The Exclusive Higher Dimensional Reductions of Heterotic and IIB Supergravity Theories

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Abstract

Superstring theory is the most promising candidate for a unified quantum theory of the fundamental interactions including gravity. The purpose of this investigation is to study toroidal compactification of the type IIB theory and implications of SL(2, R) symmetry for the reduced action. It has been shown recently that the toroidally compactified type IIB string effective action possesses an SL(2, R) invariance as a consequence of the corresponding symmetry in ten dimensions when the self-dual five-form field strength is set to zero. We will study the toroidal compactification of type IIB string theory and explore some of the consequences of SL(2, R) invariance of ten dimensional theory. The compactified theory on a d-dimensional torus respects the symmetry when specify the transformation properties of the resulting scalar and vector fields.

1 Introduction

It is recognised that dualities play a central role in our understanding of the dynamics of string theory. The intimate connections between various superstring theories and the nonperturbative features of these theories in diverse dimensions are unravelled by the web of duality relations. The S-duality transformation relates strong and weak coupling phases of a given theory in some cases, whereas in some other situation strong and weak coupling regimes of two different theories are connected. One familiar example is the heterotic string toroidally compactified from ten to four dimension and for such a theory S-duality is the generalization of the familiar electric-magnetic duality. Another situation arises in six spacetime dimensions, when the ten dimensional hetetoric string is compactified on T^4. The S-duality, on this occasion, relates the six dimensional heterotic string to type IIA theory compactified on K_3. It was conjectured that type IIB theory in ten dimensions is endowed with $SL(2,Z)$ symmetry. There is mounting evidence for this symmetry and it has played a very important role in providing deeper insight into the nonperturbative attributes of type IIB theory. The discrete subgroup of the $SL(2,R)$ group survives as an exact symmetry of the quantum theory and has been referred to as S-duality in the literature in analogy with the corresponding symmetry in N=4, D=4 heterotic string theory. Furthermore, there is an intimate connection between type IIB theory compactified on a circle and the M-theory compactified on T^2 leading to a host of interesting results. We recall that the bosonic massless excitations of type IIB theory consist of graviton, dilaton and antisymmetric tensor in the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector, denoted by $\hat{G}_{MN}, \hat{\phi}$ and $\hat{B}^{(1)}_{MN}$, respectively. The Ramond-Ramond (R-R) counterparts are 'axion', $\hat{\chi}$, another antisymmetric tensor field, $\hat{B}^{(2)}_{MN}$ and a four index antisymmetric potential, $\hat{C}_{MNPR}$, with self-dual field strength. The Lorentz indices in ten dimensions are denoted by M,N,P,... and the field are defined with a hat. The complex moduli, $\hat{\lambda} = \hat{\chi} + ie^{-\hat{\phi}}$ is known to transform non-trivially under $SL(2,R)$ and same is the case for the two second rank tensor fields $\hat{B}^{(1)}$ and $B^{(2)}$. The $SL(2,R)$ eventually breaks to the robust discrete symmetry $SL(2,Z)$. The equations of motion of type IIB supergravity theory are invariant under an $SL(2,R)$ group known as the supergravity duality group. A discrete subgroup of this group has been conjectured to be the exact symmetry of the full quantum type IIB string theory. As the string theory coupling constant transforms non-trivially under this $SL(2,R)$ transformation, this symmetry is non-perturbative. In general case it is not possible to prove this conjecture in the perturbative framework of string theory. A strong evidence in favor of this conjecture has been given by Schwarz when he showed that certain BPS saturated macroscopic string-like solutions of type IIB string theory form an $SL(2,Z)$ multiplet. These solutions, when characterized by two relatively prime integers corresponding to the charges associated with the two antisymmetric gauge fields from NS-NS and R-R sectors, are stable and do not decay further into strings with lower charges. The tensions as well as the charges associated with the strings have been shown to be given by $SL(2,Z)$ covariant expressions. The level of low energy effective action that this $SL(2,R)$ invariance of the type IIB theory survives the toroidal compactification. In fact, this is not surprising since a symmetry in a higher dimensional theory should become a part of the bigger symmetry in the lower dimensional theory, although in this case, it requires quite non-trivial calculation to prove the invariance.

2 Dimensional Reduction of the Effective Action

We shall also need to consider the low-energy limit of the type–IIB superstring for our discussion duality. The zero-slope limit of the type–IIB superstring is given by $N = 2$, $D = 10$ chiral supergravity. This theory contains a metric, a complex antisymmetric tensor, a complex scalar and a four-index antisymmetric tensor gauge field. The complex scalar parametrizes the coset $SU(1,1)/U(1)$. We denote the type–IIB fields as follows:

$$\left\{\hat{D}_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}, \hat{g}_{\hat{\mu}\hat{\nu}}, \hat{\mathcal{B}}_{\hat{\mu}\hat{\nu}}, \hat{\Phi}\right\}, \tag{1}$$

where $\hat{g}_{\hat{\mu}\hat{\nu}}$ is the Einstein-frame metric. We will start in the Einstein-frame and then switch to the string-frame metric once we have correctly identified the type–IIB dilaton field. The field equations of the type–IIB theory cannot be derived from a covariant action. The theory is invariant under $D = 10$ general coordinate transformations and under the following tensor gauge transformations:

$$\delta\hat{\mathcal{B}} = \partial\hat{\Sigma}, \qquad \delta\hat{D} = \partial\hat{\rho} + \tfrac{3}{8i}\left(\partial\hat{\Sigma}\hat{\mathcal{B}}^* - \partial\hat{\Sigma}^*\hat{\mathcal{B}}\right). \tag{2}$$

It is convenient to start by rewriting the theory using the string-frame metric, but before we have to identify the type–IIB dilaton. This is easier to do in the $SL(2,R)$ version of the theory. Accordingly, we first define the complex scalar field $\hat{\lambda} = \hat{\chi} + ie^{-\hat{\phi}}$ This definition implies that $\hat{\phi}$ is the type–IIB dilaton and will be justified below. We next consider the complex antisymmetric tensor $\mathcal{B}$. To make contact with the real $O(2)$ notation we write

$$\hat{\mathcal{B}} = \hat{\mathcal{B}}^{(1)} + i\hat{\mathcal{B}}^{(2)}, \qquad \hat{\Sigma} = \hat{\Sigma}^{(1)} + i\hat{\Sigma}^{(2)}. \tag{3}$$

Using this notation the field-strengths of the $\hat{\mathcal{B}}$ gauge fields and their gauge transformations can be written as:

$$\begin{array}{rclrcl} \hat{\mathcal{H}}^{(i)} & = & \partial\hat{\mathcal{B}}^{(i)}, & \delta\hat{\mathcal{B}}^{(i)} & = & \partial\hat{\Sigma}^{(i)}, \\ \hat{F}(\hat{D}) & = & \partial\hat{D} + \frac{3}{4}\epsilon^{ij}\hat{\mathcal{B}}^{(i)}\partial\hat{\mathcal{B}}^{(j)}, & \delta\hat{D} & = & \partial\hat{\rho} - \frac{3}{4}\epsilon^{ij}\partial\hat{\Sigma}^{(i)}\hat{\mathcal{B}}^{(j)}. \end{array} \tag{4}$$

The low energy four dimensional effective action of interest to us is

$$S = \int d^4x\sqrt{-G}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{8}\mathrm{tr}\,\partial_\mu\mathcal{M}^{-1}\partial^\mu\mathcal{M} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}\mathcal{F}^T_{\mu\nu}\mathcal{M}^{-1}\mathcal{F}^{\mu\nu}\right) \tag{5}$$

where $G = (\det G_{\mu\nu})$, $G_{\mu\nu}$ being the four dimensional metric in the string frame, ϕ is the dilaton field in $D = 4$, R is the scalar curvature corresponding to the metric $G_{\mu\nu}$. This four dimensional action is of generic form which can be obtained through toroidal compactification on T^6 of a ten dimensional string effective action. For example, if we start from the ten dimensional heterotic string, the matrix $\mathcal{M}$ which contains the scalar fields, parametrizes the coset, $\frac{O(22,6)}{O(22)\times O(6)}$ and $\mathcal{F}_{\mu\nu}$ corresponds to 28 Abelian gauge field strengths. On the other hand if we start from ten dimensional action of type II theories, then the reduced action can be identified with the one that is obtained by dimensional reductions of the NS-NS sector and now there will be only 12 gauge fields (6 from the metric and 6 from antisymmetric tensor) and $\mathcal{M}$ will contain scalars parametrizing the coset $\frac{O(6,6)}{O(6)\times O(6)}$. The superscript T denotes the transpose of a matrix. Definitions of the field strengths are

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \,, \qquad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \mathcal{A}_\mu^T \eta \mathcal{F}_{\nu\rho} + \text{cyc. in } \mu\nu\rho \tag{6}$$

where $\mathcal{A}_\mu$ is a 28 dimensional vector field containing the 28 gauge fields coming from the dimensional reduction of the ten dimensional metric, antisymmetric tensor field and $\mathrm{U}(1)^{16}$ gauge fields in the case of heterotic string. We choose $\mathcal{M}$ to be constant and put $H_{\mu\nu\rho} = 0$ and set all the gauge fields except one (denoted as $A_\mu^{(1)}$) to zero, then the action (5) reduces in the Einstein frame to,

$$\bar{S} = \int d^4x \sqrt{-g} \left(R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{4} e^{-2\phi} F^{(1)}_{\mu\nu} F^{(1)\,\mu\nu} \right) \tag{7}$$

where the Einstein metric is related to the string metric by $g_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$. R denotes the scalar curvature with respect to the Einstein metric $g_{\mu\nu}$. The gauge field $A_\mu^{(1)}$ came from one of the $\mathrm{U}(1)^{16}$ gauge fields in ten dimensions, we choose $A_\mu^{(1)}$ to come from the dimensional reduction of the ten dimensional antisymmetric tensor field.

Let us recall that the massless spectrum of the type IIB string theory in the bosonic sector contains a graviton, a dilaton and an antisymmetric tensor field as NS-NS sector states, whereas, in the R-R sector it contains another scalar, another antisymmetric tensor field and a four-form gauge field whose field-strength is self-dual. It is well known that a covariant action for self dual five index antisymmetric tensor fields in ten dimensions does not exist and we set this field strength to zero, since this field is of no relevance to us in what follows. Therefore, a consistent, covariant action can be written from which the type IIB supergravity equations of motion can be derived. We have studied the dimensional reduction of this action on a $(10-D)$ dimensional torus. When $D = 4$, the corresponding four dimensional type IIB string effective action in the Einstein frame takes the following form:

$$\begin{aligned} S_{\mathrm{II}} \; = \; \int d^4x \sqrt{-g} \Big[& R + \frac{1}{4} \mathrm{tr}\, \partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} + \frac{1}{8} \partial_\mu \log \underset{\sim}{\Delta} \partial^\mu \log \underset{\sim}{\Delta} + \frac{1}{4} \partial_\mu g_{mn} \partial^\mu g^{mn} \\ & - \frac{1}{4} g_{mn} F^{(3)\,m}_{\mu\nu} F^{(3)\,\mu\nu,\,n} - \frac{1}{4} (\underset{\sim}{\Delta})^{1/2} g^{mp} g^{nq} \partial_\mu \mathcal{B}^T_{mn} \mathcal{M} \partial^\mu \mathcal{B}_{pq} \\ & - \frac{1}{4} (\underset{\sim}{\Delta})^{1/2} g^{mp} \mathcal{H}^T_{\mu\nu\, m} \mathcal{M} \mathcal{H}^{\mu\nu}{}_p - \frac{1}{12} (\underset{\sim}{\Delta})^{1/2} \mathcal{H}^T_{\mu\nu\lambda} \mathcal{M} \mathcal{H}^{\mu\nu\lambda} \Big] \end{aligned} \tag{8}$$

Here $g = (\det g_{\mu\nu})$, where $g_{\mu\nu}$ is the four dimensional Einstein metric and R is the scalar curvature associated with $g_{\mu\nu}$. $\mathcal{M}$ is an SL(2, R) matrix defined as

$$\mathcal{M} \equiv \begin{pmatrix} \chi^2 + e^{-2\tilde{\phi}} & \chi \\ \chi & 1 \end{pmatrix} e^{\tilde{\phi}} \tag{9}$$

where χ is the R-R scalar and $\tilde{\phi} = \phi + \frac{1}{2}\log\Delta$, ϕ being the NS-NS scalar, the four dimensional dilaton and $\Delta^2 = (\det G_{mn})$, G_{mn} being the scalars coming from the dimensional reduction of the ten dimensional string metric. $g_{mn} = e^{-2\phi} G_{mn}$ and $(\underset{\sim}{\Delta})^2 = (\det g_{mn})$. $F^{(3)\,m}_{\mu\nu} = \partial_\mu A^{(3)\,m}_\nu - \partial_\nu A^{(3)\,m}_\mu$, where $A^{(3)\,m}_\mu$ is the gauge field resulting from the dimensional reduction of the string metric. $\mathcal{B}_{mn} \equiv \begin{pmatrix} B^{(1)}_{mn} \\ B^{(2)}_{mn} \end{pmatrix}$, where $B^{(i)}_{mn}$, for $i = 1, 2$

are the moduli coming from the dimensional reduction of the NS-NS and R-R antisymmetric tensor fields. $\mathcal{H}_{\mu\nu\, m} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\, m} \\ H^{(2)}_{\mu\nu\, m} \end{pmatrix}$, where $H^{(i)}_{\mu\nu\, m} = F^{(i)}_{\mu\nu\, m} - B^{(i)}_{mn} F^{(3)\, n}_{\mu\nu}$ and $F^{(i)}_{\mu\nu\, m} = \partial_\mu A^{(i)}_{\nu\, m} - \partial_\nu A^{(i)}_{\mu\, m}$, with $A^{(i)}_{\mu\, m}$ being the gauge fields resulting from the dimensional reduction of the NS-NS and R-R sector antisymmetric tensor fields. Finally,

$$\mathcal{H}_{\mu\nu\lambda} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda} \end{pmatrix}, \qquad H^{(i)}_{\mu\nu\lambda} = \partial_\mu B^{(i)}_{\nu\lambda} - F^{(3)\, m}_{\mu\nu} A^{(i)}_{\lambda\, m} + \text{cyc. in } \mu\nu\lambda. \tag{10}$$

The action (8) can be easily seen to be invariant under the following global $SL(2,R)$ transformation:

$$\begin{aligned} \mathcal{M} &\rightarrow \Lambda \mathcal{M} \Lambda^T, & \mathcal{B}_{mn} &\rightarrow (\Lambda^{-1})^T \mathcal{B}_{mn} \\ \begin{pmatrix} A^{(1)}_{\mu\, m} \\ A^{(2)}_{\mu\, m} \end{pmatrix} \equiv \mathcal{A}_{\mu\, m} &\rightarrow (\Lambda^{-1})^T \mathcal{A}_{\mu\, m}, & \begin{pmatrix} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu} \end{pmatrix} \equiv \mathcal{B}_{\mu\nu} &\rightarrow (\Lambda^{-1})^T \mathcal{B}_{\mu\nu} \\ g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad g_{mn} &\rightarrow g_{mn}, & A^{(3)\, m}_\mu &\rightarrow A^{(3)\, m}_\mu \end{aligned} \tag{11}$$

where Λ is the $SL(2,R)$ transformation matrix.

We shall consider a truncated action, rather than the full action (8). Let us, from now on, set $H^{(i)}_{\mu\nu\lambda} = 0$, $A^{(3)\, m}_\mu = 0$, $G_{mn} = \delta_{mn}$, $\Delta = 1$, $B^{(i)}_{mn} = 0$ and all the components of $A^{(1)}_{\mu\, m}$ and $A^{(2)}_{\mu\, m}$ to zero except one (we call the non-zero components of the gauge fields as $A^{(1)}_\mu$ and $A^{(2)}_\mu$ with the corresponding field-strength $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$), then the action (25) reduces to:

$$\begin{aligned} \int d^4x \sqrt{-g} \Big[R &+ \frac{1}{4} \mathrm{tr}\, \partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} + \frac{1}{8} \partial_\mu \log \underline{\Delta} \partial^\mu \log \underline{\Delta} \\ &+ \frac{1}{4} \partial_\mu g_{mn} \partial^\mu g^{mn} - \frac{1}{4} (\underline{\Delta})^{\frac{1}{6}} \mathcal{F}^T_{\mu\nu} \mathcal{M} \mathcal{F}^{\mu\nu} \Big] \end{aligned} \tag{12}$$

In the equation $\mathcal{M}$ is as given in (9) with $\tilde{\phi}$ replaced by ϕ since we have set $\Delta = 1$, and $\mathcal{F}_{\mu\nu} \equiv \begin{pmatrix} F^{(1)}_{\mu\nu} \\ F^{(2)}_{\mu\nu} \end{pmatrix}$. The action (12) is invariant under the global SL(2, R) transformation:

$$\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T, \qquad \begin{pmatrix} A^{(1)}_\mu \\ A^{(2)}_\mu \end{pmatrix} \equiv \mathcal{A}_\mu \rightarrow (\Lambda^{-1})^T \mathcal{A}_\mu \tag{13}$$

The compactifications of type IIA and type IIB theories as we go from ten to nine dimensions have been studied by Bergshoeff, Hull and Ortin and they have explored implications of various dualities for this compactification. More recently, Andrianopoli and collaborators have studied compactification of type II theories and M-theory in various dimensions. It is well known that type IIA and type IIB theories are related by T-duality below ten dimensions. In lower dimensions the S-duality combines with the T-duality leading to U-duality. For example in 8-dimensions, the U-duality group is $SL(3,Z) \times SL(2,Z)$ and various branes belong to representations of this larger group. The five dimensional string effective action, obtained by toroidal compactification of type IIB superstring action, has attracted considerable of attention in establishing Beckenstein-Hawking area-entropy relations for extremal black holes and the near extremal ones. Therefore, it is of interest to obtain type IIB effective action, through dimensional reduction, in lower dimensional spacetime and explore the implications of $SL(2,R)$ duality transformations.

Let us consider the ten dimensional action for the type IIB theory:

$$\begin{aligned}\hat{S} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{G}}\Big\{e^{-2\hat{\phi}}\left(\hat{R}+4(\partial\hat{\phi})^2-\frac{1}{12}\hat{H}^{(1)}_{MNP}\hat{H}^{(1)MNP}\right)-\frac{1}{2}(\partial\hat{\chi})^2\\ &\quad -\frac{1}{12}\hat{\chi}^2\hat{H}^{(1)}_{MNP}\hat{H}^{(1)MNP}-\frac{1}{6}\hat{\chi}\hat{H}^{(1)}_{MNP}\hat{H}^{(2)MNP}-\frac{1}{12}\hat{H}^{(2)}_{MNP}\hat{H}^{(2)MNP}\Big\}\end{aligned} \tag{14}$$

Here $\hat{G}_{MN}$ is the ten dimensional metric in the string frame, $\hat{H}^{(1)}$ and $\hat{H}^{(2)}$ are the field strengths associated with the potentials $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ respectively. It is well known that in ten dimensions, it is not possible to construct a covariant action for $\hat{C}_{MNPR}$ with self-dual field strength and therefore, we have set this field to zero throughout this paper; however, one can dimensionally reduce this field while carrying out compactification; we set it to zero for convenience. In order to express the action in a manifestly $SL(2,R)$ invariant form, recall that the axion and the dilaton parametrize the coset $\frac{SL(2,R)}{SO(2)}$. We over to the Einstein frame through the conformal transformation $\hat{g}_{MN}=e^{-\frac{1}{2}\hat{\phi}}\hat{G}_{MN}$ and the action (14) takes the form

$$\hat{S}_E=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}\Big\{\hat{R}_{\hat{g}}+\frac{1}{4}Tr(\partial_N\hat{\mathcal{M}}\partial^N\hat{\mathcal{M}}^{-1})-\frac{1}{12}\hat{H}^T_{MNP}\hat{\mathcal{M}}\hat{H}^{MNP}\Big\} \tag{15}$$

Here $\hat{R}_{\hat{g}}$ is the scalar curvature computed from the Einstein metric. Note that $det\hat{\mathcal{M}}$ is unity. The action is invariant under following transformations,

$$\hat{\mathcal{M}}\to\Lambda\hat{\mathcal{M}}\Lambda^T,\qquad H\to(\Lambda^T)^{-1}H,\qquad \hat{g}_{MN}\to\hat{g}_{MN}, \tag{16}$$

where $\Lambda\in SL(2,R)$. Let us introduce a 2×2 matrix Σ and consider a generic form of Λ with $ad-bc=1$. It is easy to check that,

$$\Lambda\Sigma\Lambda^T=\Sigma,\qquad \Sigma\Lambda\Sigma=\Lambda^{-1},\qquad \hat{\mathcal{M}}\Sigma\hat{\mathcal{M}}=\Sigma,\qquad \Sigma\hat{\mathcal{M}}\Sigma=\hat{\mathcal{M}}^{-1} \tag{17}$$

Thus Σ plays the role of $SL(2,R)$ metric and the symmetric matrix $\hat{\mathcal{M}}\in SL(2,R)$. It is evident that the second term of equation (15) can be written as

$$\frac{1}{4}Tr[\partial_N\hat{\mathcal{M}}\Sigma\partial^N\hat{\mathcal{M}}\Sigma] \tag{18}$$

The Einstein equation can be derived by varying the action with respect to the metric and the equation of motion associated with the antisymmetric tensor fields can be obtained in a straight forward manner. The $\hat{\mathcal{M}}$ -equation of motion follows from the variation of the action if we keep in mind that $\hat{\mathcal{M}}$ is a symmetric $SL(2,R)$ matrix satisfying the properties mentioned above. Thus, if we consider an infinitesimal transformation, we arrive at following relations.

$$\begin{aligned}&\Lambda=\mathbf{1}+\epsilon, &&\Lambda\in SL(2,R)\\ &\epsilon\Sigma+\Sigma\epsilon^T=0, &&\hat{\mathcal{M}}\to\epsilon\hat{\mathcal{M}}+\hat{\mathcal{M}}\epsilon^T+\mathcal{M}\end{aligned} \tag{19}$$

The desired equation of motion, derived from the above action, is

$$\partial_M(\sqrt{-\hat{g}}\hat{g}^{MN}\hat{\mathcal{M}}\Sigma\partial_N\hat{\mathcal{M}}\Sigma) - \frac{1}{6}\hat{H}^T\hat{\mathcal{M}}\hat{H} = 0 \tag{20}$$

Note that this is a matrix equation of motion and we have suppressed the indices for notational conveniences. It is worthwhile, at this stage to point out some similarities with the the global $O(d,d)$ symmetry that arises when one considers toroidal compactifications to lower spacetime dimension. The metric Σ is analogous to the metric, η, associated with the $O(d,d)$ transformations and the $\hat{\mathcal{M}}$ equation of motion bears resemblance with the corresponding M-matrix of the $O(d,d)$ case.

We consider a string effective action in D spacetime dimensions with massless fields such as graviton, $\hat{G}_{MN}$, antisymmetric tensor, $\hat{B}_{MN}$, $(M,N = 0,1,\cdots,D-1)$, dilaton, $\hat{\Phi}$ and n Abelian gauge fields, $\hat{A}^I_M$. If we compactify coordinates on a $d = D-4$ dimensional torus and assume that the backgrounds are independent of these d compact coordinates, the resulting four dimensional reduced effective action takes the following form

$$\begin{aligned} S = \int d^4x\sqrt{-g}\; e^{-\Phi}\big(R &+ g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}Tr\partial_\mu M^{-1}\partial^\mu M \\ &-\frac{1}{4}\mathcal{F}^i_{\mu\nu}\,(M^{-1})_{ij}\,\mathcal{F}^{j\,\mu\nu} - \frac{1}{12}H_{\mu\nu\lambda}\,H^{\mu\nu\lambda} - 2\Lambda\big), \end{aligned} \tag{21}$$

$\Phi = \hat{\Phi} - \frac{1}{2}\ln\det G_{\alpha\beta}$ is shifted dilaton with the spacetime dependent background fields $(G_{\alpha\beta}\,,\; A^I_\alpha \equiv \hat{A}^I_\alpha\,,\; B_{\alpha\beta} \equiv \hat{B}_{\alpha\beta})$ defining a generic point in moduli-space in the toroidal compactification of string theory. The moduli M is a $(2d+n)\times(2d+n)$ matrix valued scalar field and satisfies the condition $M\mathcal{L}M\mathcal{L} = 1$, where $\mathcal{L}$ is the $O(d,d+n)$ metric

$$\Omega^T\mathcal{L}\,\Omega = \mathcal{L}\,. \tag{22}$$

Here I_d is d-dimensional identity matrix and Ω is an element of the group $O(d,d+n)$. The field strengths appearing in (21) are

$$\begin{aligned} H_{\mu\nu\lambda} &= \partial_\mu B_{\nu\lambda} - \frac{1}{2}\mathcal{A}^i_\mu\mathcal{L}_{ij}\mathcal{F}^j_{\nu\lambda} + cyclic\ perm. \\ \mathcal{F}^i_{\mu\nu} &= \partial_\mu\mathcal{A}^i_\nu - \partial_\nu\mathcal{A}^i_\mu\ , \end{aligned} \tag{23}$$

where i, j are $O(d,d+n)$ matrix indices. $\mathcal{A}^i_\mu \equiv (\,A^{(1)\alpha}_\mu = \hat{G}_{\mu\alpha}\,, A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + \hat{B}_{\alpha\beta}A^{(1)\beta}_\mu + \frac{1}{2}\hat{A}^I_\alpha\,A^{(3)\,I}_\mu\,, A^{(3)\,I}_\mu = \hat{A}^I_\mu - \hat{A}^I_\alpha\,A^{(1)\,\alpha}_\mu\,)$ is a $(2d+n)$ component vector field. It is more convenient for the implementation of S-duality transformation to rescale the σ-model metric to Einstein metric, $g_{\mu\nu} \to e^\Phi g_{\mu\nu}$, and introduce the axion $\partial_\sigma\chi = (\eta^2/6)\sqrt{-g}\epsilon_{\mu\nu\lambda\sigma}H^{\mu\nu\lambda}$ where $\eta = e^{-\Phi}$. Then (21) can be expressed as

$$\begin{aligned} S = \int d^4x\sqrt{-g}\big(R &- \frac{1}{2\,\eta^2}g^{\mu\nu}\partial_\mu\Psi\,\partial_\nu\bar{\Psi} + \frac{1}{8}Tr(\partial_\mu M^{-1}\partial^\mu M) \\ &-\frac{1}{4}\eta\,\mathcal{F}^i_{\mu\nu}M^{-1}_{ij}\mathcal{F}^{j\,\mu\nu} + \frac{1}{4}\chi\,\mathcal{F}^i_{\mu\nu}\,\mathcal{L}_{ij}\,\tilde{\mathcal{F}}^{j\,\mu\nu} - \frac{2\Lambda}{\eta}\big)\ , \end{aligned} \tag{24}$$

where $\Psi = \chi + i\,\eta$ is a complex scalar field and

$$\tilde{\mathcal{F}}^i_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\,\mathcal{F}^{i\,\rho\sigma}. \tag{25}$$

We mention in passing that the action (21) is manifestly invariant under global $O(d, d{+}n)$ transformations:

$$
\begin{aligned}
M &\;\rightarrow\; \Omega\, M\, \Omega^T \;\;, \qquad \Omega \,\in\, O(d, d+n) && (26)\\
\Phi &\;\rightarrow\; \Phi,\;\; g_{\mu\nu} \rightarrow g_{\mu\nu},\;\; B_{\mu\nu} \rightarrow B_{\mu\nu},\;\; \mathcal{A}^i_\mu \rightarrow \Omega^i_j\, \mathcal{A}^j_\mu . && (27)
\end{aligned}
$$

The equations of motion corresponding to Ψ, $g_{\mu\nu}$ and A_μ derived from the action (24) are

$$
\frac{\nabla_\mu \nabla^\mu \Psi}{\eta^2} + i \frac{\nabla_\mu \Psi \,\nabla^\mu \Psi}{\eta^3} - \frac{i}{4} \mathcal{F}\, M\, \mathcal{F} + \frac{1}{4} \mathcal{F}\, \mathcal{L}\, \tilde{\mathcal{F}} + i\, \frac{2\,\Lambda}{\eta^2} = 0, \tag{28}
$$

$$
R_{\mu\nu} - \frac{\nabla_\mu \Psi\, \nabla_\nu \bar{\Psi}}{2\,\eta^2} + \frac{1}{8} Tr(\partial_\mu M^{-1}\, \partial_\nu M) - \frac{\eta}{2} \mathcal{F}_{\mu\lambda} M^{-1} \mathcal{F}_\nu{}^\lambda + g_{\mu\nu} \left(\frac{\eta}{8} \mathcal{F}\, M^{-1}\, \mathcal{F} - \frac{\Lambda}{\eta} \right) = 0, \tag{29}
$$

$$
\nabla_\mu \left(\eta\, (M\, \mathcal{L})_{ij} \mathcal{F}^{j\,\mu\nu} - \chi\, \tilde{\mathcal{F}}^{i\,\mu\nu} \right) = 0, \tag{30}
$$

and the Bianchi identity is

$$
\nabla_\mu \tilde{\mathcal{F}}^{i\,\mu\nu} = 0. \tag{31}
$$

The S-duality transformations correspond to

$$
\Psi \rightarrow \frac{a\,\Psi + b}{c\,\Psi + d}, \quad a\,d - b\,c = 1 \;\;, \quad a, b, \cdots \in\; \mathbf{Z}, \tag{32}
$$

$$
\mathcal{F}^i_{\mu\nu} \rightarrow c\,\eta\, (M\,\mathcal{L})_{ij}\, \tilde{\mathcal{F}}^j_{\mu\nu} + (c\,\chi + d)\, \mathcal{F}^i_{\mu\nu} \tag{33}
$$

and the metric $g_{\mu\nu}$ and moduli M remain invariant.

Explicit calculations show that under S-duality the terms in (29) and (30) remain invariant when $\Lambda = 0$, however for nonvanishing Λ these equations are not S-duality invariant. In this context, we mention that it has been observed, in specific cases, that S-duality invariance of equations of motion is broken in presence of Λ. To analize S-duality invariance, let us consider a specific transformation ($a = d = 0, b = -c = 1$)

$$
\Psi \rightarrow -1/\Psi \text{ and } \;\mathcal{F}^i_{\mu\nu} \rightarrow -\,\eta\, (M\,\mathcal{L})_{ij}\, \tilde{\mathcal{F}}^j_{\mu\nu} - \chi\, \mathcal{F}^i_{\mu\nu}. \tag{34}
$$

Now it is straightforward to find that first four terms on the left-hand-side of (29) are invariant under above transformation (34) while the last term with Λ is not. Similarly, it can also be checked that except Λ-term all other terms in (30) make an invariant combination. Thus in general, the presence of cosmological constant breaks the S-duality invariance of the string equations of motion. Furthermore, in principle the cosmological constant Λ can be generalised to a nontrivial dilaton potential $V(\Phi)$ which might be generated due to nonperturbative effects. However, the corresponding equations of motion are S-duality invariant only if $V(\Phi) = 0$. We write the equations of motion involving $V(\Phi)$ after rescaling to Einstein metric:

$$\frac{\bigtriangledown_\mu \bigtriangledown^\mu \Psi}{\eta^2} + i\frac{\bigtriangledown_\mu \Psi \bigtriangledown^\mu \Psi}{\eta^3} - \frac{i}{4}\mathcal{F}\,M\,\mathcal{F} + \frac{1}{4}\mathcal{F}\,\mathcal{L}\,\tilde{\mathcal{F}} + i\left(\frac{2\,\tilde{V}(\eta)}{\eta^2} - \frac{2}{\eta}\frac{\partial \tilde{V}(\eta)}{\partial\eta}\right) = 0,$$

$$R_{\mu\nu} - \frac{\bigtriangledown_\mu \Psi \bigtriangledown_\nu \bar{\Psi}}{2\,\eta^2} + \frac{1}{8}Tr(\partial_\mu M^{-1}\,\partial_\nu M) - \frac{\eta}{2}\mathcal{F}_{\mu\lambda}M^{-1}\mathcal{F}_\nu{}^\lambda + g_{\mu\nu}\left(\frac{\eta}{8}\mathcal{F}\,M^{-1}\,\mathcal{F} - \frac{\tilde{V}(\eta)}{\eta}\right) = 0, \tag{35}$$

where $\tilde{V}(\eta)$ is the dilaton potential reexpressed in terms of new variable $\eta = e^{-\Phi}$. We note that the above equations of motion (35) are not invariant under the transformation (34) as long as the dilaton potential $V(\Phi)$ is nonzero.

In summarizing, we have explored the consequences of S-duality transformations on the equations of motion with nonzero cosmological constant. First, we studied a four dimensional action in a general frame-work. The reduced action (21) could have been obtained from toroidal compactification of a heterotic string effective action in higher dimensions. Although these actions do not necessarily represent supersymmetric theories, S-duality invariance would have implied the absence of cosmological constant. We note that the cosmological constant term breaks S-duality for the exact conformal field theory backgrounds.

In this context, let us briefly discuss the presence of higher order terms and the consequences of the S-duality transformations in the equations of motion. We write down the next higher order term to the low energy string effective action (24) as

$$S = \int d^4x\sqrt{-g}\,\eta\,\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}\right). \tag{36}$$

In presence of the higher order term the equation of motion (29) gets an additional contribution $\frac{i}{4}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ and similarly (30) is modified with the extra term $\eta\,[G_{\mu\nu} + g_{\mu\nu}\bigtriangledown_\alpha \bigtriangledown^\alpha R]$, where

$$G_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}R_{\alpha\beta\lambda\rho}R^{\alpha\beta\lambda\rho} + 2\,R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 4\bigtriangledown_\alpha\bigtriangledown^\alpha R_{\mu\nu} + 2\bigtriangledown_\mu\bigtriangledown_\nu R - 4\,R_{\mu\alpha}R_\nu{}^\alpha + 4\,R_{\mu\alpha\nu\beta}R^{\alpha\beta}. \tag{37}$$

We have checked that under the S-duality transformation with the additional term also breaks S-duality invariance. The graviton equation along with the higher order correction term as mentioned above is also not invariant under the S-duality. Thus it can be argued that the presence of the higher order terms do not restore the S-duality invariance in the equations of motion. Notice that when we dimensionally reduce the terms involving quadratic in curvature, there will be additional terms in involving moduli and gauge fields arising from dimensional reduction. We have seen that the contribution of (36) to equations of motion already breaks the S-duality. Therefore, even if we explicitly take into account the contribution coming from moduli and extra gauge fields in the corresponding equations of motion, the S-duality invariance will not be restored.

3 Scherk–Schwarz Dimensional Reduction

In the present section, we demonstrate how the preceding conjecture is realized for generalized toroidal compactifications of heterotic string theory. In this case, the standard Kaluza-Klein reduction on a d-dimensional torus from 10 to $10-d$ dimensions produces a theory with global $O(d,d+16)$ symmetry and with a $U(1)^{2d+16}$ gauge group. The effective action can be organized to make the former U-duality symmetry manifest. Essentially the $2d+16$ gauge fields may be assembled as a vector under this symmetry, while there are $d(d+16)$ moduli scalars transforming as a traceless symmetric tensor. We will show that this global symmetry is retained in the massive theories produced by generalized Scherk-Schwarz reductions. The bosonic part of effective action may be written as:

$$\begin{aligned} S \;=\; \int d^D x\sqrt{-g}e^{-\phi}\Big\{ & R+(\nabla\phi)^2+\frac{1}{8}L_{ab}\mathcal{D}_\mu M^{bc}L_{cd}\mathcal{D}^\mu M^{da} \\ & -\frac{1}{4}F^a_{\mu\nu}L_{ab}M^{bc}L_{cd}F^{d\mu\nu}-\frac{1}{12}H^2_{\mu\nu\lambda}-\mathcal{W}(M)\Big\} \end{aligned} \tag{38}$$

where the scalar potential takes the simple form:

$$\mathcal{W}(M)=\frac{1}{12}M^{ad}M^{be}M^{cf}f_{abc}f_{def}-\frac{1}{4}M^{ad}L^{be}L^{cf}f_{abc}f_{def}\ . \tag{39}$$

The essential point is that the various mass parameters introduced by the generalized reduction can be organized as a completely antisymmetric three-index tensor f_{abc} under the $O(d,d+16)$ transformations. These parameters play a dual role in the reduced theory: first, as mass parameters defining the scalar potential potcovariant, and second as structure constants in the non-abelian gauge group of this theory, implicitly appearing in $F^a_{\mu\nu}$ and $\mathcal{D}^\mu M^{ab}$. That is the generalized reduction has produced a gauged supergravity with a nontrivial non-abelian symmetry.

A simple intuition which explains the emergence of this nonabelian symmetry is as follows: The Scherk-Schwarz reduction introduces an axionic shift which depends on internal coordinates. Now a part of gauge symmetry in the reduced theory can be thought of as local shifts of the internal coordinates. These Kaluza-Klein gauge transformations are inherited from the diffeomorphism invariance of original ten-dimensional theory. Hence consistency of this symmetry in the generalized reduction requires that these gauge transformations be accompanied by a local axionic shift. That is the latter symmetries, which are ordinarily only a part of the global U-duality group, have now been incorporated as a part of the local gauge group.

We begin with a review of the standard Kaluza-Klein reduction of low energy heterotic string theory on a d-torus. Our notation will be such that $d+D=10$ and hence this compactification yields an effective D-dimensional theory. In ten dimensions, the low energy action is

$$S=\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\Big\{\mathcal{R}+(\nabla\Phi)^2-\frac{1}{12}\mathcal{H}_{\mu\nu\lambda}\mathcal{H}^{\mu\nu\lambda}-\frac{1}{4}\Sigma_{I=1}^{16}\mathcal{F}^I{}_{\mu\nu}\mathcal{F}^{I\,\mu\nu}\Big\} \tag{40}$$

The ten-dimensional fields, Φ, $\mathcal{G}_{\mu\nu}$, $\mathcal{R}$, $\mathcal{H}_{\mu\nu\lambda}$ and $\mathcal{F}^I{}_{\mu\nu}$, denote the dilaton, string-frame metric, Ricci scalar, Kalb-Ramond three-form field strength, and the Yang-Mills field strengths, respectively. The D-dimensional counterparts of these fields will be denoted with upper case latin letters, except the dilaton, which will be ϕ. Our convention for the metric signature is $\mathcal{G} = (-, +, +, ..., +)$, and that for the curvature is $R^{\mu}{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \ldots$. We assume that the only nontrivial components of the Yang-Mills potential reside in the Cartan subalgebra of the gauge group, and hence $\mathcal{F}^I{}_{\mu\nu} = \partial_\mu \mathcal{A}^I{}_\nu - \partial_\nu \mathcal{A}^I{}_\mu$. The low energy action has been truncated to terms with at most two derivatives. Consistent with this truncation, the three-form $\mathcal{H}$ is defined by including only the Yang-Mills Chern-Simons term,

$$\mathcal{H} = d\mathcal{B} - \frac{1}{2}\Sigma_{I=1}^{16} \mathcal{A}^I \wedge \mathcal{F}^I \tag{41}$$

In component notation by the antisymmetry of $\mathcal{B}_{\mu\nu}$ and $\mathcal{F}^I{}_{\mu\nu}$, we have

$$\mathcal{H}_{\mu\nu\rho} = \partial_\mu \mathcal{B}_{\nu\rho} - \frac{1}{2}\Sigma_{I=1}^{16} \mathcal{A}^I_\mu \mathcal{F}^I{}_{\nu\rho} + \ \textit{cyclic perm.} \tag{42}$$

We wish to consider the standard Kaluza-Klein dimensional reduction of heterotic action (40) on a d-torus, to set the stage for Scherk-Schwarz reductions. After precision algebraic manipulations with the reduction formulas, we find that the reduced degrees of freedom, with simple gauge transformation properties, are given in terms of the original higher-dimensional degrees of freedom as follows:

$$\begin{aligned} A^I{}_\mu = \mathcal{A}^I{}_\mu - \mathcal{A}^I{}_M V^M{}_\mu, \qquad & B_{\mu M} = \mathcal{B}_{\mu M} + \mathcal{B}_{MN} V^N{}_\mu + \frac{1}{2}\mathcal{A}^I{}_M A^I_\mu \\ B_{\mu\nu} = \mathcal{B}_{\mu\nu} + V^M{}_{[\mu} B_{\nu]M} & - \mathcal{B}_{MN} V^M{}_\mu V^N{}_\nu - \mathcal{A}^I{}_M V^M{}_{[\mu} A^I{}_{\nu]} \end{aligned} \tag{43}$$

The vector fields $B_{\mu M}$ and $A^I{}_\mu$ with $V^M{}_\mu$ comprise the full multiplet of $2d + 16$ Abelian $U(1)$ gauge fields. Their field strengths will be denoted:

$$V^M{}_{\mu\nu} = \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu, \qquad H_{\mu\nu M} = \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M}, \qquad F^I{}_{\mu\nu} = \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu. \tag{44}$$

The reduced action in D dimensions may be decomposed as follows:

$$S = S_1 + S_2 + S_3 \tag{45}$$

where the reduced metric-dilaton-two-form action is

$$S_1 = \int d^D x \sqrt{-g} e^{-\phi} \Big\{ R + (\nabla\phi)^2 - \frac{1}{12} H^2_{\mu\nu\lambda} \Big\} , \tag{46}$$

the scalar moduli action is

$$\begin{aligned} S_2 \ = \ & \int d^D x \sqrt{-g} e^{-\phi} \Big\{ \frac{1}{4}(\nabla_\mu \mathcal{G}_{MN})(\nabla^\mu \mathcal{G}^{MN}) - \frac{1}{2}\mathcal{G}^{MN}(\nabla_\mu \mathcal{A}^I{}_M)(\nabla^\mu \mathcal{A}^I{}_N) \\ & - \frac{1}{4}\mathcal{G}^{MP}\mathcal{G}^{NQ}\Big(\nabla_\mu \mathcal{B}_{MN} + \mathcal{A}^I{}_{[M}\nabla_\mu \mathcal{A}^I{}_{N]}\Big)\Big(\nabla^\mu \mathcal{B}_{PQ} + \mathcal{A}^J{}_{[P}\nabla^\mu \mathcal{A}^J{}_{Q]}\Big)\Big\} \end{aligned} \tag{47}$$

and the gauge field action is

$$S_3 = -\frac{1}{4}\int d^D x \sqrt{-g} e^{-\phi} \Big\{ f^I_{\mu\nu} f^{I\ \mu\nu} + \mathcal{G}^{MN} h_{\mu\nu M} h^{\mu\nu}{}_N + \mathcal{G}_{MN} V^M{}_{\mu\nu} V^{N\,\mu\nu} \Big\} . \tag{48}$$

In the latter, we use the definitions

$$f^I_{\mu\nu} = F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu} \qquad\qquad h_{\mu\nu M} = H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MN} V^N{}_{\mu\nu} \tag{49}$$

where $\mathcal{C}_{MN} = \mathcal{B}_{MN} + \frac{1}{2}\mathcal{A}^I{}_M \mathcal{A}^I{}_N$. The reduced three-form field strength is

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}A^I{}_\mu F^I{}_{\nu\rho} - \frac{1}{2}V^M{}_\mu H_{\nu\rho M} - \frac{1}{2}B_{\mu M} V^M{}_{\nu\rho} + \textit{ cyclic perm.} \tag{50}$$

In addition to the original Yang-Mills Chern-Simons terms, the three-form field strength now also contains the induced Chern-Simons terms arising for the new gauge fields. At this point, we are ready to carry out the reduction of the action (40) from 10 to D dimensions with the set of mass parameters γ^M_{NP}, m^I_{MN} and β_{MNP}. First, it is convenient to split the action into three sectors: the metric-dilaton, Yang-Mills, and Kalb-Ramond three-from, and we discuss each of them separately.

The standard Kaluza-Klein reduction can be generalized by the introduction of a constant flux of the three-form or gauge field strengths on a three- or two-cycle in the internal space. These compactifications are then similar to the Type II string and M-theory reductions. Note that a constant internal flux requires that the corresponding potential necessarily depends on the internal coordinates. These fluxes, or alternatively the slopes for the internal dependence for the potentials, then appear as mass parameters in the reduced theory. There is also another set of masses related to certain components of the internal metric, but the discussion of these contributions is more involved and we will leave their discussion for the following section.

We can now reduce the action (40) using these results. This will generalize the Kaluza-Klein reduction of the low energy heterotic action. The net result of this calculation is

$$S = S_1 + S_2 + S_3 \tag{51}$$

where the individual contributions to the action are

$$S_1 = \int d^Dx\sqrt{-g}e^{-\phi}\Big\{R + (\nabla\phi)^2 - \frac{1}{2}H^2_{\mu\nu\lambda}\Big\} \tag{52}$$

for the reduced metric-dilaton-two-form part,

$$\begin{aligned} S_2 \;=\; & -\int d^Dx\sqrt{-g}e^{-\phi}\Big\{\mathcal{W}(\mathcal{G},\mathcal{A}) - \frac{1}{4}\nabla_\mu \mathcal{G}_{MN}\nabla^\mu \mathcal{G}^{MN} + \frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu \mathcal{A}^I{}_M \mathcal{D}^\mu \mathcal{A}^I{}_N \\ & + \frac{1}{4}\mathcal{G}^{MN}\mathcal{G}^{PQ}(\mathcal{D}_\mu \mathcal{B}_{MP} + \mathcal{A}^I{}_{[M}\mathcal{D}_\mu \mathcal{A}^I{}_{P]})(\mathcal{D}^\mu \mathcal{B}_{NQ} + \mathcal{A}^J{}_{[N}\mathcal{D}^\mu \mathcal{A}^J{}_{Q]})\Big\} \end{aligned} \tag{53}$$

for the moduli fields, and

$$S_3 = -\frac{1}{4}\int d^Dx\sqrt{-g}e^{-\phi}\Big\{\mathcal{G}_{MN}V^M{}_{\mu\nu}V^{N\,\mu\nu} + f^I{}_{\mu\nu}f^{I\,\mu\nu} + \mathcal{G}^{MN}h_{\mu\nu M}h^{\mu\nu}{}_N\Big\} \tag{54}$$

for the gauge field contributions. The auxiliary fields are defined according to

$$\begin{aligned} f^I{}_{\mu\nu} &= F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu} \\ h_{\mu\nu M} &= H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MN}V^N{}_{\mu\nu} \end{aligned} \tag{55}$$

The reduced three-form field strength, with all Chern-Simons contributions, is in component form

$$
\begin{aligned}
H_{\mu\nu\lambda} &= \partial_\mu B_{\nu\lambda} - \frac{1}{2}A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{2}V^M{}_\mu H_{\nu\lambda M} - \frac{1}{2}B_{\mu M}V^M{}_{\nu\lambda} \\
&+\frac{1}{2}\beta_{MNP}V^M{}_\mu V^N{}_\nu V^P{}_\lambda - m^I_{MN}A^I{}_\mu V^M{}_\nu V^N{}_\lambda + \text{ cyclic perm.}
\end{aligned} \tag{56}
$$

In the moduli action modract, the function $\mathcal{W}(\mathcal{G}_{MN}, \mathcal{A}^I{}_M)$ denotes the moduli potential, which arises because of the internal fluxes. They will in general induce an effective scalar potential, via the terms such as, $\mathcal{F}^I{}_{MN}\mathcal{F}^{I\,MN} \sim \mathcal{G}^{MN}\mathcal{G}^{PQ}m^I_{MP}m^I_{NQ}$. For the reduction scheme we find that the reduced moduli potential is

$$
\begin{aligned}
\mathcal{W}(\mathcal{G},\mathcal{A}) &= \frac{3}{4}\mathcal{G}^{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\Big(\beta_{MPR} + 2\mathcal{A}^I{}_{[M}m^I_{PR]}\Big)\Big(\beta_{NQS} + 2\mathcal{A}^J{}_{[N}m^J_{QS]}\Big) \\
&+\mathcal{G}^{MN}\mathcal{G}^{PQ}m^I_{MP}m^I_{NQ}
\end{aligned} \tag{57}
$$

Note that this moduli potential is independent of the two-form axions $\mathcal{B}_{MN}$.

We continue with the metric-dilaton sector, which is given by

$$
S_{g\phi} = \int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\Big\{\mathcal{R}(\mathcal{G}) + (\nabla\Phi)^2\Big\} \tag{58}
$$

We can expand the ten-dimensional Ricci scalar and dilaton in terms of fields in the D-dimensional space-time. The previous action becomes

$$
\begin{aligned}
S_{g\phi} &= \int d^Dx\sqrt{-g}e^{-\phi}\Big\{R + (\nabla\phi)^2 + \frac{1}{4}\mathcal{D}_\mu\mathcal{G}_{MN}\mathcal{D}^\mu\mathcal{G}^{MN} - \frac{1}{4}\mathcal{G}_{MN}V^M{}_{\mu\nu}V^{N\mu\nu} \\
&\qquad - \mathcal{G}_{MN}\mathcal{G}^{PQ}\mathcal{G}^{RS}\gamma^M_{PR}\gamma^N_{QS} - 2\mathcal{G}^{MN}\gamma^P_{MQ}\gamma^Q_{NP}\Big\}
\end{aligned} \tag{59}
$$

where we have used the following definitions:

$$
\begin{aligned}
\mathcal{D}_\mu\mathcal{G}_{MN} &= \partial_\mu\mathcal{G}_{MN} - 2\mathcal{G}_{MP}\gamma^P_{NQ}V^Q{}_\mu - 2\mathcal{G}_{NP}\gamma^P_{MQ}V^Q{}_\mu \\
V^M{}_{\mu\nu} &= \partial_\mu V^M{}_\nu - \partial_\nu V^M{}_\mu - 2\gamma^M_{NP}V^N{}_\mu V^P{}_\nu
\end{aligned} \tag{60}
$$

and where the covariant derivative of the moduli $\mathcal{G}_{MN}$ emerges because of the axionic degrees of freedom contained in the matrix $\mathcal{G}_{MN}$: we have $\mathcal{G}_{MN} = \delta_{AB}\mathcal{E}^A{}_M\mathcal{E}^B{}_N$. Hence the local derivatives of $\mathcal{G}_{MN}$ must be defined covariantly, since it contains a symmetric bilinear of adjoint fields with respect to the nonabelian Kaluza-Klein group. It can be easily verified that the determinant of $\mathcal{G}_{MN}$ does not contain any axionic fields, however, and so is a gauge singlet. That it why we can still shift the ten-dimensional dilaton to get the D-dimensional dilaton.

We can now reduce the Yang-Mills sector. The ten-dimensional action is

$$
S_{CYM} = -\frac{1}{4}\int d^{10}x\sqrt{-\mathcal{G}}e^{-\Phi}\mathcal{F}^I{}_{\mu\nu}\mathcal{F}^{I\;\mu\nu} \tag{61}
$$

Using the definition $A^I{}_\mu = \mathcal{A}^I{}_\mu - \mathcal{A}^I{}_M V^M{}_\mu$, we arrive at

$$
\begin{aligned}
S_{CYM} &= -\int d^Dx\sqrt{-g}e^{-\phi}\Big\{\frac{1}{4}(F^I{}_{\mu\nu} + \mathcal{A}^I{}_M V^M{}_{\mu\nu})(F^{I\;\mu\nu} + \mathcal{A}^I{}_M V^{A\;\mu\nu}) \\
&+\frac{1}{2}\mathcal{G}^{MN}\mathcal{D}_\mu\mathcal{A}^I{}_M\mathcal{D}^\mu\mathcal{A}^I{}_N + \mathcal{G}^{MP}\mathcal{G}^{NQ}(m^I_{MN} + \mathcal{A}^I{}_R\gamma^R_{MN})(m^I_{PQ} + \mathcal{A}^I{}_S\gamma^S_{PQ})\Big\}
\end{aligned} \tag{62}
$$

where we use

$$
\begin{aligned}
\mathcal{D}_\mu \mathcal{A}^I{}_M &= \partial_\mu \mathcal{A}^I{}_M - 2(m^I_{MN} + \mathcal{A}^I{}_P \gamma^P_{MN}) V^N{}_\mu \\
F^I{}_{\mu\nu} &= \partial_\mu A^I{}_\nu - \partial_\nu A^I{}_\mu - 2m^I_{MN} V^M{}_\mu V^N{}_\nu
\end{aligned}
\tag{63}
$$

which again follow straightforwardly by dimensional reduction. The last contribution to the action comes from the three-form kinetic terms in ten dimensions

$$
S_{NS} = -\frac{1}{12} \int d^{10}x \sqrt{-\mathcal{G}} e^{-\Phi} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda}
\tag{64}
$$

The reduction of this action produces the following action in D dimensions:

$$
\begin{aligned}
S_{NS} &= -\int d^D x \sqrt{-g} e^{-\phi} \Big\{ \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \\
&+\frac{1}{4} \mathcal{G}^{MN} (H_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MP} V^P{}_{\mu\nu})(H^{\mu\nu}{}_N - \mathcal{A}^I{}_N F^{I\ \mu\nu} - \mathcal{C}_{NQ} V^{Q\mu\nu}) \\
&+\frac{1}{4} \mathcal{G}^{MP} \mathcal{G}^{NQ} (\mathcal{D}_\mu \mathcal{B}_{MN} + \mathcal{A}^I{}_{[M} \mathcal{D}_\mu \mathcal{A}^I{}_{N]})(\mathcal{D}^\mu \mathcal{B}_{PQ} + \mathcal{A}^J{}_{[P} \mathcal{D}^\mu \mathcal{A}^J{}_{Q]}) \\
&+\frac{3}{4} \mathcal{G}^{MQ} \mathcal{G}^{NR} \mathcal{G}^{PS} (\beta_{MNP} + 2\mathcal{A}^I{}_{[M} m^I_{NP]} - 2\mathcal{C}_{T[M} \gamma^G_{NP]}) \\
&\qquad\qquad \times (\beta_{QRS} + 2\mathcal{A}^J{}_{[Q} m^J_{RS]} - 2\mathcal{C}_{U[Q} \gamma^U_{RS]}) \Big\}
\end{aligned}
\tag{65}
$$

The new definitions here are

$$
\begin{aligned}
&\mathcal{D}_\mu \mathcal{B}_{MN} = \partial_\mu \mathcal{B}_{MN} + 2m^I_{MN} A^I{}_\mu + 2\gamma^P_{MN} B_{\mu P} - \beta_{MNP} V^P{}_\mu \\
&+4\mathcal{B}_{Q[M} \gamma^Q_{N]P} V^P{}_\mu - 2\mathcal{A}^I{}_{[M} m^I_{N]P} V^P{}_\mu \\
&H_{\mu\nu M} = \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP} V^N{}_\mu V^P{}_\nu + 4\gamma^P_{MN} B_{[\mu P} V^N{}_{\nu]} + 4m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]}
\end{aligned}
\tag{66}
$$

and the reduced three-form field strength is

$$
\begin{aligned}
H_{\mu\nu\lambda} &= \partial_\mu B_{\nu\lambda} - \frac{1}{2} A^I{}_\mu F^I{}_{\nu\lambda} - \frac{1}{2} V^M{}_\mu H_{\nu\lambda M} - \frac{1}{2} B_{\mu M} V^M{}_{\nu\lambda} + \frac{1}{2} \beta_{MNP} V^M{}_\mu V^N{}_\nu V^P{}_\lambda \\
&-m^I_{MN} A^I{}_\mu V^M{}_\nu V^N{}_\lambda - \gamma^M_{NP} B_{\mu M} V^N{}_\nu V^P{}_\lambda + \ \textit{cyclic perm.}
\end{aligned}
\tag{67}
$$

where $B_{\mu M}$ and $B_{\mu\nu}$ are defined in (43), and still are the correct quantities to express the reduced action, in a manifestly gauge and U-duality symmetric way. With this, we finally find the reduced Kalb-Ramond action in D dimensions:

$$
\begin{aligned}
S_{KR} &= -\int d^D x \sqrt{-g} e^{-\phi} \Big\{ \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \\
&+\frac{1}{4} \mathcal{G}^{MN} (B_{\mu\nu M} - \mathcal{A}^I{}_M F^I{}_{\mu\nu} - \mathcal{C}_{MP} V^P{}_{\mu\nu})(B^{\mu\nu}{}_N - \mathcal{A}^I{}_N F^{I\ \mu\nu} - \mathcal{C}_{NQ} V^{Q\mu\nu}) \\
&+\frac{1}{4} \mathcal{G}^{MP} \mathcal{G}^{NQ} (\mathcal{D}_\mu B_{MN} + \mathcal{A}^I{}_{[M} \mathcal{D}_\mu \mathcal{A}^I{}_{N]})(\mathcal{D}^\mu B_{PQ} + \mathcal{A}^J{}_{[P} \mathcal{D}^\mu \mathcal{A}^J{}_{Q]}) \\
&+\frac{3}{4} \mathcal{G}^{MQ} \mathcal{G}^{NR} \mathcal{G}^{PS} (\beta_{MNP} + 2\mathcal{A}^I{}_{[M} m^I_{NP]} - 2\mathcal{C}_{T[M} \gamma^G_{NP]}) \\
&\qquad\qquad \times (\beta_{QRS} + 2\mathcal{A}^J{}_{[Q} m^J_{RS]} - 2\mathcal{C}_{U[Q} \gamma^H_{RS]}) \Big\}
\end{aligned}
\tag{68}
$$

This is the last step in the reduction of the effective action.

To reassemble the reduced terms (59), (62) and (65) into a manifestly symmetric action in D dimensions, we first need to establish the correct gauge algebra of the reduced theory and identify the gauge invariant couplings of fields. Proceeding as before, we first give the infinitesimal reduced gauge transformations. The fields not listed below explicitly are invariant under the corresponding transformations. The reduced gauge transformations are

Kalb-Ramond gauge transformations:

$$\mathcal{B}'_{MN} = \mathcal{B}_{MN} - 2\lambda_P \gamma^P_{MN}, \qquad B'_{\mu M} = B_{\mu M} + \partial_\mu \lambda_M - 2\lambda_P \gamma^P_{MN} V^N{}_\mu$$
$$B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{2}\lambda_M V^M{}_{\mu\nu} + \gamma^M_{NP}\lambda_M V^N{}_\mu V^P{}_\nu \tag{69}$$

Yang-Mills gauge transformations:

$$\mathcal{B}'_{MN} = \mathcal{B}_{MN} - 2\lambda^I m^I_{MN}, \qquad A'^I{}_\mu = A^I{}_\mu + \partial_\mu \lambda^I$$
$$B'_{\mu M} = B_{\mu M} - 2\lambda^I m^I_{MN} V^N{}_\mu, \qquad B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{2}\lambda^I F^I{}_{\mu\nu} + m^I_{MN}\lambda^I V^M{}_\mu V^N{}_\nu \tag{70}$$

Kaluza-Klein gauge transformations:

$$\begin{aligned}
\mathcal{A}'^I{}_M &= \mathcal{A}^I{}_M + 2\gamma^N_{MP}\omega^P \mathcal{A}^I{}_N + 2m^I_{MN}\omega^N \\
\mathcal{B}'_{MN} &= \mathcal{B}_{MN} + 3\beta_{MNP}\omega^P + 2\mathcal{A}^I{}_{[M} m^I_{N]P}\omega^P + O(\omega^2) \\
\mathcal{G}'_{MN} &= \mathcal{G}_{MN} + 2\gamma^P_{MQ}\omega^Q \mathcal{G}_{PN} + 2\gamma^P_{NQ}\omega^Q \mathcal{G}_{MP} + O(\omega^2) \\
V'^M{}_\mu &= V^M{}_\mu - 2\gamma^M_{NP}\omega^P V^N{}_\mu + \partial_\mu \omega^M + O(\omega^2) \\
A'^I{}_\mu &= A^I{}_\mu - 2m^I_{MN}\omega^N V^M{}_\mu + O(\omega^2) \\
B'_{\mu M} &= B_{\mu M} + 2\gamma^N_{MP}\omega^P B_{\mu N} + 2m^I_{MN}\omega^N A^I{}_\mu + 3\beta_{MNP}\omega^P V^N{}_\mu + O(\omega^2) \\
B'_{\mu\nu} &= B_{\mu\nu} + \frac{1}{2}\omega^M H_{\mu\nu M} - \frac{3}{2}\beta_{MNP}\omega^M V^N{}_\mu V^P{}_\nu \\
&\qquad - 2\gamma^P_{MN}\omega^M B_{[\mu P} V^N{}_{\nu]} - 2\omega^M m^I_{MN} A^I{}_{[\mu} V^N{}_{\nu]} + O(\omega^2)
\end{aligned} \tag{71}$$

Note that in the last set of gauge transformations, we have nontrivial transformation rules for the moduli $\mathcal{G}_{MN}$. This arises from the nontrivial couplings of the metric axions, where the metric mass parameters γ^M_{NP} were set to zero.

We define a combined gauge parameter $\hat{\omega}^a = (\omega^M, \lambda_M, \lambda^I)$ and generators: $T_a = (Z_M, X^M, Y^I)$. The algebra of the latter

$$[T_a, T_b] = i f_{ab}{}^c T_c \tag{72}$$

defines the new set of structure constants, $f^{ab}{}_c$. To compute these, we again consider the products of transformations (69–71) of the form $h^{-1}\cdot g^{-1}\cdot h\cdot g$ where h and g are two of gauge transformations with $g = \exp(i\hat{\omega}^a_1 T_a)$ and $h = \exp(i\hat{\omega}^a_2 T_a)$. Hence substituting the explicit form of the gauge transformations (69–71), we can deduce the structure constants. We evaluate $h^{-1}\cdot g^{-1}\cdot h\cdot g|\Psi\rangle$ for the set of basis states defined by the vector fields. The structure constants are

$$f^M{}_{NP} = f_{NP}{}^M = 2\gamma^M_{NP} \qquad f^I{}_{MN} = f_{MN}{}^I = 2m^I_{MN} \qquad f_{MNP} = -3\beta_{MNP} \tag{73}$$

The resulting gauge algebra is

$$
\begin{aligned}
&[X^M, X^N] = [Y^I, Y^J] = [X^M, Y^I] = 0 \\
&[X^M, Z_N] = 2i\gamma^M_{NP} X^P \qquad\qquad [Y^I, Z_M] = 2im^I_{MN} X^N \\
&[Z_M, Z_N] = -3i\beta_{MNP} X^P + 2im^I_{MN} Y^I + 2i\gamma^P_{MN} Z_P
\end{aligned} \tag{74}
$$

While the standard Cartan metric on this Lie algebra (74) is still degenerate, since the gauge algebra is not semi-simple, we can nevertheless define the metric on the gauge algebra by $\langle T_a, T_b \rangle = L_{ab}$. Formally keeping the definition of the Lie-algebra-valued gauge field one-form potential, we find that the Lie-algebra-valued gauge field strength

$$
F = dA + iA \wedge A = \frac{1}{2} F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu \tag{75}
$$

again has components which coincide with the expressions for field strengths that come from dimensional reduction: $F^a_{\mu\nu} = ({V^M}_{\mu\nu}, H_{\mu\nu M}, F^I_{\mu\nu})$. Explicitly, the components of the gauge field strength are

$$
\begin{aligned}
{F_{\mu\nu}}^M &= \partial_\mu {V^M}_\nu - \partial_\nu {V^M}_\mu - 2\gamma^M_{NP} {V^N}_\mu {V^P}_\nu \\
F_{\mu\nu M} &= \partial_\mu B_{\nu M} - \partial_\nu B_{\mu M} + 3\beta_{MNP} {V^N}_\mu {V^P}_\nu + 4m^I_{MN} {A^I}_{[\mu} {V^N}_{\nu]} + 4\gamma^P_{MN} B_{[\mu P} {V^N}_{\nu]} \\
{F_{\mu\nu}}^I &= \partial_\mu {A^I}_\nu - \partial_\nu {A^I}_\mu - 2m^I_{MN} {V^M}_\mu {V^N}_\nu
\end{aligned} \tag{76}
$$

The nonabelian Chern-Simons form can be computed as usual, to yield

$$
\begin{aligned}
\omega_{CS} &= \langle A \wedge F - \frac{i}{3} A \wedge A \wedge A \rangle \qquad (77) \\
&= \frac{1}{3} dx^\mu \wedge dx^\nu \wedge dx^\lambda \Big(\frac{1}{2} {A^I}_\mu {F^I}_{\nu\lambda} + \frac{1}{2} {V^M}_\mu H_{\nu\lambda M} + \frac{1}{2} B_{\mu M} {V^M}_{\nu\lambda} \\
&+ \frac{1}{2} \beta_{MNP} {V^M}_\mu {V^N}_\nu {V^P}_\lambda - m^I_{MN} {A^I}_\mu {V^M}_\nu {V^N}_\lambda - \gamma^M_{NP} B_{\mu M} {V^N}_\nu {V^P}_\nu + c.p. \Big)
\end{aligned}
$$

This is exactly the anomaly contribution to the reduced three-form field strength. Again the Chern-Simons terms get twisted together into a single nonabelian structure, such that, in form notation, the reduced three-form field strength is simply

$$
H = dB - \frac{1}{2} \omega_{CS} \tag{78}
$$

We can also put the moduli potential in a duality symmetric form. Lowering the last of index on the structure constants, $f_{abc} = f_{[ab}{}^d L_{d|c]}$, we can write the scalar potential entirely in terms of the moduli matrix M^{ab}:

$$
\mathcal{W}(M) = \frac{1}{12} M^{ad} M^{be} M^{cf} f_{abc} f_{def} - \frac{1}{4} M^{ad} L^{be} L^{cf} f_{abc} f_{def} \ . \tag{79}
$$

Note the additional term linear in M which was absent in the formula (**??**) obtained with nontrivial fluxes in the matter sector. This is, of course, consistent with our previous results. Using the structure constants found in eq. strc, $\gamma^M_{NP} = 0$, this linear term automatically vanishes. This new term's only nontrivial contribution here is the last interaction appearing in eq. redmetdil, which is linear in $\mathcal{G}$ and quadratic in the γ^M_{NP}.

The mass parameters γ^M_{NP}, m^I_{MN} and β_{MNP} must satisfy the constraints

$$\begin{aligned} &\gamma^M_{N[P}\gamma^N_{QR]} = 0, && \gamma^M_{MN} = 0, \\ &m^I_{Q[M}\gamma^Q_{NP]} = 0, && \beta_{R[MN}\gamma^R_{PQ]} = m^I_{[MN}m^I_{PQ]} \ . \end{aligned} \tag{80}$$

These relations will ensure that the structure constants satisfy the Jacobi identity:

$$L^{ad} f_{a[bc}f_{d|ef]} = \frac{1}{3}L^{ad}\Big(f_{abc}f_{def} + f_{abe}f_{dfc} + f_{abf}f_{dce}\Big) = 0 \tag{81}$$

It now remains to note that we can also collect the gauge kinetic terms and the moduli terms in the manifestly gauge- and duality-invariant fashion. The gauge fields transform according to $F'^a_{\mu\nu} = U^a{}_b F^b_{\mu\nu}$, where $U^a{}_b$ is an $O(d, d+16)$ matrix. Thus the moduli fields transform as

$$M'^{ab} = U^a{}_c U^b{}_d M^{cd} \tag{82}$$

under the same gauge transformations, and we see that the covariant derivative of the scalar moduli can be written

$$\mathcal{D}_\mu M^{ab} = \partial_\mu M^{ab} - f_{cd}{}^a A^c_\mu M^{db} - f_{cd}{}^b A^c_\mu M^{ad} \ . \tag{83}$$

These are indeed identical with the expressions which were obtained by reduction. With this it can be easily shown that the gauge kinetic terms can be collected into the covariant expression

$$\mathcal{F}^2 = F^a_{\mu\nu} L_{ab} M^{bc} L_{cd} F^{d\mu\nu} \tag{84}$$

Therefore, the reduced action can be again rewritten as

$$\begin{aligned} S \ = \ \int d^D x\sqrt{-g}e^{-\phi}\Big\{ & R + (\nabla\phi)^2 + \frac{1}{8}L_{ab}\mathcal{D}_\mu M^{bc} L_{cd}\mathcal{D}^\mu M^{da} \\ & - \frac{1}{4}F^a_{\mu\nu} L_{ab} M^{bc} L_{cd} F^{d\mu\nu} - \frac{1}{12}H^2_{\mu\nu\lambda} - \mathcal{W}(M)\Big\} \end{aligned} \tag{85}$$

The general massive reductions produce reduced theories with a remarkably symmetric form, where a part of the $O(d, d+16)$ duality must be gauged in order to accommodate the couplings induced by the mass terms. Here, we are lead to a slight puzzle. Quantization conditions seem to break the $O(d, d+16, R)$ symmetry of the reduced action to $O(d, d+16, Z)$. Another aspect of the reduced theory was that a part of the global $O(d, d+16, R)$ symmetry becomes a local gauge symmetry. Thus it would seem that this continuous subgroup of the U-duality group must also be exact since it corresponds to a constant gauge transformations. The puzzle is to understand the interplay of these two apparent exact symmetries. We will argue that in fact these symmetries are distinct symmetries, despite their apparent common origin in $O(d, d+16, R)$. The physically meaningful properties of the reduced theory, given by the masses and the structure constants, depend on the directions and types of fields which are excited on the internal space. In general, the internal fields can be turned on by using the tensor representations of isometries. The similarity of the reduced actions for four dimensions is essentially dictated by the fact that they are both versions of gauged $N = 4$ supergravity in which the form of the action is completely fixed given the gauge group.

4 Higher Dimensional Effective Action

It is well-known that the equations of motion of type IIB supergravity theory can not be obtained from a covariant action because of the presence of a four-form gauge field with the self-dual field strength in the spectrum. This gauge field couples to a self-dual three-brane which can give rise to string solution in $D \leq 8$. But, we are not going to consider this type of string solution and set the corresponding field-strength F_5 to zero. There are also magnetically charged string solution for type II theory in $D \leq 6$, but since we are not restricting ourselves to any particular dimensionality we will not consider those kinds of solutions also. Now as we set $F_5 = 0$, the type IIB equations of motion can be derived from the following covariant action:

$$\begin{aligned}\tilde{S}_{10}^{\rm IIB} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left[e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\right)\right.\\ &\qquad \left. -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{12}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right] \qquad (86)\end{aligned}$$

Here $\mu, \nu, \ldots = 0, 1, \ldots, D-1$ are the space-time indices and $m, n, \ldots = D, \ldots, 9$ are the internal indices. The metric $G_{\mu\nu}$, the dilaton ϕ and the antisymmetric tensor $B^{(1)}_{\mu\nu}$ (with $H^{(1)} = dB^{(1)}$) represent the massless modes in the NS-NS sector of type IIB theory. Also the scalar χ and $B^{(2)}_{\mu\nu}$ (with $H^{(2)} = dB^{(2)}$) represent the massless modes in the R-R sector. The reduced action takes the form:

$$\begin{aligned}&\frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\left[e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}G_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} + \frac{1}{4}\partial_\mu G_{mn}\partial^\mu G^{mn}\right.\right.\\ &\qquad \left. -\frac{1}{4}G^{mp}G^{nq}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq} - \frac{1}{4}G^{mp}H^{(1)}_{\mu\nu\,m}H^{(1)\,\mu\nu}_{\ p} - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\right)\\ &\qquad -\frac{1}{2}\Delta\partial_\mu\chi\partial^\mu\chi - \frac{1}{4}\Delta G^{mp}G^{nq}\left(\partial_\mu B^{(2)}_{mn} + \chi\partial_\mu B^{(1)}_{mn}\right)\left(\partial^\mu B^{(2)}_{pq} + \chi\partial^\mu B^{(1)}_{pq}\right)\\ &\qquad -\frac{1}{4}\Delta G^{mp}\left(H^{(2)}_{\mu\nu\,m} + \chi H^{(1)}_{\mu\nu\,m}\right)\left(H^{(2)\,\mu\nu}_{\ p} + \chi H^{(1)\,\mu\nu}_{\ p}\right)\\ &\qquad \left. -\frac{1}{12}\Delta\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right] \qquad (87)\end{aligned}$$

The corresponding field-strengths are given below:

$$H^{(i)}_{\mu mn} = H^{(i)}_{\mu mn} = \partial_\mu B^{(i)}_{mn}, \qquad H^{(i)}_{\mu\nu m} = F^{(i)}_{\mu\nu m} - B^{(i)}_{mn}F^{(3)\,n}_{\mu\nu} \qquad (88)$$

where $F^{(i)}_{\mu\nu\,m} = \partial_\mu A^{(i)}_{\nu\,m} - \partial_\nu A^{(i)}_{\mu\,m}$ and $F^{(3)\,m}_{\mu\nu} = \partial_\mu A^{(3)\,m}_\nu - \partial_\nu A^{(3)\,m}_\mu$ and finally

$$H^{(i)}_{\mu\nu\lambda} = \partial_\mu B^{(i)}_{\nu\lambda} - F^{(3)\,m}_{\mu\nu}A^{(i)}_{\lambda\,m} + \text{cyc. in } \mu\nu\lambda \qquad (89)$$

The reduced action (87) have an $SL(2, R)$ invariance which can be better understood by rewriting the action in the Einstein frame. The metric in the Einstein frame is related with the string metric as given in the second section. The action (87) in the Einstein frame takes the following form:

$$
\begin{aligned}
&\frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\left[R-\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}-\frac{1}{2}e^{2\tilde{\phi}}\partial_\mu\chi\partial^\mu\chi+\frac{1}{8}\partial_\mu\log\underline{\Delta}\partial^\mu\log\underline{\Delta}\right.\\
&\quad+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn}-\frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n}-\frac{1}{4}(\underline{\Delta})^{1/2}g^{mp}g^{nq}e^{-\tilde{\phi}}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq}\\
&\quad-\frac{1}{4}(\underline{\Delta})^{1/2}g^{mp}g^{nq}e^{\tilde{\phi}}\left(\partial_\mu B^{(2)}_{mn}+\chi\partial_\mu B^{(1)}_{mn}\right)\left(\partial^\mu B^{(2)}_{pq}+\chi\partial^\mu B^{(1)}_{pq}\right)\\
&\quad-\frac{1}{4}(\underline{\Delta})^{1/2}g^{mp}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu m}H^{(1)\,\mu\nu}_{\;\;p}+e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\,m}+\chi H^{(1)}_{\mu\nu\,m}\right)\left(H^{(2)\,\mu\nu}_{\;\;p}+\chi H^{(1)\,\mu\nu}_{\;\;p}\right)\right\}\\
&\quad\left.-\frac{1}{12}(\underline{\Delta})^{1/2}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}+e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\lambda}+\chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda}+\chi H^{(1)\,\mu\nu\lambda}\right)\right\}\right]
\end{aligned}
\tag{90}
$$

where we have defined $\tilde{\phi}=\phi+\frac{1}{2}\log\Delta$. Also, $G_{mn}=e^{\frac{4}{D-2}\phi}g_{mn}$ and $\Delta=e^{2\frac{(10-D)}{(D-2)}\phi}\underline{\Delta}$ with $(\underline{\Delta})^2=(\det g_{mn})$. If we define the following $SL(2,R)$ matrix then the action (90) can be expressed in the manifestly $SL(2,R)$ invariant form as

$$
\begin{aligned}
&\frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\left[R+\frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1}+\frac{1}{8}\partial_\mu\log\underline{\Delta}\partial^\mu\log\underline{\Delta}+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn}\right.\\
&\qquad-\frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n}-\frac{1}{4}(\underline{\Delta})^{1/2}g^{mp}g^{nq}\partial_\mu\mathcal{B}^T_{mn}\mathcal{M}_D\partial^\mu\mathcal{B}_{pq}\\
&\qquad\left.-\frac{1}{4}(\underline{\Delta})^{1/2}g^{mp}\mathcal{H}^T_{\mu\nu\,m}\mathcal{M}_D\mathcal{H}^{\mu\nu}_{\;\;p}-\frac{1}{12}(\underline{\Delta})^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\right]
\end{aligned}
\tag{91}
$$

Here we have defined $\mathcal{B}_{mn}\equiv\begin{pmatrix}B^{(1)}_{mn}\\ B^{(2)}_{mn}\end{pmatrix}$, $\mathcal{H}_{\mu\nu\,m}\equiv\begin{pmatrix}H^{(1)}_{\mu\nu\,m}\\ H^{(2)}_{\mu\nu\,m}\end{pmatrix}$, $\mathcal{H}_{\mu\nu\lambda}\equiv\begin{pmatrix}H^{(1)}_{\mu\nu\lambda}\\ H^{(2)}_{\mu\nu\lambda}\end{pmatrix}$. The action (91) is invariant under the following global $SL(2,R)$ transformation:

$$
\begin{aligned}
\mathcal{M}_D &\rightarrow \Lambda\mathcal{M}_D\Lambda^T, & \mathcal{B}_{mn} &\rightarrow (\Lambda^{-1})^T\mathcal{B}_{mn}\\
\begin{pmatrix}A^{(1)}_{\mu\,m}\\ A^{(2)}_{\mu\,m}\end{pmatrix}\equiv\mathcal{A}_{\mu\,m} &\rightarrow (\Lambda^{-1})^T\mathcal{A}_{\mu\,m}, & \begin{pmatrix}B^{(1)}_{\mu\nu}\\ B^{(2)}_{\mu\nu}\end{pmatrix}\equiv\mathcal{B}_{\mu\nu} &\rightarrow (\Lambda^{-1})^T\mathcal{B}_{\mu\nu}
\end{aligned}
\tag{92}
$$

where $\Lambda=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is the $SL(2,R)$ transformation matrix and a,b,c,d are constants satisfying $ad-bc=1$. If we set $G_{mn}=\delta_{mn}$, $\Delta=1$, $A^{(3)\,m}_{\mu}=A^{(i)}_{\mu\,n}=B^{(i)}_{mn}=0$, then the action (91) reduces to

$$
\begin{aligned}
&\frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\left[R+\frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1}+\frac{1}{8}\partial_\mu\log\underline{\Delta}\partial^\mu\log\underline{\Delta}\right.\\
&\qquad\left.+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn}-\frac{1}{12}(\underline{\Delta})^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\right]
\end{aligned}
\tag{93}
$$

This action is $SL(2,R)$ invariant under the transformation (92). Note that both g_{mn} and $\underline{\Delta}$ are $SL(2,R)$ invariant. Also, $\mathcal{M}_D$ in (92) is as given in (90) with $\tilde{\phi}$ replaced by ϕ, the D-dimensional dilaton as $\Delta=1$ in this case. Note also that although we have set $G_{mn}=\delta_{mn}$ and $\Delta=1$, but as they are not $SL(2,R)$ invariant, non-trivial values of

G_{mn} and Δ will be generated through $SL(2,R)$ transformation. It can be easily checked that the $SL(2,R)$ invariant action (93) gets precisely converted to the effective action considered by Dabholkar by setting the R-R fields to zero. Thus, we note that the action in the Einstein frame is a special case of the more general type II action (93) and so the solution is a particular case of a general solution that we are going to construct.

The 10-dimensional effective action is invariant under general coordinate transformations as well as the gauge transformations associated with the two antisymmetric tensor fields. When we examine the local symmetries of the theory in D-dimensions after dimensional reduction, we find that there is general coordinate transformation invariance in D-dimensions. The Abelian gauge transformation, associated with $\mathcal{A}^{\alpha}_{\mu}$, has its origin in 10-dimensional general coordinate transformations. The field strength $H^{(i)}_{\mu\nu\alpha}$ is invariant under a suitable gauge transformation once we define the gauge transformation for $F^{(i)}_{\mu\nu\alpha}$ since $\mathcal{F}^{\alpha}_{\mu\nu}$ is gauge invariant under the gauge transformation of $\mathcal{A}$-gauge fields. Finally, the tensor field strength $H^{(i)}_{\mu\nu\rho}$, defined above, can be shown to be gauge invariant by defining appropriate gauge transformations for $B^{(i)}_{\mu\nu}$:

$$\delta B^{(i)}_{\mu\nu} = \partial_{\mu}\xi^{(i)}_{\nu} - \partial_{\nu}\xi^{(i)}_{\mu}. \tag{94}$$

The D-dimensional effective action takes the following form

$$\begin{aligned} S_E \;=\; & \int d^Dx\sqrt{-g}\sqrt{\mathcal{G}}\Big\{R + \frac{1}{4}[\partial_{\mu}\mathcal{G}_{\alpha\beta}\partial^{\mu}\mathcal{G}^{\alpha\beta} + g^{\mu\nu}\partial_{\mu}\log\mathcal{G}\partial_{\nu}\log\mathcal{G} - g^{\mu\lambda}g^{\nu\rho}\mathcal{G}_{\alpha\beta}\mathcal{F}^{\alpha}_{\mu\nu}\mathcal{F}^{\beta}_{\lambda\rho}] \\ & -\frac{1}{4}\mathcal{G}^{\alpha\beta}\mathcal{G}^{\gamma\delta}\partial_{\mu}B^{(i)}_{\alpha\gamma}\mathcal{M}_{ij}\partial^{\mu}B^{(j)}_{\beta\delta} - \frac{1}{4}\mathcal{G}^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho}H^{(i)}_{\mu\nu\alpha}\mathcal{M}_{ij}H^{(j)}_{\lambda\rho\beta} \\ & -\frac{1}{12}H^{(i)}_{\mu\nu\rho}\mathcal{M}_{ij}H^{(j)\mu\nu\rho} + \frac{1}{4}Tr(\partial_{\mu}\mathcal{M}\Sigma\partial^{\mu}\mathcal{M}\Sigma)\Big\} \end{aligned} \tag{95}$$

The above action is expressed in the Einstein frame, $\mathcal{G}$ being determinant of $\mathcal{G}_{\alpha\beta}$. If we demand $SL(2,R)$ invariance of the above action, then the backgrounds are required to satisfy following transformation properties:

$$\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T, \qquad H^{(i)}_{\mu\nu\rho} \to (\Lambda^T)^{-1}{}_{ij}H^{(j)}_{\mu\nu\rho}A^{(i)}_{\mu\alpha} \to (\Lambda^T)^{-1}{}_{ij}A^{(j)}_{\mu\alpha}, \qquad B^{(i)}_{\alpha\beta} \to (\Lambda^T)^{-1}{}_{ij}B^{(j)}_{\alpha\beta} \tag{96}$$

It is evident from the D-dimensional action that dilaton and axion interact with antisymmetric tensor fields, gauge fields and the scalars due to the presence of $\mathcal{M}$ matrix in various terms and these interaction terms respect the $SL(2,R)$ symmetry. It is important know what type of dilatonic potential is admissible in the above action which respects the S-duality symmetry. The only permissible interaction terms preserving the symmetry are of the form $Tr[\mathcal{M}\Sigma]^n$. It is easy to check using the properties of Σ and $\mathcal{M}$ matrices, such as $Tr(\mathcal{M}\Sigma) = 0$ and $Tr(\mathcal{M}\Sigma\mathcal{M}\Sigma) = 2$, that

$$Tr[\mathcal{M}\Sigma]^n = 0, \qquad Tr[\mathcal{M}\Sigma]^n = 2 \tag{97}$$

For odd $n \in Z$ and even $n \in Z$ respectively. Therefore, we reach a surprizing conclusion that the presence of interaction terms of the form only adds constant term which amounts to adding cosmological constant term to the reduced action. Note that the Einstein metric is $SL(2,R)$ invariant and one can add terms involving higher powers of curvature (higher derivatives of metric) to the action and maintain the symmetry. However, we are considering the case when the gravitational part of the action has the Einstein-Hilbert term only.

5 Discussion and Conclusion

To summarize, we first argued that the magnetically charged black hole solution by Garfinkle, Horowitz and Strominger (GHS) derived in the context of $D = 4$ heterotic string theory can also be interpreted as black hole solution of $D = 4$ type IIB theory such that the gauge field appears due to compactification of the NS-NS antisymmetric field of $D = 10$ action with all R-R fields set to zero. This GHS solution arises due to the presence of an Abelian gauge field in the NS-NS sector of the theory as a consequence of compactification of the corresponding antisymmetric tensor field. It has been demonstrated that the low energy effective action of type IIB string theory compactified on torus possesses an $SL(2, Z)$ invariance if the $D = 10$ theory is endowed with the same symmetry. By exploiting this symmetry of type IIB string theory, we have constructed an infinite family of magnetically charged black hole solutions in $D = 4$. Black hole solutions in string theory having electric, magnetic and both charges associated with the gauge fields originating from the dimensional reduction of the various heterotic string states as well as the NS-NS sector states of type II string theory have been constructed before. The solutions we have constructed are characterized by two integers corresponding to the charges associated with both NS-NS sector and R-R sector gauge fields. We have studied the toroidal compactification of a truncated version when the self-dual five-form field strength is zero of the type IIB string effective action in the string frame. As we finally converted the reduced action in the Einstein frame by conformal rescaling of the metric we have recovered the $SL(2, R)$ invariance of the reduced action as a consequence of the same symmetry in ten dimensions. We have obtained the transformation properties of the various fields and compared with the recently obtained results of toroidal compactification of the same type IIB action in the Einstein frame. Since the $SL(2, R)$ matrix $\mathcal{M}_D$ involved in the process of showing the invariance does not contain the D-dimensional dilaton, the issue of strong-weak coupling duality symmetry under a Z_2 subgroup of this $SL(2, R)$ group becomes confusing. We have clarified this point and have shown how the Z_2 subgroup produces the strong-weak coupling duality in the reduced theory.

To conclude, we have constructed in this paper the $SL(2, Z)$ multiplets of macroscopic string-like solutions of type II theory in any $D < 10$. This construction is made possible by a recent observation of the $SL(2, R)$ invariance of toroidally compactified type IIB string effective action. This generalizes the construction of $SL(2, Z)$ multiplets of string-like solutions of type IIB string theory in $D = 10$ by Schwarz. Our solutions have formal similarity with the solutions in $D = 10$, but they are totally different as they involve dimensionally dependent functions. The string-like solutions in $D < 10$ are also characterized by two relatively prime integers, as their counterpart in $D = 10$, corresponding to the charges of two antisymmetric tensor fields in the theory. We have also discussed the stability of the solutions from the charge conservation and tension gap relation. As we have mentioned, there are more string-like solutions not only with electric charge but also with magnetic charge in type II theories in lower dimensions which should form multiplets of bigger symmetry group, the U-duality group. Apart from the string-like solutions, there are also other p-brane solutions in these theories which deserve a systematic study to properly identify the complete U-duality group. This will provide strong evidence for the conjecture of the U-duality symmetries in those theories.

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