Optimal Vantage Points for Crowds

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For a typically 2D distributed crowd of viewers all trying to get a line-of-sight (LOS) to the same object, the close-packed distribution favors only the few closest viewers who inadvertently occlude the line-of-sight (LOS) for the rest of the group stuck behind them. In this paper we study optimal arrangements of the viewers which guarantee everyone gets a LOS. We discuss analytical solutions and numerical simulations for N viewers of both the point-like as well as the extended line-like central object, suggesting applications ranging from focusing power sources to waiting at the airport baggage claim.

I. INTRODUCTION

Society is increasingly prone to situations where crowds of people compete for access to a resource, and while the sophistication of such situations continue to spiral upwards alongside the technology that enables it (Internet, 5G, electric vehicle charging stations, etc.), we are still dealing with situations where just simple geometry unnecessarily impedes everyone from enjoying access, such as line-of-sight (LOS) issues with the crowd around a stage, and, as many well-traveled readers must be aware, the airport baggage claim.

We make a distinction between a crowd and an audience: while the latter is typically pre-conditioned for optimal viewing, with known numbers of viewers and fixed seating arrangements in, e.g., rising rows to guarantee LOS, the former is a spontaneous phenomenon that can arise anywhere, with an unforeseen number of viewers, and typically in a 2D environment — because after all humans spend most of their time on flat planes.

This type of LOS problem is quite general, in fact: from people on the street gathering around a spectacle, to commuters in a train station trying to all read the same schedule screen, to the photographer trying to fit everyone into a group photo. The issue is that people are not merely sets of eyes – they have bodies too, which potentially occlude other people. To generalize the terminology, we are dealing with 'viewers', meaning any observer with a sensor element and a physically-extended body, so this would include cameras, drones, and other viewing apparatuses[1], and a 'central object', which could either be point-like or extended in physical space. In this terminology, any situation where a discrete group of viewers of significant size communicate with a central object via straight-line connections would give rise to the LOS problem we study.

To keep the problem manageable in a short article, we will focus on two basic situations: the point-like central object, and the line-like central object. An example of the former might be a crowd of viewers surrounding a popular speaker, while the latter we have experienced in the airport baggage claim carousel (approximated by a series of line segments). While in the former case it is merely annoying to not be able to see the point of interest, in the latter case it is in addition highly inefficient, for what typically happens is the first arrivers stand as close as they can to the conveyer belt, blocking the view of later arrivers, and thus leaving the majority of people waiting for their luggage without a LOS. This majority then needs to wait for the minority of first arrivers to take their bags first, which may take a long time given that the output of bags on the belt is essentially random.

We thus inquire whether there there might be a simple solution heretofore unrecognized. Given that the close-pack greedy configuration of viewers is clearly not optimal, what is the best arrangement that both guarantees everyone obtains a clear LOS and minimizes the spread of the crowd? And is there a simple practical rule we can employ such that, if viewers were to be aware to leave a bit more space around themselves, this optimal arrangement may be achieved? This may not only give everyone a better viewing experience, improving the efficiency of the situation, but also lower collective stress and confusion in such crowds.

II. N OBSERVERS OF A CENTRAL POINT: THE PROPHET

We start with the simplest situation in which N observers attempt to obtain a LOS to a central point, a scenario we dub "The Prophet" as it is reminiscent of a group of people gathering around a great person, all eager to get a

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[1] Conversely, it would also include observers with an emission element such as lasers, for then the problem is reversed to optimize the delivery of power to a central object. We will return to this point later.

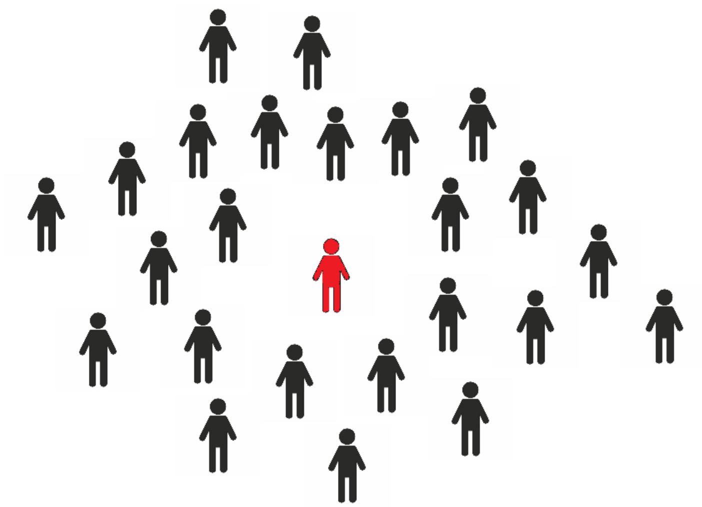


FIG. 1: *Schematic representation of 'The Prophet'*

glimpse. Figure 1 shows a schematic of the situation. For more definite handling, we model viewers as circles with fixed radius r: this could either be a physical boundary or virtual 'personal space' around each observer that must be honored – viewers are not allowed to intersect either each other or the central point, nor can they occlude each others' lines of sight.

Beyond the trivial cases of N=1 and N=2, where the best solution is for each viewer to be right adjacent to the central point, securing LOS for all viewers is always possible by having everyone form a circle around the central point, (see Figure 2) but this is clearly not optimal, as some viewers can move closer without occluding others. The optimal arrangement in this case actually depends on how much you weigh the importance of being close to the central point, as we discuss next.

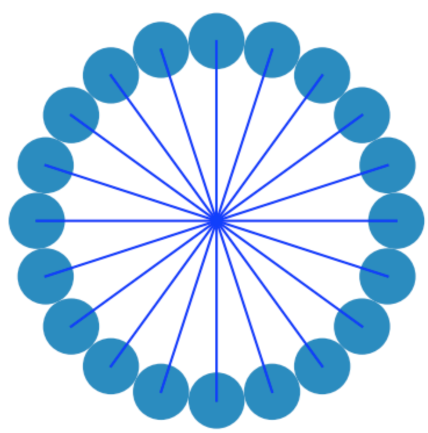


FIG. 2: *All circular solutions work, but are not optimal, e.g. any viewer can move closer.*

II.1. Weighting Metric

As we mentioned above, for $N >= 3$ it turns out that the optimal arrangement of viewers around the central point depends on how much value one assigns to being closer to the central point, defined by a 'weighting function' $W(d)$ which, as a function of each viewer's distance d to the central point, when summed over all the viewers' distances in the configuration, $\sum_i W(d_i)$, gives a collective potential (or loss) quantifying the goodness of one configuration relative to another.

For an ordinary distance weighting such as $W(d) = d$, for example, one minimizes the loss $\sum_{i=1}^{N} W(d_i)$ which is proportional to the average distance to the central point; this is a fairly intuitive idea, as this tends to symmetrize the distribution of viewers such that none of them are unreasonably far away. Imagining physical examples, this type of weighting would make sense in the situation of say people all trying to see the same sign, or for say a group photo where each person's face should occupy roughly the same angular spread.

For an inverse-distance weighting on the other hand, e.g. by the inverse-square $W(d) = d^{-2}$, one would wish to *maximize* the quantity $\sum_{i=1}^{N} W(d_i)$ which is mathematically proportional to the 'power' exchange between each viewer and the central point, as power falls with the square of distance (e.g. for physical sources of light, sound, and gravity). This kind of weighting would make sense, for example, in the situation where each viewer was actually a laser or heat lamp and one was trying to maximize the power delivered to a sample at the central point. Note that maximizing this power is not necessarily equivalent to minimizing the average viewer distance, as we will now demonstrate.

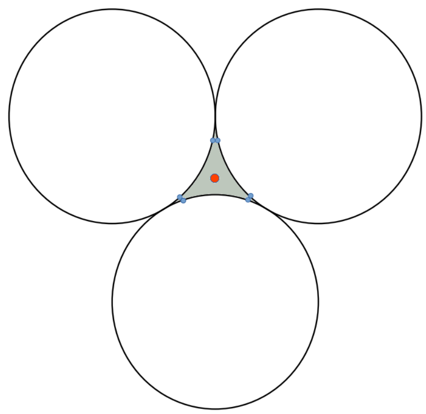


FIG. 3: *Optimal center point positions for 3 viewers: the red central point, at the Fermat Point, is optimal for distance-weighting, while any of the 6 blue points are optimal for inverse-square-distance weighting.*

Consider, for example, N=3 (Figure 3). Clearly all three viewers will want to get as close the central point as possible, so that they end up being mutually tangent ("close-packed"). However, where will the central point be within the shaded region? For distance weighting $W(d) = d$, where one is trying to minimize the loss $\sum_i W(d_i)$, i.e. average distance to the central point, it should come as no surprise that the central point will be exactly in the red center point, i.e. the "Fermat Point" of the equilateral triangle formed by the three viewer centers. For the inverse-square-distance weighting, $W(d) = d^{-2}$, however, maximizing the power $\sum_i W(d_i)$ results in a 6-fold degenerate solution where the central point ends up being tangent to one of the viewers, slightly displaced from the nearest corner of the shaded region (see Appendix A for derivation) — the blue points in Figure 3. This makes the counterintuitive prediction, for example, that if one has three spherical lightbulbs in a triangular configuration, the brightest spot is not in the center where all three bulbs' luminosities contribute equally, but displaced off to the side at any of the six special 'max power' points — one can compute the power is about 5% higher there (again see Appendix A).

Clearly then, the choice of the weighting function $W(d)$ is important and generally produces quite different predictions, but we do not consider the general case in this article. We will only consider the weighting functions $W(d) = d$ and $W(d) = d^{-2}$, due to their direct physical interpretation and relevance to the main practical LOS problems we wish to investigate.

II.2. Monte Carlo simulation

Given that already for N=3 non-trivial solutions arise, we can only expect that an analytical solution will become difficult if not impossible for larger values of N. Thus we employ Monte Carlo (MC) simulations[1] of the points' configurations in order to guide our understanding of optimal arrangements, hopefully enabling us to recognize a pattern leading to an analytical solution for any N.

In our initial MC design, we set the central point at the origin and considered just randomly choosing locations for N viewers for each trial and computing their power or loss. However, this was inefficient in that the vast majority of random configurations of N viewers were not permissable (either the viewers intersected each other, or occluded each other's LOS), and those which were permitted were quite disperse, i.e. far from optimal, requiring an unmanageable number of such trials to have hopes of landing on an optimal solution.

Thus, we settled on first randomly placing the N viewers in a permissible but disperse configuration, and then gradually relaxing them towards the optimal solution via a pseudo-physical simulation wherein each viewer is pulled towards the origin by a 'force', and likewise repelled from nearby viewers by another, configurable force. At each step of the simulation, each viewer's position is updated based on the vector sum of these attractive and repulsive forces, as long as the new position doesn't intersect another viewer or origin, or occlude another viewer's LOS. There are numerous static parameters of the simulation, including the number of steps, the size of each viewer's radius and size of the area around the origin in which the viewers are initialized, as well as dynamic parameters such as the relative strength of the attractive and repulsive forces, and the distance dependence of said forces. In practice, we found it useful to adaptively reduce the magnitude of the dynamic parameters as the configuration relaxed towards the origin, till the inter-viewer repulsion was zero, and run the simulation over thousands of random initial locations to select the configuration with the best fitness (either minimal total distance or maximal power, depending on the weighting scheme) . Note the final best result is scale- and parameter-independent: it is just the optimal arrangement of circles whose centers are connected to the origin along unobstructed line segments. Besides being a mathematical technique to find optimal solutions, the MC may be also interpreted as how actual people might behave in closing in on a point of interest and may reproduce some of the unplanned phenomena that occur in the process, such as viewers getting in each other's way and adjusting their positions accordingly. Watching the simulation relax to an optimal configuration with each run was thus similar to watching an actual crowd converging, as sometimes viewers got 'stuck' in non-optimal positions or repelled away into better positions, though there was of course no actual decision-making on the part of the viewers.

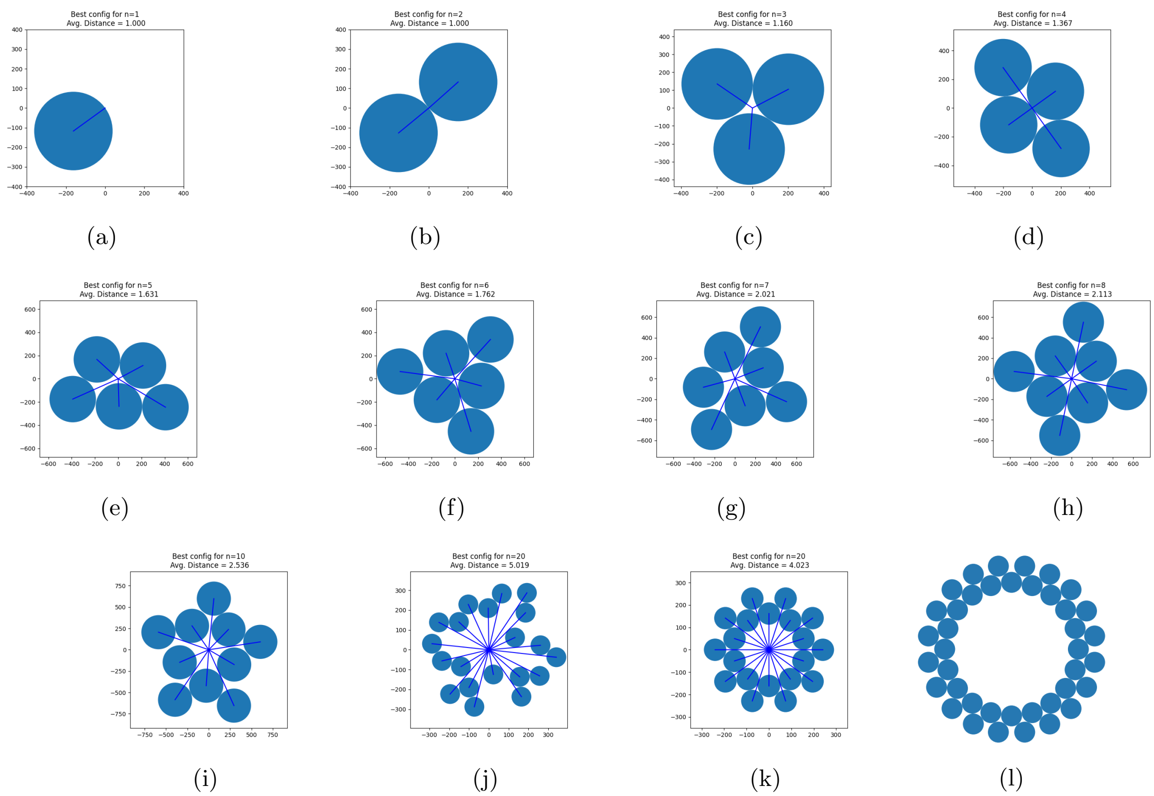
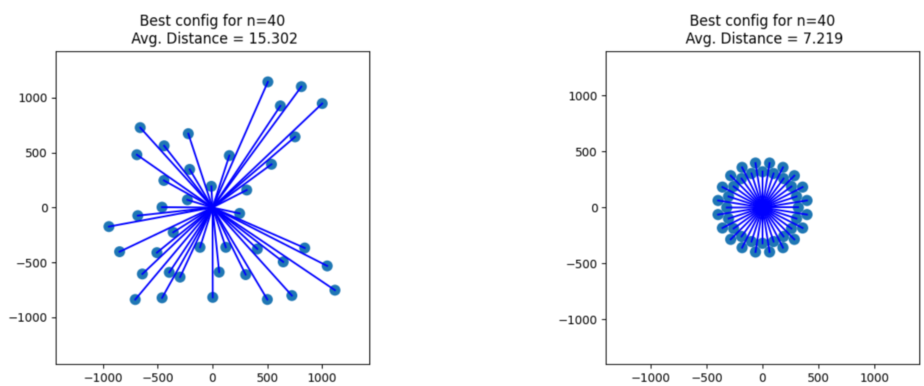


FIG. 4: *Best MC results for distance-weighted configurations: (a)-(i) show best out of 1000 trials, essentially the same as the ideal configuration represented in (l) ((j) had difficulty converging to beat the ideal solution (k)). Axes units and orientation arbitrary. Quoted average distances in units of viewer radii.*

Figure 4 shows resulting optimal configurations for distance-weighted configurations in our MC, after running O(1000) trials for each value of N. We found this level of simulation sufficient to see a clear pattern arise which is easy to understand: equal (or near equal) distancing from the origin, requiring viewers to form a 'double-packed ring'. The pattern is near exact up to N=10, but beyond that the simulation has trouble relaxing viewers from random initial positions into that double ring – they get stuck, e.g. in Figure 4(j), even when run for many thousands of iterations. We suspect it would take many times more to chance upon the optimal configuration, which we inferred and show in Figure 4(k), that does indeed have the lowest average distance from the center point. Thus in general, for N viewers we conjecture that the optimal configuration is in fact this double ring, i.e. one close-packed circle of $\lfloor N/2 \rfloor$ viewers, with an outer circle of the remaining viewers tucked into the 'cracks'[2] just behind the inner circle (Figure 4(l)), i.e. this is akin to the classic 'group photo' arrangement of people's heads for wide angle shots, which afterall has been known since (photographical) antiquity. We are content with the MC results supporting this conjecture for now, and a rigorous proof will wait for a future article.

This close-packed double ring solution may leave something to be desired, however, for although it apparently minimizes the average distance to the central point, it forces viewers to be very close to their neighbors and leaves large open spaces elsewhere. If a more uniform distribution of viewers is preferred, one can increase the repulsion between viewers which will naturally push some closer to the central point while others farther away. In Figure 5(a), for example, we see 40 viewers thus distributed with an average distance to the center of ≈ 15.3 radii, instead of the smaller average distance of ≈ 7.2 radii (see Appendix for derivation) attained with the optimal double-ring distribution in Figure 5(b). There is a continuum of such distributions as one tunes the repulsive force to give the desired inter-viewer distance, which as seen tends to distribute viewers more homogenously in space. As a rule crowds could implement in practice, this requires that people be aware to leave more than a few 'person-widths' distance to the next person — recent 6-foot social-distancing standards[3] may in fact naturally enforce this. For crowds of much more than $O(100)$ viewers, physical distances may render the issue moot: those who are close enough to see what they are looking at can distribute themselves more-or-less fairly, while those too far away might give up, such that spontaneous events like this have a natural self-sustaining crowd size.



(a) *Here a constant inter-viewer repulsive force spaces out the viewers.*

(b) *The ideal close-packed solution minimizes inter-viewer distance.*

FIG. 5: *Increasing inter-viewer repulsion distributes viewers more uniformly in space, at the cost of greater distance to the center.*

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[2] Technically, we must demand there be an infinitesimal space between first-ring viewers so as to just barely open LOS to the second-ring viewers. Thus the total configuration is infinitesimally-close to a close-pack. We drop this technicality in all subsequent discussion and treat all tangential viewers as though they can allow exactly one ray to pass between them.

[3] We are referring here to COVID-19 protocols established by various government authorities, e.g. [3].

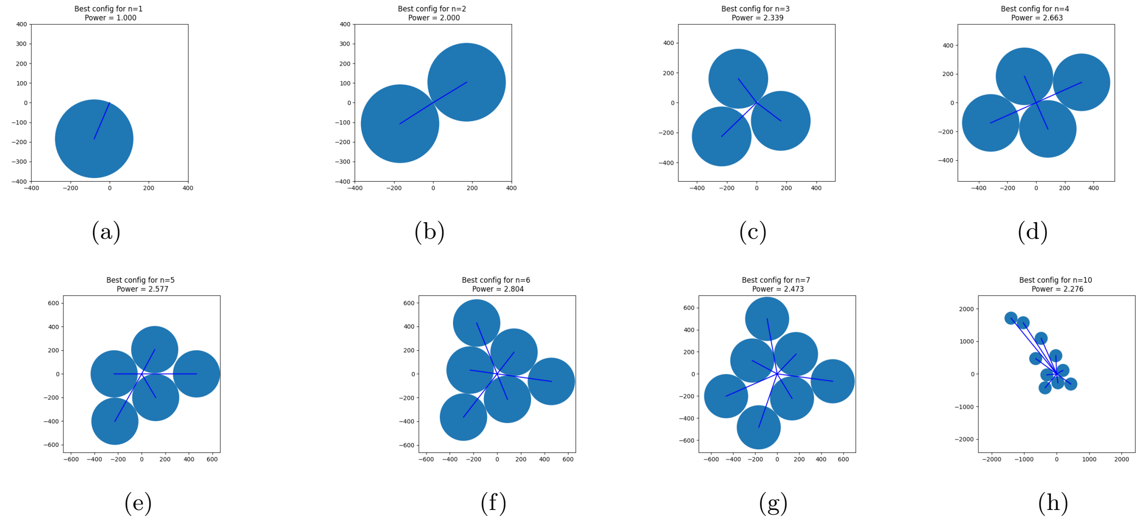


FIG. 6: *Best MC results for inverse-square-distance-weighted configurations. Scale and orientation arbitrary.*

Now turning to inverse-square-distance weighting, we will see the situation is more complicated. We run the same MC as before, but this time optimizing to maximize the 'power' $\sum_{i=1}^{N} d^{-2}$. Results for small N appear in Figure 6. Aside from the confirmation of the asymmetric surprise for N=3, the configurations look about the same as the distance-weighted case for $N <= 6$ (Figure 6(a)-(f)). For $N > 6$, however, they start to radically diverge: whereas distance weighting prefers to keep all viewers at close to the same distance from the origin, inverse-square-distance weighting places emphasis on several viewers being as close as possible while allowing most others to be much farther away. The best configuration from a power-delivery perspective is in fact for N=6 where all viewers are close-packed, and one can compute the power here as 2.8125 (arbitrary units relative to one source, i.e. Figure 6(a))[4]. For higher values of N the maximal power decreases as viewers must move farther away from the origin in order to accommodate the requirement that all viewers have a LOS. A practical implication of this would appear to be that, e.g. for 2D heating applications with multiple circular-shaped lasers, one might arrange 6 sources in the configuation shown in Figure 6(f) for maximal power, and not bother deploying more sources which would just waste power. Of course for non-circular shapes, and more concentrated source geometries (e.g. cylinder-like lasers), as well as distribution of sources in the 3rd dimension as well, the limit is higher, but exists similarly nonetheless. Beyond $N > 6$, an analytical solution for inverse-square-distance-weighted configurations seems difficult to ascertain from the simulations. This is in part due to a singularity-type feature that, for any finite-size angular window from the central point, it is possible to accommodate an infinite number of viewers distributed arbitrarily far away (see Figure 7). Thus, once one obtains the maximal power configuration for N=6, it makes sense to just open a small window to open LOS to all subsequent viewers (which will, however, then contribute negligable power), and we see hints of this behavior in the best MC solutions found for N=7 and N=10 (Figure 6(g),(h) respectively). This exact posited solution, i.e. opening an infinitesimal angular window to accommodate all viewers above the sixth, was not teased out of the MC because of the finite size of the simulation (both in space and number of runs), but we infer it from the nature of the best solutions seen thus far. In some sense then, the optimal arrangements for $N > 6$ are not very interesting, physically speaking.

[4] We assume, based on simulation and intuition, that the central point is at the geometric center of the configuration. Then simple geometry gives the sum of inverse-square distances as $3 \cdot (\frac{2\sqrt{3}}{3})^{-2} + 3 \cdot (\frac{4\sqrt{3}}{3})^{-2} = 2\frac{13}{16} = 2.8125$.

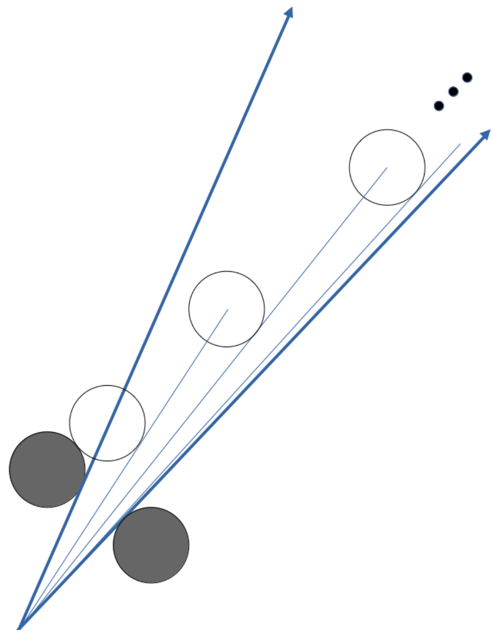


FIG. 7: *Between any two viewers with finite size gap between them, delineated by the blue rays, one can place any number of viewers between them maintaining LOS to the center point, as shown.*

III. N OBSERVERS OF A LINE: THE BAGGAGE CLAIM

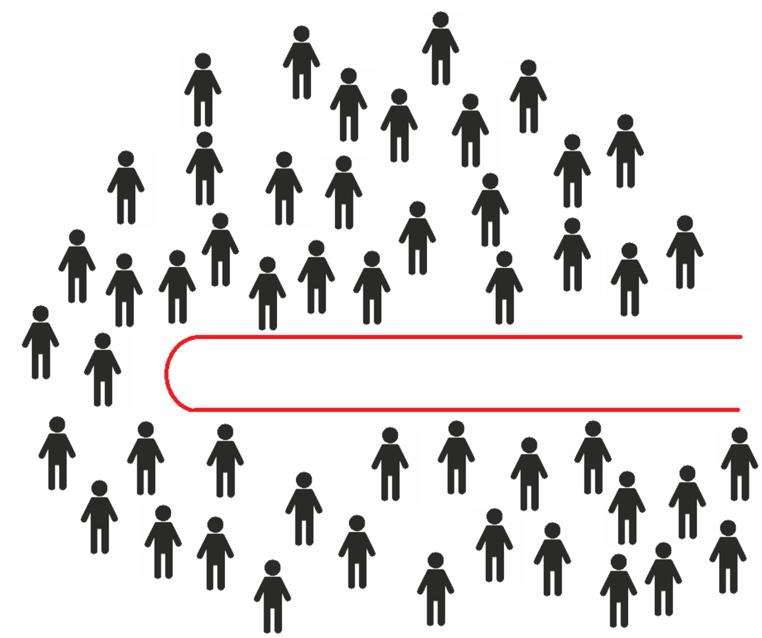


FIG. 8: *Schematic representation of "The Baggage Claim". Luggage is imagined to travel along the curve defined by the red line, so that observers strive to obtain a LOS to any point on this line.*

We now consider the more complex situation of viewers observing an extended central object. Motivated by the real life situation of multiple people waiting for their luggage at the airport baggage claim carousel, as schematically shown in Figure 8, we approximate the central object as a central line segment of finite length, and only consider viewers on one side of this line. Here there appears to be much more leeway than in the last section, for it is not necessary that all observers are viewing the same point on the line segment — it suffices that each observer has a LOS to any point on the line, so in general they will all have different ‘foci’ on the line segment. However, we also constrain viewers to be within the horizontal span of the line segment, effectively constrained to be within two walls, so that not all viewers can approach arbitrarily close – some will have to view behind others – and this makes the problem non-trivial.

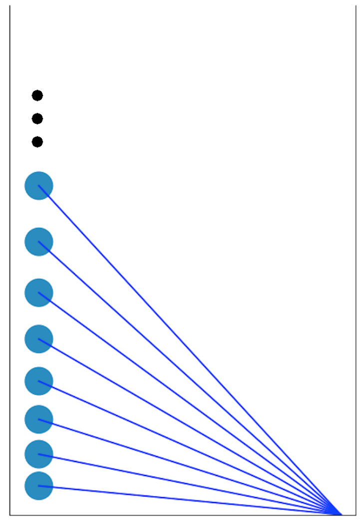


FIG. 9: *An arbitrary number of viewers can get a LOS to the line (x-axis) by lining up against the wall and focusing on the same point.*

As in the case of the central point, here too there always exists an easy solution by lining up people along a wall, all focusing on the point at the far end of the line segment , as shown in Figure 9. If the viewers focus on a closer point, then we can repeat multiple columns of such viewers (the reader may look ahead to Figure 11(b) for an example). These solutions, however, distribute viewers exponentially far away (see Appendix C), and we should be able to do much better.[5]

We proceed with the same type of MC setup as in the last section, i.e. randomly-initialized positions which relax to equilibrium under the action of attraction to the ‘focus line’ and repulsion from other viewers, except that each viewer will be drawn towards its own dynamic focus on the focus line, instead of a fixed point. At any step of the simulation, each viewer’s focus is the point on the line closest to it that affords an unobstructed LOS, and this can change as the simulation progresses. Eventually, though, the viewers’ positions will relax to an equilibrium where they cannot get any closer to the line without obstructing other viewers, and this defines one run of the MC. Over thousands of runs, we show bests results below.

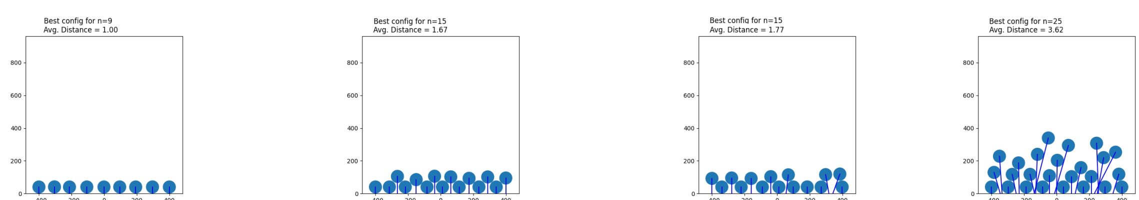
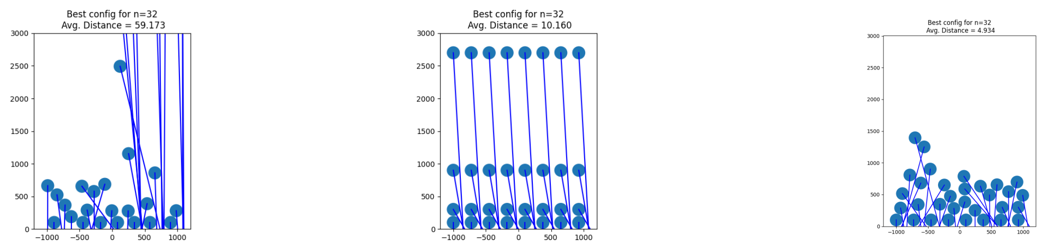


FIG. 10: *Best MC results for small numbers of viewers of the baggage claim (x-axis, arbitrary units). Average distance is in units of viewer radii.*

For small numbers of viewers, everyone gets right up to the line of course (see Figure 10(a)). As the number of viewers increases pass the point where the horizontal span can fit them all, a second row is formed right behind the

[5] Note however the diabolical case where the line segment is so narrow that only one viewer can fit — effectively the ‘hallway’ leading to the professor’s door — then this type of solution is the only one available

first, again as closely as possible (Figure 10(b)), in something reminiscent of the double-packed ring solution we saw in the last section. In some runs of the simulation, viewers can 'get stuck' competing for vantage points, as we see in (see Figure 10(c)) with two viewers trying to converge on the same space – one could alleviate this situation by allowing some random 'jumps' when viewers get stuck, similar to real dynamics of crowds. As the number of viewers increases, the distance weighting scheme becomes important, just as it did for the central point problem of the last section. We will here concentrate on the distance-weighted configurations, rather than the inverse-square-distance weightings, as we remind the reader that these latter tend to generate uninteresting (and possibly unethical) solutions where just one person from the back row grudgingly moves in order to open a tiny angular window for all subsequent viewers, because they are too far away to contribute much to the total power. Continuing to add more viewers, Figure 10(d) shows one final configuration for 25 viewers which begins to show something of a pattern. However, proceeding in this way adding more viewers does not easily lead to sensible configurations, as the dynamics become too complicated (see Figure 11(a)). It helps to initialize viewer positions more intelligently, e.g. in exponential stacks like in Figure 11(b), but even then convergence to an optimal solution is quite difficult to achieve dynamically, as we see in Figure 11(c). Thus we change our MC to specialize for solutions of the type we suspect are optimal for this situation.



(a) *Random initialization tends to leave some viewers stranded at large distances away.*

(b) *Initializing viewer positions in exponentially-distributed stacks already helps.*

(c) *The final configuration evolved from (b) is still somewhat messy and clearly non-optimal.*

FIG. 11: *Better initializing the positions of the viewers improves results.*

As a first change to our MC, we will remove the restriction of left- and right-boundaries for now (the 'walls'), and allow the focus line to be infinitely long; instead of bringing in a certain number of viewers, we bring in a density of viewers expressed in number of 'layers', e.g. Figure 10(a) would be classified as having 1 layer, Figure 10(b) as having two layers, etc.

(a)

(b)

FIG. 12: *Simplest cases of 1- and 2-layers of viewers on the infinite line*

Now, based on what we've seen so far we can easily reason out the solution for 1 and 2 layers of viewers: these are just the close-packed configurations in Figure 12(a)-(b), as there is no further space for optimization. To generalize to higher numbers of layers, we posit several principles consistent with the expected pattern for a general optimal solution:

- the first layer has the minimal space between viewers allowing LOS for all other viewers (parsimony)
- each layer-1 viewer has a stack of viewers above it in the same relative positions as the next (translation-invariance, so the pattern can easily repeat over the whole line)
- each viewer is tangential to at least one other viewer (optimality – no space is wasted)
- the last viewer in each stack just barely has LOS to a point on the line (optimality – no space is wasted)

Note that the optimal configurations in Figure 12 clearly conform to these principles. We claim that given the above assumptions, the form of the solution is unique and we can explicitly solve for it for any number of layers of viewers. The final solution can be cast in the form $d(N) = (d,\ p_2,\ p_3, ...\ p_N)$, where d is the distance between layer-1 viewers, and p_N is the position of the Nth viewer relative to the first. Given that, the positions of the viewers is fixed and pre-computable, and by periodicity determines the configuration of each identical stack up and down the line.

We expect the general case to look something like Figure 13 in order to conform to the above principles. To verify, we run the MC on a segment of the infinite line, randomly generating millions of configurations of viewers for each case of N layers, only keeping the best configurations consistent with our constraints. Instead of fully random initialization, we first initialize first layer viewers on the line with a random spacing between them, then build up the stack above each viewer one at a time, choosing a random angle to 'bend' the stack at each layer as long as LOS for all viewers is maintained. The fitness of each configuration is the average viewer distance plus a reward for not over-spacing first layer viewers, and optimizing this results in the best configuration with minimally-spaced first-layer viewers. Figure 14 shows our results for 3 to 10 layers of viewers.

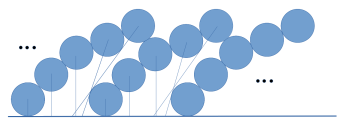
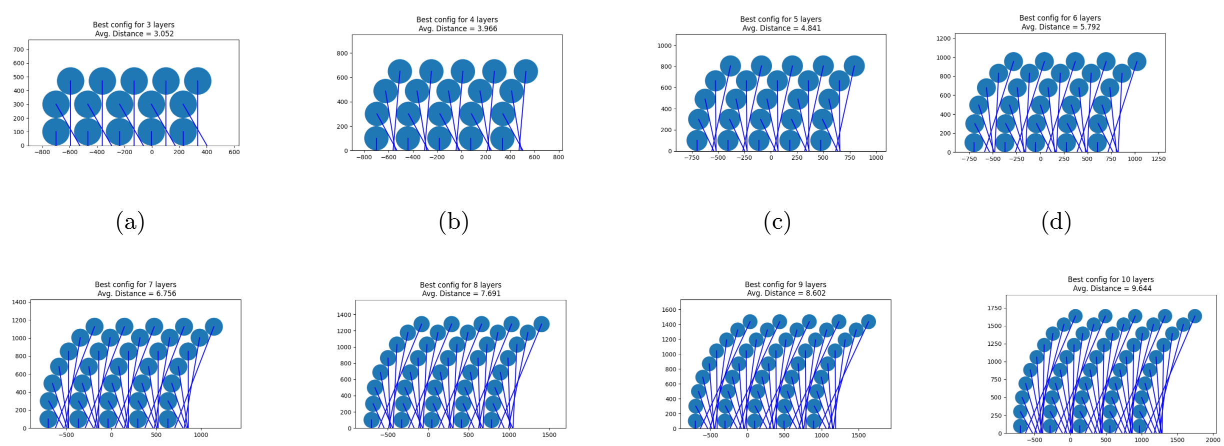


FIG. 13: *Conjectured type of optimal configuration for multiple layers of viewers on the infinite line*



(a) (b) (c) (d)

(e) (f) (g) (h)

FIG. 14: B*est MC results for stacks of 3 to 10 layers of viewers of the infinite line (only 5 stacks of the repeating pattern shown)*

We notice a few patterns in these configurations. First, the general shape is like a 'curved queue', the angle of curve increasing with N. Second, the spacing between the base of the queues increases with N. And finally, the average distance to the focus line rises sub-linearly (Figure 15). This is good news for baggage claims: an easy solution exists which doesn't leave people standing too far away. Moreover, the space between the queues[6] serves as an avenue for people to quickly collect their luggage as it goes by.

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[6] These are not literal queues in the sense that layer-N viewers have to wait for layer-(N-1) viewers to collect their bags first; we merely name them so because they are reminiscent of queues at, say the bakery. But some queue-like property could be retained to prioritize two viewers whose bags both appear at the same time on the belt.

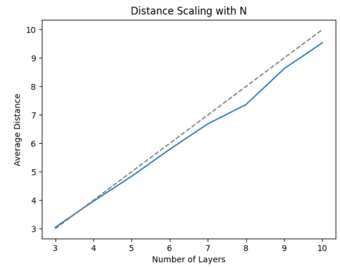

FIG. 15: *Average distance (in units of viewer radius) for the Baggage Claim Problem scales sub-linearly with the number of layers, N. (1-1 scaling marked by dashed line)*

IV. CAVEATS AND APPLICATIONS

Having now obtained heuristic solutions for the point-like and line-like central object LOS problem, let us make a few notes, observations, and generalizing remarks.

First, with regards to other central object geometries such as circles, squares, and more complex convex shapes, to the extent those shapes are approximated by piece-wise connected line segments, the optimal configurations of those objects should probably look like smooth piece-wise connections of our line segment solution, i.e. curved rows of the curved queues we saw in Figure 14. For convex central objects or situations with boundaries, there there will be complications which depend on the degree of convexity — the smaller the radius of curvature of the boundary, the more significant these effects will be. We did not simulate enough of these types of situations to give an impression of what the solutions would look like, so this remains an open question that probably depends quite strongly on the type of boundaries[7]. Finally, if the size of the central object is negligible compared to the viewer radius, the central-point like solution should be a good approximation, i.e. the double-ring of viewers (Figure 4(l)) should be the best configuration.

Speaking of viewer size, another issue to consider is unequal viewer radii or other viewer geometries besides circles. Of course simulation and certainly exact computation becomes much more difficult in those cases, and may not merit the extra work, unless one knows in advance of special geometrical viewer features (e.g. pencil-like, or pancake-like). Without such special features, however, we are, after all, considering *crowds*, which by definition are unpredictable in size and number — so in general the best one can do is assume uniform viewer geometry, which in 2D is, without loss of generality, a circle.

Of course, if we extend the situation to 3D, we expect the solutions to change, approximately, in the naive way. That is, viewers become spheres, and e.g. the central point/sphere-like situation admits a 'double-shell' optimal solution[8]. Presumably the line/cylinder-like central object situation generalizes to 'spiral queues' of viewers. We consider these situations less pertinent to everyday life, but perhaps such situations would be more common when our social interactions extend equally to all dimensions as in e.g. zero-gravity environments[9].

We concede that our heuristic solutions for the point-like and line-like central object, assuming a certain symmetry or periodicity, may not actually be the exact optimal solutions — however, to the extent that our MC simulations failed to produce anything better, and asymptotically seemed to approach our heuristic solutions, we are confident enough to make the conjecture and look forward to a rigorous mathematical proof in a future publication. Moreover, since our motivation is purely practical in nature, addressing common day situations such as crowds around stages and baggage claim carousels, there is the question of whether we really need an exact solution. As long as one can reliably generate numerical solutions as we have in our MC, or even put forward easy-to-implement configurations (i.e. the double-ring, the curved queue), this may be enough or even preferable from the standpoint of those in charge of crowd experience optimization. Over the last hundred or so years, people have probably not even been thinking about whether standing in two staggered rows was optimal for group photos, but naturally settled on that at the behest of the photographer. If travelers can likewise be coaxed into realizing that curved queues are far more efficient than random packing (e.g. through suggestive floor lighting), our work will not have been in vain.

Finally, we mention other applications beyond the spontaneous crowd striving for LOS. For passive, distance-weighted viewing akin to a crowd viewing a central object, one could consider applications in communication networks

[7] Note that other problems with geometrically-competing elements, e.g. circle-packing, also suffer from complexities in bounded spaces (see [2] for recent review and innovation).

[8] In this sense, sports stadiums could double their seating with a shell of seats *behind* the ones already intact, though this would probably not be cost-effective.

[9] To paraphrase NASA Astronaut Doug Wheelock, "A room is suddenly a lot bigger in zero-gravity"[4]

(e.g. satellites needing to retain LOS with a central transmitter), transportation or logistics networks (hub-and-spoke configurations where one wants to minimize average spoke length), acoustics (e.g. designing sound-absorptive barriers which maximize exposure to the acoustic wave-front), and security (redundant viewing for camera systems). For active, inverse-square-distance-weighted viewing akin to power delivery, we already noted the pack-of-six optimal configuration (Figure 6(j)) for e.g. lasers heating a central point-like sample, or the 2-layer close pack configuration (Figure 12(b)) for e.g. heating an assembly line.

V. CONCLUSIONS

In this article we explored optimal viewer configurations for the central point and line scenarios, discovering that distance weighting plays an important role. While many possible distance weighting schemes are possible, we focused on average distance and average inverse-square-distance weightings due their direct physical interpretations.

For the central point ("The Prophet") problem, numerical MC simulation and analytic computation confirm that a 'double-ring' of observers always optimizes the average distance case for small N, similar to how people arrange themselves for group photographs. In this case, for N viewers the average distance grows like N. For the inverse-square-distance case there there is a similar pattern up to six viewers forming a close pack around the central point, after which additional viewers are just positioned farther and farther away as their contribution to the power is negligible. We obtained two surprising results: for N=3 the maximal power point is off-center and six-fold degenerate, and for N=6 the power is maximal. This would seem to imply that for applications focusing e.g. power from circular sources on a central point, the most powerful configuration is a close-pack of six, where the power is almost three times that of a single tangential source.

For the line problem ("The Baggage Claim"), our naive MC was confounded by the additional complexities and boundary effects. We were, however, able to infer an optimal configuration pattern for long lines wherein viewers tend to form 'curved queues' giving an average viewer distance slightly less than the length of each queue, a result that guarantees reasonable access for all. In that this solution follows from various reasonable heuristics, beats all of our MC results, and looks almost close-packed, we conjecture this is the optimal configuration for a long central line, and for finite central lines likely dominates towards the center away from boundary effects. An analytic proof is deferred to a future work. It may also be easily implementable in practice if people remember not to queue up in straight lines, but curved ones. Curved queues might become part of our common culture. Alternatively, the airport (or wherever) could have suggestive floor patterns to guide people's standing positions.

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Appendix A: Optimal Configuration for N=3

In the case of 3 observers on a focus point, we derive the optimal configuration for both distance-weighting and inverse-square-distance weighting.

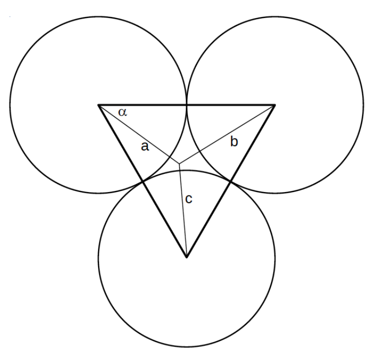

FIG. 16: *Geometry for finding point of maximal power / minimal distance for N=3 viewers of a central point*

Given that the observers will necessarily close-pack into the co-tangential configuration as shown in Figure 3, the general focus point coordinates can be described by any two of the three distances a, b, c to the obvserver centers at the vertices of the equilateral triangle, constrained to lie in the convex-triangular region in between the observers' circles (see Figure 16).

For distance-weighting, we wish to minimize the quantity

$$\text{Total Distance} = a + b + c \tag{A.1}$$

or equivalently, the sum of squares

$$\text{T2} \equiv a^2 + b^2 + c^2 \tag{A.2}$$

Taking the observer radius to be unity without loss of generality, and by expressing the Law of Cosines twice, i.e.

$$b^2 = a^2 + 4 - 4a\cos(\alpha) \tag{A.3}$$

$$c^2 = a^2 + 4 - 4a\cos(\pi/3 - \alpha), \tag{A.4}$$

we can express (A.2) in terms of a and b only,

$$\text{T2} = \frac{3}{2}(a^2 + b^2) + 2 - \frac{\sqrt{3}}{2}\sqrt{16a^2 - (b^2 - a^2 - 4)^2} \tag{A.5}$$

Looking for a local minimum, we set $\frac{\partial T2}{\partial a} = \frac{\partial T2}{\partial b} = 0$ to find easily the solution $a = b = c$, which when substituted back into (A.2) and minimized, gives $a = 2/\sqrt{3}$, i.e. the Fermat point.

Now we consider inverse-square-distance weighting, where we will obtain the surprising conclusion that the optimal point is *not* at the geometrical center as one would naively guess. In this case we want to find e.g. a, b so as to maximize the quantity

$$\text{Power} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \tag{A.6}$$

As before we can express (A.6) in terms of a and b only,

$$\text{Power} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{a^2 + b^2 + 4 - \sqrt{3(16a^2 - (b^2 - a^2 - 4)^2)}} \tag{A.7}$$

We wish to maximize this power. Note that because we have a boundary constraint, in general the maximum will occur either on the boundary or in the interior of the region, so we need to check values on the boundary as well as possible local maxima in the interior. First searching for local maxima in the interior, we set the partial derivatives with respect to a and b to zero,

$$\frac{\partial \text{Power}}{\partial a} = -\frac{2\left(2a - \frac{\sqrt{3}\left(4a\left(-a^2+b^2-4\right)+32a\right)}{2\sqrt{16a^2-\left(-a^2+b^2-4\right)^2}}\right)}{\left(-\sqrt{3}\sqrt{16a^2 - \left(-a^2 + b^2 - 4\right)^2} + a^2 + b^2 + 4\right)^2} - \frac{2}{a^3} = 0 \tag{A.8}$$

$$\frac{\partial \text{Power}}{\partial b} = -\frac{2\left(\frac{2\sqrt{3}b\left(b^2-a^2-4\right)}{\sqrt{16a^2-\left(b^2-a^2-4\right)^2}} + 2b\right)}{\left(-\sqrt{3}\sqrt{16a^2 - \left(b^2 - a^2 - 4\right)^2} + b^2 + a^2 + 4\right)^2} - \frac{2}{b^3} = 0 \tag{A.9}$$

and the above system is solved for $a = b$. This allows us to simplify Eqn. A.7 to

$$\text{Power}_{|a=b} = \frac{2}{a^2} + \frac{1}{a^2 + 2 - 2\sqrt{3(a^2 - 1)}} \tag{A.10}$$

Proceeding to set $\frac{\partial \text{Power}_{|a=b}}{\partial a} = 0$ and simplifying, one will arrive at the following polynomial equation:

$$9a^{10} - 60a^8 + 304a^6 + 1280a^4 + 2048a^2 - 1024 = 0 \tag{A.11}$$

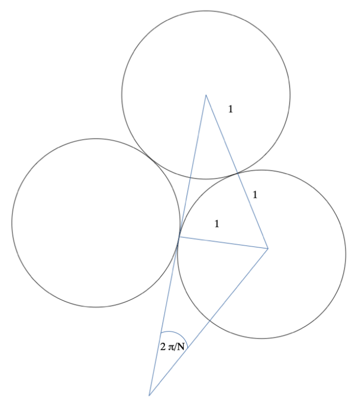

FIG. 17: *Geometry for averaging distances to center point*

which can be factored into several pieces:

$$(a^2-4)(3a^2-4)(3a^6-4a^4+64a^2-64)=0 \tag{A.12}$$

The first two factors give positive roots $a=b=2$ (outside the region, in fact located at the third vertex which is a singularity) and $a=b=c=2/\sqrt{3}\approx 1.1547$ (the Fermat point at the center of the region, which is however a *minimum* for the Power: $9/4=2.25$). The remaining factor is a cubic in a^2 and also admits a positive root we can find analytically,

$$a=b=\frac{\sqrt{2}}{3}\sqrt{2-70\left(\frac{2}{137+81\sqrt{29}}\right)^{\frac{1}{3}}+\left(4(137+81\sqrt{29})\right)^{\frac{1}{3}}}\approx 1.0077 \tag{A.13}$$

which is indeed a local maximum along the line $a=b$, but within the entire region is but a saddle point; the true maximum lies slightly to the left or right of this point on the boundary of the region, i.e. the blue points in Figure 3, which we find by going back to Eqn. A.7, setting e.g. $a=1$, and maximizing over b. We numerically find $b\approx 1.015$, $c\approx 1.611$, and the Power here is about 2.36, roughtly 5% higher than at the central Fermat Point.

Appendix B: Optimal Configurations for Distance Weightings

For the central point problem, the optimal configuration is for two packed circles of viewers as shown in Figure 4(l). By simple geometry (Figure 17), the average distance is (for N even)

$$\frac{1}{2}\left(\frac{1}{\sin 2\pi/N}+\sqrt{3}+\sqrt{\frac{1}{\sin^2 2\pi/N}-1}\right)=\frac{1}{2}\left(\csc 2\pi/N+\sqrt{3}+\cot 2\pi/N\right) \tag{B.1}$$

while for N odd it is

$$\frac{1}{2N}\left(\frac{N+1}{\sin 2\pi/(N+1)}+(N-1)\left(\sqrt{3}+\sqrt{\frac{1}{\sin^2 2\pi/(N+1)}-1}\right)\right) \tag{B.2}$$

$$=\frac{1}{2N}\left((N+1)\csc 2\pi/(N+1)+(N-1)\left(\sqrt{3}+\cot 2\pi/(N+1)\right)\right) \tag{B.3}$$

Thus, for example for $N=40$ the average distance is about 7.219.

Appendix C: Exponential Stacks of Viewers

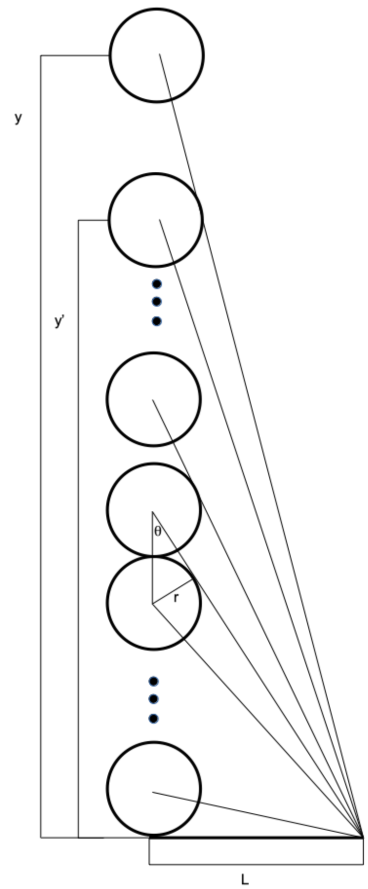

FIG. 18: *Stack of viewers exponentially-distributed.*

Here we briefly derive the fact that a vertical stack of viewers all focusing on the same point must be exponentially-distributed. Consider the setup of Figure 7 : we can compact the first few viewers to be right next to eachother, but eventually we must space them out, see Figure 18. The point at which they must start to space out can be found from simple geometry, i.e. when the nth viewer's LOS is tangential to the (n-1)st viewer, the next, (n+1)st must be shifted up by more than the viewer diameter. The angle the nth viewer's LOS subtends to the vertical is thus $\cos\theta = 1/2$, or $\theta = \pi/6$. The same angle also defines the ratio $\tan\pi/6 = \frac{L}{n(2r+1)}$, thus $n = \frac{L\sqrt{3}}{(2r+1)}$. So for example, for $r = 1$ and $L = 5$, the third viewer must already be shifted upwards. Far up the stack where the viewers are already quite spaced out, so $y >> L$, the distance between two adjacent viewers at positions y and y' can be shown to satisfy, by the same geometrical argument,

$$dy \equiv y - y' = \frac{ry}{L} \tag{C.1}$$

which as a difference equation is approximated by the exponential solution $y = e^{\frac{r}{L}m}y(0)$, with $m = 1, 2, 3, ...$ the relative viewer position.

[1] https://github.com/eryk-kersting/vantage_points
[2] P. Amore et al. (Universidad de Colima), "Efficient algorithms for the dense packing of congruent circles inside a square," 2102.01537.
[3] https://www.cdc.gov

[4] Douglas Wheelock, private communication at IBM Research visit, Yorktown Heights, NY.