Generalized Quantum Evidence Theory on Interference Effect *

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Abstract. In this paper, CET [2, 3] is generalized to quantum framework of Hilbert space in an open world, called generalized quantum evidence theory (GQET). Differ with classical GET, interference effects are involved in GQET. Especially, when a GQBBA turns into a classical GBBA, interference effects disappear, so that GQB and GQP functions of GQET degenerate to classical GBel and GPl functions of classical GET, respectively.

Keywords: Generalized quantum evidence theory · Generalized quantum mass function · Generalized quantum belief function · Generalized quantum Plausibility function · Complex evidence theory · Complex mass function · Uncertainty reasoning · Interference effect · Quantum decision.

1 Generalized quantum evidence theory

In this section, CET [2, 3] is generalized to quantum framework of Hilbert space in an open world, called generalized quantum evidence theory (GQET).

Definition 1. (Quantum FOD). Let $|\Phi\rangle$ be a quantum FOD (QFOD), making of a set of mutually exclusive and collectively non-empty events, each of which is expressed as an orthonormal basis $|\phi_g\rangle$ in a Hilbert space:

$$|\Phi\rangle = \{|\phi_1\rangle, \ldots, |\phi_g\rangle, \ldots, |\phi_n\rangle\}.$$  

Definition 2. (Quantum proposition). The power set of $|\Phi\rangle$ is denoted as:

$$2^{|\Phi\rangle} = \{\emptyset, \{|\phi_1\rangle\}, \{|\phi_2\rangle\}, \ldots, \{|\phi_g\rangle\}, \{|\phi_1\phi_2\rangle\}, \ldots, \{|\phi_1\phi_2\ldots\phi_g\rangle\}, \ldots, |\Phi\rangle\},$$

in which $\emptyset$ denotes an unknown event.

Eq. (2) can be simply represented as:

$$2^{|\Phi\rangle} = \{\emptyset, |\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_g\rangle, |\phi_1\phi_2\rangle, \ldots, |\phi_1\phi_2\ldots\phi_g\rangle, \ldots, |\Phi\rangle\}.$$ 

$|\psi_j\rangle$ is defined as a quantum proposition, when $|\psi_j\rangle \in 2^{|\Phi\rangle}$.

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Definition 3. (Generalized quantum mass function). A generalized quantum mass function (GQMF) $Q_M$ in QFOD $|\Phi\rangle$, also called a GQBBA, mapping from $2^{(|\Phi\rangle)}$ to $\mathbb{C}$, is defined by

$$Q_M : 2^{(|\Phi\rangle)} \rightarrow \mathbb{C},$$

and satisfies:

$$Q_M(|\psi_j\rangle) = m(|\psi_j\rangle)e^{i\theta(|\psi_j\rangle)}, \quad |\psi_j\rangle \in 2^{(|\Phi\rangle)},$$

$$\sum_{|\psi_j\rangle \in 2^{(|\Phi\rangle)}} Q_M(|\psi_j\rangle) = 1,$$

in which $i = \sqrt{-1}$, $m(|\psi_j\rangle) \in [0, 1]$ denotes the magnitude of $Q_M(|\psi_j\rangle)$, and $\theta(|\psi_j\rangle)$ denotes a phase term.

$Q_M(|\psi_j\rangle)$ is also represented as:

$$Q_M(|\psi_j\rangle) = x_j + y_j i, \quad |\psi_j\rangle \in 2^{(|\Phi\rangle)},$$

and its magnitude is expressed as:

$$|Q_M(|\psi_j\rangle)| = m(|\psi_j\rangle) = \sqrt{x_j^2 + y_j^2},$$

where $\sqrt{x_j^2 + y_j^2} \in [0, 1].$

Definition 4. (Quantum focal element). If $|Q_M(|\psi_j\rangle)|$ or $m(|\psi_j\rangle) > 0$, $|\psi_j\rangle$ is defined as a quantum focal element. The value of $|Q_M(|\psi_j\rangle)|$ or $m(|\psi_j\rangle)$ represents the degree to which QBBA supports $|\psi_j\rangle$.

Definition 5. (Quantum evidential combination rule). Let $Q_{M_h}$ and $Q_{M_k}$ be two independently GQBBA with propositions $|\psi_p\rangle$ and $|\psi_q\rangle$ in QFOD $|\Phi\rangle$, respectively. Quantum evidential combination rule (QECR), denoted as $Q_M = Q_{M_h} \oplus Q_{M_k}$ is defined by:

$$Q_M(|\psi_j\rangle) = \frac{\sum_{|\psi_p\rangle \cap |\psi_q\rangle = \psi_j} Q_{M_h}(|\psi_p\rangle)Q_{M_k}(|\psi_q\rangle)}{1 - K_Q}, \quad |\psi_j\rangle \in 2^{(|\Phi\rangle)},$$

with

$$K_Q = \sum_{|\psi_p\rangle \cap |\psi_q\rangle = |\emptyset\rangle} Q_{M_h}(|\psi_p\rangle)Q_{M_k}(|\psi_q\rangle),$$

in which $K_Q$ is the conflict coefficient between $Q_{M_h}$ and $Q_{M_k}$.

Definition 6. (Generalized quantum belief function) Let $Q_M$ be a GQBBA with proposition $|\psi_j\rangle \in 2^{(|\Phi\rangle)}$. The generalized quantum belief (GQB) function, denoted as GQBel is defined by:

$$GQBel(|\psi_j\rangle) = \begin{cases} \sum_{|\psi_p\rangle \subseteq |\psi_j\rangle} Q_M(|\psi_p\rangle), & |\psi_j\rangle \neq \emptyset, \\ Q_M(|\psi_j\rangle), & |\psi_j\rangle = \emptyset. \end{cases}$$


Let $|\phi_p\rangle, |\phi_q\rangle \subseteq |\psi_j\rangle (|\psi_j\rangle \neq \emptyset)$ with $s$ items. The amplitude of $\text{GQBel}(|\psi_j\rangle)$ is calculated as:

$$|\text{GQBel}(|\psi_j\rangle)| = \sum_{|\phi_p\rangle \subseteq |\psi_j\rangle} |\text{Qm}(|\psi_p\rangle)| + 2 \sum_{p=1}^{s-1} \sum_{q=p+1}^s |\text{Qm}(|\psi_p\rangle)||\text{Qm}(|\psi_q\rangle)| \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)]$$

$$= \sum_{|\phi_p\rangle \subseteq |\psi_j\rangle} \text{m}(|\psi_p\rangle) + 2 \sum_{p=1}^{s-1} \sum_{q=p+1}^s \text{m}(|\psi_p\rangle)\text{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)].$$

When $\cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)] = 0$, Eq. (11) converges to classical GBel [1], such that:

$$|\text{GQBel}(|\psi_j\rangle)| = \sum_{|\phi_p\rangle \subseteq |\psi_j\rangle |\psi_j\rangle \neq \emptyset} \text{m}(|\psi_p\rangle).$$

**Definition 7.** (Interference effect in GQB function) Let $Q_M$ be a GQBBA with propositions $|\psi_p\rangle, |\phi_q\rangle \subseteq |\psi_j\rangle (|\psi_j\rangle \neq \emptyset)$ with $s$ items. The interference effect for $|\psi_j\rangle$ in GQB function is defined by:

$$\text{Int}_{\text{GQB}}(|\psi_j\rangle) = 2 \sum_{p=1}^{s-1} \sum_{q=p+1}^s \text{m}(|\psi_p\rangle)\text{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)].$$

where $\text{m}$ and $\theta$ are magnitude and phase terms of $Q_M$.

When GQBBA only consists of singletons, the amplitude of $\text{GQBel}(|\psi_{12...n}\rangle)$ is calculated as:

$$|\text{GQBel}(|\psi_{12...n}\rangle)| = \sum_{|\phi_p\rangle \subseteq |\psi_{12...n}\rangle} \text{m}(|\phi_p\rangle) + \text{Int}_{\text{GQB}}(|\psi_{12...n}\rangle);$$

and $\text{Int}_{\text{GQB}}(|\psi_{12...n}\rangle)$ is calculated as:

$$\text{Int}_{\text{GQB}}(|\psi_{12...n}\rangle) = 2 \sum_{p=1}^{n-1} \sum_{q=p+1}^n \text{m}(|\psi_p\rangle)\text{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)].$$

**Definition 8.** (Generalized quantum Plausibility function) Let $Q_M$ be a GQBBA with proposition $|\psi_j\rangle \in 2^{|\psi\rangle}$. The generalized quantum Plausibility (GQP) function, denoted as GQPl is defined by:

$$\text{GQPl}(|\psi_j\rangle) = \begin{cases} \sum_{|\phi_u\rangle \cap |\psi_j\rangle \neq \emptyset} Q_M(|\psi_u\rangle), & |\psi_j\rangle \neq \emptyset, \\ Q_M(|\psi_j\rangle), & |\psi_j\rangle = \emptyset. \end{cases}$$

Let $|\phi_u\rangle, |\phi_v\rangle \cap |\psi_j\rangle \neq \emptyset (|\psi_j\rangle \neq \emptyset)$ with $t$ items. The amplitude of $\text{GQPl}(|\psi_j\rangle)$ is calculated as:

$$|\text{GQPl}(|\psi_j\rangle)| = \sum_{|\phi_u\rangle \cap |\psi_j\rangle \neq \emptyset} |Q_M(|\psi_u\rangle)| + 2 \sum_{u=1}^{t-1} \sum_{v=u+1}^t |Q_M(|\psi_u\rangle)||Q_M(|\psi_v\rangle)| \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)]$$

$$= \sum_{|\phi_u\rangle \cap |\psi_j\rangle \neq \emptyset} \text{m}(|\psi_u\rangle) + 2 \sum_{u=1}^{t-1} \sum_{v=u+1}^t \text{m}(|\psi_u\rangle)\text{m}(|\psi_v\rangle) \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)].$$
When \( \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)] = 0 \), Eq. (17) converges to classical GPl [1], such that:

\[
|\text{GQPl}(|\psi_j\rangle)| = \sum_{|\phi_u\rangle \cap |\psi_j\rangle \neq \emptyset, |\psi_j\rangle \neq \emptyset} m(|\psi_u\rangle).
\] (18)

**Definition 9.** *(Interference effect in GQP function)* Let \( Q_M \) be a GQBBA with propositions \( |\phi_u\rangle, |\phi_v\rangle \cap |\psi_j\rangle \neq \emptyset (|\psi_j\rangle \neq \emptyset) \) with \( t \) items. The interference effect for \( |\psi_j\rangle \) in GQP function is defined by:

\[
\text{Int}_{\text{GQP}}(|\psi_j\rangle) = 2 \sum_{u=1}^{t-1} \sum_{v=u+1}^t m(|\psi_u\rangle)m(|\psi_v\rangle) \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)].
\] (19)
Bibliography