Synthesizing Regular, Iterative, and Recursive Integration by Parts

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Abstract

We give a template for all forms of integration by parts: single use, iterative, and recursive.

Introduction

This article provides fast ways to use integration by parts (IBP). Text books typically give examples of three instances where IBP is useful: regular (one use of IBP); iterative (more than one use); and recursive (or reduction) formulations. The three are treated separately [2]. We start with a single use example and show how a template can be used for it. This template also works for the iterative and recursive cases; we show this with further examples. If the program works, our template will be easily recalled and give fast ways to use IBP for all typical IBP problems. This template is based on tabular integration by parts [1].

Single Use

There are several integration problems that require integration by parts. An example of a standard function that does not have any other technique that works is to integrate $\ln x$. We will start from scratch with the goal of integrating $\ln x$.

One might recall that IBP is a consequence of the product rule for differentiation. That is $(uv)' = u'v + v'u$ gives

$$\int (uv)' = \int u'v + \int v'u$$

and this in turn implies

$$\int u'v = uv - \int v'u.$$

We can infer from this derivation that IBP is for integrands that are products of two functions: these functions should be chosen so the integrations and differentiations involved are possible, if not easy. In the case of $\ln x$, we can make 1 a function; it is easily integrated and $\ln x$ is easily differentiated. We assign $u' = 1$ and $v = \ln x$ If $u' = 1$ then $u = \int u' = \int 1 = x$ and $v = \ln x$ gives $v' = 1/x$. So $\int u'v = \int \ln x = x \ln x - \int xx^{-1} = x \ln x - x$. As usual, differentiation confirms that this is correct.

Here's a faster way to remember IBP. Make the following template, Table 1. Table 2 shows the template's use for $\int \ln x$. Reading the table, the diagonal

	A	B	C
1		$\int$	d/dx
2		(u')	
3	$+$	(u)	(v)
4	$-$		(v')

Table 1: Read off $\int u'v = uv - \int v'u$. The uv is given by the multiplication of row 3 cell entries.

$B2$ and $C3$ gives the start up integral $\int (1) \ln x$, row 3 gives the uv part or $x \ln x$. Now ask if the integrand formed from multiplying the cell contents of $B3$ and $C4$ is integrable. If so, integrate it. In this case the integral is $\int xx^{-1} = \int 1$, an easy integration. The answer is $\int \ln x \text{ dx} = x \ln x - x$ (plus C of course). We are reminded to use the negative sign via cell A4.

Here's another example: $\int \tan^{-1}(x) \text{ dx}$. We can integrate $\int \frac{x}{1+x^2} \text{ dx}$ with the usual substitution: $u = 1 + x^2$, $du = 2x$, giving

$$\frac{1}{2} \int \frac{2x \text{ dx}}{1+x^2} = \frac{1}{2} \int \frac{\text{du}}{u} = \frac{1}{2} \ln(1 + x^2).$$

Thus the answer is $\int \tan^{-1}(x) \text{ dx} = x \tan^{-1}(x) - \frac{1}{2} \ln(1 + x^2)$.

	A	B	C
1		$\int$	d/dx
2		$1(u')$	
3	$+$	$x(u)$	$\ln x(v)$
4	$-$		$1/x(v')$

Table 2: It is fast to fill in the template for 1 and $\ln(x)$.

	A	B	C
1		$\int$	d/dx
2		$1(u')$	
3	$+$	$x(u)$	$\tan^{-1}(x)(v)$
4	$-$		$1/(1+x^2)(v')$

Table 3: Is $\int \frac{x}{(1+x^2)}$ easily integrated?

Iterative Use

Integrating $\ln x$ requires one instance of IBP, but many problems require multiple or iterative uses of IBP.

One can find an easy integral using this template. For example, in Table 4 we calculate $\int x^7$ using the two functions x^3 and x^4. We know the answer is $x^8/8$. With iterative IBP one idea is to chose a function that when repeatedly differentiated yields the zero function. The template allows for the continuation of the pattern of the single use idea. We, in effect, convert the integral to the sum of uv rows. Table 4 shows these ideas.

The product of xs in each row is x^8, so finding the dot product of the coefficients in each column should give us 1/8. Using Maple we get a confirmation of this, Figure 1.

In Table 5, we've split up x^7 using x^4 and x^3. We, of course, get different coefficients, but Maple will show the dot product of the coefficients in each column is 1/8. It seems to work. Let's try a more challenging integration.

For a single use of IBPT we ask a question, but for the iterative use we actually get the integral. Consider the integral $\int x^2 \sin(x)$ dx.

We quickly make our template for a single use, Table 6. We ask whether or not $-\int 2x \cos x$ is easily integrated. It isn't so we ramp up to the iterative

	A	B	C
1		$\int$	d/dx
2		$x^3(u')$	
3	$+$	$\frac{x^4}{4}(u)$	$x^4(v)$
4	$-$	$\frac{x^5}{20}$	$4x^3(v')$
5	$+$	$\frac{x^6}{120}$	$12x^2$
6	$-$	$\frac{x^7}{840}$	$24x$
7	$+$	$\frac{x^8}{8*840}$	24
8	$-$		0

Table 4: A proof of concept for iterative IBP via a template (IBPT).

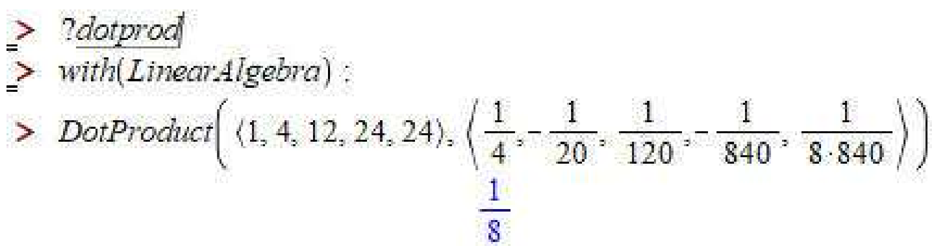

Figure 1: Maple computes the dot product of the coefficients in columns B and C. As expected, we get 1/8.

mode and continue integrating column B and differentiating column C.

The structure of the table allows for a new round to start. We are, in effect, just resetting the u' and v rows. The iterative continuation is given in Table 7. We arrive at $-x^2 \cos x + 2x \sin x + 2 \cos x$ and taking derivatives we get a confirmation.

	A	B	C
1		$\int$	d/dx
2		$x^4(u')$	
3	$+$	$\frac{x^5}{5}(u)$	$x^3(v)$
4	$-$	$\frac{x^6}{30}$	$3x^2(v')$
5	$+$	$\frac{x^7}{210}$	$6x$
6	$-$	$\frac{x^8}{8*210}$	6
7	$+$		0

Table 5: Different partitions of 7 yield the same dot product result.

	A	B	C
1		$\int$	d/dx
2		$\sin x(u')$	
3	$+$	$-\cos x(u)$	$x^2(v)$
4	$-$		$2x(v')$

Table 6: The single use question is asked and then the iterative mode clicks in.

Here's another example of iterative integration by parts using a template. Integrate x^2e^x. It's easy. Referring to Table 8, The answer is $x^2e^x - 2xe^x + 2e^x$.

	A	B	C
1		$\int$	d/dx
2		$\sin x(u')$	
3	+	$-\cos x(u)$ becomes new u'	$x^2(v)$
4	−	$-\sin x$ becomes new u	$2x(v')$ becomes new v
5	+	$\cos x$	2 becomes new v'
6	−		0

Table 7: The iterative mode yields the complete answer.

	A	B	C
1		$\int$	d/dx
2		$e^x(u')$	
3	+	$e^x(u)$	$x^2(v)$
4	−	e^x	$2x(v')$
5	+	e^x	2
6	−		0

Table 8: Integrating x^2e^x using iterative integration by parts with template.

Recursive

Recursive formulations are needed for such integrals as $\int \cos^n(x)$ dx, $\int \sin^n(x)$ dx, and $\int e^{ax}\cos(bx)$ dx. The idea is to repeatedly (iteratively) use IBP until a constant times the original integrand appears. One can then solve for the original integrand. Thus with a single use if the original integrand appears, stop; if it doesn't continue with the iterative mode until such an integrand appears. Examples will give the idea.

Create a recursive formula that allows for integrating $\int cos^n(x)$ dx. Using Table 9, we contemplate whether the pattern

$$\int \cos^n(x)\ \mathrm{dx} = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\ \mathrm{dx} \tag{1}$$

has a right side integral that could give a recursive formula – a reduction in the power of $\cos(x)$. Yes, we can rewrite (1) this integral:

$$\int \cos^{n-2}(x)\sin^2(x)\ \mathrm{dx} = \int \cos^{n-2}(x)(1-\cos^2(x))\ \mathrm{dx}$$

and this gives

$$\int \cos^{n-2}(x)(1-\cos^2(x))\ \mathrm{dx} = \int \cos^{n-2}(x)\ \mathrm{dx} - \int \cos^{n}(x)\ \mathrm{dx}$$

	A	B	C
1		$\int$	d/dx
2		$\cos(x)(u')$	
3	$+$	$\sin(x)(u)$	$\cos^{n-1}(x)(v)$
4	$-$		$(n-1)\cos^{n-2}(x)(-\sin(x))(v')$

Table 9: A single use IBP for a possible recursive formulation.

With a little algebra this yields

$$\int \cos^{n}(x)\ \mathrm{dx} = \frac{\sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\ \mathrm{dx}}{n}, \tag{2}$$

a power reducing formula.

We can use a notational system to avoid writing a lot. In the spirit of synthetic division, the formula in (2) is

$$\{n\} : \frac{1}{n}[\text{ n-1} \quad 1] + \frac{n-1}{n}\{\text{ n-2 }\}$$

where the brackets indicate first cos to a power times sin to a power and curly braces mean apply the formula for the enclosed natural number. So reads $\cos^4(x)$ is

$$\{4\} : \frac{1}{4}[\ 3 \quad 1] + \frac{3}{4}\{\ 2\ \}$$

and then applying the formula for a power of 2

$$\{4\} : \frac{1}{4}[\ 3 \quad 1] + \frac{3}{4}\{\frac{1}{2}[\ 1 \quad 1] + \frac{1}{2}\{0\}\}$$

yielding

$$\frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left(\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right).$$

We check this with Maple, Figure (2).

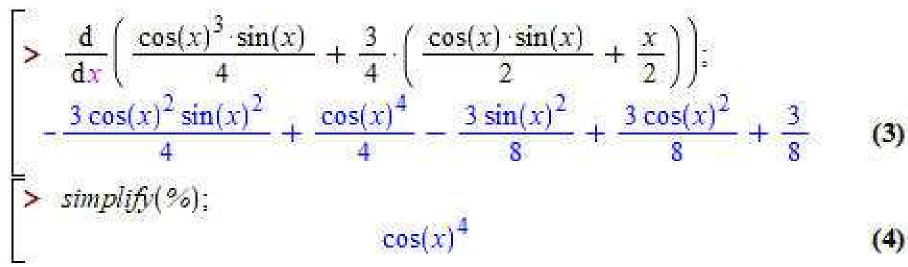

Figure 2: Maple confirms the formula for the fourth power of cosine is correct.

Here is the notation for $\cos(x)^5$:

$$\{5\} : \frac{1}{5}[\ 4 \quad 1] + \frac{4}{5}\{\frac{1}{3}[\ 2 \quad 1] + \frac{2}{3}\{\int 1\}\}$$

giving

$$\frac{\cos^4(x)\sin(x)}{5} + \frac{4}{5}\left(\frac{\cos^2(x)\sin(x)}{3} + \frac{2}{3}\sin(x)\right).$$

Maple in Figure 3 confirms that this is correct.

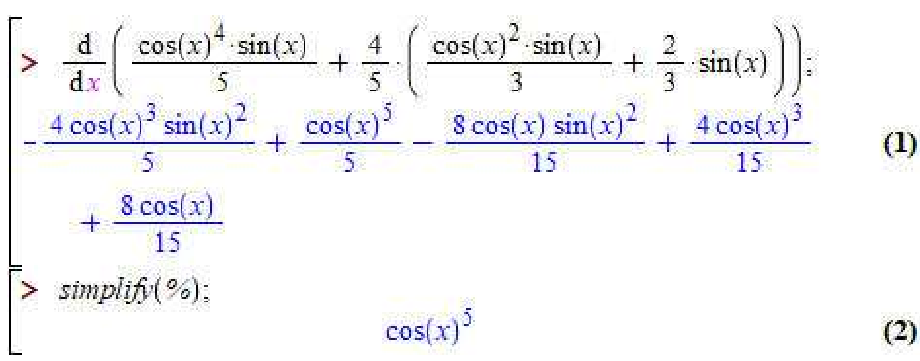

Figure 3: Maple confirms the formula for the fifth power of cosine is correct.

Using our template allows for some flexibility. We can continue down the template iteratively or stop for a single use – as we did with the first recursive formulation. We see this flexibility in our next example.

Create a reduction formula for $\int e^{ax} \cos bx$ dx. Table 10 asks us to consider the $v'u$ integral. It isn't in the right form. We need the original integral times something to be repeated. So, as indicated in Table 11, we continue for another iteration.

With a little algebra we arrive at our formula. We first read off from this Table

$$\int e^{ax} \cos bx \text{ dx} = \frac{e^{ax} \sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \text{ dx}.$$

The goal is not to reach zero in column C (as with usual iterations) but to see a repeat of the original integral. We ask a question of the product given by cells B3 and C4: does the integral formed from this product repeat the original with a multiplier? It does.

	A	B	C
1		$\int$	d/dx
2		$\cos bx(u')$	
3	$+$	$\frac{1}{b} \sin bx(u)$	e^{ax}(v)
4	$-$		$ae^{ax}(v')$

Table 10: Single use IBP doesn't deliver back the original integral.

Next, letting $J_n = \int e^{ax} \cos(bx)$ dx, we find

$$J_n + \frac{a^2}{b^2} J_n = \frac{e^{ax} b \sin(bx) + e^{ax} a \cos(bx)}{b^2}. \tag{3}$$

Now

$$J_n + \frac{a^2}{b^2} J_n = J_n(1 + \frac{a^2}{b^2}) = J_n(\frac{a^2 + b^2}{b^2})$$

and with (3) this gives

$$J_n = \frac{e^{ax} b \sin(bx) + e^{ax} a \cos(bx)}{a^2 + b^2}.$$

Figure 4 provides a check that this formulation is correct using Maple.

	A	B	C
1		$\int$	d/dx
2		$\cos bx(u')$	
3	$+$	$\frac{1}{b}\sin bx(u)$	e^{ax}(v)
4	$-$	$-\frac{1}{b^2}\cos bx$	$ae^{ax}(v')$
5	$+$	"	a^2e^{ax}

Table 11: Another iteration and we have the needed re-occurrence of the original integral.

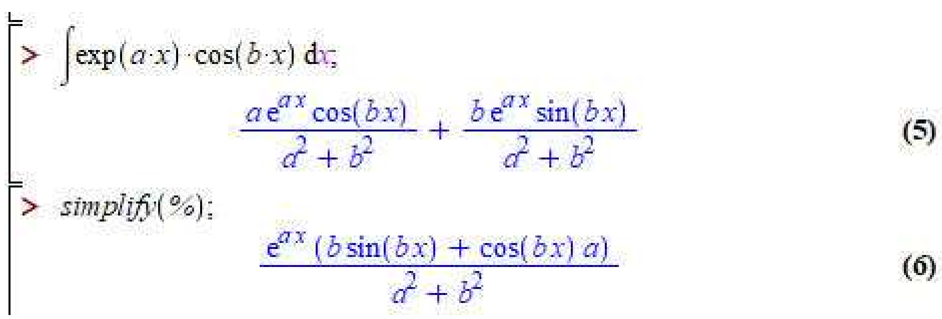

Figure 4: Maple confirms that this reduction formulation is correct.

Conclusion

There are many integral problems that can be generated from these templates. One can break in at any row and go down and over equals straight across minus another down and over. So, for example, using Table 5, the integral given by B4 times C5 equals B5 times C5 - integral of B5 times C6. This makes it clear why this template works; one is just resetting with each move down the columns the same structure as the first embedded IBP.

Another interesting aspect of these tables is one can use them in reverse, going up rather than going down. So, using the same Table 5, what is the fourth derivative of $x^8/8(210)$, it is four up from where it occurs: x^4. See Figure 5. Going up using column C, we find nesting integrals – an usual form of integration. See Figure 6.

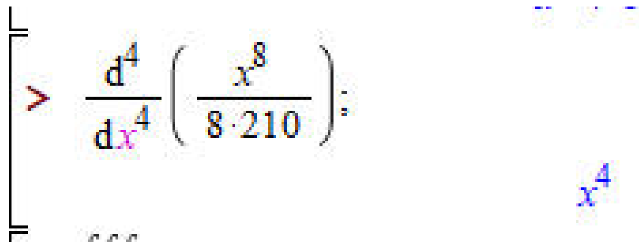

Figure 5: As integration is the inverse of differentiation, such results can be observed.

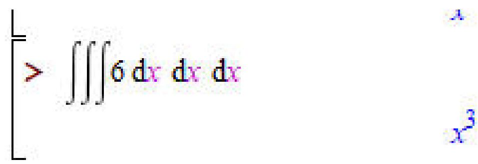

Figure 6: Nesting intervals – all with same dx are an unusual form.

References

[1] Larson, R. and Edwards, B. H. (2010). *Calculus*, 9th ed., New York: Brooks/Cole.

[2] Thomas, G. B. (1968). *Calculus and Analytic Geometry*, 4th ed., Reading, Massachusetts: Addison-Wesley.