Rigorous proof for Riemann hypothesis as Pseudo-zeroes to Zeroes conversion that obeys trigonometric identities

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Abstract The 1859 Riemann hypothesis propose all nontrivial zeros in Riemann zeta function are located on sigma = 1/2 critical line. Employing a simple observation to geometrically and mathematically define the unique location of all nontrivial zeros permits rigorous proof for this hypothesis to fully materialize. Solving this hypothesis using correct and complete mathematical arguments is finalized by deriving Dirichlet Sigma-Power Law with its computed Pseudo-zeroes which can all be converted to Zeroes. Pseudo-zeroes uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (1 - sigma) that is twice present in this law to be integer 1. By applying Euler formula to Dirichlet eta function [proxy for Riemann zeta function], we obtain simplified Dirichlet eta function whereby its computed Zeroes uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (-sigma) that is twice present in this function to be integer -1. This law and function will both obey trigonometric identities that comply with Principle of Maximum Density for Integer Number Solutions and manifest Principle of Equidistant for Multiplicative Inverse.

Keywords Dirichlet Sigma-Power Law · Pseudo-zeroes · Riemann hypothesis · Zeroes

Mathematics Subject Classification (2010) 11M26 · 11A41

1 Introduction


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Fig. 1: INPUT for $\sigma = \tfrac{1}{2}, \tfrac{1}{5}$, and $\tfrac{3}{5}$. $\zeta(s)$ has countable infinite set of Completely Predictable trivial zeros which are located at $\sigma = \text{all negative even numbers}$ and [conjectured] CIS of Incompletely Predictable nontrivial zeros which are located at $\sigma = \tfrac{1}{2}$ given by various $t$ values.

Fig. 2: OUTPUT for $\sigma = \tfrac{1}{2}$ as Gram points. This Figure represents schematically depicted polar graph of $\zeta(\tfrac{1}{2} + it)$ plotted along critical line for real values of $t$ running from 0 to 34, horizontal axis: $\text{Re}\{\zeta(\tfrac{1}{2} + it)\}$, and vertical axis: $\text{Im}\{\zeta(\tfrac{1}{2} + it)\}$. There are presence of Origin intercept points which are totally absent in Figures 3 and 4 [with identical axes definitions but, respectively, adjusted to $\sigma = \tfrac{2}{5}$ and $\sigma = \tfrac{3}{5}$]

solving Riemann hypothesis and explaining closely related two types of Gram points. Some of the composed materials in our current stand-alone research paper are adapted from relevant parts of [1] especially on these derived equations. Inspiration to geometrically and mathematically define the locations of nontrivial zeros and two types of Gram points stem from simple observation that Figures 3 and 4 with associated shifts of Varying Loops in their $\zeta(\sigma + it)$ Polar Graphs will manifest Principle of Equidistant for Multiplicative Inverse.

Preliminary information to help explain utilized nomenclature of this paper: Antiderivative $F(n)$ [which is simply another function] denotes solution from performing integration on function $f(n)$; viz, $\int f(n)dn = F(n) + C$ with $F(n) = f(n)$. In this paper, we intrinsically treat and analyze in a de novo fashion all simple and complex single-variable function $f(n)$ or $F(n)$ and their simple and complex single-variable equation $f(n) = 0$ or $F(n) = 0$ as unique Completely Predictable or Incompletely Predictable mathematical objects. Complex $f(n)$ of interest consist of Riemann zeta function $\zeta(s)$, its proxy Dirichlet eta function $\eta(s)$, and simplified Dirichlet eta function $\text{sim-}\eta(s)$ which is obtained by applying Euler formula to $\eta(s)$. Law is a set of rules created by state institutions which make laws through the authority of the state. Symbolically linked to German mathematician Gustav Lejeune Dirichlet (February 13, 1805 - May 5, 1859), complex $F(n)$ named Dirichlet Sigma-Power Law [DSPL] is solution from $\int \text{sim-}\eta(s)dn$ whereby the word “Law” utilized in DSPL represent a convenient terminology to describe this function – viz, there is resemblance to Zipf’s law via power law functions in $\sigma$ from $s = \sigma + it$ being exponent of a power function as similar format to $n^\sigma$, logarithm scale use, and harmonic $\zeta(s)$ series connection. Respectively, we use the terms Zeroes and Pseudo-zeroes to collectively refer to $f(n)$’s and $F(n)$’s x-axis intercept points, y-axis intercept points and Origin intercept points.
Specific to $f(n)$'s Zeros incorporated by the listed functions $\{\zeta(s), \eta(s) \text{ and } \text{sim}-\zeta(s)\}$ and $F(n)$'s Pseudo-zeros incorporated by the listed function \{DSPL\} with both located at $\sigma = \frac{1}{2}$ critical line; virtual Zeros and virtual Pseudo-zeros incorporated by corresponding same listed functions will both be located at corollary $\sigma \neq \frac{1}{2}$ non-critical lines. All these Zeros, virtual Zeros, Pseudo-zeros and virtual Pseudo-zeros are classified as Incompletely Predictable entities. A variable e.g. $n$ in $f(n) = \zeta(s)$, $n_1$ in $f(n_1)$, and $n_2$ in $f(n_2)$ represents a mathematical model state and may change during simulation. A parameter, e.g. $\sigma$ and $t$ in $\zeta(s)$, is normally a constant in a single simulation used to describe objects statically and is changed only when we need to adjust our model behavior. A single-variable function e.g. $f(n_1)$ or multiple-variable function e.g. $f(n_1, n_2)$ is a set of input-output pairs that follows a few particular rules. An expression usually contains number(s), parameter(s), variable(s) and operator(s). A particular function e.g. $f(n_1)$ is an expression involving variable $n_1$ that is defined for interval $[a,b]$. An equation is an assertion that two expressions are equal from which one can determine a particular quantity. An algorithm is a precise step-by-step plan for a computational procedure that possibly begins with an input value and yields an output value in a finite number of steps. A complex algorithm e.g. for generating prime numbers or composite numbers is only defined at two end-points $a,b$ (but not defined for interval $[a,b]$ as it is not a function). A formula is a fact or a rule written with mathematical symbols, and usually connects two or more quantities with an equal to sign. The terms ‘variable’, ‘parameter’, ‘function’, ‘algorithm’, ‘equation’ and ‘formula’ could sometimes be loosely used to describe near-identical situations.

Overall objective of this paper given as aesthetic arguments to support one-and-only-one possible $\sigma$ [conjectured to be $= \frac{1}{2}$] value location for all nontrivial zeros of $\zeta(s)$: Solitary $\sigma = \frac{1}{2}$ value that denotes critical line is an element of the infinitely many $\sigma$ values in critical strip $0 < \sigma < 1$. With $s = \sigma \pm it$; DSPL
is derived from sim-η(σ) [via \( \int \sin - \eta(\sigma) d\eta \)], which is derived from η(σ), which is in turn derived from \( \zeta(\sigma) \). \( \zeta(\sigma) \) is a harmonic series which is divergent while η(σ) is an alternating harmonic series which is convergent. Hence, we must instead use proxy η(σ) for \( \zeta(\sigma) \) in the critical strip. Both sim-η(σ) and DSPL contain periodic functions sine or cosine with one solitary \( \sigma \)-valued type of Origin intercept points but infinitely many different \( \sigma \)-valued types of x-axis intercept points and y-axis intercept points whereby at \( \sigma = \frac{1}{2} \) critical line, the [mathematical] nontrivial zeros (Gram[\( x=0,y=0 \)] points), Gram[\( y=0 \)] points and Gram[\( x=0 \)] points correspond to the [geometrical] Origin intercept points, x-axis intercept points and y-axis intercept points. All these mentioned intercept points are known as Zeros at \( \sigma = \frac{1}{2} \) critical line and virtual Zeros at \( \sigma \neq \frac{1}{2} \) non-critical lines in sim-η(σ); and Pseudo-zeros at \( \sigma = \frac{1}{2} \) critical line and virtual Pseudo-zeros at \( \sigma \neq \frac{1}{2} \) non-critical lines in DSPL whereby the later entities from DSPL can be precisely converted into the former entities from sim-η(σ). Setting η(σ) = 0 as true mathematical definition for nontrivial zeros would logically dictate that there be one unique [and not more than one non-unique] \( \sigma \) value for this definition. This is justified by the analytical observation that, unlike sim-η(σ) and DSPL, η(σ) does not contain periodic functions sine or cosine that could intrinsically allow more than one non-unique \( \sigma \) values to be used on mathematical definition for nontrivial zeros. In other words, sim-η(σ) and DSPL that do contain periodic functions sine or cosine intrinsically allow more than one non-unique \( \sigma \) values satisfying sim-η(σ) = 0 or DSPL = 0. This observation could be advocated to represent another (potentially different) mathematical definition for nontrivial zeros. Our overall objective is to rigorously prove in a geometrical and mathematical sense that, apart from solitary unique \( \sigma = \frac{1}{2} \) value, the more than one non-unique \( \sigma \neq \frac{1}{2} \) values that coincidentally comply with sim-η(σ) = 0 or DSPL = 0 equations [but not with \( \eta(\sigma) = 0 \) equation] do not represent the true mathematical definition for nontrivial zeros but instead validly represent the true mathematical definition for (non-existent) virtual nontrivial zeros. Finally, we recognize the \( \eta(\sigma) \neq 0 \) situation arising from substituting \( \sigma \neq \frac{1}{2} \) values will validly represent the true mathematical definition for (non-existent) virtual nontrivial zeros.

We use three main kinds of logical reasoning: (I) In deductive reasoning (top-down logic or mathematical logic), the true conclusion [theorem] without epistemic uncertainty is reduced reductively by applying general rules which hold over entirety of a closed domain of discourse. This action narrows range under consideration until only the true conclusion remains; (II) In inductive reasoning (bottom-up logic or generalization from empirical evidence), the final conclusion with epistemic uncertainty is reached by generalizing or extrapolating from specific cases to general rules; and (III) In abductive reasoning (inference to the best explanation), a selected cogent set of preconditions is often used to develop a conjecture or hypothesis which then can be tested by additional reasoning or data. In other words, given a true conclusion and a rule, abductive reasoning attempts to select some possible premises that, if true also, can non-uniquely support the new conclusion.

Within the context of a mathematical model, these three kinds of logical reasoning can succinctly describe the model: Construction or creation of structure of the model is abduction. Assigning values (or probability distributions) to parameters of the model is induction. Executing or running the model is deduction.

**Derived f(n) = 0 and F(n) = 0 equations** – see complete sets of \( \sigma = \frac{1}{2} \) and \( \frac{\pi}{4} \) examples given in [1], p. 27-28, 29-30 and subsection 1.2 – comply with exact Dimensional analysis (DA) homogeneity at \( \sigma = \frac{1}{2} \) critical line and inexact DA homogeneity at \( \sigma \neq \frac{1}{2} \) non-critical lines. Nontrivial zeros are synonymous with Gram[\( x=0,y=0 \)] points which is one type of Gram points. Together with Gram[\( y=0 \)] points and Gram[\( x=0 \)] points as remaining two types of Gram points, these three types of Gram points are intrinsically incorporated or contained in their complex equations (viz, Complex Containers) as Incompletely Predictable entities whereby their actual location [but not actual positions] are also intrinsically incorporated in their complex equations – see subsection 1.1. We observe Eqs. (1.1), (1.3), (1.5), (1.6), (1.7) and (1.8) that comply with exact DA homogeneity at \( \sigma = \frac{1}{2} \) all have fractional exponents \( \frac{1}{2} \). Eqs. (1.2) and (1.4) that comply with inexact DA homogeneity at \( \sigma = \frac{\pi}{4} \) have fractional exponents \( \frac{1}{2} \) in the former and \( \frac{3}{4} \) in the later that are mixed with fractional exponents \( \frac{3}{2} \).

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{4}} 2^{\frac{1}{2}} \cos(\sqrt{\ln(2n)} + \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{4}} 2^{\frac{1}{2}} \cos(\sqrt{\ln(2n-1)} + \frac{1}{4} \pi) = 0 \tag{1.1}
\]

With exact DA homogeneity, Eq. (1.1) is f(n) sim-η(σ) at \( \sigma = \frac{1}{2} \) that will incorporate all nontrivial zeros [as Zeros]. There is total absence of [non-existent] virtual nontrivial zeros [as virtual Zeros].
With inexact DA homogeneity, Eq. (1.2) is \( f(n) \) sim-\( \eta(s) \) at \( \sigma = \frac{5}{2} \) that will incorporate all [non-existent] virtual nontrivial zeros [as virtual Zeros]. There is total absence of nontrivial zeros [as Zeroes].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1) - \frac{1}{4} \pi) = 0 \tag{1.2}
\]

(1.2)

With exact DA homogeneity, Eq. (1.3) is \( f(n) \) DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all nontrivial zeros [as Pseudo-zeroes to Zeros conversion]. There is total absence of nontrivial zeros [as virtual Zeros].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1) - \frac{1}{4} \pi) + C = 0 \tag{1.3}
\]

(1.3)

With inexact DA homogeneity, Eq. (1.4) is \( F(n) \) DSPL at \( \sigma = \frac{5}{2} \) that will incorporate all nontrivial zeros [as virtual Pseudo-zeroes to virtual Zeros conversion].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1) - \frac{1}{4} \pi) + C = 0 \tag{1.4}
\]

(1.4)

With exact DA homogeneity, Eq. (1.5) can also be equivalently written as

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \sin(t \ln(2n-1)) = 0 \tag{1.5}
\]

(1.5)

With exact DA homogeneity, Eq. (1.6) is \( f(n) \) Gram\( [y=0] \) points-sim-\( \eta(s) \) at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\( [y=0] \) points [as Zeroes]. There is total absence of virtual Gram\( [y=0] \) points [as virtual Zeros].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1) - \frac{1}{4} \pi) + C = 0 \tag{1.6}
\]

(1.6)

With exact DA homogeneity, Eq. (1.7) is \( f(n) \) Gram\( [x=0] \) points-sim-\( \eta(s) \) at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\( [x=0] \) points [as Zeroes]. There is total absence of virtual Gram\( [x=0] \) points [as virtual Zeros].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1)) = 0 \tag{1.7}
\]

(1.7)

With exact DA homogeneity, Eq. (1.8) is \( f(n) \) Gram\( [x=0] \) points-DSPL at \( \sigma = \frac{1}{2} \) that will incorporate all Gram\( [x=0] \) points [as Pseudo-zeroes to Zeros conversion]. There is total absence of virtual Gram\( [x=0] \) points [as virtual Pseudo-zeroes to virtual Zeros conversion].

\[
\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} \cos(t \ln(2n) - \frac{3}{4} \pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} \cos(t \ln(2n-1) - \frac{3}{4} \pi) + C = 0 \tag{1.8}
\]

(1.8)
equations. Using [selective] trigonometric identity for linear combination of sine and cosine function whenever applicable to relevant \( n = 0 \) and \( F(n) = 0 \) equations, we outline exact DA homogeneity at \( \sigma = \frac{1}{2} \) critical line (depicted by Figure 2) for Gram\([x=0,y=0]\) points (nontrivial zeros) as Eq. (1.1), Gram\([y=0]\) points as Eq. (1.5) and Gram\([x=0]\) points as Eq. (1.7). However, \( f(n) = 0 \) equations for Gram\([y=0]\) points as Eq. (1.5) and Gram\([x=0]\) points as Eq. (1.7) with exact DA homogeneity at \( \sigma = \frac{1}{2} \) critical line are not amendable to treatments using trigonometric identity with implication that their corollary situation endowed with inexact DA homogeneity at \( \sigma \neq \frac{1}{2} \) non-critical lines (depicted by Figures 3 and 4) will only manifest solitary [unmixed] \( \neq \frac{1}{2} \) fractional exponents. Here, we will not provide examples of \( f(n) = 0 \) equations for this corollary situation.

**Abbreviations:** CP = Completely Predictable; IP = Incompletely Predictable. We arbitrarily chose single cosine wave with format \( R \cos(n \pm \alpha) \) where \( R \) is the scaled amplitude and \( \alpha \) is the phase shift. We analyze \( f(n) = 0 \) and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for nontrivial zeros situation where \( R = 2^{\frac{1}{2}}(2n)^{-\frac{1}{2}} \) or \( 2^{\frac{1}{2}}(2n - 1)^{-\frac{1}{2}} \) in \( f(n) \)'s Eq. (1.1) and \( R = \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \) in \( F(n) \)'s Eq. (1.3). For \( f(n) \)'s and \( F(n) \)'s equations in relation to nontrivial zeros, Gram\([y=0]\) points and Gram\([x=0]\) points; all their geometrically generated approximate Areas of Varying Loops \( \propto \) precise Areas of Varying Loops.

**Remark 1.** Scaled amplitude \( R \) is validly treated as a proportionality factor. Whereas for nontrivial zeros \( F(n) \) Eq. (1.3) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (1.1) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) in Eq. (1.6) for Gram\([y=0]\) points \( F(n) \) Eq. (1.6) with \( R = \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \) and Gram\([x=0]\) points \( F(n) \) Eq. (1.8) with \( R = \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \), we observe the former \( R \) to be the negative of the later \( R \). However, this observation is context-sensitive because when Eq. (1.6) is written in its equivalent format above, the former \( R \) is identical to the later \( R \). Both \( R \) are now just given by \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \).

**Remark 2.** Whereas for Gram\([y=0]\) points \( F(n) \) Eq. (1.6) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (1.5) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) in Eq. (1.6) for Gram\([y=0]\) points \( F(n) \) Eq. (1.6) with \( R = \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \), we observe the former \( R \) to be the negative of the later \( R \). However, this observation is context-sensitive because when Eq. (1.6) is written in its equivalent format above, the former \( R \) is identical to the later \( R \). Both \( R \) are now just given by \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \).

**Remark 3.** Whereas for Gram\([x=0]\) points \( F(n) \) Eq. (1.8) that exactly represent precise Areas of Varying Loops and \( f(n) \) Eq. (1.7) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude \( R \) in Eq. (1.8) for Gram\([x=0]\) points \( F(n) \) Eq. (1.8) with \( R = \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \), we observe the former \( R \) to be the negative of the later \( R \). However, this observation is context-sensitive because when Eq. (1.6) is written in its equivalent format above, the former \( R \) is identical to the later \( R \). Both \( R \) are now just given by \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n)^{\frac{1}{2}} \) or \( \frac{1}{2^{\frac{1}{2}}(i^2 + \frac{1}{4})^{\frac{1}{2}}} (2n - 1)^{\frac{1}{2}} \).

Finally, we analyze \( f(n) = 0 \) and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for nontrivial zeros situation where phase shift \( \alpha = \frac{1}{4}\pi \) in nontrivial zeros \( f(n) \) Eq. (1.1) and \( -\frac{1}{4}\pi \) in nontrivial zeros \( F(n) \) Eq. (1.3); and \( F(n) = 0 \) equations at \( \sigma = \frac{1}{2} \) critical line for Gram\([y=0]\) points and Gram\([x=0]\) points situations where phase shift \( \alpha = -\frac{1}{4}\pi \) (or \( \frac{3}{4}\pi \) when written in its equivalent format above) in Gram\([y=0]\) points \( F(n) \) Eq. (1.6) and \( -\frac{3}{4}\pi \) in Gram\([x=0]\) points \( F(n) \) Eq. (1.8). Always being \( \frac{1}{2}\pi \) out-of-phase with each other, trigonometric functions
sine and cosine are cofunctions with \( \sin n = \cos \left( \frac{\pi}{2} - n \right) \) or \( \cos n = \sin \left( \frac{\pi}{2} - n \right) \) or \( \sin (n + \frac{\pi}{2}) \),
\[
\frac{d(\sin n)}{dn} = \cos n, \quad \frac{d(\cos n)}{dn} = -\sin n.
\]
\[\int \sin n \cdot dn = -\cos n + C = \sin \left( n - \frac{\pi}{2} \right) + C.\]

Remark 4. \( \int f(n) \, dn = F(n) + C \) where \( F'(n) = f(n) \).

1.1 Completely Predictable and Incompletely Predictable entities

Cardinality of a given set: With increasing size, arbitrary Set \( X \) can be countable finite set (CFS), countable infinite set (CIS) or uncountable infinite set (UIS). Cardinality of Set \( X \), \( |X| \), measures number of elements in Set \( X \). E.g. Set negative Gram\([y=0]\) point has CFS of negative Gram\([y=0]\) point with \( |\text{negative Gram}[y=0]\text{ point}| = 1 \), Set even Prime number has CIS of even Prime number with \( |\text{even Prime number}| = 1 \), Set Natural numbers has CIS of Natural numbers with \( |\text{Natural numbers}| = \aleph_0 \), and Set Real numbers has UIS of Real numbers with \( |\text{Real numbers}| = \mathfrak{c} \) (cardinality of the continuum).

The word “number” [singular noun] or “numbers” [plural noun] in reference to prime and composite numbers, nontrivial zeros and two types of Gram points can interchangeably be replaced with the word “entity” [singular noun] or “entities” [plural noun]. We outline an innovative method to classify certain appropriately chosen equations or algorithms in two ways by using relevant locational properties of its output. This output consist of generated entities either from function-based equations or from algorithms. Our classification sys-
tem is formalized by providing definitive definitions for Completely Predictable (CP) entities obtained from CP equations or algorithms, and Incompletely Predictable (IP) entities obtained from IP equations or algorithms.

CP simple equation or algorithm will generate CP numbers. A generated CP number is locationally defined as a number whose position is independently determined by simple calculations without needing to know related positions of all preceding numbers in neighborhood. IP complex equation or algorithm will generate IP numbers. A generated IP number is locationally defined as a number whose position is independently determined by complex calculations with needing to know related positions of all preceding numbers in neighborhood.

Container is a useful analogical term that metaphorically group CP entities (e.g. even and odd numbers) and IP entities (e.g. nontrivial zeros, prime and composite numbers) to be exclusively located in, respectively, Simple Container and Complex Container.

Simple properties are inferred from a sentence such as “This simple equation or algorithm by itself will intrinsically incorporate actual location [and actual positions] of all CP numbers”. Examples: simple equations $E = (2 \times i)$ and $O = (2 \times i) \cdot 1$ for $i = all$ real numbers $\geq 0$ or $i = all$ integers $\geq 0$ will respectively and intrinsically incorporate or generate CIS of all CP even number $E_i = 0, 2, 4, 6,...$ and CIS of all CP odd numbers $O_i = 1, 3, 5, 7,...$ whereby even number $(n)$ is defined as “Any integer that can be divided exactly by 2 with last digit always being 0, 2, 4, 6 or 8” and odd number $(n)$ is defined as “Any integer that cannot be divided exactly by 2 with last digit always being 1, 3, 5, 7 or 9”. Congruence $n \equiv 0 \pmod{2}$ holds for even $n$ and congruence $n \equiv 1 \pmod{2}$ holds for odd $n$. We note the zeroth even number is given by $E_0 = 0$.

Complex properties, or meta-properties, are inferred from a sentence such as “This complex equation or algorithm by itself will intrinsically incorporate actual location [but not actual positions] of all IP numbers”. Examples: complex algorithms $P_{i+1} = P_{i} + pGap$ and $C_{i+1} = C_{i} + cGap$ for $i = 1, 2, 3,... \infty$ with $P_{1} = 2$ and $C_{1} = 4$ will respectively and intrinsically incorporate CIS of all IP prime number $2, 3, 5, 7,...$ and CIS of all IP composite numbers $4, 6, 8, 9,...$ whereby prime numbers are defined as “All Natural numbers apart from 1 that are evenly divisible by itself and by 1” and composite numbers are defined as “All Natural numbers apart from 1 that are evenly divisible by numbers other than itself and 1”. E.g. via computed Pseudo-zeroes that can be converted to Zeros at $\sigma = \frac{1}{2}$ critical line, complex equations DSPL will intrinsically incorporate CIS of all IP nontrivial zeros [given as $t$ values rounded off to six decimal places]: 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178,... and complex Gram[y=0] points-DSPL will intrinsically incorporate CIS of all IP Gram[y=0] points [given as $t$ values rounded off to six decimal places]: 0, 3.436218, 9.666908, 17.845599, 23.170282, 27.670182,...

1.2 Exact and Inexact Dimensional Analysis Homogeneity for Equations

Respectively for ‘base quantities’ such as length, mass and time; their fundamental SI ‘units of measurement’ meter (m) is defined as distance travelled by light in vacuum for time interval 1/299 792 458 s with speed of light $c$ = 299,792,458 m s$^{-1}$, kilogram (kg) is defined by taking fixed numerical value Planck constant $h$ to be 6.626 070 15 X $10^{-34}$ Joules-second (Js) [whereby Js is equal to kg m$^2$ s$^{-1}$] and second (s) is defined in terms of $\Delta VCs$ = $\Delta (^{133}\text{Cs})_{101}$s = 9,192,631,770 s$^{-1}$. Derived SI units such as $J$ and ms$^{-1}$ respectively represent ‘base quantities’ energy and velocity. The word ‘dimension’ is commonly used to indicate all those mentioned ‘units of measurement’ in well-defined equations.

Dimensional analysis (DA) is traditionally an analytic tool with DA homogeneity and non-homogeneity (respectively) denoting valid and invalid equation occurring when ‘units of measurements’ for ‘base quantities’ are “balanced” and “unbalanced” across both sides of the equation. E.g. equation $2m + 3kg = 5m$ is valid but equation $2m + 3kg = 5mkg$ is invalid (respectively) manifesting DA homogeneity and non-homogeneity.

We conveniently adopt concepts from DA with this justifiable action being mathematically correct and valid. Let $(2n)$ and $(2n-1)$ be ‘base quantities’ in DSPL formatted in simplest forms as equations. E.g. DA on exponent $\frac{1}{2}$ in $(2n)^{\frac{1}{2}}$ when depicted in simplest form is desirable for our purpose but DA on exponent $\frac{1}{2}$ in equivalent $(2^n)^{\frac{1}{2}}$ not depicted in simplest form is undesirable for our purpose. Fractional exponents as ‘units of measurement’ given by $(1 - \sigma)$ for equations when $\sigma = \frac{1}{2}$ coincide with exact DA homogeneity; and $(1 - \sigma)$ for equations when $\sigma = 1$ coincide with inexact DA homogeneity. Respectively, exact DA homogeneity at
σ = 1 2 denotes Σ(all fractional exponents) as 2(1 − σ) equates to [exact] integer 1; and inexact DA homogeneity at σ ̸= 1 2 denotes Σ(all fractional exponents) as 2(1 − σ) equates to [inexact] fractional number ̸= 1 [Range: 0 < 2(1 − σ) < 1 and 1 < 2(1 − σ) < 2]. Computations based on exact and inexact DA homogeneity in DSPL will explicitly give rise to corresponding σ = 1 2 critical line Gram points (given as Pseudo-zeroes t-values which can be converted to Zeroes t-values) and σ ̸= 1 2 non-critical lines virtual Gram points (given as virtual Pseudo-zeroes t-values which can be converted to virtual Zeroes t-values).

Note: For calculations involving 2(1 − σ) above or 2(−σ) below, it is inconsequential whether σ values in these fractional exponents are depicted in simplest form or not in simplest form. Performing exact and inexact DA homogeneity on sim-η(s) is equally valid. Again, (2n) and (2n−1) are ‘base quantities’. Fractional exponents as ‘units of measurement’ are now given by (−σ). Respectively, exact DA homogeneity at σ = 1 2 denotes Σ(all fractional exponents) as 2(−σ) equates to [exact] integer −1; and inexact DA homogeneity at σ ̸= 1 2 denotes Σ(all fractional exponents) as 2(−σ) equates to [inexact] fractional number ̸= −1 [Range: −2 < 2(−σ) < −1 and −1 < 2(−σ) < 0]. Computations using sim-η(s) [when interpreted as Riemann sum] will explicitly give rise to corresponding σ = 1 2 critical line Gram points (given as Zeroes t-values) while representing exact DA homogeneity and σ ̸= 1 2 non-critical lines virtual Gram points (given as virtual Zeroes t-values) while representing inexact DA homogeneity.

1.3 Overall Summary

The Overall Summary here literally represents a useful symbolic and executive summary of this paper.

Legend: DA = Dimensional analysis; NTZ = nontrivial zeros; Three types of Gram points = Gram [x=0,y=0] points (or NTZ) as Origin intercept points + Gram [y=0] points (‘usual’ Gram points) as x-axis intercept points + Gram [x=0] points as y-axis intercept points; Two types of virtual Gram points = virtual Gram [y=0] points as x-axis intercept points + virtual Gram [x=0] points as y-axis intercept points (where mathematical virtual Gram [x=0,y=0] points or virtual NTZ do not exist as valid geometric Origin intercept points); f(n)’s Zeros = Three types of Gram points; f(n)’s virtual Zeros = Two types of virtual Gram points; F(n)’s Pseudo-zeroes = Three types of Pseudo-Gram points; F(n)’s virtual Pseudo-zeroes = Two types of virtual Pseudo-Gram points.

Implications: Sets of f(n)’s Gram points as f(n)’s Zeros and F(n)’s Pseudo-Gram points as F(n)’s Pseudo-zeroes occur when σ = 1 2; and sets of f(n)’s virtual Gram points as f(n)’s virtual Zeros and F(n)’s virtual Pseudo-Gram points as F(n)’s virtual Pseudo-zeroes occur when σ ̸= 1 2. All these mentioned sets which are constituted by their corresponding subsets defined as Origin intercept points, x-axis intercept points and y-axis intercept points are classified as having IP entities. Generated by corresponding functions or laws, we note these IP entities [when σ = 1 2] and IP virtual entities [when σ ̸= 1 2] are mathematically defined as equations with relevant functions = 0 or relevant laws = 0; and geometrically defined as relevant Origin intercept points, x-axis intercept points and y-axis intercept points.

The f(n) η(s) acts as proxy function for f(n) ζ(s). The former [and not the later] must be analytically used in 0 < σ < 1 critical strip that contains σ = 1 2 critical line because the later only converges when σ > 1 [viz, unlike the former, the later does not converge in the critical strip]. With Euler formula application, f(n) η(s) give rise to f(n) sim-η(s). Antiderivative F(n) DSPL is solution to integration of f(n) sim-η(s). Named after famous German mathematician Bernhard Riemann (September 17, 1826 - July 20, 1866), original 1859-dated Riemann hypothesis propose all NTZ [classified as one type of Gram points] in ζ(s) to be located on σ = 1 2 critical line. We supply the following definitive Lemma, Proposition, Corollary and Theorem that provide rigorous proof for Riemann hypothesis [conjecture involving NTZ as Origin intercept points at σ = 1 2 critical line] and precise explanation for closely related two types of Gram points [which are explained as x-axis intercept points and y-axis intercept points at σ = 1 2 critical line].

Lemma on Gauss Areas of Varying Loops. Conforming to the solitary σ = 1 2 critical line [and not the infinitely many σ ̸= 1 2 non-critical lines e.g. σ = 1 3 or 3 2] whereby σ forms part of relevant fractional exponents from base quantities (2n) and (2n−1) in sim-η(s) [as Riemann sum with variable n classically involving all integers ≥ 1] or DSPL [as definite integral with variable n classically involving all real numbers ≥ 1]; square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when
applied to combined base quantities \((2n)\) and \((2n-1)\) in sim-\(\eta(s)\) or DSPL will generate the maximum number of integer solutions (constituted by all integers \(\geq 1\)) that uniquely comply with Principle of Maximum Density for Integer Number Solutions while also manifesting Principle of Equidistant for Multiplicative Inverse.

**Proof.** \(\int \text{sim-}\eta(s)\ dn = \text{DSPL}\). Whereas the two subsets of rational roots as integers and irrational roots as irrational numbers can be generated by combined base quantities \((2n)\) and \((2n-1)\) from sim-\(\eta(s)\) [with variable \(n\) involving all integers \(\geq 1\)], so must these two exact same subsets be generated by combined base quantities \((2n)\) and \((2n-1)\) from DSPL [with variable \(n\) involving all real numbers \(\geq 1\)]. Thus in sim-\(\eta(s)\) or DSPL, its computed CIS rational roots (subset) as integers [rational numbers] + computed CIS irrational roots (subset) as irrational numbers = computed CIS total roots. These two mutually exclusive subsets belong to UIS real numbers. Using subset rational roots as integers at \(\sigma = \frac{1}{2}\) critical line, and by comparing and contrasting this subset with [different] subset rational roots as integers at \(\frac{1}{2} \sigma = \frac{1}{2}\) or \(\frac{3}{2}\) non-critical lines corollary situation; we will show that square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when applied to combined base quantities \((2n)\) and \((2n-1)\) from sim-\(\eta(s)\) or DSPL, giving rise to maximum number of integer solutions (constituted by all integers \(\geq 1\)) must uniquely comply with Principle of Maximum Density for Integer Number Solutions (see Remark 7 from subsection 1.4) while also manifesting Principle of Equidistant for Multiplicative Inverse (see Remark 11 from subsection 1.5). We also apply concepts from elegant Gauss Circle Problem and Primitive Circle Problem into our supporting materials on this aptly-named Gauss Areas of Varying Loops to justifiably obtain correct and complete set of mathematical arguments that fully support this Lemma. *The proof is now complete for Lemma on Gauss Areas of Varying Loops.*

**Proposition on Gram Points Location.** Rigidly complying with exact DA homogeneity, sim-\(\eta(s)\) can uniquely incorporate three types of Gram points onto \(\sigma = \frac{1}{2}\) critical line.

**Proof.** \(f(n)\ \eta(s)\) is derived from \(f(n)\ \zeta(s)\). \(f(n)\ \eta(s)\) give rise to \(f(n)\ \text{sim-}\eta(s)\) with Euler formula application. Conveniently defined term of exact DA homogeneity denote [exact] integer \(\neg 1\) derived from \(\sigma\) (all fractional exponents) = \(2(\neg \sigma)\) act as surrogate marker in sim-\(\eta(s)\) on \(\sigma = \frac{1}{2}\) situation. Generated by function sim-\(\eta(s)\), we note these three types of Gram points [when \(\sigma = \frac{1}{2}\)] are mathematically defined as equations sim-\(\eta(s)\) = 0, Gram[y=0] points-sim-\(\eta(s)\) = 0 and Gram[x=0] points-sim-\(\eta(s)\) = 0; and geometrically defined as relevant Origin intercept points, x-axis intercept points and y-axis intercept points. Thus, the three types of IP Gram points [IP Zeroes] are mathematically and geometrically defined to be located at \(\sigma = \frac{1}{2}\) critical line. Based solely on this definitive definition, we can uniquely incorporate the three types of IP Gram points [IP Zeroes] onto \(\sigma = \frac{1}{2}\) critical line of sim-\(\eta(s)\). *The proof is now complete for Proposition on Gram Points Location.*

**Corollary on Gram Points Location.** Rigidly complying with inexact DA homogeneity, sim-\(\eta(s)\) cannot incorporate three types of Gram points onto \(\sigma \neq \frac{1}{2}\) non-critical lines.

**Proof.** We again note the relationship between \(f(n)\ \zeta(s)\), \(f(n)\ \eta(s)\) and \(f(n)\ \text{sim-}\eta(s)\). Conveniently defined term of inexact DA homogeneity denote [inexact] fractional (non-integer) number \(\neg 1\) derived from \(\Sigma\) (all fractional exponents) = \(2(\neg \sigma)\) act as surrogate marker in sim-\(\eta(s)\) on \(\sigma \neq \frac{1}{2}\) situations. Generated by function sim-\(\eta(s)\), we note these virtual entities [when \(\sigma \neq \frac{1}{2}\)] are mathematically defined as equations sim-\(\eta(s)\) = 0, Gram[y=0] points-sim-\(\eta(s)\) = 0 and Gram[x=0] points-sim-\(\eta(s)\) = 0; and geometrically defined as relevant (non-existant) Origin intercept points, x-axis intercept points and y-axis intercept points. Thus, the two types of IP virtual Gram points [IP Virtual Zeroes] are mathematically and geometrically defined to be located on \(\sigma \neq \frac{1}{2}\) non-critical lines. Based solely on this definitive definition, we can uniquely incorporate the two types of IP virtual Gram points [IP Virtual Zeroes] onto \(\sigma \neq \frac{1}{2}\) non-critical lines of sim-\(\eta(s)\). We additionally apply inclusion-exclusion principle [which will logically and explicitly prove this Corollary as stated]: Since exclusive presence of Gram points and absence of virtual Gram points on critical line denotes exclusive absence of Gram points and exclusive presence of virtual Gram points on non-critical lines; then Gram points and virtual Gram points as mutually exclusive entities must mathematically and geometrically be incorporated, respectively, onto unique (solitary) critical line and non-unique (infinitely many) non-critical lines of sim-\(\eta(s)\). *The proof is now complete for Corollary on Gram Points Location.*

**Theorem Grand Riemann.** Rigidly complying with exact DA homogeneity, NTZ being validly classified as one type of Gram points existing at \(\sigma = \frac{1}{2}\) critical line will mathematically and geometrically confirm the 1859-dated Riemann hypothesis as originally stated using \(f(n)\ \zeta(s)\) [and with additionally also involving \(f(n)\) DSPL] to be rigorously true and provide precise explanation for two types of Gram points.
Proof. As already alluded to earlier, $\eta(s)$ must mathematically act as proxy function for $\zeta(s)$ in $0 < \sigma < 1$ critical strip which contains $\sigma = \frac{1}{2}$ critical line; sim-$\eta(s)$ is obtained by applying Euler formula to $\eta(s)$; and $\int \text{sim-} \eta(s) dn = \text{DSPL}$. Respectively, the conveniently defined terms of exact and inexact DA homogeneity in DSPL denote [exact] integer 1 for Pseudo-Gram points and [inexact] fractional (non-integer) number $\neq 1$ for virtual Pseudo-Gram points. These exact and inexact DA homogeneity given by $\sum(\text{all fractional exponents}) = 2(1 - \sigma)$ act as surrogate markers in DSPL on corresponding $\sigma = \frac{1}{2}$ and $\sigma \neq \frac{1}{2}$ situations with Pseudo-Gram points and virtual Pseudo-Gram points both computed from DSPL as equations with DSPL = 0, Gram[y=0] points-DSPL = 0 and Gram[x=0] points-DSPL = 0. NTZ or Gram[x=0,y=0] points as one type of Gram points, and Gram[y=0] points and Gram[x=0] points as remaining two types of Gram points, are entities which were already mathematically and geometrically defined (and explained) to be located at $\sigma = \frac{1}{2}$ critical line of sim-$\eta(s)$ [existing as IP Zeros] and DSPL [existing as IP Pseudo-zeros which can be converted to IP Zeros]. Both x-axis and y-axis as two-dimensional lines are constituted by solitary type of $\sigma = \frac{1}{2}$ geometrical x-axis intercept points (existing as Gram[x=0] points) plus infinitely many different types of $\sigma \neq \frac{1}{2}$ geometrical x-axis intercept points (existing as virtual Gram[y=0] points); and solitary type of $\sigma = \frac{1}{2}$ geometrical y-axis intercept points (existing as Gram[y=0] points) plus infinitely many different types of $\sigma \neq \frac{1}{2}$ geometrical y-axis intercept points (existing as virtual Gram[x=0] points). But Origin as solitary one-dimensional point can only be constituted by the one-and-only-one type of $\sigma = \frac{1}{2}$ geometrical Origin intercept points existing as already incorporated $\sigma = \frac{1}{2}$ critical line’s NTZ. Since NTZ and virtual NTZ must be mutually exclusive entities, and mathematical NTZ must correspond to geometrical Origin intercept points; then NTZ cannot exist at $\sigma \neq \frac{1}{2}$ non-critical lines and the (perceived) infinitely many $\sigma \neq \frac{1}{2}$ non-critical lines’ [non-existent] virtual NTZ cannot represent extra incorporated entities as geometrical Origin intercept points. Having now involved DSPL together with this one-and-only-one-mathematical-possibility option at Origin, our correct and complete mathematical arguments [that also incorporate Proposition on Gram Points Location and Corollary on Gram Points Location] will fully support Riemann hypothesis to be true and provide precise explanation for two types of Gram points. $\eta(s)$ is an alternating harmonic series that is convergent. We recognize $\eta(s) = 0$ arising from substituting $\sigma = \frac{1}{2}$ values will validly represent true mathematical definition for nontrivial zeros while the corollary $\eta(s) \neq 0$ arising from substituting $\sigma \neq \frac{1}{2}$ values will validly represent true mathematical definition for (non-existent) virtual nontrivial zeros. The proof is now complete for Theorem Grand Riemann$\zeta$. 

Rigorous proof for Riemann hypothesis depicted as simultaneously satisfying two mutually inclusive conditions at [unique] solitary $\sigma = \frac{1}{2}$ critical line and [non-unique] infinitely many non-critical lines $\sigma \neq \frac{1}{2}$: 

Condition I. With rigid manifestation of exact DA homogeneity, Set NTZ obtained as Pseudo-zeros which are converted to Zeros with $\text{NTZ} = \{0\}$ is located on $\sigma = \frac{1}{2}$ critical line when $2(1 - \sigma)$ as $\sum(\text{all fractional exponents}) = \text{integer 1 in DSPL}$. Rigid manifestation of exact DA homogeneity will also occur at critical line location for Set NTZ obtained as Zeros with $2(-\sigma)$ as $\sum(\text{all fractional exponents}) = \text{integer - 1 in sim-$\eta(s)$}$. 

Condition II (which is unavoidably expressed with ambivalence). With rigid manifestation of inexact DA homogeneity, imaginary [non-existent] Set NTZ obtained as [non-existent] virtual Pseudo-zeros which are converted to [non-existent] virtual Zeros with imaginary [non-existent] $\text{NTZ} = \{0\}$ is not located on $\sigma \neq \frac{1}{2}$ non-critical lines when $2(1 - \sigma)$ as $\sum(\text{all fractional exponents}) = \text{fractional number $\neq 1$ in DSPL}$. Rigid manifestation of inexact DA homogeneity will also occur at non-critical lines location for imaginary [non-existent] Set NTZ obtained as [non-existent] virtual Zeros with $2(-\sigma)$ as $\sum(\text{all fractional exponents}) = \text{fractional number $\neq -1$ in sim-$\eta(s)$}$. 

Footnote 1: $\int \text{sim-} \eta(s) dn = \text{DSPL}$ gives rise to $\sigma = \frac{1}{2}$ Pseudo-zeros and $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeros [which can be converted to corresponding Zeros and virtual Zeros]. DSPL symbolizes the end-product proof on Riemann hypothesis. We crucially observe the [solitary] one-dimensional point Origin to be uniquely, serendipitously and intrinsically associated with the $\sigma = \frac{1}{2}$ critical line [and not the infinitely many $\sigma \neq \frac{1}{2}$ non-critical lines]. Existing as mutually exclusive entities and with both defined to be located at $\sigma \neq \frac{1}{2}$ non-critical lines; the [directly obtained] sim-$\eta(s)$’s virtual Zeros and also the [indirectly obtained] DSPL’s virtual Pseudo-zeros [which can be converted to virtual Zeros] cannot exist as valid geometrical entities at Origin. Therefore, the perceived (non-existent) virtual NTZ [as virtual Zeros] cannot exist as valid geometrical entities at Origin. Using logical deduction, only NTZ [as Zeros] can exist as valid geometrical entities at Origin.
1.4 Gauss Areas of Varying Loops and incorporating Principle of Maximum Density for Integer Number Solutions

Legend: C = UIS complex numbers, R = UIS real numbers, Q = CIS rational numbers that include fractional numbers and rational roots, R-Q = UIS total irrational numbers, A = CIS algebraic numbers, R-A = UIS transcendental irrational numbers, Z = CIS integers which are literally fractional numbers with denominator 1, W = CIS whole numbers, N = CIS natural numbers, E = CIS even numbers, O = CIS odd numbers, P = CIS prime numbers, and C = CIS composite numbers. CIS N = Set E [whereby we did not include the zeroth even number E₀ = 0] + Set O; CIS N = CIS P + CIS C + CFS Number 1; and CIS N ⊂ CIS W ⊂ CIS Z ⊂ CIS Q ⊂ UIS R ⊂ UIS C. CIS A as C (including R) = CIS Q that include fractional numbers and rational roots + CIS irrational roots whereby both rational and irrational roots are derived from non-zero polynomials.

The following refined definitions are useful: UIS total irrational numbers = CIS irrational roots (numbers) + UIS transcendental irrational numbers whereby transcendental irrational numbers >> irrational numbers. Whereas CIS rational roots (numbers), CIS irrational roots (numbers) and UIS transcendental numbers are treated separately as mutually exclusive numbers; so must the existing algebraic functions that generate CIS rational roots (numbers) and CIS irrational roots (numbers), and the existing transcendental functions that generate UIS transcendental numbers be treated separately as mutually exclusive functions.

An algebraic function [such as rational functions, square root, cube root function, etc] satisfies a polynomial equation. A transcendental function [such as exponential function, natural logarithm, trigonometric functions, hyperbolic functions, gamma, elliptic, zeta functions, etc] is an analytic function that does not satisfy a polynomial equation. Thus, a transcendental function “transcends” algebra since it cannot be expressed in terms of a finite sequence of algebraic operations consisting of addition, subtraction, multiplication, division, powers, and fractional powers or root extraction. All integers, rational numbers, rational or irrational roots of real and complex numbers are algebraic numbers e.g. a root of polynomial xⁿ + 1 = 0 is algebraic.

An interesting property of irrational number \( \sqrt{2} \) is \( \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 \). This is related to the property of silver ratios. \( \sqrt{2} \) can also be expressed in terms of copies of imaginary unit i using only square root and arithmetic operations, if the square root symbol is interpreted suitably for complex numbers i and i: \( \sqrt{i} + i\sqrt{i} \) and \( \sqrt{-i} - i\sqrt{-i} \). Multiplicative inverse (reciprocal) of \( 2\sqrt{2} \) or \( \sqrt{2} \) is \( 2^{-\frac{1}{2}} \) or \( \frac{1}{\sqrt{2}} \) which is a unique [irrational number] constant since \( \sqrt{2} = \frac{\sqrt{2}}{2} = \sqrt{\frac{\sqrt{2}}{\sqrt{2}}} = \cos \frac{\pi}{4} = \sin \frac{\pi}{4} \). Transcendental numbers such as \( \frac{\pi}{4} \) (given by Leibniz series \( \frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \cdots \approx 0.78539816 \)) and \( \frac{\pi^2}{6} \) (given by \( \zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \approx 1.64493406684822643647 \)), respectively, encode complete set of alternating odd and, by default, alternating even numbers; and natural numbers. Also known as alternating zeta function, Dirichlet eta function \( \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \) when expanded, will intrinsically encode complete set of alternating natural numbers e.g. \( \eta(1) = \ln(2) \) (given by \( \sum_{n=1}^{\infty} \frac{-1^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{1}{2n} [\zeta(n) - 1] + \frac{1}{2} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \approx 0.69314718056 \). Equivalent Euler product formula for \( \zeta(s) \) with product over prime numbers [instead of summation over natural numbers] will intrinsically encode complete set of prime and, by default, alternating
nating composite numbers is encoded in converging alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^k}{p_k} \approx -0.2696063519 \) (transcendental number) when fully expanded whereby \( p_k \) is \( k \)th prime number.

Equation of a circle centered at Origin with radius \( r \) and precise Area is \( \pi r^2 \) is given in Cartesian coordinates as \( x^2+y^2=r^2 \). The number of integer lattice points \( N(r) \) on and inside a circle [viz, pairs of integers \((m,n)\) such that \( m^2+n^2 \leq r^2 \)] can be exactly determined by following two equations whereby \( N(r) \) is considered the most accurate surrogate marker of approximate Area for a given circle. Named after German mathematician Carl Friedrich Gauss (April 30, 1777 - February 23, 1855), Gauss Circle Problem is the problem of determining how many integer lattice points as approximate Area for a given circle. For \( i \) and \( r = 0, 1, 2, 3, \ldots \), in terms of a sum involving the floor function, \( N(r) \) can be expressed as equation \( N(r) = 1 + 4 \sum_{i=0}^{\infty} \left( \left\lfloor \frac{r^2}{4i+1} \right\rfloor - \left\lfloor \frac{r^2}{4i+3} \right\rfloor \right) \)

whereby it is a consequence of Jacobi’s two-square theorem which follows almost immediately from the Jacobi triple product. A much simpler sum appears if sum of squares function \( r_2(n) \) is defined as number of ways of writing number \( n \) as sum of two squares is used. Then, we have alternative equation \( N(r) = \sum_{n=0}^{r^2} r_2(n) \). The first few \( N(r) \) values for \( r = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots \) are 1, 5, 13, 29, 49, 81, 113, 149, \ldots whereby these are IP entities complying with relationship: [simple] equation for precise Area of circle = \( \pi r^2 \) is proportional to above two most accurate and equivalent [complex] equations for approximate Area of circle = \( N(r) \).

The identity \( N(x) - \frac{r_2(x^2)}{2} = \pi x^2 + \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} \) is implicitly related to number of integer lattice points, \( N(r) \), where \( J_1(2\pi x \sqrt{n}) \) is first discovered by English mathematician Godfrey Harold Hardy (February 7, 1877 - December 1, 1947)[2].

\[
N(r) \text{ is closely connected with Leibniz series since } \frac{1}{4} \left[ \frac{N(r)}{r^2} - \frac{1}{r^2} \right] = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} \ldots \\
\frac{1}{r} \pm E(r) = \frac{1}{4} [\pi + 2 \Phi(-1, 1, 1, r)] \pm \frac{E(r)}{r} = \frac{1}{4} [\pi + \psi(1)(3 + 2r)] - \psi(1)(1 + 2r)] \pm \frac{E(r)}{r}, \text{ where } E(r) \text{ is an error term, } \Phi(z,s,a) \text{ is a Lerch transcendent and } \psi(x) \text{ is a digamma function, so taking the limit } r \to \infty \text{ gives } \frac{\pi}{4} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots. \text{ Gauss showed } N(r) = \pi r^2 + E(r), \text{ where } |E(r)| < 2 \sqrt{2} \pi r.
\]

Primitive Circle Problem as least accurate surrogate marker of approximate Area for a given circle involves calculating the number of coprime integer solutions \((m,n)\) to the inequality \( m^2 + n^2 \leq r^2 \). If the number of such solutions is denoted \( V(r) \) then the values of \( V(r) \) for \( r \) taking small integer values are 0, 4, 8, 16, 32, 48, 72, 88, 120, 152, 192, \ldots. Using the same ideas as usual Gauss Circle Problem and the fact that probability two integers are coprime is \( \frac{6}{\pi^2} \), it is relatively straightforward to show \( V(r) = \frac{6}{\pi^2} + O(r^{1+\epsilon}) \). We solve problematic part of Primitive Circle Problem by reducing the exponent in the error term. This exponent is presently best known to be \( 221 / 304 + \epsilon \) when we validly assume Riemann hypothesis to be true in this paper.

**Remark 6.** Let \( A \) denote Area of a given circle with radius \( r \). The computed precise \( A \) using \( A = \pi r^2 \) method, computed approximate \( A \) using [most accurate] approximate \( N(r) \) method of Gauss Circle Problem and computed approximate \( A \) using [least accurate] approximate \( A(r) \) method of Primitive Circle Problem will explicitly confirm \( A \propto r^2 \) for all three methods.

We translate above concepts from Gauss Circle Problem and Primitive Circle Problem to solve Gauss Areas of Varying Loops. This exercise will fully support the following Remarks in this subsection. For \( n = \) all integers \( \geq 1 \) in sim-\( \eta(s) \) or \( n = \) all real numbers \( \geq 1 \) in DSPL; their base quantities (2n) and (2n-1) will, respectively, generate CIS even numbers commencing from 2 and CIS odd numbers commencing from 1. These base quantities are subjected to algebraic function square roots at \( \sigma = \frac{1}{4} \) critical line [when \( \sigma = \frac{1}{4} \)] and cube roots or twice cube roots at \( \sigma = \frac{1}{5} \) or \( \sigma = \frac{1}{7} \) non-critical lines [when \( \sigma \neq \frac{1}{4} \)] thus giving rise to corresponding subset of rational roots and subset of irrational roots. Relevant to Remark 7, we now concentrate on the combined (2n)’s and (2n-1)’s obtained integer lattice points \( \geq 1 \) in deriving the solitary subset of rational roots for the \( n = 1 \) to 100 range in sim-\( \eta(s) \) or DSPL when:

1. \( \sigma = \frac{1}{2} \) viz, involving a **neither even nor odd function** with no symmetry by applying fractional exponent
or square root on ten perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 giving rise to (maximum) ten rational roots as consecutive integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

(II) $\sigma = \frac{1}{2}$ viz, involving a odd function with Origin symmetry by applying fractional exponent $\frac{1}{2}$ or cube root on four perfect cubes 1, 8, 27, 64 giving rise to (non-maximum) four rational roots as consecutive integer solutions 1, 2, 3, 4.

(III) $\sigma = \frac{3}{4}$ viz, involving an even function with y-axis symmetry by applying fractional exponent $\frac{3}{4}$ or squared cube root on four perfect cubes 1, 8, 27, 64 giving rise to (non-maximum) four rational roots as non-consecutive integer solutions 1, 4, 9, 16.

**Remark 7.** Only at $\sigma = \frac{1}{2}$ critical line which involves applying fractional exponent $\frac{1}{2}$ or square root to all perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100... give rise to maximum number of rational roots as consecutive integer solutions 1, 2, 3, 4, 5, 6, 7, 8, 9, 10... (viz, all integers $\geq 1$). This observation uniquely result in compliance with Principle of Maximum Density for Integer Number Solutions at $\sigma = \frac{1}{2}$ critical line.

The sim-$\eta(s)$ or DSPL is classified as complex function or law with single variable n and parameters $\sigma$, t. All their derived equations [Eqs. (1.1) to (1.8)] have (2n)- or (2n-1)-complex term with algebraic functions consisting of powers, fractional powers, root extraction (equating to scaled amplitude R in Remarks 1, 2 and 3) [that are dependent on parameter $\sigma$] and (2n)- or (2n-1)-complex term with transcendental functions consisting of sine, cosine, single cosine wave, single sine wave, natural logarithm [that are independent of parameter $\sigma$].

**Remark 8.** In sim-$\eta(s)$ or DSPL, abbreviated Term-(2n) = (2n)-complex term with algebraic functions X (2n)-complex term with transcendental functions; and abbreviated Term-(2n-1) = (2n-1)-complex term with algebraic functions X (2n-1)-complex term with transcendental functions. Corresponding to Areas of Varying Loops = 0, Term-(2n) must precisely cancel Term-(2n-1) in order to obtain $\sigma = \frac{1}{2}$ f(n)'s Zeroes and F(n)'s Pseudo-zeroes or to obtain $\sigma \neq \frac{1}{2}$ f(n)'s virtual Zeroes and F(n)'s virtual Pseudo-zeroes.

**Remark 9.** The complex function $F(n) = $ [representive of precise Area under the Curve] will generate the most accurate precise Areas of Varying Loops [when all rational and irrational roots from combined base quantities (2n) and (2n-1) are utilized] and the least accurate precise Areas of Varying Loops [when only rational roots from combined base quantities (2n) and (2n-1) are utilized]. The complex function $f(n) = $ sim-$\eta(s)$ when interpreted as Riemann sum [representive of approximate Area under the Curve] will generate the most accurate approximate Areas of Varying Loops [when all rational and irrational roots from combined base quantities (2n) and (2n-1) are utilized] and the least accurate approximate Areas of Varying Loops [when only rational roots from combined base quantities (2n) and (2n-1) are utilized].

**Remark 10.** Our [metaphoric] varying radius $r$ in sim-$\eta(s)$ or DSPL is defined as Term-(2n) – Term-(2n-1) whereby perpetually recurring $r = 0$ will correspond to Areas of Varying Loops = 0 in order to obtain $\sigma = \frac{1}{2}$ f(n)'s Zeroes and F(n)'s Pseudo-zeroes or to obtain $\sigma \neq \frac{1}{2}$ f(n)'s virtual Zeroes and F(n)'s virtual Pseudo-zeroes. In sim-$\eta(s)$ or DSPL, its computed CIS rational roots (subset) as integers [rational numbers] + computed CIS irrational roots (subset) as irrational numbers = computed CIS total roots. Whether involving the most accurate method using total roots or the least accurate method using rational roots to determine DSPL's precise or sim-$\eta(s)$'s approximate Areas of Varying Loops, we explicitly conclude all these obtained Areas of Varying Loops $\propto$ varying radius $r$ will repeat in a perpetually dynamic, cyclical and Incompletely Predictable manner.

1.5 Simple observation on Shift of Varying Loops in $\zeta(\sigma + it)$ Polar Graph and incorporating Principle of Equidistant for Multiplicative Inverse

For single-term trigonometric function $f(n) = \sin(n)$, it is an odd function with Origin symmetry since $-f(n) = f(-n)$ for all n. The $f(n) = \sin(n)$ has an infinite number of Completely Predictable (CP) x-axis intercept points (Zeroes) and a solitary unique Origin intercept point (Zero) since it belongs to a class of odd functions that is defined at n = 0 and must pass through the Origin. Otherwise, the other class of odd functions such as $f(n) = \sin(\frac{n}{2})$ with infinite number of CP x-axis intercept points (Zeroes) but without Origin intercept point [since $\sin(\frac{n}{2})$ is undefined at n = 0] can remain symmetrical about the Origin without actually passing through it.

For single term trigonometric function $f(n) = \cos(n)$ with symmetry about the y-axis, it is an even function
since f(n) = f(−n) for all n. It has an infinite number of CP x-axis intercept points (Zeroes). Being undefined at n = 0, it will never have Origin intercept point.

For dual terms trigonometric functions f(n) = cos(n) - sin(n) and f(n) = cos(n) + sin(n), they are neither even nor odd without any symmetry. They both have an infinite number of CP x-axis intercept points (Zeroes). Being undefined at n = 0, they will never have Origin intercept point.

Special properties are given below for Addition and Multiplication.

The sum or difference of two even functions is even. The sum or difference of two odd functions is odd.

The sum or difference of an even and odd function is neither even nor odd unless one function is zero; viz, there is (exactly) one function that is both even and odd, and it is the zero function f(n) = 0.

The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even function and an odd function is an odd function.

**Trigonometric identity for linear combination of sine and cosine function:** Here, we use the notation f(x) instead of f(n). The trigonometric identity for linear combination of sine and cosine function \(ac\cos(x) + b\sin(x)\) can be freely, arbitrarily and interchangeably written as either [simple] single cosine wave \(R\cos(x - \alpha)\) or [simple] single sine wave \(R\sin(x + \alpha)\) whereby \(R\) is the scaled amplitude and \(\alpha\) is the phase shift. \(R = \sqrt{a^2 + b^2} = (a^2 + b^2)^{\frac{1}{2}}\). Since \(\sin(\alpha) = \frac{b}{\sqrt{a^2 + b^2}}\) and \(\cos(\alpha) = \frac{a}{\sqrt{a^2 + b^2}}\), then \(\alpha = \tan^{-1}\frac{b}{a}\). Below, we conveniently assign \(\sqrt{2}\) to equivalently denote \(2^\frac{1}{2}\).

With \(a = 1, b = -1, R = \sqrt{2}; \cos(x) - \sin(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = 2\sin\left(x + \frac{3\pi}{4}\right)\).

With \(a = -1, b = 1, R = \sqrt{2}; -\cos(x) + \sin(x) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 2\cos\left(x - \frac{3\pi}{4}\right)\).

With \(a = 1, b = 1, R = \sqrt{2}; \cos(x) + \sin(x) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = 2\sin\left(x + \frac{3\pi}{4}\right)\).

With \(a = -1, b = -1, R = \sqrt{2}; -\cos(x) - \sin(x) = \sqrt{2}\cos\left(x + \frac{3\pi}{4}\right) = 2\sin\left(x - \frac{3\pi}{4}\right)\).

\[\int f(x)dx = F(x) + C\] with \(F'(x) = f(x)\). With \(|a| = 1\) and \(|b| = 1\), consider single-term [simple] trigonometric functions: \(f(x) = a\cos(x)\) which belongs to an even function and \(f(x) = b\sin(x)\) which belongs to an odd function. Whereas all linear combination of [simple] \(\cos(x)\) and [simple] \(\sin(x)\) as sum or difference such as \(f(x) = \cos(x) + \sin(x)\) and \(f(x) = \cos(x) - \sin(x)\) belong to neither even nor odd functions, then their corresponding \(F(x)\) being linear combination of [simple] \(\cos(x)\) and [simple] \(\sin(x)\) as sum or difference must also belong to neither even nor odd functions. With both \(f(x)\) and \(F(x)\) considered as [simple] functions and relevant trigonometric identities being applied, they can intrinsically and arbitrarily be expressed as either [simple] single cosine wave or [simple] single sine wave containing a phase shift \(\pm \frac{\pi}{4}\) or \(\pm \frac{3\pi}{4}\) and a scaled amplitude \(\sqrt{2} \approx 2^\frac{1}{2}\) which is base 2 endowed with exponent \(\frac{1}{2}\). Respectively, \(F(x)\) and \(f(x)\) have an infinite number of x-axis intercept points called Pseudo-zeroes and Zeroes but with nil Origin intercept point.

Figure 1 depict complex variable \(s = \sigma \pm it\) as INPUT with x-axis denoting real part \(\text{Re}\{s\}\) associated with \(\sigma\), and y-axis denoting imaginary part \(\text{Im}\{s\}\) associated with \(t\). \(\text{Gram}[x=0,y=0]\) points, \(\text{Gram}[y=0]\) points and \(\text{Gram}[x=0]\) points are three types of IP Gram points [viz, Zeroes] occurring at \(\sigma = \frac{1}{2}\) critical line (Figure 2) based on, respectively, Origin intercept points, x-axis intercept points and y-axis intercept points. They can be dependently computed from \(\eta(s)\) [proxy \(\zeta(s)\)] and \(\text{Im}\{\zeta(s)\}\) whereby \(\text{Im}\{\zeta(s)\}\) is obtained by applying Euler formula to \(\eta(s)\). \(\text{Gram}[x=0,y=0]\) points are synonymous with NTZ and \(\text{Gram}[y=0]\) points are synonymous with ‘usual’ Gram points, Virtual \(\text{Gram}[y=0]\) points and virtual \(\text{Gram}[x=0]\) points are two types of IP virtual Gram points [viz, virtual Zeroes] occurring at \(\sigma \neq \frac{1}{2}\) non-critical lines based on, respectively, x-axis intercept points and y-axis intercept points – see Figure 3 for \(\sigma = \frac{1}{2}\) and Figure 4 for \(\sigma = \frac{3}{2}\). They can be dependently computed from these exact same functions.

As polar graphs with x-axis denoting real part \(\text{Re}\{\zeta(s)\}\) and y-axis denoting imaginary part \(\text{Im}\{\zeta(s)\}\) generated by \(\zeta(s)\)’s output as real values of \(t\) running from 0 to 34, Figure 2 [uniquely] represents solitary \(\sigma = \frac{1}{2}\) critical line situation. This Figure depicts complete presence of Origin intercept points. Figures 3 and 4 [non-uniquely] represent corresponding \(\sigma = \frac{1}{2}\) non-critical line situation and \(\sigma = \frac{3}{2}\) non-critical line situation.
These two figures depict complete absence of Origin intercept points. Thus, there are infinite types of spirals (Varying Loops) possibilities associated with each \( \sigma \) value arising from all infinite \( \sigma \) values in critical strip \([0 < \sigma < 1] \) by which the unique and solitary \( \sigma = \frac{1}{2} \) value also belong.

Let \( \delta = \frac{1}{10} \). This will generate in Figure 3 and Figure 4 the \( \delta \) induced shift of \([\text{finite}] \) Varying Loops in reference to Origin; viz, the simple relationship of \([\text{more negative}] \) left-shift given by \( \zeta(\frac{1}{2} - \delta + i\tau) \) [Figure 3] \( < \) \([\text{neutral}] \) nil-shift given by \( \zeta(\frac{1}{2} + i\tau) \) [Figure 2] \( < \) \([\text{more positive}] \) right-shift given by \( \zeta(\frac{1}{2} + \delta + i\tau) \) [Figure 4] \( \) will always be consistently true.

**Remark 11.** Given \( \delta = \frac{1}{10} \), the \( \sigma = \frac{1}{2} - \delta \) non-critical line (represented by Figure 3) and \( \sigma = \frac{1}{2} + \delta \) non-critical line (represented by Figure 4) are equidistant from \( \sigma = \frac{1}{2} \) critical line (represented by Figure 2). We refer to the important comment that precede Remark 8. Then additive inverse operation of \( \sin(\delta) + \sin(-\delta) \) \( = 0 \) indicating symmetry with respect to Origin \([\text{or cos(} \delta \text{) - cos(-} \delta \text{)} = 0 \) indicating symmetry with respect to \( y \)-axis\] is not applicable to our complex single sine wave \([\text{or single cosine wave}] \) since \((2n)\)- or \((2n-1)\)-complex term with transcendental functions consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are independent of parameter \( \sigma \). However, \((2n)\)- or \((2n-1)\)-complex term with \( \) algebraic functions consisting of powers, fractional powers, root extraction \( \) (equating to scaled amplitude \( R \) in Remarks 1, 2 and 3) are dependent on parameter \( \sigma \). Let \( x = (2n) \) or \( \frac{1}{(2n)} \) or \((2n-1) \) or \( \frac{1}{(2n-1)} \).

With multiplicative inverse operation of \( x^\delta \cdot x^{-\delta} = 1 \) or \( \frac{1}{x^\delta} = \frac{1}{x^{-\delta}} = 1 \) that is applicable, this must imply intrinsic presence of Multiplicative Inverse in \( \sin(\theta) \) or DSPL for all \( \sigma \) values when this function or law rigidly obey the relevant trigonometric identity. We named this phenomenon Principle of Equidistant for Multiplicative Inverse. We additionally note by letting \( \delta = 0 \), we always generate Figure 2 that represents \( \sigma = \frac{1}{2} \) critical line.

**Zeros and Pseudo-zeroes:** There are three types of stationary points in a given periodic \( f(n) \) involving sine and/or cosine functions that could act as \( x \)-axis intercept points via three types of \( f(n) \)'s Zeros with corresponding three types of \( F(n) \)'s Pseudo-zeros: maximum points e.g. with \( f(n) \) or \( F(n) = \sin n + 1 \); and points of inflection e.g. with \( f(n) \) or \( F(n) = \sin n \) \([\text{which also has Origin intercept point as a Zero or Pseudo-zero}] \). A fourth type of \( f(n) \)'s Zeros and \( F(n) \)'s Pseudo-zeros consist of non-stationary points occurring e.g. with \( f(n) \) or \( F(n) = \sin n + 0.5 \).

With \((j - i) = (l - k) = 2\pi \) \([\text{viz, one Full cycle}] \), let a given Zero be located in \( f(n) \)'s interval \([i,j] \) \([\text{viz, one Full cycle}] \) and its corresponding Pseudo-zero be located in \( F(n) \)'s Pseudo-interval \([k,l] \) \([\text{viz, one Full cycle}] \) viz, \( k < \sigma < l \). For this Zero and Pseudo-zero characterized by either point of inflection or non-stationary point; both will comply with preserving positivity \([\text{going from} \{\text{-ve}\} \text{below} \ x\text{-axis to} \{\text{+ve}\} \text{above} \ x\text{-axis}] \) as explained using the Zero case \([\text{with the Pseudo-zero case following similar lines of explanations}] \). This can be stated as follow for interval \([i,j] \): If \( j > i \), then computed \( f(j) \) > computed \( f(i) \). In particular, the condition \("If \( i \geq 0 \), then computed \( f(i) \geq 0 \)" \) must not be present for these two particular types of Zero to validly exist in interval \([i,j] \). With reversal of inequality signs, the converse situation for \( j < i \) and corresponding \( l < k \) will be equally true in preserving negativity \([\text{going from} \{\text{+ve}\} \text{above} \ x\text{-axis to} \{\text{-ve}\} \text{below} \ x\text{-axis}] \). All these are useful observed properties for Zeros and Pseudo-zeroes.

**Preservation or conservation of Net Area Value and Total Area Value with definitions[1], p. 10 - 13:** \( \int f(n) \, dn = F(n) + C \) with \( F'(n) = f(n) \). Consider a nominated function \( f(n) \) for interval \([a,b] \). We define Net Area Value (NAV) calculated using its antiderivative \( F(n) \) as the net difference between positive area value(s) [above horizontal \( x \)-axis] and negative area value(s) [below horizontal \( x \)-axis] in interval \([a,b] \); viz, NAV = all +ve value(s) + all -ve value(s). Again calculated using \( F(n) \), we define Total Area Value (TAV) as the total sum of (absolute value) positive area value(s) [above horizontal \( x \)-axis] and (absolute value) negative area value(s) [below horizontal \( x \)-axis] in interval \([a,b] \); viz, TAV = all +ve value(s) + all -ve value(s). Calculated NAV and TAV are precise using antiderivative \( F(n) \) obtained from integration of \( f(n) \) but are only approximate when using Riemann sum on \( f(n) \). For \( f(n) \)'s interval \([a,b] \) whereby \( a = \text{initial Zero} \) and \( b = \text{next Zero} \), and \( F(n) \)'s Pseudo interval \([c,d] \) whereby \( c = \text{initial Pseudo-zero} \) and \( d = \text{next Pseudo-zero} \); then compliance with preservation or conservation of NAV and TAV will simultaneously occur in both \( f(n) \)'s Zeros and \( F(n) \)'s Pseudo-zeroes given by their sine and/or cosine functions only when Zero gap = \( (b - a) \) = Pseudo-zero gap = \( (d - c) = 2\pi \) \([\text{viz, involving one Full cycle}] \). For our purpose, NAV = 0 condition is validly preserved or preserved.
for \( f(n) \) sim-\( \eta(s) \)'s IP Zeroes and \( F(n) \) DSPL’s IP Pseudo-zeroes at parameter \( \sigma = \frac{1}{2} \). Ditto for \( f(n) \) sim-\( \eta(s) \)'s IP virtual Zeroes and \( F(n) \) DSPL’s IP virtual Pseudo-zeroes at parameter \( \sigma \neq \frac{1}{2} \); viz, NAV = 0 condition is validly preserved or conserved for \( f(n) \) sim-\( \eta(s) \)'s IP virtual Zeroes and \( F(n) \) DSPL’s IP virtual Pseudo-zeroes.

For all our complex functions and complex equations in this paper, \( s = \sigma \pm it \) whereby one would commonly or routinely invoke \( s = \sigma + it \) for discussion. For all \( f(n) \) and \( F(n) \) general equations depicted below without trigonometric identity application, we note the presence of mixed sine and cosine terms in all these general equations except for \( f(n) \)'s Gram[y=0] points-sim-\( \eta(s) \) and \( f(n) \)'s Gram[x=0] points-sim-\( \eta(s) \).

I. **NTZ or Gram \([x=0,y=0]\) points** as geometrical Origin intercept points are mathematically defined by

\[
\sum \text{Re}(\eta(s)) = \text{Re}\{\eta(s)\} + \text{Im}\{\eta(s)\} = 0,
\]

General equation for \( f(n) \)'s sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{\sigma}} (t \ln(2n)) = \frac{(2n-1)^{\sigma}}{(2n-1)^{\sigma} - (t \ln(2n))} + \frac{(2n-1)^{\sigma}}{(2n-1)^{\sigma} - (t \ln(2n))} + C \]

(1.9)

General equation for \( F(n) \)'s DSPL as Pseudo-zeroes to Zeros conversion is given by

\[
\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left[ (2n)^{\sigma} (t \ln(2n)) + (t \ln(2n)) \right] = 0
\]

(1.10)

II. **Gram\([y=0]\) points** as geometrical y-axis intercept points are mathematically defined by

\[
\sum \text{Re}(\eta(s)) = \text{Re}\{\eta(s)\} + 0, \text{ or simply } \text{Im}\{\eta(s)\} = 0.
\]

General equation for \( f(n) \)'s Gram\([y=0]\) points-sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{\sigma} \sin(t \ln(2n)) = 0
\]

(1.11)

General equation for \( F(n) \)'s Gram\([y=0]\) points-DSPL as Pseudo-zeroes to Zeros conversion is given by

\[
\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left[ (2n)^{\sigma} (t \ln(2n)) + t \cos(t \ln(2n)) \right] - \frac{(2n-1)^{\sigma} ((\sigma - 1) \sin(t \ln(2n)) + t \cos(t \ln(2n))) + C \]

(1.12)

III. **Gram\([x=0]\) points** as geometrical x-axis intercept points are mathematically defined by

\[
\sum \text{Re}(\eta(s)) = 0 + \text{Im}\{\eta(s)\}, \text{ or simply } \text{Re}\{\eta(s)\} = 0.
\]

General equation for \( f(n) \)'s Gram\([x=0]\) points-sim-\( \eta(s) \) as Zeroes is given by

\[
\sum_{n=1}^{\infty} (2n)^{\sigma} \cos(t \ln(2n)) = 0
\]

(1.13)

General equation for \( F(n) \)'s Gram\([x=0]\) points-DSPL as Pseudo-zeroes to Zeros conversion is given by

\[
\frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot \left[ (2n)^{\sigma} (t \ln(2n)) - (\sigma - 1) \cos(t \ln(2n)) \right] - \frac{(2n-1)^{\sigma} ((\sigma - 1) \sin(t \ln(2n)) - (\sigma - 1) \cos(t \ln(2n))) + C \]

(1.14)

Cartesian Coordinates \((x,y)\) is related to Polar Coordinates \((r,\theta)\) with \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \). In anti-clockwise direction, it has four quadrants defined by the + or - of \((x,y)\); viz, Quadrant I as \((+,+), \) Quadrant II as \((-,+), \) Quadrant III as \((-,-), \) and Quadrant IV as \((+,-). \) NTZ are Origin intercept points or Gram \([x=0,y=0]\) points. With 'gap' being synonymous with 'interval', NTZ gap is given by initial NTZ t-value minus next NTZ t-value. Running a Full cycle from \( 0\pi \) to \( 2\pi \), size of each IP Varying Loop in Figure 2 is proportional to magnitude of its corresponding IP NTZ varying gap. We note the \( 2\pi \) here as observed in Figure 2 [on Gram points], Figure 3 [on virtual Gram points] and Figure 4 [on virtual Gram points] refers to IP Varying Loops.
transversed by parameter $t$ with NTZ (Gram $[x=0, y=0]$ points) corresponding to $t$ values as Origin intercept on Origin’s solitary (0,0) part (point); Gram $[y=0]$ points and virtual Gram $[y=0]$ points corresponding to $t$ values as x-axis intercept on x-axis’ (+ve) $0\pi$ part and (-ve) $\pi$ part; and Gram $[x=0]$ points and virtual Gram $[x=0]$ points corresponding to $t$ values as y-axis intercept on y-axis’ (+ve) $\frac{\pi}{2}$ part and (-ve) $\frac{3\pi}{2}$ part. The virtual NTZ entities do not exist; viz, Origin intercept points do not occur in Figure 3 and Figure 4.

With $\eta(s)$ being the proxy function for $\zeta(s)$, NTZ are mathematically defined by $\eta(s) = 0$ or $\sin\eta(s) = 0$. Riemann hypothesis is the original 1859-dated conjecture that all NTZ are located on $\sigma = \frac{1}{2}$ critical line of $\zeta(s)$. Mathematically proving all NTZ location on critical line as denoted by solitary $\sigma = \frac{1}{2}$ value equates to geometrically proving all Origin intercept points occurrence at solitary $\sigma = \frac{1}{2}$ value. Both result in rigorous proof for Riemann hypothesis.

**Remark 12.** Locations of first 10,000,000,000,000 NTZ on critical line have previously been computed to be correct. Hardy[2], and with Littlewood[3], showed infinite NTZ on $\sigma = \frac{1}{2}$ critical line by considering moments of certain functions related to $\zeta(s)$. This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of NTZ located away from this line (when $\sigma \neq \frac{1}{2}$).

Furthermore, it is literally a mathematical impossibility (“mathematical impasse”) to be able to computationally evaluate definite integrals Eq. (1.3) or Eq. (1.10), Eq. (1.6) or Eq. (1.12) and Eq. (1.8) or Eq. (1.14) using $\Delta$ deducing exact DA homogeneity in DSPL symbolizes rigorous proof for Riemann hypothesis which is depicted from $f(n)$ given by sim-$\eta(s)$, NTZ are mathematically defined by $\eta(s) = 0$ or $\sin\eta(s) = 0$. Riemann hypothesis is the original 1859-dated conjecture that all NTZ are located on $\sigma = \frac{1}{2}$ critical line of $\zeta(s)$. Mathematically proving all NTZ location on critical line as denoted by solitary $\sigma = \frac{1}{2}$ value equates to geometrically proving all Origin intercept points occurrence at solitary $\sigma = \frac{1}{2}$ value. Both result in rigorous proof for Riemann hypothesis.

The monumental task of solving Riemann hypothesis is completed by deriving $F(n)$ DSPL from $f(n)$ sim-$\eta(s)$ with its computed Pseudo-zeros and virtual Pseudo-zeros which can all be converted to corresponding Zeros and virtual Zeros since $F(n)$’s IP Pseudo-zeros and IP virtual Pseudo-zeros (t values) = $f(n)$’s IP Zeros and IP virtual Zeros (t values) + $\frac{\pi}{2}$ whereby both $f(n)$ and $F(n)$ have parameters $\sigma$ and $t$. Correctly deducing exact DA homogeneity in DSPL symbolizes rigorous proof for Riemann hypothesis which is depicted as Pseudo-zeros to Zeros conversion that obeys relevant trigonometric identities.

With $n = 1, 2, 3, \ldots, \infty$ and therefore $\Delta n = 1$, we note $f(n)$ can analogically be interpreted as Area under the Curve (AUC) [right infinite-interval] Riemann sum $\sum_{n=1}^{\infty} f(n) \Delta n = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} f(n) + \sum_{n=3}^{\infty} f(n) + \sum_{n=5}^{\infty} f(n) + \ldots$ + $\sum_{n=1}^{\infty} f(n)$.

Corresponding solution to exact AUC improper integral $\int_{n=1}^{\infty} f(n) \, dn$ can be validly expanded as $\sum_{n=1}^{6} f(n) \, dn + \sum_{n=1}^{6} f(n) \, dn + \sum_{n=1}^{8} f(n) \, dn + \ldots + \sum_{n=1}^{\infty} f(n) \, dn = [F(n) + C]^2_1 + [F(n) + C]^3_2 + [F(n) + C]^3_3 + \ldots + [F(n) + C]^\infty_{n-1}$ which, for all sufficiently large $n$ as $n \to \infty$, will manifest divergence by oscillation (viz. for all sufficiently large $n$ as $n \to \infty$, this cumulative total will not diverge in a particular direction to a solitary well-defined limit value such as $\sin\frac{\pi}{2} = 1$ or less well-defined limit value such as $+\infty$).

Evaluation of definite integrals Eq. (1.3) or Eq. (1.10), Eq. (1.6) or Eq. (1.12) and Eq. (1.8) or Eq. (1.14) using limit as $n \to +\infty$ for $0 < t < +\infty$ enable countless computations resulting in $t$ values for (respectively) CIS nontrivial zeros, CIS Gram$[y=0]$ points and CIS Gram$[x=0]$ points [all as Pseudo-zeros to Zeros conversion]. Larger $n$ values used for computations will correspond to increasing accuracy of these entities (which are all transcendental numbers).

**Remark 13.** Whereas exact AUC from $F(n)$ given by DSPL $\int_{n=1}^{\infty} \sin - \eta(s) \, ds$ and approximate AUC from $f(n)$ given by sim-$\eta(s)$ = $\sum_{n=1}^{\infty} \sin - \eta(s)$ [when interpreted as Riemann sum] are proportional; the Zeros when indirectly derived from DSPL [as Pseudo-zeros converted to Zeros] and the Zeros when directly derived from sim-$\eta(s)$ must agree with each other at $\sigma = \frac{1}{2}$ critical line.

2 Riemann zeta function, Dirichlet eta function, simplified Dirichlet eta function and Dirichlet Sigma-Power Law

An L-function consists of a Dirichlet series with a functional equation and an Euler product. Examples of L-functions come from modular forms, elliptic curves, number fields, and Dirichlet characters, as well as more
generally from automorphic forms, algebraic varieties, and Artin representations. They form an integrated component of ‘L-functions and Modular Forms Database’ (LMFDB) with far-reaching implications. In perspective, \( \zeta(s) \), being the simplest example of an L-function, is a function of complex variable \( s = \sigma \pm it \) that analytically continues sum of infinite series

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.
\]

The common convention is to write \( s = \sigma + it \) with \( t = \sqrt{-1} \), and with \( \sigma \) and \( t \) real. Valid for \( \sigma > 0 \), we write \( \zeta(s) \) as \( Re\{\zeta(s)\} + iIm\{\zeta(s)\} \) and note that \( \zeta(\sigma + it) \) when \( 0 < t < +\infty \) is the complex conjugate of \( \zeta(\sigma - it) \) when \( -\infty < t < 0 \).

Also known as alternating zeta function, \( \eta(s) \) must act as "proxy" for \( \zeta(s) \) in critical strip (viz. \( 0 < \sigma < 1 \)) containing critical line (viz. \( \sigma = \frac{1}{2} \)) because \( \zeta(s) \) only converges when \( \sigma > 1 \). This implies \( \zeta(s) \) is undefined to left of this region in critical strip which then requires \( \eta(s) \) representation instead. They are related to each other as \( \zeta(s) = \gamma \cdot \eta(s) \) with proportionality factor \( \gamma = \frac{1}{(1-2^{-s})} \) and \( \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \).

Eq. (2.1) is defined for only \( 1 < \sigma < \infty \) region where \( \zeta(s) \) is absolutely convergent with no zeros located here. In Eq. (2.1), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents \( \zeta(s) \rightleftharpoons \) all prime and, by default, composite numbers are (intrinsically) encoded in \( \zeta(s) \).

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \cdot \Gamma(1-s) \cdot \zeta(1-s) \tag{2.2}
\]

With \( \sigma = \frac{1}{2} \) as symmetry line of reflection, Eq. (2.2) is Riemann’s functional equation valid for \( -\infty < \sigma < \infty \). It can be used to find all trivial zeros on horizontal line at \( t = 0 \) occurring when \( \sigma = -2, -4, -6, -8, -10, \ldots, \infty \) whereby \( \zeta(s) = 0 \) because factor \( \sin \left( \frac{\pi s}{2} \right) \) vanishes. \( \Gamma \) is gamma function, an extension of factorial function [a product function denoted by \( ! \) notation whereby \( n! = n(n-1)(n-2) \ldots (n-(n-1)) \)] with its argument shifted down by 1, to real and complex numbers. That is, if \( n \) is a positive integer, \( \Gamma(n) = (n-1)! \).

\[
\zeta(s) = \frac{1}{(1-2^{-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{2.3}
\]

Eq. (2.3) is defined for all \( \sigma > 0 \) values except for simple pole at \( \sigma = 1 \). As alluded to above, \( \zeta(s) \) without \( \frac{1}{(1-2^{-s})} \) viz. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \rightleftharpoons \eta(s) \). It is a holomorphic function of \( s \) defined by analytic continuation and is mathematically defined at \( \sigma = 1 \) whereby analogous trivial zeros with presence for \( \eta(s) \) [but not for \( \zeta(s) \)] on vertical straight line \( \sigma = 1 \) are found at \( s = 1 \pm \frac{2\pi k}{\ln(2)} \) where \( k = 1, 2, 3, 4, \ldots, \infty \).

Euler formula can be stated as \( e^{it} = \cos t + i \cdot \sin t \). Euler identity (where \( n = \pi \)) is \( e^{it} = \cos t + i \cdot \sin t = -1 + 0 \) [or stated as \( e^{it} + 1 = 0 \)]. The \( n^t \) of \( \zeta(s) \) is expanded to \( n^t = n^{(\sigma+it)} = n^\sigma e^{it \ln(n)} \) since \( n^t = e^{\ln(n^t)} \).
Apply Euler formula to $n^r$ result in $n^r = n^\sigma (\cos(t \ln(n)) + t \cdot \sin(t \ln(n)))$. This is written in trigonometric form [designated by short-hand notation $n^r(Euler)$] whereby $n^\sigma$ is modulus and $t \ln(n)$ is polar angle (argument).

Apply $n^r(Euler)$ to Eq. (2.3) to obtain $f(n)$ general sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ NTZ versus non-existent $\sigma \neq \frac{1}{2}$ virtual NTZ[1], section 4, p. 24 - 28. This is given as Eq. (1.9) and with relevant trigonometric identity application at $\sigma = \frac{1}{2}$ as Eq. (1.1). Integrate $f(n)$ general sim-$\eta(s)$ to obtain $F(n)$ general DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeros versus non-existent $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeros. Pseudo-zeros and non-existent virtual Pseudo-zeros can be converted to Zeros (NTZ) and non-existent virtual Zeros (virtual NTZ). This is given as Eq. (1.10) and with relevant trigonometric identity application at $\sigma = \frac{1}{2}$ as Eq. (1.3).

We provide $f(n)$ general Gram[\(y=0\)] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[y=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[y=0] points[1], section 5, p. 28 - 30. This is given as Eq. (1.11) but we are unable to apply relevant trigonometric identity. Integrate $f(n)$ general Gram[y=0] points-sim-$\eta(s)$ to obtain $F(n)$ general Gram[y=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeros versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeros. Pseudo-zeros and virtual Pseudo-zeros can be converted to Zeros (Gram[y=0] points) and virtual Zeros (virtual Gram[y=0] points). This is given as Eq. (1.12) and with relevant trigonometric identity application at $\sigma = \frac{1}{2}$ as Eq. (1.6).

We provide $f(n)$ general Gram[\(x=0\)] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[x=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[x=0] points[1], section 5, p. 28 - 30. This is given as Eq. (1.13) but we are unable to apply relevant trigonometric identity. Integrate $f(n)$ general Gram[x=0] points-sim-$\eta(s)$ to obtain $F(n)$ general Gram[x=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeros versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeros. Pseudo-zeros and virtual Pseudo-zeros can be converted to Zeros (Gram[x=0] points) and virtual Zeros (virtual Gram[x=0] points). This is given as Eq. (1.14) and with relevant trigonometric identity application at $\sigma = \frac{1}{2}$ as Eq. (1.8).

3 Conclusions

Critical line of Riemann zeta function is denoted by $\sigma = \frac{1}{2}$ whereby all nontrivial zeros are proposed to be located in the 1859 Riemann hypothesis. Treated as Incompletely Predictable problems, we gave a relatively elementary proof on Riemann hypothesis while also explaining the existence of three types of Gram points and two types of virtual Gram points by analyzing the complex (meta-) properties of relevant Dirichlet Sigma-Power Laws viz, (1) exact DA homogeneity [occurring only once at $\sigma = \frac{1}{2}$ critical line] in these Laws with ability to convert their computed Pseudo-zeros to Zeros resulting in nontrivial zeros (Origin intercept points or Gram[x=0,y=0] points) as one type of Gram points plus two remaining types of Gram points; and (2) inexact DA homogeneity [occurring infinitely often at $\sigma \neq \frac{1}{2}$ non-critical lines] in these Laws with ability to convert their computed virtual Pseudo-zeros to virtual Zeros resulting in two types of virtual Gram points.

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