A Barrier to the Vaidya Event Horizon

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Abstract
A diagonal form of the Vaidya solution for a spherical symmetric accreting or evaporating black hole is analysed. A divergent potential barrier against a low mass freefalling particle is shown to exist at the Schwarzschild critical radius.

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In classical general relativity, from the perspective of distant observers, stationary black holes are objects existing in the infinite future. This stems from diverging time dilation at the critical radius. However, these objects either exist in a vacuum or are the source of an electromagnetic field if charged. By contrast, astrophysical black holes exist in an environment containing matter accreting onto it, and also emit radiation due to evaporation (Hawking, 1974). More realistic models of black holes due to Vaidya (1951) exchange mass-energy with their environment and consequently these solutions are non-stationary.

The Vaidya metric is traditionally represented in the Eddington-Finkelstein form

$$ds^2 = \left(1 - \frac{2M(z)}{r}\right)dz^2 + 2dzdr - r^2 d\Omega^2 \tag{1}$$

where, given null *time* variables u or v, $z = -v$ for an (advanced) accreting black hole, or $z = u$ for the (retarded) evaporating case. However, this form of the metric hides important features such as the ability of falling particles to reach the horizon in a finite time relative to stationary distant observers. This feature is far from apparent in the form of equation (1). Moreover, care must be taken to include energy-momentum tensor fields of the surrounding environment, and these fields are sufficient to perturb the solution in such a way as to allow this early arrival at the critical radius before the final evaporation point. That freefalling matter close to the critical radius is best modelled as null radiation makes equation (1) a natural choice of coordinates for this application.

However, in order to elucidate important features of the Vaidya solution it is advantageous to diagonalise equation (1). This has the benefit of making the coordinates time-symmetric and more meaningful as far as physical measurement is concerned. More recently Berezin *et al* (2017) have recast this metric in diagonal form. A diagonal Vaidya metric may be written

$$ds^2 = f_0(t, r; \alpha)dt^2 - \left(1 - \frac{2M(t,r)}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{2}$$

where f_0 is given by

$$f_0(t,r;\alpha)=\left(1-\frac{2M(r,t)}{r}\right)^{-1}\left|\frac{1}{2}-\frac{2M(r,t)}{r}+\frac{\sqrt{1-8\alpha}}{2}\right|^{1+\sqrt{1-8\alpha}}\left|\frac{1}{2}-\frac{2M(r,t)}{r}-\frac{\sqrt{1-8\alpha}}{2}\right|^{1-\sqrt{1-8\alpha}}. \tag{3}$$

The parameter, α, is accretion rate or evaporation rate as appropriate, which at any instant Berezin and colleagues have chosen to make linear in the null variable. This simplifies things by setting $\alpha(=-dm/dz)$ as a constant. It is immediately seen that equation (3) is valid for $\alpha\in[0,1/8]$.

Inspection of equation (3) shows that space-time is partitioned into four regions bounded by three concentric timelike hyper-tubes, whose radii vary with *t*. Here we label these radii as $r_c<r_1<r_2$ reflected by where the corresponding factors appear in equation (3). The critical radius, r_c, is where the event horizon appears in the Schwarzschild solution. The function $f_0=0$ at r_1 and r_2, and we necessarily see divergent time dilation at these points. In the Schwarzschild limit $r_1\to r_c$ and $r_2\to\infty$. Given that at sufficiently large distance the modelling of accreting matter as radiation is inaccurate, coupled with outgoing Hawking-radiation being relatively weak, it is difficult to see the physical significance of the zero at r_2. Moreover, the exponent on the corresponding factor in equation (3) is small. So we confine our interest to a region a range (r_c,R), $R<r_2$.

The radial freefall time, Δt_F, from a specified radius, $R>r_1$ to r_1 is represented by

$$\Delta t_F=\int_{r_1}^{R}\sqrt{\frac{-g_{rr}}{g_{00}}}dr \qquad r_1<R<r_2.$$

In the Schwarzschild limit, this integral is known to diverge. Moreover similar freefall times also diverge for Kerr-Newman metrics. However, taking $g_{00}=f_0(r,t;\alpha)$ from equation (3) we see convergence due to the exponent in the second factor $1+\sqrt{1-8\alpha}<2$. Therefore an inbound null trajectory will exhibit an inflection point at r_1 instead of approaching it asymptotically as in stationary cases. Beyond r_1 the null cones open out again, allowing the inbound null trajectory to reach r_c for $t<\infty$.

By holding *t* constant, it is useful to plot f_0 against *r* to see the relevant features and to compare with the Schwarzschild limit ($\alpha\to0$). This is shown in figure 1. The most obvious feature is the almost abrupt change in the direction of gradient seen at r_1, and we see a large negative gradient in the interval (r_c,r_1). The geodesic equation for a test particle, where $U^r=0$ momentarily, gives the radial component of four-acceleration as

$$\dot{U}^r=\frac{1}{2}g^{rr}\left(g_{00,r}\right)=\frac{1}{2}g^{rr}f_{0,r}.$$

From equation (2) we notice that g_{rr} (and therefore g^{rr}) is negative for $r > r_c$. The positive gradient of f_0 above r_1 makes $\dot{U}^r$ negative thus representing attraction as expected. The change in direction of $f_{0,r}$ at r_1 implies gravitational repulsion in the interval (r_c, r_1) with f_0 as an effective potential. The question is, does the actual potential diverge at r_c?

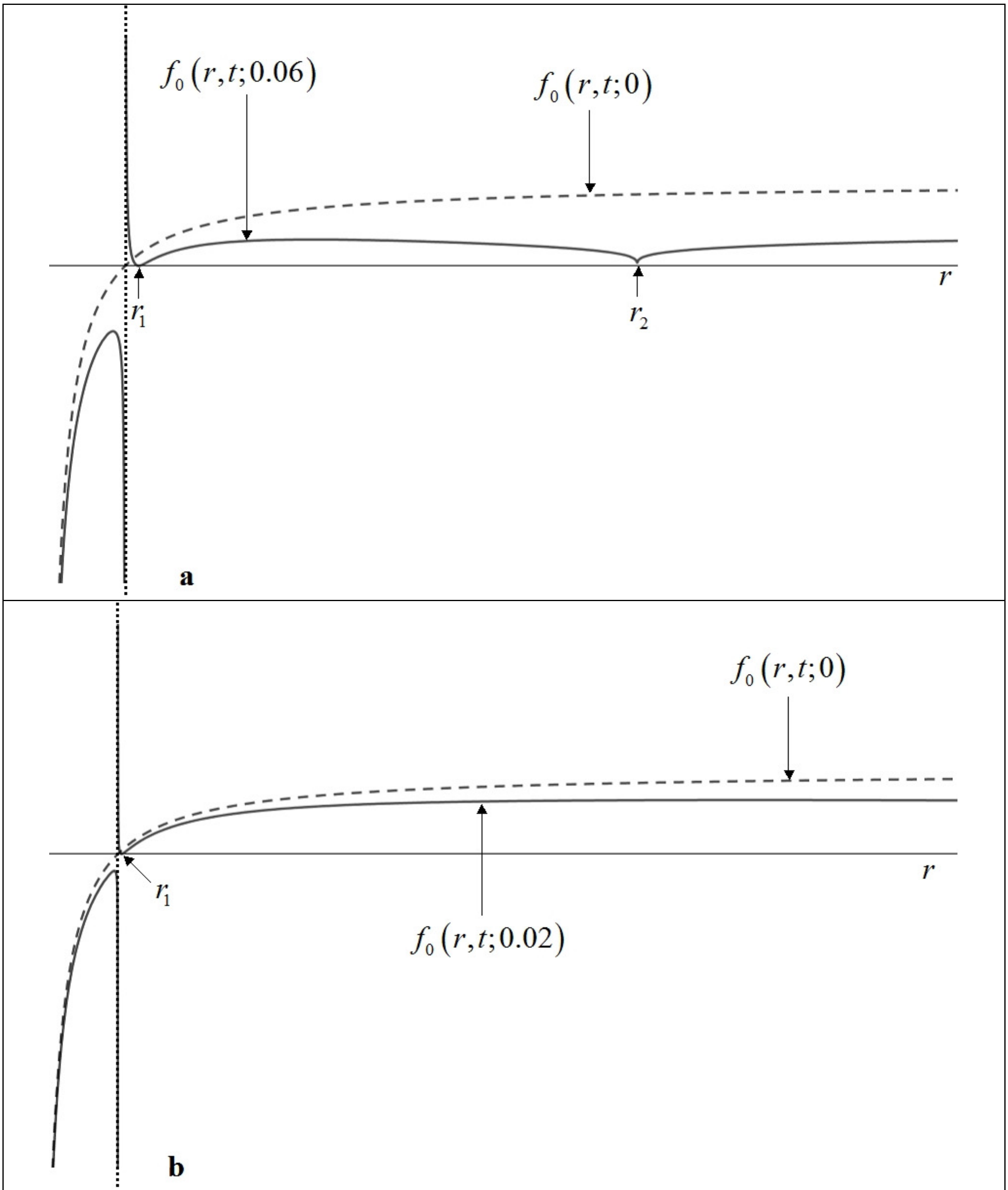

Figure 1: Graph of $f_0(r,t;\alpha)$ from equation (3) (solid line). The dashed line is the corresponding graph in the Schwarzschild case ($\alpha = 0$). The vertical dotted line marks the position of the critical radius, r_c. **a** $\alpha = 0.06$, **b** $\alpha = 0.02$.

The best way to answer this is to consider the Hamiltonian for a low mass particle of intrinsic mass m in a general gravitational field. Here we start by representing the intrinsic mass in terms of four-momentum, p_a, in a general space-time: this is given by $m^2 = p^a p_a$. The derivation is as follows; we start by decomposing the right hand side into temporal and spatial components

$$
\begin{aligned}
m^2 &= p_a p_b g^{ab} \\
&= p_0 p_0 g^{00} + 2 p_0 p_i g^{0i} + p_i p_j g^{ij}, \quad i, j = r, \theta, \phi
\end{aligned}
$$

Solving the quadratic for the Hamiltonian, $H = p_0$, gives

$$
H = -\frac{p_i g^{0i}}{g^{00}} + \sqrt{\left(\frac{p_i g^{0i}}{g^{00}}\right)^2 - \frac{p_i p_j g^{ij} - m^2}{g^{00}}} \,. \tag{4}
$$

Now considering only the radial component of momentum, the Hamiltonian becomes

$$
H = -\frac{p_r g^{0r}}{g^{00}} + \sqrt{\left(\frac{p_r g^{0r}}{g^{00}}\right)^2 - \frac{p_r{}^2 g^{rr} - m^2}{g^{00}}} \,.
$$

As $r \to r_c$, $g^{00} \to 0$ for $m > 0$ therefore $H \to \infty$. For $m = 0$, we have $g^{rr} \to 0$ implying that $p^r \to 0$ as $r \to r_c$. For material particles ($m > 0$) and photons ($m = 0$) this indicates a divergent potential barrier at the critical radius.

For a realistic black hole that has reached equilibrium with its environment, the interval $r_1 - r_c$ is very small, most likely less than a Planck length. The potential barrier is therefore *harder* than during the initial collapse phase when all the mass of a star is yet to be consumed. But even there, the energy required to reach the increasing critical radius still diverges in the spherically symmetric case. If this is correct then what are the implications? There are certainly some intriguing possibilities. Most important, if this mechanism extends to asymmetric collapse, it is an immediate resolution of the information paradox. It appears that a void forms in a collapsing star with all the consumed matter being squeezed onto its inner surface. Inside the inner surface (at $r = r_c(t)$) there would be no matter present.

However, this is where we need to be cautious. If there is no mass-energy flux at r_c then the Vaidya solution is not valid, and this would limit the otherwise infinite positive potential, and allow material to leak across the boundary and enter the void. At present it is difficult to calculate the proportion of material crossing this boundary. If the total mass entering the void is low enough to have negligible back-reaction then it may be attracted back towards the potential minimum at r_1. Other mechanisms for leakage might, for example, included quantum tunnelling, particularly if momenta are reduced to levels where associated wavelengths are comparable to r_c.

That said there is the possibility that initially inbound material would bounce back across the potential minimum and oscillate back and forth in a damped fashion. During periods when the system is in equilibrium, mass-energy influx and outgoing Hawking radiation are at a minimum. If this minimum approaches zero then we are back to the

Schwarzschild solution. Taking a pragmatic view, all we know about black holes is outside the critical radius. So, notwithstanding all of the consistent mathematical models of black hole interiors, they remain in the realms of speculation. An object with all its mass concentrated at its critical radius instead of a central singularity would, to an outside observer, appear similar to any black hole satisfying more traditional models. In this, admittedly rather crude model, the main attractor is not a central (or a ring) singularity at $r = 0$, but a sphere at or just outside the critical radius. And when the system approaches the Schwarzschild limit, the potential barrier is replaced by diverging time dilation at r_c and the consequential asymptotic behaviour of inbound null geodesics.

The author is currently unaware of a similar model being proposed elsewhere or if so, whether it has been rigorously tested in the theoretical arena. Unless there is something seriously wrong with what is being suggested here then it is tempting to propose the following conjecture

Conjecture: Given a spacelike Cauchy hypersurface, *S*, with no regions of radius, *R*, containing mass, $M > \frac{1}{2}\left(R + a^2 + Q^2 / \left(4\pi\varepsilon_0\right)\right)$ for specific angular momentum, *a*, and charge, *Q*, then the laws of physics combine to prevent such regions forming to the future of *S*.

If this is regarded as a serious possibility then it raises questions as to whether it can be tested. If true then all consumed material is confined to a uniform layer just above the critical radius with a thickness of no more than a few Planck lengths. Moreover it is possible that scattered photons can be emitted perpendicular to this surface and escape to infinity. Whether such photons carry information characterising this layer is an open question. If detectable, the image of a black hole would appear with a relatively bright spot in the centre of the image. Initial steps toward imaging black holes have already been taken with the Event Horizon Telescope (EHT) (Wielgus *et al*, 2020), but these observations are to date, in their infancy. It can be hoped that similar observations in the future, employing some descendent of the EHT, would make this kind of testing possible.

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