A Nonlinear Generalisation of Quantum Mechanics

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Abstract

A new definition for quantum-mechanical momentum is proposed which yields novel nonlinear generalisations of Schrödinger and Klein-Gordon equations. It is thence argued that the superposition and uncertainty principles as they stand cannot have general validity.

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1 Introduction

1.1 A brief history

In the foundations of modern physics there is nothing more controversial than the *measurement problem* of quantum mechanics[1]: How should one

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reconcile the fact that on one hand, the wave function evolves deterministically according to the Schrödinger equation as a linear superposition of different states, while on the other hand, outcome of measurements are always a single definite state?

This problem is a mere reappearance of the old issue of *wave-particle duality*, for as long as the wavefunction evolves one is dealing with a wave, while as soon as a measurement is performed one observes a single particle, a single dot on the screen for example. The reconciliation of the wave and particle pictures therefore is a crucial problem whose resolution is necessary for making any progress regarding the measurement problem.

The pursuit of such reconciliation is at least as old as the quantum theory itself[1]. It began with Mie's ideas[2] and was pursued seriously by Einstein after his successful explanation of the photoelectric effect[3]. Although in a letter to Besso[4] Einstein admitted that **'***[...] All the fifty years of conscious ruminations have not gotten me closer to an anwser for the question: "What are light quanta?" These days any rascal may believe that he knows, but he deludes himself.***'**, his idea of *particles being concentrations (lumps) of continuous fields*[3][5] is still the best concrete idea we have for our attempts along such lines. This idea was a persistent theme of Einstein's attempts and was pursued through his grand programme of *Unified Field Theory*[5].

In fact all the realist founders of quantum mechanics especially Schrödinger himself initially thought in terms of *wavepackets*: a particle was assumed to be a 'parcel' of matterwaves (an *envelope*) that moved together as a whole by a group velocity equal to the velocity of the particle meanwhile the inner waves (in the parcel) oscillated by a frequency equal to the phase velocity of de Broglie's matterwaves. But this view soon faced a serious problem[6]: wavepackets which were supposed to be particles did not maintain their integrity due to linearity and *dispersion*[2]. This objection and the consequent 'victory' of the idealist founders like Bohr was serious enough to discourage even de Broglie for some twenty years [7]. de Broglie's interest in the problem was revitalised by Bohm's theory[8]; the theory that is now called *de Broglie-Bohm* or *Pilot wave* theory, according to which –roughly– the particle and wave pictures are both maintained simultaneously and the wave 'guides' the particle[9] by the *guiding equation*

$$\mathbf{p} = \nabla S$$

where S is the phase of the wavefunction (classically *Hamilton's principal function*)[10]. de Broglie-Bohm theory in its description of the double-slit experiment, by considering the analogy between water waves and matterwaves[9], relies on the idea of the fluid behaviour of the wavefunction, which was proposed and elaborated by Madelung[11] shortly after Schrödinger's work. This fluid picture of Madelung will prove to be useful in our discussions in this paper.

de Broglie himself however considered pilot wave theory a 'degenerate' form of his early attempts[7]; such attempts evolved to what de Broglie

[1] We use 'quantum *theory*' to include also the so-called *old quantum theory*.

[2] *Non*linearity and dispersion however solves this problem by allowing the existence of *soliton* solutions. This is one of our achievements in this paper.

in his late years dubbed as *the theory of the double solution*, according to which, particles were to be described by a *nonlinear equation* which has the Schrödinger equation as its approximation, hence the return of the Einsteinian theme. But neither Einstein nor de Broglie were able to derive such an equation. In this paper we derive such an equation that

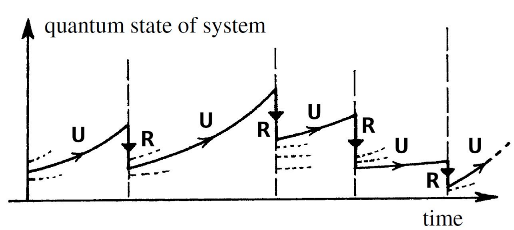
- Is a generalisation of Schrödinger equation: it simplifies to Schrödinger equation by a certain mathematical condition, hence fulfilling de Broglie's maxim.
- Unlike the so-called Nonlinear Schrödinger equation[12] and the Ghirardi-Rimini-Weber (GRW) theory[13], it contains no new arbitrary parameters.
- Is *nonlinear and dispersive*, hence allows for soliton solutions[14].

1.2 Motivation

According to the current understanding, Schrödinger equation governs the evolution of the wavefunction *between* measurements and measurements 'reset' the probabilities. If we compare this statement with the issue of wave-particle duality, we realise that observation of particles violates unitarity and conservation of probability

$$\frac{\partial|\psi|^2}{\partial t} + \nabla\cdot\left(|\psi|^2\frac{\nabla S}{m}\right) = 0, \tag{1}$$

In other words we maintain that it is the conservation of probability that obstructs the application of Schrödinger equation to the collapse process. Hence any reconciliation of wave and particle pictures must go beyond the law of conservation of probability. It now temporarily serves us to recall Penrose's description of quantum mechanics[1] in terms of **U**-process[3] and **R**-process[4]. According to Penrose the *complete* evolution of the quantum state must look like[5]



If we recall the fluid picture of Madelung and hence think of the above graph as the motion of a quantum fluid of probability, we can presume that *the **R**-processes act as the sinks of the probability fluid.* But the current (orthodox) quantum mechanics crucially misses this point because

[3] Unitary evolution.

[4] Reduction (collapse of the wavefunction).

[5] The graph has a mere explanatory role and we do not consider it to accurately describe reality, for in our view there is not any such *discontinuous* **R**-process.

the fluid that it describes is an *incompressible* one; in other words one *without* sinks (sources)[15][16]. This condition in non-relativistic quantum mechanics is mathematically expressed by

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{p} = \nabla \cdot \mathbf{k} = 0. \tag{2}$$

The conclusion is that a complete picture of reality in which the wave and particle aspects are united hence the measurement problem is resolved –or at least a serious attempt is made for its resolution– must *dispose of the incompressibility of the quantum probability fluid.* It is an interesting fact –to the best of our knowledge– that relaxation of this significant restriction has not yet been considered.

Methodologically, we know it is quite difficult –if not impossible– to begin with the condition (2) and arrive at a nonlinear equation. We must instead reverse the process and by starting from a firm physical motivation, find a nonlinear equation which simplifies to the Schrödinger equation by imposing (2).

Nevertheless a key observation can be made immediately from (2) that the current *operatorial* scheme of quantum mechanics cannot handle the incompressibility condition (1) and in fact it yields a contradiction: as

$$\hat{\mathbf{p}} = -i\hbar\nabla$$

condition (2) for a test wavefunction ψ reads

$$\nabla \cdot (\hat{\mathbf{p}}\psi) = (\nabla \cdot \hat{\mathbf{p}})\psi = -i\hbar\nabla^2\psi = 0 \tag{3}$$

which is evidently an absurdity since derivation of (1) assumes $\nabla^2\psi \neq 0$, otherwise $\partial_t \rho = 0$. We will discuss this observation later in detail and argue that this problem is caused by the associativity of quantum mechanics' operator algebra.

2 Harmonisation

It is the received wisdom that Planck-Einstein-de Broglie law[6] $p^\nu = \hbar k^\nu$ belongs to the era of 'old quantum mechanics' and that in the realm of quantum mechanics the right (and more fundamental) perspective is to solve the Schrödinger equation for any case at hand. Although from an instrumentalist point of view this perspective has been quite successful, in this paper we advocate another perspective which will prove to be more fruitful with regard to the foundational questions of quantum mechanics. Our perspective is that *quantum mechanics is basically 'all' about* $p^\nu = \hbar k^\nu$. To adopt such perspective we need to first scrutinise our understanding of its essential ingredient k^ν. By any rigorous math-

[6] The metric signature $(+, -, -, -)$ is used everywhere in this paper. Greek indices run over 0 to 3 (four dimensions).

ematical definition[7] it is required that a wave be defined as a field[8] on spacetime which satisfies a certain equation, without any explicit reference to its four-wavevector. On the other hand, according to our perspective $p^\nu = \hbar k^\nu$ is a fundamental law of nature and appearance of k^ν in such a law suggests that we must enforce all waves to acquire a mathematically well-defined four-wavevector. Therefore we must find a definition for the four-wavevector of a wave ψ in terms of the ψ *itself*; we shall call this process *harmonisation*. Considering the simplest case of a complex harmonic wave[9,10],

$$\psi = Ae^{-ik_\mu x^\mu},$$

if we apply the gradient operator to both sides we have,

$$\partial_\mu \psi = -ik_\mu \psi = -i\psi k_\mu \tag{4}$$

we realise that there are *three* possibilites for harmonisation:

1. Linear-operatorial approach, in which $\hat{k}_\mu \in End\big(L^2(\mathbb{R})\big)$ such that

   $$\psi \xmapsto{\hat{k}_\mu} i\partial_\mu \psi$$

   i.e.

   $$\hat{k}_\mu\big(\psi\big) := i\partial_\mu \psi$$

   plus the *eigenvalue hypothesis*

   $$\hat{k}_\mu(\psi) = k_\mu \psi = \psi k_\mu$$

   An immediate elaboration is needed here. It is astonishing how rich and delicate notation can be. Equation (4) is all that we have to begin with. It is valid only as it stands, and as it stands k_μ is a vector, *not an operator* on the L^2 space of wavefunctions[11]. It is not impermissible to promote k_μ to an operator but to be logically consistent we cannot change the fundamental equation (4) that we started with. Therefore if we choose to promote k_μ to an operator –as we do in this approach– we must keep (4) as it stands, and that is why we need the additional eigenvalue hypothesis. What we just expressed is not anything controversial; it is one of the *axioms*[18] of quantum mechanics, only in light of a different perspective.

[7] For example, an entity ϕ which satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \nabla^2 \phi,$$

where v is the speed of propagation of the wave. Or, an entity ϕ which satisfies the Schrödinger equation.

[8] We only consider scalar fields in this paper. Sufficient conditions of smoothness are also assumed implicitly.

[9] Note that this is not the most general harmonic wave one can write. Moreover notice that in these definitions only forward-in-time waves are considered. It is not clear whether this preference of time direction affects the theory in a decisive manner; Woit has elaborated on this issue in the context of QFT, see *Appendix A* to [17].

[10] Technically plane waves are not physically legitimate as they are not square-integrable, but this issue can be rigorously avoided; see[18]. We do not involve in such technical details hereafter in this paper.

[11] This must not be confused with the way one thinks of vectors as linear operators on space of forms; that is a totally different matter

2. Nonlinear-operatorial[12] approach, in which $\check{k}_\mu \in {L^2}^{L^2\backslash\{0\}}$ i.e. $\check{k}_\mu : L^2(\mathbb{R})\backslash\{0\} \to L^2(\mathbb{R})$ such that

$$\psi \xmapsto{\check{k}_\mu} i\frac{\partial_\mu \psi}{\psi}$$

i.e.

$$\check{k}_\mu(\psi) := i\frac{\partial_\mu \psi}{\psi} \neq \big(k_\mu \psi = \psi k_\mu\big)$$

thus **without** the need to add the eigenvalue hypothesis; because with the eigenvalue hypothesis in this case we would have

$$i\frac{\partial_\mu \psi}{\psi} = k_\mu \psi$$

$$\Rightarrow \partial_\mu \psi = -k_\mu \psi^2$$

which is incompatible with (4).

3. Neoclassical nonlinear approach, in which $\psi \in L^2(\mathbb{R})\backslash\{0\}$ and $k_\mu \in \mathbb{C}^{\mathbb{R}^4}$ i.e. $k_\mu : \mathbb{R}^4 \to \mathbb{C}$ such that

$$x^\nu \xmapsto{k_\mu} i\frac{\partial_\mu \psi(x^\nu)}{\psi(x^\nu)}$$

or

$$k_\mu(x^\nu) := i\frac{\partial_\mu \psi(x^\nu)}{\psi(x^\nu)} \neq \big(k_\mu \psi = \psi k_\mu\big)$$

again **without** the need to add the eigenvalue hypothesis, for the same reason stated above. As we shall see later the eigenvalue hypothesis is automatically included as a special case in this neoclassical approach in the sense that it leads to a generalisation of Schrödinger and Klein-Gordon eigenvalue problems. The assumption $\psi \in L^2(\mathbb{R})\backslash\{0\}$ is made only to make connection with the Born Principle, as our starting motivation requires. But after our theory is developed it will become conceivable that the structure (Hilbert Space) that comes with this assumption might not be necessary; especially the necessity of the inner product structure is hard to see given that in this approach –among other things– we do not need to define the notion of self-adjointness (which requires an inner product to be defined).

The second and third approach might seem identical but there are crucial differences which make them not only different but also *incomparable*: in the nonlinear-operatorial approach –like linear-operatorial– ψ and $\partial_\mu \psi$ are not considered 'independent'; once ψ is given one just puts it into the input of $\check{k}_\mu$ to get momentum. This is essentially the reason that in the derivation of Schrödinger equation from the variational principle, one only varies with respect to ψ (and ψ^*). In the neoclassical approach however, ψ and $\partial_\mu \psi$ are considered 'independent'. Indeed this is the reason for calling it *neoclassical* because it reminds one of the way position and momentum

[12] We do not restrict our notion of 'operator' here to one which has the same domain and codomain.

are treated in analytical mechanics. Another crucial difference of the second and third approach will be explained in Appendix A.
The linear-operatorial approach is familiar and well-studied, being the foundation for orthodox quantum mechanics. The nonlinear-operatorial approach can handle the incompressibility condition, but in finding a wave equation it faces a serious problem[13]; therefore we do not pursue it in this paper. The neoclassical approach is the only alternative that can both handle the incompressibility condition and yield a generalisation of Schrödinger equation. It is therefore the neoclassical approach on which we are focused in this paper as our main proposal. To the best of our knowledge this approach is new: it is the first time that

$$p_\mu = i\hbar \frac{\partial_\mu \psi}{\psi} \tag{5}$$

is proposed as the *definition* of quantum-mechanical momentum. Although de Broglie-Bohm theory comes quite close to our approach, it misses the point by dispensing with the imaginary part of the momentum in order to make the same predictions as the Schrödinger picture. To be precise, if we write the wavefunction in polar form

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) e^{iS(\mathbf{x},t)/\hbar}$$

where $R, S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we see that according to our proposed definition of quantum-mechanical momentum we have

$$\mathbf{v} = \frac{\nabla S}{m} - \frac{i\hbar}{m} \frac{\nabla R}{R} \tag{6}$$

while in de Broglie-Bohm theory only the first term is considered[19], however in our view there is no a priori theoretical reason why one should do so, on the contrary we will soon argue that neglecting the second term comes from a metaphysical position. By omission of this second term one is losing some of the 'information' encoded in the wavefunction. Indeed this can be better seen if we think of S as the Hamilton's principal function in classical mechanics (Hamilton-Jacobi theory[10]). We know that quantum mechanics must tell us more about reality than what classical mechanics does, however by leaving something (the guiding equation) unchanged that *already* exists in classical mechanics we cannot hope to fully achieve this expectation. Inclusion of the imaginary part is an important point of departure for our theory compared to de Broglie-Bohm theory, whose guiding equation is a special case of our definition by letting

$$\nabla R = 0, \quad R \neq 0$$

resulting in

$$\exists f : \mathbb{R} \to \mathbb{R}, \text{ such that } R = f(t) \neq 0$$

which means that de Broglie-Bohm theory by assumption only includes wavefunctions with amplitudes that are uniform in space. The orthodox quantum mechanics is not different with regard to this limitation because by its conservation of probability (1) it implicitly assumes $\mathbf{v} = \nabla S/m$.

[13]The problem is explained in *Appendix A*.

The basic reason that de Broglie-Bohm and orthodox quantum theory neglect the second term is that being bound by the eigenvalue hypothesis they think of eigenvalues as what is actually observed in measurements hence they require the eigenvalues to be real numbers. We believe that this is the last remnant of 'classical thinking' in quantum mechanics, that presupposes only real numbers exist. It is true that in measurements one only observes real numbers but that can well be a limitation of our understanding: that we cannot observe imaginary numbers is not a reason they cannot exist. In fact our proposal of including the second term in (6) is quite aligned with the work of *Renou et al.*[20] who are discussing the possibility of empirically testing the 'reality' of complex numbers. By loosening this restriction our approach proves to yield novel insights which are obscured by narrowing physical entities to linear self-adjoint operators.

Even if we ignore this important point, on the conceptual side, de Broglie-Bohm theory never promotes (5) to *the definition* of momentum in quantum mechanics as the theory only *augments* Schrödinger's theory with the *additional* guiding equation.

One aspect of incompleteness of mathematics of quantum theory as it stands nowadays is therefore shown here. The incompleteness that (6) reveals provides a reason to think that the definition (5) by successfully handling the incompressibility condition as we shall see, is the right definition for quantum-mechanical momentum by virtue of its generality.

3 Neoclassical Nonlinear theory

3.1 Generalisation of Schrödinger equation

Similar to the familiar derivation of the Schrödinger equation from the law of conservation of energy, we apply the Planck-Einstein-de Broglie law $p^\mu = \hbar k^\mu$ to the logarithmic approach to get

$$\mathbf{p} = -i\hbar\nabla(\log\psi) \quad \text{and} \quad E = i\hbar\frac{\partial}{\partial t}(\log\psi) \tag{7}$$

Substituting (7) in the law of conservation of energy

$$E = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + V,$$

yields

$$i\hbar\frac{\partial}{\partial t}\log\psi = -\frac{\hbar^2}{2m}(\nabla\log\psi)^2 + V \tag{8}$$

To get the Schrödinger equation, notice that (8) is equivalent to

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{|\nabla\psi|^2}{\psi} + V\psi} \tag{9}$$

which differs from the Schrödinger equation only by the term

$$\frac{|\nabla\psi|^2}{\psi},$$

which we now show only in the *special case* that **k** is a solenoidal field, is equal to the corresponding term in Schrödinger equation. Consider the incompressibility condition (2)

$$\nabla \cdot \mathbf{k} = 0 \tag{10}$$

Which, by our definition (5) is

$$\nabla \cdot (\frac{\nabla \psi}{\psi}) = \frac{\psi \nabla^2 \psi - |\nabla \psi|^2}{\psi^2} = 0$$

therefore

$$\frac{|\nabla \psi|^2}{\psi} = \nabla^2 \psi \tag{11}$$

In other words, Schrödinger equation is a special case of the nonlinear equation derived in this paper. Condition (11) reveals an important but *hidden assumption* of the current theory of quantum mechanics about wavefunctions; yet it is not unexpected at all, for it is equivalent –if we use the right definition of momentum– to the condition (2) which was already a well-known fact. What obscured this equation so far to be explicitly stated is the incomplete definition of momentum in orthodox quantum mechanics, which, as we remarked in the introduction cannot handle the condition satisfactorily.
We can now explicitly see how linearity of Schrödinger equation arises from nonlinearity of (9), and how an eigenvalue problem which is a marker of quantum *discreteness* and quantum 'jumps' is only a special case to a nonlinear *but continuous* reality. In this light the superposition 'principle' is only an special-case feature of nature and has a limited domain of applicability.
The observation in which *an eigenvalue problem arises from a more general **nonlinear** equation* is quite a generic one and worthy of emphasis. As a simple example consider how *Helmholtz equation*

$$\nabla^2 \phi = -k^2 \phi$$

can be an approximation to the following nonlinear equation

$$\nabla \cdot (\frac{\nabla \phi}{\phi}) = -k^2,$$

for $\phi \neq 0$,
As

$$\nabla \cdot (\frac{\nabla \phi}{\phi}) = \frac{1}{\phi} \nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi^2} = -k^2$$

Multiplying both sides by ϕ yields

$$\nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi} = -k^2 \phi$$

If we apply the *approximation*

$$\frac{|\nabla \phi|^2}{\phi} \approx 0$$

we are led to the original Helmholtz equation.

3.2 Generalisation of Klein-Gordon equation

Our definition of momentum (5) can be readily substituted in $E^2 = p^2c^2 + m^2c^4$ to yield

$$\boxed{\frac{1}{c^2}(\frac{\partial \psi}{\partial t})^2 - |\nabla \psi|^2 + (\frac{mc}{\hbar})^2 \psi^2 = 0} \tag{12}$$

which can also be written as

$$-\frac{\langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = (\frac{mc}{\hbar})^2 \tag{13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the complexified Minkowski spacetime.
Similar to the case for non-relativistic equation (3), this equation as well is reduced to the Klein-Gordon equation by the special-relativistic generalisation of condition (2)

$$\partial_\mu p^\mu = \partial_\mu k^\mu = 0 \tag{14}$$

which is

$$\frac{\psi \Box \psi - \langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = 0$$

$$\Rightarrow \frac{\langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = \frac{\Box \psi}{\psi}$$

Substitution in the alternative form (13), we have

$$\frac{\Box \psi}{\psi} = -(\frac{mc}{\hbar})^2$$

multiplication of both sides by ψ yields

$$\left(\Box + (\frac{mc}{\hbar})^2\right) \psi = 0$$

i.e. the Klein-Gordon equation.

3.3 Comparison with other relevant nonlinear equations

A nonlinear generalisation of Schrödinger equation reminds one of the Nonlinear Schrödinger equation[12]

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2 \psi + k|\psi|^2 \psi$$

and possibly other similar equations like Gross-Pitaevskii[21].
On mathematical side, our proposed equation is totally different from such equations in that it is a *first-order* PDE.
On physical side, all such equations are based on various approximations[22][23] hence with no simple and elegant derivation. Being approximations and *not generalisations* they all owe their physical role to a more *fundamental* (as opposed to *emergent*) equation. In case of the Gross-Pitaevskii equation for example, the nonlinearity is an emergent one: involving scattering

length a_s which is not a fundamental constant of physics, the nonlinear term is due to interaction of particles and not a fundamental term present for single particles.
Our proposition on the contrary is based on no approximation nor additional assumption: it is based on the most general form of the de Broglie hypothesis that is possible to express in terms of the wavefunction itself using differential calculus.
For the very reason of not having simple derivations and being approximate and not fundamental, such equations have not the potential of being extended to relativistic quantum mechanics while special relativity is easily applied to our proposed fundamental definition of momentum (5) to yield the elegant generalisation of Klein-Gordon equation (12).
Along totally different lines that similar to our approach follow nonlinearity as having a fundamental role in quantum physics, the term $\Box\psi/\psi$ in (13) resembles $\Box f/f$ in de Broglie's theory of double solutions[7]. Basically de Broglie considers two waves; usual ψ and the u-wave. de Broglie thought of this new u-wave as representing a 'mobile singularity' intended to represent the particle aspect. The f function is the amplitude of the u-wave. Although de Broglie himself –like the theory that bears his name– also missed the point by neglecting the new (second) term in (6), he did correctly realise the significance of $\Box f/f$ by stating that
'*The departure from the older mechanics is always bound up with the presence of the term* $\Box f/f$.'[7]. Apart from this remark, our theory is completely different from de Broglie's theory and does not appeal to the redundant notion of u-wave.

4 Implications

It must have become clear by now that our proposal is neither an alternative nor an interpretation of any existing quantum theory. It is a generalisation and as such it has novel physical consequences.

4.1 Generation or Destruction of Probabilities

Although our physical motivation was based on probabilities and measurements, after the development of our proposal we now face difficulties in maintaining such concepts. Our proposal cannot say anything about the Born Principle as the principle is a metaphysical one: it is about *meaning*, not mathematics. The very statement of the Born Principle that '$|\psi|^2$ is the probability distribution of a *superposed* quantum state switching to a single definite *eigenvalue* by performance of a measurement' is blurred by our proposal: according to our view there are neither superpositions nor eigenvalues in general; both are too special cases to deserve reference of a fundamental *principle* of nature. Possible redundancy of concepts like superposition and eigenvalues in turn makes the meaning of 'measurement' and 'probability of outcome of a measurement' unclear. In this light therefore, it is expected that we must look for a new meaning for ψ. In this paper however we follow Newton's maxim of *hypotheses non fingo* and leave the question open for further meticulous investigations. Accordingly we temporarily assume that $|\psi|^2$ *is some sort of probability* in order to demonstrate the following consequence. If we leave (almost) intact the assumption of $\rho = |\psi|^2$ representing probabilities, comparison of (1) and (6) implies that the rate of generation or destruction of probabilities denoted by π is given by

$$\pi = \frac{i\hbar}{m}\nabla \cdot \left(|\psi|^2 \frac{\nabla R}{R}\right) \tag{15}$$

As we know from dynamics of compressible fluids[16], $\pi > 0$ means generation of probability, and $\pi < 0$ its destruction. Naturally $\pi = 0$ is the situation in which probability is not generated and destructed i.e. conserved.

4.2 Nonlinearity and Possibility of Solitons

Notice that equation (9) is dispersive: Consider for example the case for a free particle

$$i\frac{1}{\psi}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{|\nabla \psi|^2}{\psi^2}$$

in which we have set $\hbar = m = 1$ for simplicity. By our definition (5) the above equation yields the dispersion relation

$$\omega = -\frac{1}{2}|\mathbf{k}|^2,$$

Which is in fact identical with the dispersion relation that Schrödinger equation yields for a free particle. Unlike Schrödinger equation however, our equation is nonlinear. As we mentioned in the introduction it is known that dispersion and nonlinearity *together* allow for the possibility of existence of solitons[14]. It is therefore possible to revive the old notion of *wavepackets as particles* should such solutions actually occur.

4.3 Uncertainty Principle

Born Principle is not the only conceptual problem revealed by our approach. There is another –perhaps more significant– issue which has a better theoretical status in that it can be mathematically analysed. The fundamental conceptual point of departure for quantum mechanics (in the Heisenberg picture) compared to classical mechanics is the *canonical commutation relation*

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{16}$$

where $\hat{q}_i, \hat{p}_j \in End(L^2)$ i.e. in quantum mechanics momentum and position are linear operators which act on the space of square-integrable functions. The set of all such endomorphisms together with addition, (functional) composition and scalar multiplication forms an associative algebra. But this algebra is not a commutative one[14] as expressed by (16). The associativity of this algebra is the reason for the failure (3) of

[14] As such $\hat{q}_i$ and $\hat{p}_j$ can be considered as $\infty \times \infty$ matrices and matrix multiplication is not commutative.

orthodox quantum mechanics in handling the incompressibility condition: as the first equality in (3) shows, by giving the same mathematical status (operator) to both divergence and **p** quantum mechanics allows for the two to become a laplacian via functional composition as the operation of the algebra. In our theory however, *divergence and* $\boldsymbol{p}$ *do not have same mathemematical status*; first one is a linear operator, second one is a vector.
According to our theory $q_i, p_j : \mathbb{R}^4 \to \mathbb{C}$ i.e. in our theory momentum and position of a certain point in phase space are elements of the field of complex numbers (scalars) and by the commutative property of a field they *do commute*, viz. in our approach (for simplicity we restrict to one dimension)

$$p = -\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial x}$$

and

$$x = -\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial p}$$

therefore

$$px = -\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial x}x$$

as long as all the constituents of this equation are elements of a field –which they are– we have

$$-\frac{i\hbar x}{\psi}\frac{\partial\psi}{\partial x} = xp$$

therefore in this neoclassical approach $xp = px$.
This is not to say that uncertainty 'principle' is *wrong*; it cannot be, as it as has been empirically verified[24][25][26], but that it is not a fundamental *principle* of nature due to its lack of universal validity; it applies only when the operatorial definition of momentum is held, which need not generally be the case. Therefore as Einstein remarked that '*It is the theory which decides what we can observe*'[27] more accurate experiments must be done both to better understand the incompressibility condition and to test the general validity of the Heisenberg uncertainty principle.

A Failure of nonlinear-operatorial approach

As we know from the linear-operatorial approach (orthodox quantum mechanics), the square of momentum in the law of conservation of energy acted on ψ

$$E\psi = \frac{p^2}{2m}\psi + V\psi$$

translates to the quantum-mechanical momentum operator *acting twice*; nabla acting twice becomes laplacian, which is why we have the laplacian in Schrödinger equation. Mathematically,

$$p^2\psi \to \hat{p}o\hat{p}(\psi) = \hat{p}\big(\hat{p}(\psi)\big)$$

as the operation of operator algebra is functional composition. In the nonlinear-operatorial approach we are still keeping the operator algebra and this *acting twice* still holds, viz.

$$p^2\psi = \hat{p}\big(\hat{p}(\psi)\big)$$

but now the definiton is different

$$\hat{p}(\psi) = -i\hbar\frac{\nabla\psi}{\psi} \tag{17}$$

let

$$\phi = \hat{p}(\psi) = -i\hbar\frac{\nabla\psi}{\psi},$$

as

$$\hat{p}(\phi) = -i\hbar\frac{\nabla\phi}{\phi}$$

we must have

$$\hat{p}(\phi) \stackrel{?}{=} -i\hbar\frac{\nabla\cdot\left(-i\hbar\frac{\nabla\psi}{\psi}\right)}{-i\hbar\frac{\nabla\psi}{\psi}}$$

which is seriously problematic as a vector cannot be in a denominator. So definition (17) cannot be maintained in a non-linear *operatorial* approach.

B A conjecture

It is natural to expect however that *it is in this realm* (non-linear operatorial approach) that the uncertainty principle continues to rule. Therefore if we want to create a nonlinear picture that continues to use operators, we must find an alternative for (17) such that it would have the definition of momentum in conventional quantum mechanics as its 'special case'. But this 'special case' must be clearly stated mathematically. Suppose one succeeds in finding a definition for quantum-mechanical momentum which would not face the problem mentioned above. Call such definition $\breve{p}_j$, then a candidate for quantification of *nonlinearity* of this operator would be

$$\mathcal{M} = \sup\{\|\big(\breve{p}_j - (-i\hbar\partial_j)\big)\psi\|_2\}, \quad \forall\psi \in L^2(\mathbb{R})$$

Thus when nonlinearity of all operators is zero, orthodox quantum mechanics is recovered. In this way, if we assume any such $\breve{p}_j$ exists, the nonlinear-operatorial theory built upon it would be a proper generalisation of quantum mechanics: the same way one recovers classical mechanics when position and momentum commute, one would recover linear-operatorial quantum mechanics when $\mathcal{M} = 0$. Assuming $\mathcal{M}$ is a viable measure of nonlinearity of quantum-mechanical operators, or any other measure that can satisfy our expectations, then by comparison to (16) one is lead to a key question: *Could this measure be a constant, universal for all operators?*

Our conjecture is that, as $[\mathcal{M}] =$ *momenum* and the only fundamental constant with such dimension that we know of is *Planck momentum*, it is possible that

$$\mathcal{M} = m_P c = \sqrt{\frac{\hbar c^3}{G}}, \tag{18}$$

Therefore if this conjecture is true, *when both $\hbar$ and G are significant linearity of quantum mechanics no longer holds.* This conjecure if proved to be viable (both mathematically and physically) would have momentous consequences for quantum gravity.

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