM)^2 - \frac{1}{T_P^2} \right)^{-2} \quad (17)$$

$$S = 4\pi GM^2 \Rightarrow \dot{S} = \frac{dS}{d\tilde{S}} = (8\pi GM) \left((8\pi GM)^2 - \frac{1}{T_P^2} \right)^{-1/2} \Rightarrow$$

$$\ddot{S} = \frac{d^2S}{d\tilde{S}^2} = (8\pi G) \left((8\pi GM)^2 - \frac{1}{T_P^2} \right)^{-1} - (8\pi G)^3 M^2 \left((8\pi GM)^2 - \frac{1}{T_P^2} \right)^{-2} \quad (18)$$

Hence, after some straightforward algebra, the quantity $(T_P\ddot{S})^2 - (\ddot{M})^2 = -\mathcal{T}^2$ can be expressed directly in terms of M as

$$(T_P\ddot{S})^2 - (\ddot{M})^2 \equiv -\mathcal{T}^2(M) = -T_P^2 (\frac{T_P}{M})^2 (8\pi GMT_P)^2 \left((8\pi GMT_P)^2 - 1 \right)^{-3} \quad (19)$$

Using the definition of the Hawking temperature $T_H = (8\pi GM)^{-1}$, the above expression can be also be rewritten as

$$(T_P\ddot{S})^2 - (\ddot{M})^2 \equiv -\mathcal{T}^2(M) = -T_P^2 (\frac{T_P}{M})^2 (\frac{T_P}{T_H})^2 \left((\frac{T_P}{T_H})^2 - 1 \right)^{-3} \quad (20)$$

From eqs-(19) one learns that $\mathcal{T}^2 \to 0$ as $M \to \infty$ as one would expect. A trivial example in classical mechanics is that if one were to exert a finite force on an infinitely massive object its acceleration will be zero. One also infers that the right-hand side of eq-(19) is ≤ 0 if, and only if, the mass $M \geq M_o = \frac{M_P}{8\pi}$ is greater or equal to the minimal mass M_o. Note that when the Schwarzschild black hole reaches the minimal mass M_o (it has attained the maximal Planck temperature $T_H = T_P$) one has that $\mathcal{T}^2(M = M_o) = \infty$ blows up due to the divergence of the last factor in (19) because when $M = M_o = \frac{M_P}{8\pi} \Rightarrow 8\pi GM_oT_P - 1 = GM_P^2 - 1 = 0$, due to $T_P = M_P$ and $G = L_P^2 = M_P^{-2}$ in the natural units $\hbar = c = k_B = 1$.

Also if $T_H > T_P$ were to exceed T_P then $\mathcal{T}^2 < 0$, and $\mathcal{T}$ would have been imaginary, which is the analog of a tachyon $m^2 < 0$. Note that T in eq-(9) plays an analogous role as a velocity; $\mathcal{T}$ in eqs-(15,20) plays an analogous role as a proper acceleration, and $T_{eff} = T\gamma(T)$ plays the analogous role as energy.

The Fulling-Davies-Unruh effect [9] states that for a uniformly accelerating observer, the vacuum state of an inertial observer is seen as a mixed state in thermodynamic equilibrium with a non-zero temperature bath. The uniformly accelerated observer (detector) will be immersed in a bath of thermal radiation with a temperature $T = \frac{a}{2\pi}$. In this case one has a precise relation between temperature and acceleration, whereas in Thermal Relativity one has a *correspondence* between T and velocity, instead.

Therefore, the thermal analog of the tangent spacetime interval for a Schwarzschild black hole is given by

$$(d\sigma)^2 \ = \ (dM)^2 \ (\frac{d\tilde{S}}{dM})^2 \ \left(1 \ - \ \frac{\mathcal{T}^2(\tilde{S}(M))}{T_P^2} \right) \ =$$

$$(dM)^2 \ (\frac{dM}{d\tilde{S}})^{-2} \ \left(1 \ - \ \frac{\mathcal{T}^2(\tilde{S}(M))}{T_P^2} \right) \ =$$

$$T_P^{-2} \ (dM)^2 \ [\ (8\pi G M T_P)^2 - 1 \ ] \ \left(1 \ - \ \frac{\mathcal{T}^2(\tilde{S}(M))}{T_P^2} \right) \tag{21}$$

where $-\mathcal{T}^2$ is explicitly given by eq-(19) in terms of M.

When $M = M_o$, the first parenthesis in eq-(21) goes to 0, while the second one goes to $-\infty$ even faster, so that the interval $(d\sigma)^2 \to -\infty \times 0 \to -\infty < 0$ blows up and becomes negative definite at the location of the minimal mass $M = M_o = \frac{M_P}{8\pi}$ (at $T = T_P$). This reminds us of the *spacelike* singularity at $r = 0$ of a Schwarzchild black hole due to the exchange roles of t and r as a result of the sign changes in the g_{tt}, g_{rr} metric components once one crosses the horizon.

On the other hand, the value of M_h such that $\mathcal{T}^2(M_h) = T_P^2$ leading to $d\sigma^2 = 0$ is the thermal analog of the black hole horizon beyond which the roles of t and r are exchanged. Upon setting $\mathcal{T}^2(M_h) = T_P^2$ in eq-(19) yields for M_h the value of

$$M_h \ = \ \frac{M_P}{8\pi} \ \sqrt{1 + (8\pi)^{2/3}} \ > \ M_o = \frac{M_P}{8\pi} \tag{22}$$

One finds that M_h is greater but of the same order of magnitude as the minimal mass M_o.

To sum up, one has explicitly checked that the thermal analog of the proper acceleration $-\mathcal{T}^2 \leq 0$ is spacelike if $M \geq M_o$. Both the effective $T_{eff} = \infty$, and $\mathcal{T} = \infty$ *blow* up when one reaches the maximal temperature $T = T_H = T_P$ at the reduced Planck mass $M = \frac{1}{8\pi G T_P} = \frac{M_P}{8\pi}$, corresponding to the *minimal* mass M_o at the point when the Schwarzschild black hole evaporation process *stops*.

Despite postulating a maximal bound to the thermal analog of velocity and given by the Planck temperature T_P, we found that there is *no maximal* bound to the thermal analog of proper acceleration since the $\mathcal{T}(M_o) = \infty$ blows up when the black hole has reached its minimal mass. Eq-(19) reflects the thermal relativistic corrections to the "acceleration" rate of mass-loss of a Schwarzschild black hole due to Hawking evaporation.

To recapitulate, the thermal analog of "velocity" boost transformations leaving invariant the thermal tangent space interval $(d\sigma)^2 = (d\sigma')^2$ in eq-(13) are of the form

$$S' \ = \ S \ cosh(\xi_T) \ - \ T_P^{-1} \ M \ sinh(\xi_T), \tag{23a}$$

$$T_P^{-1} \ M' \ = \ T_P^{-1} M \ cosh(\xi_T) \ - \ S \ sinh(\xi_T) \tag{23b}$$

$$T_P \ \dot{S}' \ \equiv \ T_P \ \frac{dS'}{d\tilde{S}} \ = T_P \ \dot{S} \ cosh(\xi_T) \ - \ \dot{M} \ sinh(\xi_T), \ \ (\dot{S} \equiv \frac{dS}{d\tilde{S}}, \ \dot{M} \equiv \frac{dM}{d\tilde{S}}) \tag{23c}$$

$$\dot{M}' \ \equiv \ \frac{dM'}{d\tilde{S}} \ = \ \dot{M} \ cosh(\xi_T) \ - \ T_P \ \dot{S} \ \sinh(\xi_T), \tag{23d}$$

and where the thermal "velocity" boost rapidity parameter ξ_T is defined as

$$tanh(\xi) \ \equiv \ \frac{T}{T_P} \ \Rightarrow \ cosh(\xi) \ = \ \frac{1}{\sqrt{1 - (\frac{T}{T_P})^2}}, \ \ sinh(\xi) \ = \ \frac{\frac{T}{T_P}}{\sqrt{1 - (\frac{T}{T_P})^2}} \tag{24}$$

such that $\xi_T \rightarrow \infty$ as $T \rightarrow T_P \Rightarrow tanh(\xi_T) \rightarrow 1$.

One can verify that $(d\sigma)^2$ in eq-(13) is invariant under the thermal velocity boost transformations of eqs-(23). This is the analog of the Lorentz transformations leaving invariant the norms $x_\mu x^\mu = t^2 - x_i^2$, and $p_\mu p^\mu = E^2 - p_i^2$, $i = 1, 2, 3$ of the four-vectors x^μ, p^μ.

A Kerr-Newman black hole is described in terms of the mass M, charge Q and angular momentum J. Given the first law $dM = TdS + \phi dQ + \omega dJ$, where ϕ is the electrostatic potential and ω is the angular velocity, it allows us to write the thermodynamic length interval as

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ = \ (T_P dS)^2 \ - \ (TdS + \phi dQ + \omega dJ)^2 \tag{25}$$

and leads to a metric with diagonal and off-diagonal terms. The role of the intensive variables $T^{-1} = \frac{\partial S}{\partial M}, \phi = -T\frac{\partial S}{\partial Q}, \omega = -T\frac{\partial S}{\partial J}$, in the thermodynamic distance (25) can be interpreted as mere parameters, whereas the extensive variables S, Q, J represent the "coordinates" of the "points" of the thermodynamic manifold.

By holding $J, Q \ fixed \ dJ = dQ = 0$, the first law reduces to $dM = TdS$ and one ends up with the similar expression found before in eq-(2) of the form

$$dM \ = \ \gamma(T) \ T \ \frac{d\mathbf{s}}{T_P} \ = \ \gamma(T) \ T \ d\tilde{S} \tag{26}$$

where $\gamma(T)$ is the corresponding thermal dilation factor associated with the Hawking temperature $T_H(M, Q, J)$ for the Kerr-Newman black hole and written in terms of the inner r_- and outer horizon radius r_+ as

$$T_H(M,Q,\frac{J}{M}) \ = \ \frac{1}{2\pi} \ \frac{\sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2}}{2(GM)^2 \ - \ Q^2 \ + \ 2GM \ \sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2}} \ =$$

$$\frac{1}{4\pi} \frac{r_+ \ - \ r_-}{r_+^2 \ + \ (\frac{J}{M})^2}, \ \ r_\pm \ = \ GM \pm \sqrt{(GM)^2 - Q^2 - (\frac{J}{M})^2} \qquad (27)$$

After using eqs-(26,27) one finds that the Thermal Relativity corrections to the Kerr-Newman black hole entropy is given by the integral,

$$\tilde{S} - \tilde{S}_o \ = \ \int_{M_o}^{M} \ dM \ \sqrt{ \ T_H^{-2}(M,Q,\frac{J}{M}) \ - \ T_P^{-2}}, \ \ T_H(M,Q,\frac{J}{M}) \neq 0 \qquad (28)$$

Q, J are fixed in eq-(28) and treated as mere parameters independent on M. The above and more complicated integral furnishes the sought-after *corrections* $\tilde{S}$ to the original expression for Kerr-Newman black hole entropy given by

$$S_{KN}(M,Q,J) \ = \ \frac{A}{4G} \ = \ \frac{\pi \ (\ r_+^2 \ + \ (\frac{J}{M})^2 \ )}{G} \qquad (29)$$

Setting $J = 0$ in the integral (28) furnishes the corrections to the Reissner-Nordstrom black hole entropy.

Note that the integral eq-(28) blows up when $T_H = 0$ in the extremal case $r_+ = r_-$. However, there is a caveat because when one sets $r_+ = r_- \Rightarrow (GM)^2 - Q^2 - (\frac{J}{M})^2 = 0$ it leads to a *constraint* among M, Q, J. Consequently if one fixes Q, J, one also has to fix M so that the integration (28) over the variable M *no* longer makes sense. Therefore, the integral (28) is valid only when $T_H(M,Q,\frac{J}{M}) \neq 0$, it is not valid in the extremal black hole case.

In the most general case one must proceed by recurring to the explicit expression $S_{KN} = S_{KN}(M,Q,J)$ (29) which implies

$$dS_{KN} \ = \ \frac{\partial S_{KN}}{\partial M} dM \ + \ \frac{\partial S_{KN}}{\partial Q} dQ \ + \ \frac{\partial S_{KN}}{\partial J} dJ \qquad (30)$$

so that eq-(25) can be rewritten as

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 - (dM)^2 \ = \ T_P^2 \ (\ \frac{\partial S_{KN}}{\partial M} dM + \frac{\partial S_{KN}}{\partial Q} dQ + \frac{\partial S_{KN}}{\partial J} dJ \ )^2 - (dM)^2 \qquad (31)$$

leading to a 3×3 non-Hessian metric with diagonal and off-diagonal terms $(d\mathbf{s})^2 = g_{ab}(M,Q,J) dZ^a dZ^b$, with $Z^a = M, Q, J$. Once the Riemannian metric is known one can find the connection and curvature. Similar to the Ruppeiner thermodynamic geometry one would expect that the curvature singularities should also signal critical behaviors and instabilities. Note however that the 3×3 metric (31) $differs$ from the 2×2 Ruppeiner Hessian metric $(ds)^2_R = -\partial_i \partial_j S_{KN}(M, Q, J) dZ^i dZ^j$, with $Z^i = Q, J$.

Eq-(31) could be *integrated* if one knew the M-dependence of $Q(M), J(M)$. For instance, in string theory the Regge trajectories display the J versus the m^2 relation $J \sim \alpha' m^2 + \alpha_o$ given in terms of the Regge slope α', the inverse of the string tension, and the Regge intercept α_o. Then a "path" in the thermodynamic manifold can be chosen by selecting the functions $Q = Q(M), J = J(M)$, and in doing so, one will be able to integrate (31) with respect to M and determine the thermodynamic length $\mathbf{s}$ of such "path" in terms of M. Similar arguments follow if one knew the dependence of two other variables in terms of the third one, for example, like $M = M(Q); J = J(Q)$. Moreover, the geodesic paths determined in terms of the analog of the torsionless Levi-Civita connection, and corresponding to the metric which defines the interval (31), will generate the solutions $M(\tilde{S}), Q(\tilde{S}), J(\tilde{S})$ which extremize the proper entropy $\tilde{S} = T_P^{-1}\mathbf{s}$. One would have to distinguish the maxima from the minima in this case.

So far we have studied the asymptotically flat Schwarzschild and Kerr-Newman black holes within the context of Thermal Relativity. Asymptotically Anti de Sitter (de Sitter) spacetimes are more subtle to study since the mass turns out to be related to the enthalpy which can be written as $M = H = U + pV$. This framework is known as black hole thermodynamics in the extended phase space or black hole chemistry [10]. Without the negative cosmological constant, the asymptotically flat black hole for example has the vanishing pressure, and hence the notion of thermodynamic pressure and volume are absent.

To finalize, there are other impending avenues to explore. Like to find what is the thermal-relativistic analog of proper force (phase space) and to write the thermal version of a *cotangent* space interval, instead of the tangent space interval displayed in eq-(13). Also required is to extend our analysis to the case of Reny and Tsallis entropy [11]. More importantly, to find the full dictionary between the Riemannian, Finslerian geometric description of spacetime and the thermal-relativistic description of the thermodynamic manifold. In other words, using the language of Category theory, to establish the functorial map between one category to the other.

A prior Spacetime/Thermodynamic manifold correspondence is already manifest in Thermal Quantum Field Theory where the period in imaginary time is related to the inverse temperature $\beta = \frac{1}{T}$. In Relativistic Thermodynamics one introduces the so-called inverse four-temperature $\beta^\mu = T^{-1} v^\mu$, which is proportional to the four-vector $v^\mu = \frac{dx^\mu}{d\tau}$, since the relativistic thermodynamic equilibrium conditions demand that β^μ is a Killing vector field with the remarkable consequence that all Lie derivatives of all physical observables along the four-temperature flow must then vanish [12]. For this reason we believe that the introduction of $complexified$ spacetime coordinates (real and imaginary times) might reveal more important clues of the interplay between spacetime relativity with thermal relativity.

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