Do causality violations initiating a compactly generated Cauchy horizon require exotic matter?

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Abstract

This short paper summarises the proof that a causality violation formed from a bounded region requires matter that violates the null energy condition. Therefore the answer to the question in the title is yes. This is one of the results on which Stephen Hawking based his *Chronology protection conjecture*. It is conjectured that the required negative mass density fields cannot cover sufficiently large regions of space-time to make causality violations a reality. This lends support to the idea that the evolution of the universe is an entirely unitary process. This work is intended to be more accessible than Hawking's original, which was cast in terms of the Newman-Penrose formalism.

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1. Introduction

Time travel into the past has been a staple of science fiction since the late nineteenth century. For almost a century the question remained: is it really possible? It was not until the early 1990s that researchers found themselves in a position to seriously investigate this question. However, throughout the twentieth century a number of solutions from general relativity appeared that exhibit closed timelike curves (CTCs). Many have speculated that such traversable loops in space-time would be unstable due to vacuum fluctuations being indefinitely amplified by repeated propagation around the loop. Some of the proposed solutions have other problems too, for example the rotating cylinder (van Stockum, 1937) and the rotating universe (Gödel, 1949) are non-compact and therefore are deemed unrealistic. Other solutions such as, wormholes (Morris, Thorne and Yurtsever, 1988), warp drive (Alcubierre, 1994) and the Krasnikov tube (Krasnikov, 1998) require exotic material that, by definition, violates reasonable energy conditions.

In this work we consider a causality violation evolving, without topological transformation, from a past compact subset of a non-compact spacelike surface, *S*. Here a causality violation is defined as a metric admitting a closed non-spacelike curve. The solution is described in general terms and is examined with the aid of the Raychaudhuri equation to see whether it could be realised using normal matter. Such a solution was investigated by Hawking (1992) and shown to be possible as long as fields violating the null energy condition (NEC:

$T_{ab}k^ak^b \geq 0$ for any null 4-vector, k^a) are present. In this work we do not engage in a discussion as to whether NEC violations are possible. Such violations are known to exist in the form of the Casimir effect and certain squeezed quantum states of microscopic 4-volumes. However, over sufficiently large macroscopic volumes of space-time it seems that NEC violations are smeared out (Flanagan and Wald, 1996).

In the following section we present a short discussion of the Raychaudhuri equation describing a congruence of world lines for particles in a gravitating body of fluid. This is briefly described without proof. For an accessible derivation of Raychaudhuri's equation, the reader is referred to Dadhich (2005).

A summary of Hawking's original proof that such a solution is not possible without violating the NEC is presented in section 3. We refer to this as the *classical causality theorem*, and its proof is presented in a form intended to be more accessible than Hawking's original that made use of the Newman-Penrose formalism. The equivalent of Raychaudhuri's equation in this formalism is known as the *Newman-Penrose equation for the convergence* ρ (Hawking 1992, p605). All mathematical descriptions in this work use the Penrose signature $(+---)$, which is my preference over the more popular $(-+++)$. This explains differing signs of certain terms compared with other authors.

2. The Raychaudhuri equation

This equation describes, in general terms, how matter behaves in a space-time manifold that is itself modified by its presence. The behaviour of matter fields is couched in terms of congruencies, families of timelike and null curves, which may be fluid flow lines or histories of photons. These can be expressed in terms of rates of change of expansion (volume changes), vorticity and shear. Raychaudhuri's equation determines rate of change of expansion, and this is central to the derivation of theorems concerning singularities due to divergences of density at *focal* points. We consider a congruence of world lines for a collection of test bodies in space-time that we imagine as initially expanding or contracting. The Raychaudhuri equation encapsulates the idea that nearby material bodies, falling freely under their own gravity will converge provided that we are dealing with matter that has positive density as measured locally. Ignoring non-gravitational interactions, if their gravitational potential is greater in magnitude than their kinetic energy then all of these bodies will converge to a focal point at some future time. If, on the other hand, the kinetic energy is greater, then one can infer that all such bodies emerged from a focal point in the past. These focal points, referred to as conjugate points if a Jacobi field exists which vanishes at these points (O'Neill, 1983, p270), would constitute a singular breakdown of the theory (Dadhich, 2005, p1).

Employing the notation of Hawking and Ellis (1973, p84) the Raychaudhuri equation is given by

$$\frac{d\theta}{d\lambda} = R_{ab}V^aV^b - 2\sigma^2 + 2\omega^2 - \frac{\theta^2}{3} + \dot{V}^a_{\ ;a} \tag{1}$$

The expansion, θ, is differentiated with respect to an affine parameter, λ, which is usually proper time for timelike geodesics. The terms $-2\sigma^2$ and $2\omega^2$ are shear and vorticity scalars respectively. The shear term contributes positive mass-energy to the matter fields and therefore promotes convergence explaining the minus sign. The vorticity term, on the other hand, has the opposite sign. On physical grounds this is to be expected because it encourages expansion by analogy with centrifugal force. When vorticity is non-zero, one can imagine geodesics winding around each other in space-time. The last term is due to non-gravitational acceleration therefore this term vanishes when we consider particles on geodesic trajectories, i.e. $\dot{V}^a \equiv V^a{}_{;b} V^b = 0$. The only other term not explicitly dependent on the expansion is the first term on the right hand side, $R_{ab} V^a V^b$ where V^a is timelike 4-velocity along the congruence of geodesics.

In the following section we use a geometric optics version of Raychaudhuri's equation, which applies only to null geodesics. These have 4-velocity, k^a. This equation is given by

$$\frac{d\hat{\theta}}{d\lambda} = R_{ab} k^a k^b - 2\hat{\sigma}^2 + 2\hat{\omega}^2 - \frac{\hat{\theta}^2}{2} \tag{2}$$

Since local 4-acceleration vanishes for null geodesics also, then there is no corresponding term as seen in equation (1). It is also noted that the *expansion-squared* term has 2 on its denominator as opposed to 3. This reflects the reduced dimensionality of null cones in which the null geodesics are confined. For details see (Hawking and Ellis, 1973, p88; Wald, 1984, p222).

An important step involved in the proof of the classical causality theorem is the elimination of the vorticity term. Hawking and Ellis (1973, p97) and Wald (1984, p226) provide a general proof of this when conjugate points are present. However, using a simple physical argument this may be justified by noting that it vanishes whenever the angular momentum of constituent particles is zero. We note that in the specific case of our time machine model (figure 1), any null geodesic, γ_1, in the Cauchy horizon may be traced indefinitely into its own past and converge asymptotically and without oscillation towards the initial causality violation known as the *fountain*, γ. Assuming that angular momentum is non-zero then in order for γ_1 to converge to γ in the past then, due to the conservation of angular momentum, the angular velocity must diverge in proportion to the reciprocal radius of gyration, and this can only be avoided by setting $\hat{\omega} = 0$ along γ_1.

3. An asymptotically flat causality violation

Before stating and proving the classical causality theorem we describe the formation of a causality violation from the familiar circumstances of a time-orientable space-time to the past of a non-compact spacelike surface, *S*, containing the initial data. The structure consists of three regions delimited by two non-compact hypersurfaces: the surface, *S*, and to its future, the Cauchy horizon, $H^+(S)$, which is a null hypersurface generated by the fountain. The future Cauchy development, $D^+(S)$, is the domain of dependence of *S*, which is bounded to the future by $H^+(S)$ in the presence of a singularity or a causality violation. In general the boundary of $D^+(S)$ is given by $\partial D^+(S) = H^+(S) \cap S$. For the space-time manifold, *M*,

$D^+(S)$ is defined as the set of all points $p \in M$ such that every past-inextendible non-spacelike curve through p intersects S (Hawking and Ellis, 1973, p201). The region to the future of $H^+(S)$ may be described as a *domain of unpredictability* because it contains elements that are independent of the data on S.

The above description is often considered in the context of singularities (Hawking and Ellis, 1973, figure 53, p288) where it may be applied in the case of gravitational collapse. In our case the Cauchy horizon is formed by the presence of an initial causality violation (figure 1).

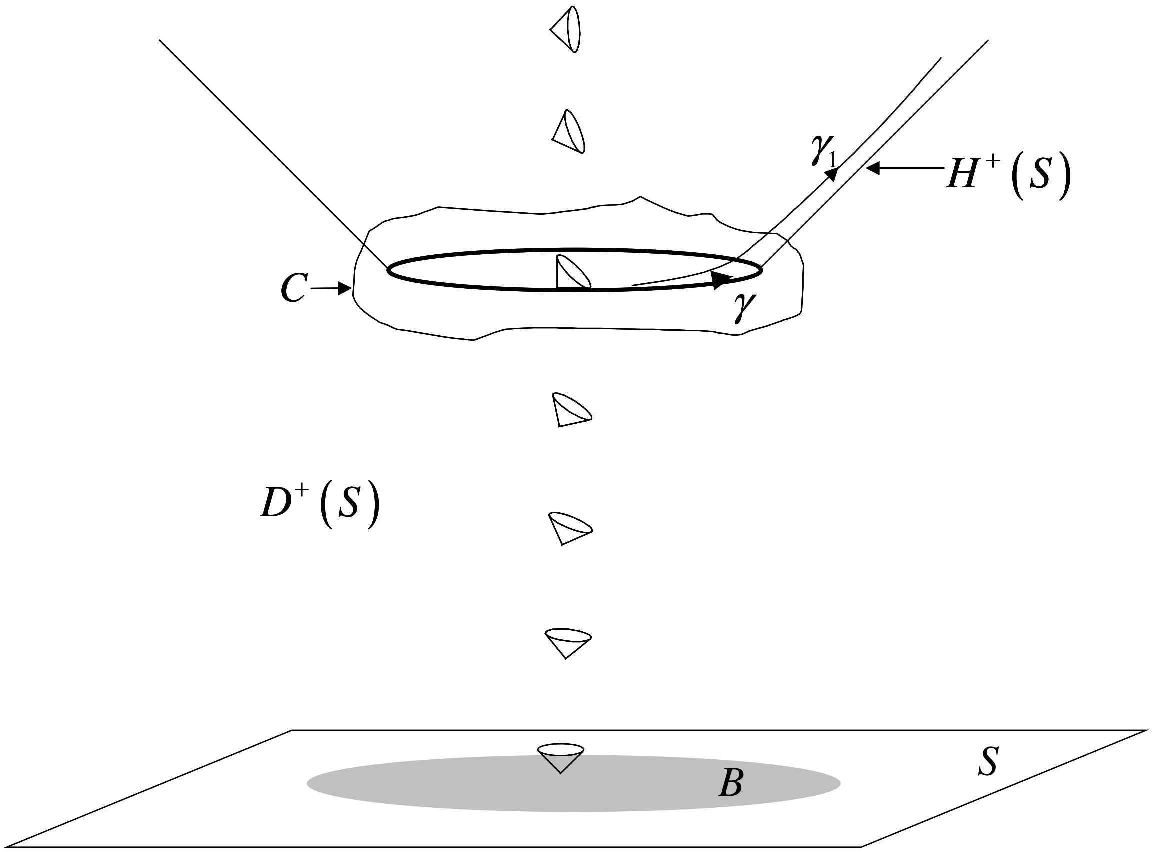

Figure 1: All of the material used to construct the causality violation is in the compact region $B \subseteq S$ (grey patch). As time progresses it is arranged so that the light cones tip within an equatorial plane. Eventually it reaches the *fountain*, γ, which is the initial closed null curve. To the future of $H^+(S)$ the null cones allow signals and matter to wind back from the future.

In figure 1 we see that construction of a causality-violating region begins with material confined to a compact region, $B \subseteq S$. As the system evolves material is configured to cause the light cones to tip in a defined equatorial plane to the point where the leading side of the cone can be continuously projected into its own past. This initial causality violation is the earliest set of events marking the boundary of the causality-violating region. Tipping of the null cones is a real phenomenon and occurs to a significant degree in, for example, rapidly rotating neutron stars and rotating black holes. It is variously known as dragging of the inertial frames or the Lense-Thirring effect. In real world situations it is always accompanied by closing up of the null cones, and this competing effect prevents the situation depicted in figure 1, where the leading side of the null cone loops back to its own past.

Figure 1 shows a particular null geodesic, γ_1, approaching the fountain asymptotically as one traces it into the past. All such geodesics form a set of generators for $H^+(S)$, and it is a feature that any null geodesic in $H^+(S)$ smoothly approaches the fountain with no conjugate points in the past. Null geodesics can be indefinitely traced into their past and remain confined within the compact region, *C*. Knowledge of the Raychaudhuri equation and its properties should alert us to the situation that such features are not possible without violations of the energy conditions. This is the essence of Hawking's proof and is one of the reasons that led him to his chronology protection conjecture.

In what follows we state and prove the classical causality theorem. The statement of the theorem does not follow the description made by Hawking but it does include the essential features. Moreover, this does not disprove the existence of causality violations, only that matter violating the NEC is required to generate them.

Classical causality theorem: Any compactly generated Cauchy horizon formed from an initial causality violation to the future of a non-compact spacelike surface, *S*, requires a violation of the null energy condition (Hawking, 1992, p606).

The structure of the proof is as follows:

- The Raychaudhuri equation with $T_{ab}k^a k^b > 0$ implies that there are conjugate points in the null past of any point in $H^+(S)$ generated by matter satisfying the NEC implying that $\hat{\theta} \leq 0$.
- The geometry of $H^+(S)$ being generated from a compact region implies that $\hat{\theta} > 0$.
- This constitutes a contradiction proving the classical causality theorem.

Proof of the classical causality theorem:

For reasons previously discussed we use the torsionless ($\hat{\omega} = 0$) version of the Raychaudhuri equation for null geodesics presented in terms of the energy-momentum tensor, T_{ab}.

$$\frac{d\hat{\theta}}{d\lambda} = -\kappa T_{ab}k^a k^b - 2\hat{\sigma}^2 - \frac{\hat{\theta}^2}{2}. \tag{3}$$

We can now show that for matter fields satisfying the null energy condition, the expansion in $H^+(S)$ is non-positive ($\hat{\theta} \leq 0$). For suppose in contradiction that there exists a point on a particular null geodesic where $\hat{\theta} > 0$. We can integrate the following differential inequality consistent with the Raychaudhuri equation. Assuming $d\lambda > 0$

$$
\begin{aligned}
\frac{d\hat{\theta}}{d\lambda} \le -\frac{\hat{\theta}^2}{2} \Rightarrow -\frac{2d\hat{\theta}}{\hat{\theta}^2} \ge d\lambda& \\
\Rightarrow -2\int \frac{d\hat{\theta}}{\hat{\theta}^2} \ge \lambda + \lambda_0& \\
\Rightarrow \frac{2}{\hat{\theta}} \ge \lambda + \lambda_0& \\
\Rightarrow \hat{\theta} \le \frac{2}{\lambda + \lambda_0}&.
\end{aligned}
$$

Tracing λ back in a negative time direction we find that as it approaches $\lambda \le -\lambda_0$ then $\hat{\theta}$ becomes infinite in the negative direction. This shows that the expansion is infinite within a finite distance into the past. This contradicts the confinement of past-directed null geodesics in the compact set, *C*. Therefore the expansion is non-positive ($\hat{\theta} \le 0$).

Now we establish a contradiction to this condition ($\hat{\theta} \le 0$). Starting with the geometric definition of the expansion, $\hat{\theta}$, in the Raychaudhuri equation (equation (3))

$$\frac{dA}{d\lambda} = \hat{\theta} A \tag{4}$$

where *A* may be regarded as the cross sectional area of a torus in *C* enclosing the fountain, γ, and λ is an affine parameter on γ increasing in the positive time direction. We may consider a small part of the cross section, δA, containing the fountain, γ, and an affine parameter, *z*, on γ to increase in the negative time direction. With these modifications the defining equation for the expansion now reads

$$\frac{d}{dz}\delta A = -\hat{\theta}\,\delta A\,. \tag{5}$$

Now we may take the small cross section in the limit of zero measure, $\delta A \to dA$, and integrate with respect to *A* over the set $\beta_z\left(H^+(S) \cap C\right)$ where

$$\beta_z : H^+(S) \cap C \to H^+(S) \cap C$$

is any differentiable map. For ease of visualisation we could imagine using the identity map where we just integrate over $H^+(S) \cap C$ on both sides. The corresponding integral is therefore

$$\frac{d}{dz}\int_{\beta_z\left(H^+(S)\cap C\right)} dA = -\int_{\beta_z\left(H^+(S)\cap C\right)} \hat{\theta}\, dA\,.$$

Because β_z maps $H^+(S) \cap C$ into itself, the integral on the left hand side must be ≤ 0. This is because as we trace back in time (increasing *z*) we see a contraction in the cross section, *A*

($dA/dz \le 0$). This shows that the right hand side must also be ≤ 0 (Hawking and Ellis, 1973, p297), and therefore $\hat{\theta} \ge 0$. Moreover because both hypersurfaces, *S*, and $H^{+}(S)$, are non-compact and that $H^{+}(S)$ is generated from within a compact region, *C*, then the expansion must be strictly positive ($\hat{\theta} > 0$). This contradicts the condition that $\hat{\theta} \le 0$, which is a consequence of all the related matter fields satisfying the null energy condition. This completes the proof of the classical causality theorem.

□

3. Concluding remarks

What has been shown in this work is that closed timelike curves require matter violating the null energy condition. Like Hawking himself emphasised, it does not disprove the existence of CTCs because quantum mechanics allows the existence of NEC violating fields in, for example, the Casimir effect and certain squeezed vacuum states. Moreover it can be shown that such fields can persist for an extended *time period*, *T*, where the averaged null energy condition (ANEC) is non-trivially violated (Hawking, 1992; Thorne, 1993). This may be expressed as

$$\int_T T_{ab} k^a k^b d\lambda < 0 .$$

However it has also been shown that when integrated over a suitably extended 4-volume then violation of an appropriately defined ANEC is smeared out (Flanagan and Wald, 1996). So it looks like Hawking's instincts are right after all and therefore this universe is a safe place for historians. Violation of the NEC being the strongest infringement of the energy conditions means that the denial of all other recognised energy conditions are a requirement to generate a causality violation. Moreover the matter violating the weak energy condition possesses a 4-momentum whose temporal component is directed into the past (negative energy). It can be argued therefore that the implied correspondence between causality violations and negative energy density is to be expected. In other words it requires a causality violation in order to generate one. This adds further support to the conjecture that the evolution of the universe as a whole is purely unitary.

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