Linear and non-linear refractive indices in Riemannian and topological spaces

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The refractive index and curved space relation is formulated using the Riemann-Christoffel curvature tensor. As a consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor, the refractive index should be a second rank tensor. The second rank tensor of the refractive index describes a linear optics. In case of a non-linear optics, if susceptibility is a fourth rank tensor, then the refractive index is a sixth rank tensor. In a topological space, the linear and non-linear refractive indices are related to the Euler-Poincare characteristic. Because the Euler-Poincare characteristic is a topological invariant then the linear and non-linear refractive indices are also topological invariants.

I. INTRODUCTION

What is really happened if light passes through a medium? This question becomes more interesting nowadays related to conceptual development and technological innovation. One of the very important idea to understand this question is the refractive index, i.e. a measure of the bending of a ray of light when passing from one medium into another[1]. The refractive index of a medium is an optical parameter, since it exhibits the optical properties of the material[2].

The refractive index, n, is defined as velocity of light of a given wavelength in empty space or vacuum (c) divided by its velocity in a substance, v[3,4]

$$n = \frac{c}{v} \tag{1}$$

It describes how matter affects light propagation, through the electric permittivity, ε, and the magnetic permeability, μ[5]

$$n = \sqrt{\left(\frac{\varepsilon}{\varepsilon_0}\right)\left(\frac{\mu}{\mu_0}\right)} = \sqrt{(\varepsilon_r)(\mu_r)} \tag{2}$$

where ε_0 and μ_0 are the permittivity and the permeability of vacuum respectively, ε_r and μ_r is relative permittivity and relative permeability of non-vacuum medium respectively, which the values are relative i.e. they depend on the characteristics of medium[5,6].

In the most substrates, the refractive index decreases by increasing temperature[3]. A denser material generally tends to have a larger refraction index[7]. The refractive index in an fibre optic can be changed due to external forces such as the tensile force, the bending force[8].

Mathematically, the refractive index is a zeroth rank tensor (scalar) and it can not be a first rank tensor (vector), but it can be a second rank tensor, a third rank tensor or a higher rank tensor (which is well known as non-linear phenomena of second order, third order, etc)[9]. The refractive index is the zeroth rank tensor, if the medium or material is isotropic[10–12].

Generally, the refractive index is written as the second rank tensor, a 3×3 matrix, if the material is linear[13]. It can be the third rank tensor or the fourth rank tensor if the material is non-linear[14].

The refractive index has a large number of applications. It is mostly applied to identify a particular substance, to confirm its purity or to measure its concentration. In pharmaceutical industry, it can be used in determination of drug concentration. It is also used to calculate a focusing power of lenses and a dispersive power of prisms, to estimate a thermophysical properties of hydrocarbons and petroleum mixtures[3].

II. THE LINEAR REFRACTIVE INDEX IN THE RIEMANNIAN SPACE

Let a function, f, be defined and differentiable at a point, r, in a certain region of space (i.e. f defines a differentiable scalar field), then the gradient of f in the one-dimensional spherical coordinate is defined by

$$\vec{\nabla} f \equiv \frac{df}{dr}\ \hat{r} \tag{3}$$

In the tensorial notation, eq.(3) can be rewritten as[15]

$$\vec{\nabla} f = \text{grad}\ f = f_{,j} = \frac{\partial f}{\partial x^j} \tag{4}$$

where $f_{,j}$ is the covariant derivative of f with respect to x^j. Here, $\vec{\nabla} f$ defines a vector field i.e. the gradient of a scalar field is a vector field[15,16].

The relation of the curvature of space and the refractive index can be defined as[17–19]

$$\frac{1}{R} = \hat{N}\ .\ \vec{\nabla}\ \ln\ n(r) \tag{5}$$

where R is a radius of curvature, $\hat{N}$ is an unit vector along the principal normal or has the same direction with $\vec{\nabla}$ ln $n(r)$ and $n(r)$ is the space dependent refractive index. Eq.(5) tells us that the rays are therefore bent in the direction of increasing refractive index[17]. The illustration of eq.(5) is given in **Figure 1** below

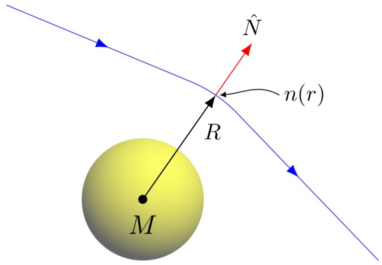

Fig. 1 The illustration of the space-dependent refractive index, $n(r)$, as a function of the curvature radius, R.

Let us analyse eq.(5). First, let us define that

$$\hat{N} \equiv \frac{\vec{\nabla} n(r)}{|\vec{\nabla} n(r)|} \tag{6}$$

Using notation of gradient operator in (3) and substituting (6) into (5), we obtain

$$\begin{aligned} \frac{1}{R} &= \frac{\vec{\nabla} n(r)}{|\vec{\nabla} n(r)|} \cdot \vec{\nabla} \ln n(r) \\ &= \frac{\frac{dn(r)}{dr}\ \hat{r}}{|\frac{dn(r)}{dr}\ \hat{r}|} \cdot \frac{d}{dr}\ \hat{r} \int \frac{1}{n(r)}\ dn(r) \end{aligned} \tag{7}$$

where $\hat{r}$ is a unit vector which its magnitude i.e. $|\hat{r}|$ is 1 and because $\hat{r}.\hat{r} = |\hat{r}||\hat{r}| \cos 0^0 = 1$, then eq.(7) becomes

$$\frac{1}{R} = \frac{1}{n(r)} \left(\frac{dn(r)}{dr}\right)^2 \left(\left|\frac{dn(r)}{dr}\right|\right)^{-1} \tag{8}$$

If we assume that the derivative of a function $n(r)$ always takes a positive value then

$$\left|\frac{dn(r)}{dr}\right| = \frac{dn(r)}{dr} \tag{9}$$

So, eq.(8) becomes

$$\frac{1}{R} = \frac{1}{n(r)}\ \frac{dn(r)}{dr} \tag{10}$$

where $n(r)$ can be e.g. exponential, logarithmic, quadratic, linear functions.

Now, we would like to write eq.(5) in the tensorial notation. But, why do we need to write eq.(5) in the tensorial notation? Eq.(5) is the curvature in one-dimensional space. If we would like to generalize the curvature in more than two-dimensional spaces, then we deal with the Riemann-Christoffel curvature tensor symbol, i.e. the fourth rank tensor, R_{mijk}.

In tensorial notation, $\hat{N}$ and $\vec{\nabla}$ can be written as N_k and $\partial/\partial x^j$ respectively and because of the Riemann-Christoffel curvature tensor is the fourth rank tensor, then *the refractive index in eq.(5) should be written as the second rank tensor*. We obtain the relation between the curvature tensor and the refractive index tensor as below[20]

$$R_{mijk} = g \left(N_k\ \frac{\partial \ln\ n_{mi}}{\partial x^j}\right) \tag{11}$$

where g is the determinant of the metric tensor.

Eq.(11) implies that the second rank tensor of the refractive index in curved space, which is described by the Riemann-Christoffel curvature tensor, related naturally to linear optics.

III. THE NON-LINEAR REFRACTIVE INDEX IN THE RIEMANNIAN SPACE

How about the form of the non-linear refractive index i.e. the refractive index related to the non-linear optics? In optics, *non-linear properties of materials are usually described by non-linear susceptibilities*[21]. Mathematically, the optical response can be expressed as a relation between the polarization density[22,23], $\vec{P}$, and the electric field, $\vec{E}$.

In the linear case, a relation between the polarization density and the electric field is simply expressed as[24,25]

$$\vec{P} = \varepsilon_0\ \chi^{(1)}\ \vec{E} \tag{12}$$

where ε_0 is the permittivity of vacuum space, $\chi^{(1)}$ is the first order susceptibility or linear susceptibility and it is a scalar, whereas the polarization and the electric field are vectors.

In the non-linear case[26–28], the polarization density can be modelled as a power series of the electric field as below[24,25,29]

$$\begin{aligned} \vec{P} &= \varepsilon_0 \left[\chi^{(1)}\ \vec{E}^1 + \chi^{(2)}\ \vec{E}^2 + \chi^{(3)}\ \vec{E}^3 + ...\right] \\ &= \vec{P}^1 + \vec{P}^2 + \vec{P}^3 + ... \end{aligned} \tag{13}$$

where $\vec{E}^1 = \vec{E}$, $\vec{E}^2 = \vec{E}\ \vec{E}$, $\vec{E}^3 = \vec{E}\ \vec{E}\ \vec{E}$, etc. $\vec{P}^1$ is called the linear polarization, while $\vec{P}^2$, $\vec{P}^3$ are called the second and third non-linear polarizations, respectively. Thus, *the polarization is composed by linear and non-linear components*[25]. The first susceptibility term, $\chi^{(1)}$, corresponds to the linear susceptibility (dimensionless). The subsequent non-linear susceptibilities, $\chi^{(a)}$, where $a > 1$, have units of (meter/volt)$^{a-1}$[30,31]. The quantities $\chi^{(2)}$ and $\chi^{(3)}$ are known as the second order and third order susceptibilities, respectively. These electric susceptibilities, $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, are the second, third and fourth rank tensors, respectively[24]. In optical Kerr effect, the third order susceptibility, $\chi^{(3)}$, related to the non-linear refractive index[10].

Now, we have a question: if the non-linear refractive index is related to the third order susceptibility, $\chi^{(3)}$, and the third order susceptibility is the fourth rank tensor[24] then how to define the non-linear refractive index which is related to the fourth rank tensor of the third order susceptibility?

For a linearly polarized monochromatic light in an isotropic medium or a cubic crystal, the non-linear refractive index, n_2, can be expressed by[29]

$$n_0 = 12\pi \ (n_2)^{-1} \ \mathrm{Re} \ \chi^{(3)} \tag{14}$$

where n_0 is a linear refractive index[32] and Re $\chi^{(3)}$ is a real part[33] of the third order non-linear susceptibility. We see from eq.(14), the linear refractive index is a function of the non-linear refractive index.

We see from eq.(11), the linear refractive index, n_0, is the second rank tensor, n_{mi}, and refer to Jatirian, et al.[24] the third order susceptibility, $\chi^{(3)}$, is the fourth rank tensor, $\chi^{(3)}_{pqrs}$, so we can write eq.(14) as below

$$n_{mi} = 12\pi \ n_{mi}^{pqrs} \ \chi^{(3)}_{pqrs} \tag{15}$$

where $n_{mi}^{pqrs} = (n_2)^{-1}$. So,

$$n_2 = n_{pqrs}^{mi} \tag{16}$$

It means that *the non-linear refractive index should be the sixth rank tensor (a mixed tensor of second rank cotravariant and fourth rank covariant).*

Substituting (15) into (11), we obtain

$$R_{mijk} = g \left[N_k \ \frac{\partial}{\partial x^j} \ \ln \left(12\pi \ n_{mi}^{pqrs} \ \chi^{(3)}_{pqrs} \right) \right] \tag{17}$$

Eq.(17) shows that in the non-linear optics, the Riemann-Christoffel curvature tensor is related to the sixth rank tensor of the refractive index (actually, the non-linear refractive index is the inverse of n_{mi}^{pqrs} as given in eq.(16)).

IV. THE REFRACTIVE INDEX IN THE TOPOLOGICAL SPACE (GLOBAL GEOMETRY)

Riemannian geometry is the study of Riemannian manifold, where *the Riemannian manifold is a pair of smooth manifold plus Riemannian metric tensor*[34]. The Riemannian geometry, which was the high dimensional generalization of Gauss intrinsic surface theory, gives a geometrical structure which is entirely *local*[35]. Local geometry is the study of small pieces of a manifold[36]. A manifold is a topological space that locally resembles Euclidean space near each point[37]. A topological space may be defined as a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods[38]. Global geometry is, as the word suggests, the study of the total manifold including, for example, the number of holes[36].

Let Ω_{mi} be *the curvature form*

$$\Omega_{mi} = \sum \ R_{mijk} \ dx^j \wedge dx^k \tag{18}$$

where R_{mijk} be the Riemann-Christoffel curvature tensor, x^j, x^k are local coordinates and $\wedge$ is a notation for the exterior (wedge) product (it satisfies the distributive, anti-commutative and associative laws)[39,40]. *This curvature form, Ω_{mi}, is an anti-symmetric matrix of 2-forms*[41,42]*. In differential geometry, the curvature form describes curvature of a connection on a principal bundle. It can be considered as an alternative to or generalization of the curvature tensor in Riemannian geometry*[43].

In the case of the linear optics, if we substitute eq.(11) into eq.(18), we obtain

$$\Omega_{mi} = \sum \ g \left(N_k \ \frac{\partial \ln n_{mi}}{\partial x^j} \right) \ dx^j \wedge dx^k \tag{19}$$

It means that the curvature form, Ω_{mi}, is related with the second rank tensor of the linear refractive index, n_{mi}.

In the case of the non-linear optics, if we substitute eq.(17) into eq.(18), we obtain

$$\Omega_{mi} = \sum \ g \left[N_k \ \frac{\partial}{\partial x^j} \ \ln \left(12\pi \ n_{mi}^{pqrs} \ \chi^{(3)}_{pqrs} \right) \right] dx^j \wedge dx^k \tag{20}$$

It means that the curvature form, Ω_{mi}, is related with the six rank tensor of the non-linear refractive index (actually, the non-linear refractive index is *the inverse of* n_{mi}^{pqrs}).

Let us define *the pfaffian of* Ω as below[39,44]

$$\mathrm{pf} \ \Omega \equiv \sum \ \epsilon_{m_1 i_1 \ldots m_{2n} i_{2n}} \ \Omega_{m_1 i_1} \wedge \ldots \wedge \Omega_{m_{2n} i_{2n}} \tag{21}$$

where Ω is *the curvature matrix*[45] i.e. any *even-size complex* $2n \times 2n$ *anti-symmetric matrix* (if Ω is *an odd size* complex anti-symmetric matrix, the corresponding pfaffian is defined to be *zero*), $\epsilon_{m_1 i_1 \ldots m_{2n} i_{2n}}$ is the $2n$-th rank Levi-Civita tensor which has value +1 or -1 according as its indices form an even or odd permutation of $1, \ldots, 2n$, and its otherwise zero, and the sum is extended over all indices from 1 to $2n$. Here, $m_1 < i_1$, ... , $m_{2n} < i_{2n}$ and $m_1 < m_2 < \ldots < m_{2n}$[39,44].

Shortly, the pfaffian of Ω (21) can be rewritten as

$$\mathrm{pf} \ \Omega = \sum \ \epsilon_{mi} \ \Omega_{mi} \tag{22}$$

Related with the Riemann-Christoffel curvature tensor, the pfaffian of Ω can be written, by substituting eq.(18) into (22), as

$$\mathrm{pf} \ \Omega = \sum \ \epsilon_{mi} \ \sum R_{mijk} \ dx^j \ \wedge \ dx^k \tag{23}$$

In the case of the linear optics, the pfaffian of Ω (22) becomes

$$\mathrm{pf} \ \Omega = \sum \epsilon_{mi} \ \sum \ g \left(N_k \ \frac{\partial \ln n_{mi}}{dx^j} \right) dx^j \wedge dx^k \tag{24}$$

In the case of the non-linear optics, the pfaffian of Ω (22) becomes

$$\mathrm{pf} \ \Omega = \sum \ \epsilon_{mi} \sum \ g \left[N_k \ \frac{\partial}{\partial x^j} \ \ln \left(12\pi \ n_{mi}^{pqrs} \chi^{(3)}_{pqrs} \right) \right] dx^j \wedge dx^k \tag{25}$$

The Gauss-Bonnet-Chern theorem[46–48] says that[39,47]

$$(-1)^n \frac{1}{2^{2n}\pi^n n!}\int_{M^{2n}} \mathrm{pf}\ \Omega = \chi(M^{2n}) \tag{26}$$

where n is a natural number, $\chi(M^{2n})$ is *the Euler-Poincare characteristic*[49,50] *of the even dimensional oriented compact Riemannian manifold,* M^{2n}. The Euler-Poincare characteristic is *a topological invariant*[39].

In the case of the linear optics, the Gauss-Bonnet-Chern theorem (26) becomes

$$\chi(M^{2n}) = (-1)^n \frac{1}{2^{2n}\pi^n n!}\int_{M^{2n}} \sum \epsilon_{mi} \sum g\left(N_k\ \frac{\partial \ln n_{mi}}{dx^j}\right) dx^j \wedge dx^k \tag{27}$$

In the case of the non-linear optics, the Gauss-Bonnet theorem (26) becomes

$$\chi(M^{2n}) = (-1)^n \frac{1}{2^{2n}\pi^n n!}\int_{M^{2n}} \sum \epsilon_{mi} \sum g\left[N_k\ \frac{\partial}{\partial x^j}\ \ln\left(12\pi\ n_{mi}^{pqrs}\chi^{(3)}_{pqrs}\right)\right] dx^j \wedge dx^k \tag{28}$$

Related with the Riemannian-Christoffel curvature tensor, the Gauss-Bonnet-Chern theorem can be written, by substituting (23) into (26), as

$$\chi(M^{2n}) = (-1)^n \frac{1}{2^{2n}\pi^n n!}\int_{M^{2n}} \sum \epsilon_{mi} \sum R_{mijk}\ dx^j\ \wedge\ dx^k \tag{29}$$

This eq.(29) relates the Riemannian geometry which is local geometry with the topological space which is global geometry. The Riemann-Christoffel curvature tensor is a local invariant and the Euler-Poincare characteristic is a global invariant.

We see from eqs.(27)-(28), the linear and non-linear refractive indices are related with the Euler-Poincare characteristic. Because the Euler-Poincare characteristic is the topological invariant[51,52] then *the linear and non-linear refractive indices should be the topological invariants.*

V. DISCUSSION AND CONCLUSION

In one and two dimensional spaces, the Gaussian curvatures are $1/R$ (a circle) and $1/R^2$ (a sphere), respectively. Georg Friedrich Bernhard Riemann, a student of Johann Carl Friedrich Gauss, generalize the Gauss curvatures for more than two-dimensional space. The result is the Riemann-Christoffel curvature tensor where the Christoffel symbol is used in the formulation of the generalized curvature.

Riemannian geometry gives a geometrical structure which is entirely local, i.e. the small pieces of a manifold. A manifold is a topological space i.e. a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods. Global geometry is, as the word suggests, the study of the total manifold.

In relation with the refractive index, because the refractive index is related with the curvature of one and two-dimensional spaces, and this curvature can be generalized for more than two-dimensional space, then the refractive index should be able to be formulated in more than two dimensional curved space. It gives *the second rank tensor* of the refractive index as the consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor. The second rank tensor of the refractive index describes the linear optics. It implies that the Riemann-Christoffel curvature tensor is related naturally to the linear optics.

Because the linear refractive index can be related with the non-linear refractive index and the third order of the susceptibility, where the linear refractive index is the second rank tensor and the third order of susceptibility is the fourth rank tensor, then the non-linear refractive index should be *the sixth rank tensor*. It implies that the Riemann-Christoffel curvature tensor is also related with the non-linear optics, i.e. the non-linear refractive index.

The Riemann-Christoffel curvature tensor can be generalized to the curvature form. So, the linear and non-linear refractive indices also can be generalized to the curvature form. Because the curvature form can be generalized to the curvature matrix, in pfaffian form, then the linear and non-linear refractive indices also can be written in relation with the pfaffian of the curvature matrix.

The pfaffian of the curvature matrix is defined to be zero if the curvature matrix is an odd-size complex antisymmetric matrix. But, the pfaffian of the curvature matrix is defined to be *non-zero* if the curvature matrix is *an even-size* complex antisymmetric matrix. It has the consequence that the related curvature form, eq.(18), is non-zero. It means that *the curvature is non-zero, i.e. the space is curved. Clearly, the size of (complex antisymmetric) matrix is related with the curvature of space. Even and odd-size (complex antisymmetric) matrices are related with a curved and flat space, respectively.*

The Gauss-Bonnet-Chern theorem is defined in eq.(26), where the pfaffian of the curvature matrix is related with the Euler-Poincare characteristic. M^{2n} indicates *even-dimensional* oriented compact Riemannian manifold, where the Riemannian manifold is smooth manifold plus Riemannian metric tensor. The pfaffian of the curvature matrix itself consists of the Riemann-Christoffel curvature tensor as shown in eq.(29), where the Riemann-Christoffel curvature tensor is anti-symmetric tensor (matrix). The Riemann-Christoffel curvature tensor based on the Riemann metric tensor which is symmetric matrix. This is the reason why, the pfaffian can be integrated using M^{2n} as the integration limit.

The non-zero Euler-Poincare characteristic is the consequence of the non-zero pfaffian of the curvature matrix.

The non-zero pfaffian of the curvature matrix is related with the even-size of complex antisymmetric matrix using in the curvature matrix. Physically, the non-zero Euler-Poincare characteristic is related with the even-dimensional curved space.

We see from eq.(27)-(28), the linear and non-linear refractive indices are related with the Euler-Poincare characteristic. The non-zero Euler-Poincare characteristic is related with the non-zero refractive indices. *Because the non-zero Euler-Poincare characteristic is related with the even-dimensional curved space, then the non-zero refractive indices should "live" in the even-dimensional curved space.* The Euler-Poincare characteristic is the topological invariant. *Because the Euler-Poincare characteristic is the topological invariant then the linear and non-linear refractive indices are also the topological invariants.*

The non-zero Euler-Poincare characteristic requires the even dimensional oriented compact Riemannian manifold. What is the consequence to the pfaffian of the curvature matrix, the Euler-Poincare characteristic, the linear and non-linear refractive indices formulation, if the Riemannian manifold is even dimensional, but unoriented non-compact? Can the Gauss-Bonnet-Chern theorem be formulated using the even dimensional unoriented non-compact Riemannian manifold?

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[44]Howard E. Haber, *Notes on antisymmetric matrices and the pfaffian*, http://scipp.ucsc.edu/~haber/webpage/pfaffian2.pdf, January 2005.

[45]The curvature matrix, Ω, is also an anti-symmetric matrix of 2-forms formed from the exterior derivative and exterior product of *the connection 1-form*, ω, which is a 2×2 matrix of 1-forms constructed in a certain way in *a vector bundle*, E.
Ω *is a generalization of* Ω_{ij}, in the sense that Ω can be defined for any projection $\pi : E \to M$ of any vector bundle, E, over a manifold, M, whereas Ω_{ij} is defined only for a space-time consisting of a manifold M plus a metric.
Ω will be identical to Ω_{ij} if E is *the tangent bundle* of a space-time and the connection 1-form, ω, is chosen to be $\omega^i_j = \Gamma^i_{jk}\, dx^k$ where Γ^i_{jk} is *the Christoffel symbol* constructed from the metric tensor[43].

[46]Shiing-Shen Chern, *From Triangles to Manifolds*, The American Mathematical Monthly, Vol. 86, No.5. (May, 1979), pp.339-349.

[47]Spalluci E. et al (2004), *Pfaffian*. In: Duplij S., Siegel W., Bagger J. (eds), Concise Encyclopedia of Supersymmetry, Springer, Dordrecht.

[48]*Gauss-Bonnet formula expresses the global invariant, $\chi(M)$, as the integral of a local invariant*, which is perhaps the most desirable relationship between local and global properties[46].
For *even-dimensional* oriented compact Riemannian manifold, M^{2n}, the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem[47].

[49]Milosav M. Marjanovic, *Euler-Poincare Characteristic - A Case of Topological Convincing*, The Teaching of Mathematics, 2014, Vol. XVII, 1, pp. 2133.

[50]The Euler-Poincare characteristic starts from *Euler's polyhedron formula (a number)* which appeared first in a note submitted by Euler to the Proceedings of the Petersburg Academy of 1752/53. Henri Poincare who defined *an integer* to be *a topological property* of all other geometric objects. *The Euler-Poincare characteristic is a stable topological property*[49].

[51]Topological Invariant. Encyclopedia of Mathematics. https://encyclopediaofmath.org/wiki/Topological_invariant.

[52]Topological invariant is any property of a topological space that is *invariant* under *homeomorphisms*[51]. Homeomorphisms are, roughly speaking, the mappings that preserve all the topological properties of a given space.

[53]Robert W. Boyd, *Nonlinear Optics*, Third Edition, Academic Press, Inc. 2008.

[54]Eiichiro Komatsu, *Curvature*, http://www.as.utexas.edu/astronomy/education/spring05/komatsu/lecture15.pdf.