# Principle of Equivalence on Massless Bodies

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**Abstract**

The validity of the principle of equivalence on massive bodies has been confirmed empirically with great accuracy. However, can the principle of equivalence be applied to massless bodies? For a massless body, what is the meaning of the ratio of its gravitational to inertial mass? In this essay, we show a mechanism to extend the principle of equivalence to massless bodies satisfactorily.

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## I. Introduction

Einstein used the following mathematical formulation to describe the principle of equivalence:

$$acceleration = \frac{gravitational\ mass}{inertial\ mass}(intensity\ of\ the\ gravitational\ field).$$

If the acceleration is always the same for a given gravitational field regardless a body's nature and condition, then the ratio of the gravitational to the inertial mass must be a constant, usually one by a suitable choice of units [1]. It has been confirmed by experiments with great accuracy. However, the principle only states to apply to a massive body.

Can the principle be applied to a massless body? By definition, a massless body has zero gravitational and inertial mass. The ratio of its gravitational to inertial mass would be undetermined. In order to solve this dilemma, we analyze the equivalence principle on a massive body, then we use the insights obtained to extend the principle of equivalence to a massless body. This essay emphasizes conceptual issues, instead of detailed calculations.

## II. Massive System of Objects

The equivalence principle states the equality of a system's total gravitational and inertial mass. In the following paragraphs we will derive this statement by applying the equivalence principle to the parts of the system. First, let us make a postulate regarding the movement of a system of objects as follows:

> *Postulate 1*: If a system of objects accelerates under the sole influence of a gravitational field, then all the parts of the system move with an identical acceleration.

To demonstrate it, let us start with a simple massive system $S_1$, which consists of two kinds of objects: A and B. Object A has gravitational mass $G_a$ and inertial mass $I_a$, while object B has gravitational mass $G_b$ and inertial mass $I_b$. Suppose there are $m$ As and $n$ Bs inside the system.

Based on *Postulate 1*, inside system $S_1$, which accelerates under the sole influence of gravity, the As and Bs move with an identical acceleration. According to the equivalence principle, A's acceleration depends on the ratio of its gravitational to inertial mass. So does B's. Since A and B have the same acceleration we have:

$$\frac{G_a}{I_a} = \frac{G_b}{I_b} = k \neq 0, \tag{1}$$

where $k$ is a constant. From Eq. (1) we find:

$$G_a = kI_a$$

$$G_b = kI_b. \tag{2}$$

By manipulating Equations (2) we have:

$$\frac{mG_a+nG_b}{mI_a+nI_b} = k, \tag{3}$$

where $mG_a + nG_b$ is the total gravitational mass of the system $S_1$; $mI_a + nI_b$ is the total inertial mass of $S_1$. Since for a massive system, $mI_a + nI_b \neq 0$, $mG_a + nG_b \neq 0$, the ratio of $S_1$'s gravitational to inertial mass is $k$ too. By a suitable choice of units, $k = 1$, so it becomes:

$$mG_a + nG_b = mI_a + nI_b \neq 0, \tag{4}$$

which is exactly the statement of the equivalence principle applied to the system $S_1$ itself.

Equation (3) proves that if the equivalence principle can be applied to all the components inside $S_1$, then it is applicable to $S_1$ in its entirety. We can simply prove the conclusion to be true for massive systems consisting of any number of different kinds of objects. Therefore, we can generalize the conclusion to a new postulate:

*Postulate 2*: The identical acceleration of all the parts of a system determines the acceleration of the system in its entirety.

## III. Massless System of Objects

Since a massless system's gravitational mass and inertial mass are zero, obviously, Eq. (3) becomes undetermined. However, based on *postulate 2*, if we can calculate the identical acceleration of all its parts, then we can determine the acceleration of the massless body. Hence, we can define the ratio of a massless system's gravitational to inertial mass as being equivalent to the same ratio for the system's parts.

Let us continue with a simple massless system $S_2$, which again consists of two kinds of objects: A and B. Suppose there are $m$ As and $n$ Bs inside the system. If equations (2) hold for the massless system $S_2$, by manipulating Equations (2) we get:

$$mG_a + nG_b = k(mI_a + nI_b) = 0, \tag{5}$$

where $mG_a + nG_b = 0$, $mI_a + nI_b = 0$, hence, $k$ can be any number.

Equation (5) indicates that for a massless body, the ratio of its gravitational to inertial mass is not fixed. However, the ratio can be determined by its parts. That is, if the following ratios hold

$$\frac{G_a}{I_a} = \frac{G_b}{I_b} = k \neq 0, \tag{6}$$

then the ratio of the body $S_2$'s gravitational to inertial mass is $k$.

Before we find out what values $k$ can be, let us manipulate Eq. (5) to obtain:

$$\frac{G_a}{G_b} = -\frac{n}{m}$$

$$\frac{I_a}{I_b} = -\frac{n}{m}, \tag{7}$$

where we know that since $m$ and $n$ are positive, then $-\frac{n}{m}$ must be negative. Equations (7) point to an interesting consequence that one of $G_a$ or $G_b$ must be a negative gravitational mass, and also one of $I_a$ or $I_b$ must be a negative inertial mass. The concept of negative mass is relatively new. It is abstract and of opposite sign to normal mass. There has not been any conclusive physical evidence that it ever existed. However, that has not hindered people's interest in studying it [2-6]. A conclusion from this research is that the existence of negative mass does not violate the laws of classical physics.

We can prove that for any pair of $G_a$ and $G_b$, which satisfy the Eq. (5), there exist many $I_a$, $I_b$ and $k$, which satisfy both Eq. (5) and Eq. (6). Therefore, the ratio $k$ in Eq. (6) may not be constant. We can also prove this conclusion for massless systems consisting of any number (at least two) of different kinds of objects. Here is the conclusion of the extended principle of equivalence on a massless body:

> *Conclusion*: If all parts of a massless system have an identical ratio of gravitational to inertial mass, then the massless system in its entirety has the same ratio of gravitational to inertial mass, but the ratio is not fixed.

Let us apply the extended principle of equivalence to sample massless systems which can have different ratios of gravitational to inertial mass. Suppose there exists a massless system consisting of 1 object A and 4 objects B, then there are some suitable values for $G_a$, $I_a$, $G_{b,}$ $I_b$, as shown in Fig. 1.

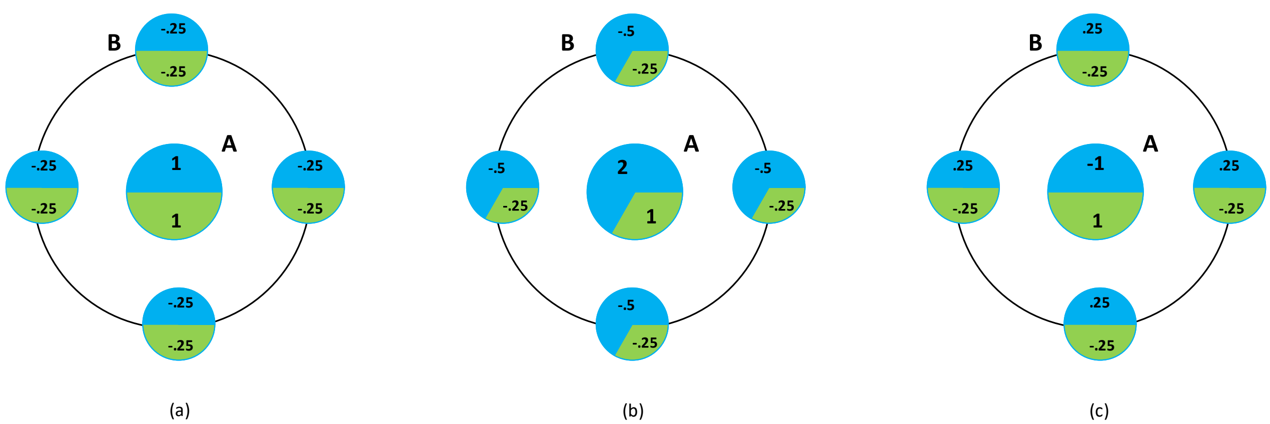


FIG. 1. A massless system consists of 1 object A and 4 objects B. The blue slices represent gravitational mass while the green slices represent inertial mass. Depending on the nature of the objects A and B, the massless system's ratio of gravitational to inertial mass is different. (a) The ratio is 1 since the ratio of gravitational to inertial mass is 1, for all A and Bs. (b) The ratio is 2 since the ratio of gravitational to inertial mass is 2, for all A and Bs. (c) The ratio is -1 since the ratio of gravitational to inertial mass is -1, for all A and Bs.

a) $G_a = 1, I_a = 1, G_b = -.25, I_b = -.25.$

The massless system's ratio of gravitational to inertial mass is 1.

b) $G_a = 2, I_a = 1, G_b = -.5, I_b = -.25.$

The ratio is 2.

c) $G_a = -1, I_a = 1, G_b = .25, I_b = -.25.$

The ratio is -1. The negative ratio indicates a repulsive acceleration in a gravitational field.

Next let us put the sample massless bodies into a gravitational field and see how they move. If the velocities of those massless bodies are the same, then the deflection angles caused by the gravitational field are proportional to their accelerations, according to classical physics. From the extended principle of equivalence, we conclude that a body's acceleration is proportional to the ratio of its gravitational to inertial mass. Thus, a body's deflection angle is proportional to the ratio of its gravitational to inertial mass. For the samples above, the different ratios dictate that those massless bodies have different deflection angles, as shown in Fig. 2.

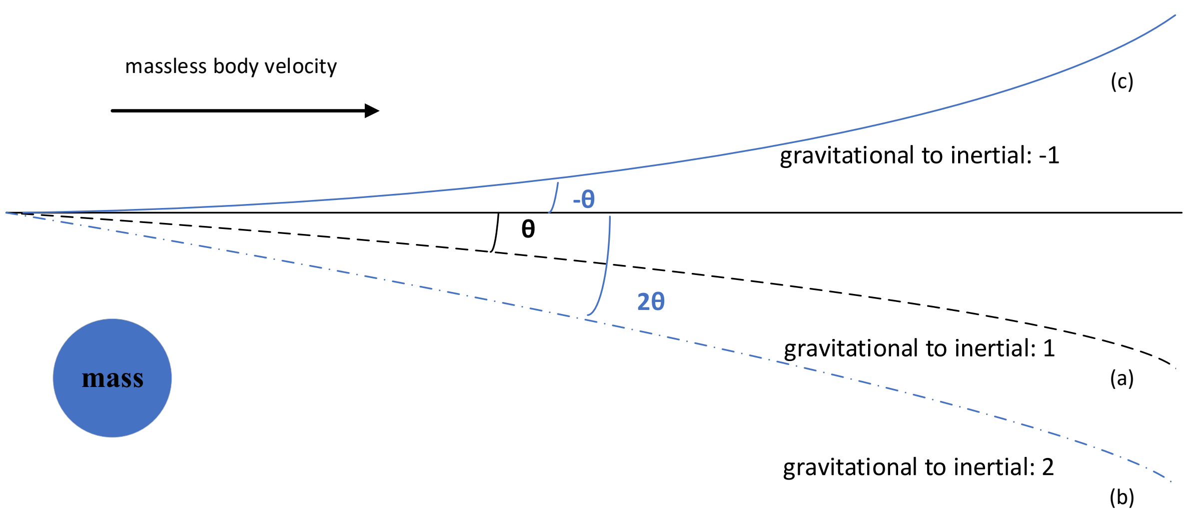


FIG. 2. Massless bodies with the same velocity pass over a mass and are deflected by the gravitational field of the mass. The horizontal line indicates the trajectory if the mass is not present. (a) The dashed black line indicates a massless body's trajectory, if the ratio of its gravitational to inertial mass is 1, with a deflection angle $\theta$ from the horizontal line. (b) The dashed blue line indicates a massless body's trajectory, if the ratio is 2, with a deflection angle $2\theta$. (c) The solid blue line indicates a massless body's trajectory, if the ratio is -1, with a deflection angle $-\theta$.

For instance, in case (a), the massless body, whose gravitational to inertial mass ratio is 1, has a deflection angle $\theta$ when it passes through the gravitational field; in case (b) the massless body has a deflection angle $2\theta$; in case (c) the massless body has a deflection angle $-\theta$, which is interpreted as an upward deflection.

## IV. Conclusion

Since a massless body has zero gravitational and inertial mass, the principle of equivalence does not apply to it. However, this essay argues that if all parts of a system move with an identical acceleration, then the system itself should move with the same acceleration; therefore, the parts determine the system's acceleration.

We have proven that for a massive system of objects, the ratio of the gravitational to inertial mass of the parts determines the ratio for the system itself. So, the parts of the system determine the acceleration of the system in its entirety. Generalizing the same reasoning to a massless system of objects, we have successfully extended the principle of equivalence to massless systems.

Furthermore, for a massless system of objects, the essay has proved it is possible for all of its parts to have an identical gravitational to inertial mass ratio, so all the parts can have an identical acceleration under the sole influence of gravity. Therefore, the massless system itself can have the same ratio and acceleration as the parts. But the ratio can vary depending on the parts' nature.

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