Quaternion Space-Time and Fields*

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In this work, we use the concept of quaternion time and demonstrate that it can be applied for description of four-dimensional space-time intervals. Real quaternions form a normed division algebra and we suggest that this is the main advantage of quaternions over other mathematical representations of space-time. First, we use the quaternion norm for the description of the measurement process. We demonstrate that the quaternion time interval together with the finite speed of light signal propagation allow for a simple intuitive understanding of the time interval measurement by a moving observer. We derive a quaternion form of Lorentz time dilation and show that the norm of the quaternion time corresponds to the traditional expression of the Lorentz transformation. We determine that the space-time interval in the observer reference frame is given by a conjugate quaternion expression, which is essential for proper definition of the quaternion derivative in the observer reference frame. Then, we use quaternion division to define the four-dimensional differentiation. Finally, we apply quaternion gradients of the commutator and anti-commutator types to an arbitrary quaternion potential, which leads to generic quaternion field expressions. We apply the resulting formalism to the electromagnetic and gravitational potentials and show that the traditional field expressions are obtained under simplifying assumptions, while the new additional field terms need further study and experimental verification.

I. INTRODUCTION

We begin by proposing the real quaternions [1], [2], [3], [4], [5], [6], as an alternative to the traditional mathematical formalism of four-dimensional space-time used in special relativity theory [7], [8], [9], [10], [11], which attempted to describe time dilation and space contraction predicted by Lorentz [12], [13].

Previously, bi-quaternions were applied to special relativity [14] and showed initial promise in developing a unified field theory [15]. However, unlike real quaternions, bi-quaternion mathematics is not a division algebra.

We develop a complex polar form of the quaternion time interval and demonstrate that it describes transition time from one physical state to another, while the norm of the quaternion time interval describes the experimentally measured value of the time interval, which corresponds to the Lorentz time dilation.

We deduce that the conjugate quaternion time interval corresponds to the time interval in the observer reference frame, which is essential for the correct definition of quaternion differentiation by the observer.

We use quaternion differentiation of a generic quaternion potential in order to define the quaternion form of a generic quaternion field.

Jack [16], [17] demonstrated a new approach of applying quaternion differentiation to derive quaternion Maxwell equations, then, Dunning-Davies and Norman [18] suggested using a similar method for the gravitational field. Application of electromagnetic analogy to gravitational fields and forces was previously extensively studied [19], [20], [21], [22].

We apply the new definition of the generic quaternion field to electromagnetic and gravitational interactions and show that it reproduces the known results for the vector fields, while introducing additional scalar and vector components that need further investigation.

II. QUATERNION SPACE-TIME

Historically, Rodrigues [1] introduced quaternions while searching for a method to describe rotation of three-dimensional solids. His discovery can be considered the precursor to quaternion algebra, which was formally introduced and extensively studied by Hamilton [2], [3], who came across quaternions while searching for well-defined division in the three-dimensional space. Hamilton was quoted saying "Time is said to have only one dimension, and space to have three dimensions... The mathematical quaternion partakes of both these elements" [4]. In Hamilton's definition of quaternions, time is real scalar and space is a three-dimensional imaginary vector.

The key advantages of real quaternion algebra over other mathematical approaches is that it has a positive Euclidean norm, it describes both rotation and propagation in three-dimensional space, and constitutes a division algebra with well-defined multiplication and division. This is fundamentally different from the four-dimensional mathematics of Poincare [7], Minkowski [8], [9], and Einstein [10] used in the special theory of relativity, where only one-dimensional inertial motion is described, no rotation is present, negative norm of the space-time interval is possible, and no four-dimensional division is defined. Consequently, quaternion algebra deserves further investigation as an alternative mathematical formalism of space-time physics.

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Since the algebra of real quaternions is the only four-dimensional division algebra, we introduce the four-dimensional quaternion manifold,

$$\mathcal{T}^4 = (\,\hat{\tau}_0\,, \vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3) = (\,\hat{\imath}_0\tau_0\,, \vec{\imath}_1\tau_1, \vec{\imath}_2\tau_2, \vec{\imath}_3\tau_3)\,, \qquad (1)$$

which we identify with time in order to facilitate an intuitive physical interpretation of the quaternion mathematics [5].

Here, $\hat{\imath}_0$, is a real scalar unity interval and, $\vec{\imath}_1, \vec{\imath}_2, \vec{\imath}_3$, are purely imaginary unit vectors, and $\tau_0, \tau_1, \tau_2, \tau_3 \in \mathbb{R}$, are real scalars. The relationships between the Euclidean quaternion units, $\hat{\imath}_0, \vec{\imath}_1, \vec{\imath}_2, \vec{\imath}_3$, are essential for the present theory and are defined according to Hamilton [2] as,

$$\begin{cases} \hat{\imath}_0\,\hat{\imath}_0 = \hat{\imath}_0 = 1\,, \\ \vec{\imath}_1\,\vec{\imath}_1 = \vec{\imath}_2\,\vec{\imath}_2 = \vec{\imath}_3\,\vec{\imath}_3 = \vec{\imath}_1\,\vec{\imath}_2\,\vec{\imath}_3 = -\imath_0 = -1\,, \\ \vec{\imath}_1\,\vec{\imath}_2 = \vec{\imath}_3, \quad \vec{\imath}_2\,\vec{\imath}_3 = \vec{\imath}_1, \quad \vec{\imath}_3\,\vec{\imath}_1 = \vec{\imath}_2\,, \\ \vec{\imath}_2\,\vec{\imath}_1 = -\vec{\imath}_3, \quad \vec{\imath}_3\,\vec{\imath}_2 = -\vec{\imath}_1, \quad \vec{\imath}_1\,\vec{\imath}_3 = -\vec{\imath}_2\,. \end{cases} \qquad (2)$$

In the current work, we develop the quaternion formalism in vacuum, therefore, we use the absolute value of the speed of light in vacuum, c, as a scalar coefficient of proportionality between space and time. This allows us to express four-dimensional space-time in terms of four-dimensional quaternion time,

$$\mathcal{T}^4 = \left(\,\hat{\imath}_0\tau_0\,, \vec{\imath}_1\frac{x_1}{c}, \vec{\imath}_2\frac{x_2}{c}, \vec{\imath}_3\frac{x_3}{c}\,\right). \qquad (3)$$

Thus, using quaternion unit intervals (2) and the speed of light in vacuum, c, we were able to express four-dimensional space-time in terms of quaternion time.

III. QUATERNION SPACE-TIME COORDINATES AND INTERVALS

Next, we use quaternion space-time in order to establish coordinate point locations in the space-time coordinate system.

Using (3) we define a point location in the quaternion space-time coordinate system as,

$$\boldsymbol{\tau} = (\tau_0,\ \vec{\tau}_v\,) = \left(t_0,\ \frac{\vec{x}}{c}\right)\,, \qquad (4)$$

where we define a pure imaginary space vector location,

$$\vec{x} = (\,\vec{\imath}_1 x_1, \vec{\imath}_2 x_2, \vec{\imath}_3 x_3\,)\,, \qquad (5)$$

and the real scalar time,

$$\tau_0 = \hat{\imath}_0 t_0 = t_0\,. \qquad (6)$$

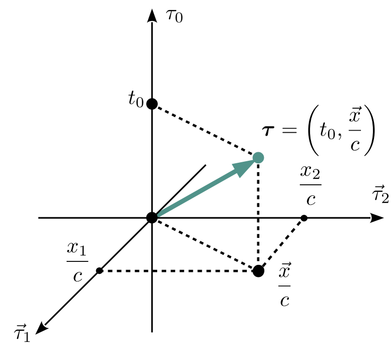

FIG. 1. A three-dimensional representation of the quaternion time-point.

Note from (4) that t_0 is the time at the zero-point space location, $\vec{x} = 0$.

The space-time coordinate point (4) is defined relative to the quaternion zero-point,

$$\mathbf{0} = \left(\,0\,, \vec{0}\,\right) = (\,\hat{\imath}_0 0\,, \vec{\imath}_1 0, \vec{\imath}_2 0, \vec{\imath}_3 0\,)\,. \qquad (7)$$

Consequently, the quaternion space-time coordinate point (4) is described by a four-dimensional quaternion interval starting at the zero-point and ending at the coordinate point.

Applying the definition of the quaternion space-time coordinates, we use the quaternion time-point (4) for description of a time event of a physical process at a space location, $\vec{x}$.

The norm of the quaternion time interval, or its absolute value, can be defined as,

$$\tau = |\boldsymbol{\tau}| = \sqrt{\boldsymbol{\tau}\,\bar{\boldsymbol{\tau}}} = \sqrt{\bar{\boldsymbol{\tau}}\,\boldsymbol{\tau}}\,, \qquad (8)$$

where we use the conjugate quaternion time defined as,

$$\bar{\boldsymbol{\tau}} = (\,\tau_0\,,\ -\vec{\tau}_v\,) = \left(t_0,\ -\frac{\vec{x}}{c}\right)\,, \qquad (9)$$

Since the quaternion norm is positive real scalar, we identify the length of the quaternion time interval with the measured time duration of a physical process.

In Fig. 1, we demonstrate a diagram of a quaternion space-time point using a three-dimensional representation, where we neglect for simplicity the fourth dimension, $\vec{\tau}_3 = 0$.

IV. POLAR REPRESENTATION OF QUATERNION TIME INTERVALS

Note that the quaternion time interval signifies a transition in space-time from the zero-point to a space location, $\vec{x}$.

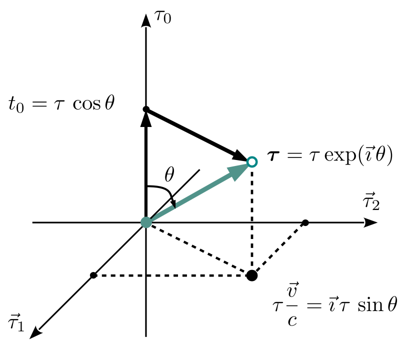

FIG. 2. Polar quaternion form of the Lorentz time interval dilation.

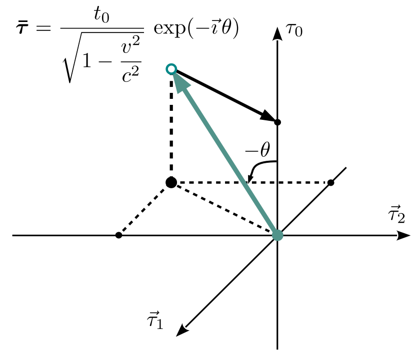

FIG. 3. Conjugate polar form of the Lorentz time interval dilation..

To describe this motion, we introduce a vector velocity,

$$\vec{v} = \frac{\vec{x}}{\tau}\,, \tag{10}$$

where, $\vec{x}$, is a space interval and, $\tau = |\boldsymbol{\tau}|$, is the absolute value of the time interval given by (8). Note that we previously defined an alternative quaternion velocity expression [5].

Then, we write quaternion time in terms of its norm and vector velocity,

$$\begin{cases} \boldsymbol{\tau} = (\tau_0,\, \vec{\tau}_v\,) = \left(t_0\,, \dfrac{\vec{v}}{c}\tau\right)\,, \\ \\ \bar{\boldsymbol{\tau}} = (\tau_0,\, -\vec{\tau}_v\,) = \left(t_0\,, -\dfrac{\vec{v}}{c}\tau\right) \end{cases} \tag{11}$$

where we note a feedback form of the quaternion time interval with the correction term determined by the velocity relative to the speed of light.

We introduce a purely imaginary unit-vector,

$$\vec{\imath} = \frac{\vec{x}}{x} = \frac{\vec{v}}{v}\,, \tag{12}$$

which signifies the direction of motion.

Finally from (11) and (12), we express the quaternion time interval in polar form,

$$\boldsymbol{\tau} = \tau\, \left(\cos\theta\,, \vec{\imath}\sin\theta\right) = \tau\, \exp\left(\vec{\imath}\theta\right), \tag{13}$$

where the angle, θ, is a function of the velocity, $\vec{v}$, and is defined as,

$$\begin{cases} \cos\theta = \dfrac{t_0}{\tau} = \sqrt{1-\dfrac{v^2}{c^2}}\,, \\ \\ \sin\theta = \dfrac{v}{c} \end{cases} \tag{14}$$

Then from (13) and (17), we obtained the full polar form of the time interval transformation,

$$\boldsymbol{\tau} = \frac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\, \exp\left(\vec{\imath}\theta\right)\,. \tag{15}$$

Similarly, we can express the quaternion conjugate time interval as,

$$\bar{\boldsymbol{\tau}} = \frac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\, \exp\left(-\vec{\imath}\theta\right)\,. \tag{16}$$

Thus, we can consider expression (15) and its conjugate (16) as the quaternion form of the Lorentz time dilation.

Now, we can easily determine the norm of the quaternion time interval from (15) and (16),

$$\tau = |\boldsymbol{\tau}| = |\bar{\boldsymbol{\tau}}| = \frac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\,, \tag{17}$$

which we immediately recognize as the traditional real scalar form of the Lorentz time dilation. As can be seen the time quaternion interval depends on both the speed, v/c, and direction of motion, $\vec{\imath}$. On the other hand, the measured time interval is a function of the speed only.

In Fig. 2 and Fig. 3, we demonstrate diagrams of a quaternion space-time interval and its conjugate using a three-dimensional representation.

Therefore, we were able to obtain one of the main mathematical results of the special theory of relativity by using quaternion formulation of the space-time interval and its absolute value.

V. PHYSICAL INTERPRETATION OF QUATERNION SPACE-TIME INTERVALS

We will now elaborate on the physical meaning of the quaternion time interval defined by (4) and (17) . Let us assume the existence of time sources such as clocks, and time detectors such as observers, with recording instruments.

Assume that there is a stationary clock located on a train platform, which we consider a signal source. First we perform an experiment in the source reference frame of the stationary clock, where the location of the clock we define as the zero-point of space, $\vec{x} = 0$. Also, let us consider an observer with a video camera passing the platform on a train at midnight, when the time on the platform clock is zero. We assume that the train is moving along a straight track with a constant vector velocity, $\vec{v}$. The observer synchronizes the camera clock with the platform clock at midnight and then starts filming the time on the platform clock while simultaneously recording the time-stamp of the camera.

After synchronization, the starting time for both the platform clock and the observer camera is zero. The observer stops filming when the camera records time, t_0, appearing on the platform clock. Then, what is the time-stamp on observer's camera at the end of the recording? Due to the finite speed of light propagation, we expect that the time on the platform clock will appear delayed relative to the time-stamp on the observer's camera. Also, we expect that the delay is a function of the train speed relative to the speed of light as the light signal from the clock is chasing the observer on the moving train.

Let us define the quaternion time-point at the end of the interval as, $\boldsymbol{\tau} = (t_0, \tau\vec{v}/c)$. The quaternion time interval of the recording is given by the difference,

$$\boldsymbol{\tau} - \mathbf{0} = \left(t_0 \,, \tau\frac{\vec{v}}{c} \right) = \boldsymbol{\tau} \,, \tag{18}$$

In Fig. 2, we demonstrate the diagram of a quaternion space-time interval in the source reference frame.

Let us suggest that the measured time interval on the camera time-stamp is a real scalar value, equal to the quaternion norm of the interval (17),

$$|\boldsymbol{\tau}| = \sqrt{\boldsymbol{\tau}\,\bar{\boldsymbol{\tau}}} = \sqrt{\bar{\boldsymbol{\tau}}\,\boldsymbol{\tau}} = \frac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \tau \,, \tag{19}$$

which is the Lorentz time dilation generally accepted as a verified experimental result.

Next, let us consider the same experiment in the observer's reference frame. Clearly, we expect to obtain the same experimental result even though the platform is now moving away from the observer with a constant velocity $-\vec{v}$. The starting time of the measurement and the clock synchronization time is zero, as in the source reference frame. However, the end time-point is now given by the conjugate quaternion $\boldsymbol{\tau'} = (t_0, -\tau\vec{v}/c)$ due to imaginary space inversion when changing from the source to the observer reference frame,

$$\boldsymbol{\tau'} - \mathbf{0} = \left(t_0 \,, -\tau\frac{\vec{v}}{c} \right) = \bar{\boldsymbol{\tau}} \,. \tag{20}$$

In Fig. 3, we demonstrate a diagram of a quaternion space-time interval in the observer reference frame.

Let us now calculate the measured time-interval duration in the observer reference frame,

$$|\bar{\boldsymbol{\tau}}| = \sqrt{\boldsymbol{\tau}\,\bar{\boldsymbol{\tau}}} = \sqrt{\bar{\boldsymbol{\tau}}\,\boldsymbol{\tau}} = \frac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \tau \,. \tag{21}$$

As expected, the measured time duration by the observer remains the same as in the source reference frame, despite the conjugate form of the time interval.

Therefore, the physical interpretation of the quaternion time interval can be deduced directly from its definition. It describes the time interval measured by the observer, moving with a constant velocity, $\vec{v}$, from the zero-point to a location, $\vec{x}$. Here, t_0, is time on the stationary zero-point clock and, $\tau = |\boldsymbol{\tau}|$, is the time interval duration measured by the moving observer. Note that the conjugate form of the space-time interval in the observer reference frame is critically important for the correct definition of quaternion differentiation in the observer reference frame. This physical interpretation is similar to the relativistic Doppler effect approach [11], however using quaternion mathematical formalism.

VI. QUATERNION POTENTIAL DIFFERENTIATION AND GENERIC FIELDS

Next, we take advantage of quaternion division in order to define proper quaternion differential operators in the source and observer reference frames. Thus, using the definition of the quaternion multiplicative inverse,

$$\begin{cases} \boldsymbol{\tau}^{-\mathbf{1}} = \dfrac{\bar{\boldsymbol{\tau}}}{\tau^2} \\[2ex] \bar{\boldsymbol{\tau}}^{-\mathbf{1}} = \dfrac{\boldsymbol{\tau}}{\tau^2} , \end{cases} \tag{22}$$

we define the quaternion differential operators corresponding to the gradient operators in the three-dimensional space,

$$\begin{cases} \bar{\boldsymbol{\nabla}} = \dfrac{1}{c}\,\dfrac{d}{d\boldsymbol{\tau}} = \left(\dfrac{\partial}{c\partial t_0}, -\vec{\imath}_1\dfrac{\partial}{\partial x_1}, -\vec{\imath}_2\dfrac{\partial}{\partial x_2}, -\vec{\imath}_3\dfrac{\partial}{\partial x_3} \right) \\[2ex] \boldsymbol{\nabla} = \dfrac{1}{c}\,\dfrac{d}{d\bar{\boldsymbol{\tau}}} = \left(\dfrac{\partial}{c\partial t_0}, +\vec{\imath}_1\dfrac{\partial}{\partial x_1}, +\vec{\imath}_2\dfrac{\partial}{\partial x_2}, +\vec{\imath}_3\dfrac{\partial}{\partial x_3} \right) . \end{cases} \tag{23}$$

We can write the four-dimensional gradients in the simplified quaternion notation as,

$$\begin{cases} \bar{\boldsymbol{\nabla}} = \left(\dfrac{\partial}{c\partial t_0} \,, -\vec{\nabla} \right) \\[2ex] \boldsymbol{\nabla} = \left(\dfrac{\partial}{c\partial t_0} \,, \ \vec{\nabla} \right) , \end{cases} \tag{24}$$

Thus, the correct form of the quaternion differential operator assumes the conjugate form, $\bar{\boldsymbol{\nabla}}$, in the source reference frame, with a minus sign in front of the vector part of the operator. On the other hand in the observer reference frame, the expression for the differential operator has the traditional form , $\boldsymbol{\nabla}$, due to the conjugate form of the space-time interval, $\bar{\tau}$, in the denominator.

Since we are primarily interested in the reference frame of the measuring apparatus, which is the observer reference frame, we will use the form of the derivative operator given by $\boldsymbol{\nabla}$.

Let us introduce a quaternion four-potential corresponding to an arbitrary physical interaction, similar to [22],

$$\boldsymbol{\phi} = (\phi_0\,,\,\vec{\phi}_v\,) = \left(\phi_0\,,\,\frac{\vec{v}}{c}\phi\right) \tag{25}$$

where, ϕ_0, is the static potential at the signal source, while, $\vec{\phi}_v = \vec{v}/c\phi$, is the vector potential due to the motion of the source relative to the observer. As usual, we can define the potential measured by the observer as the quaternion norm,

$$\phi = |\boldsymbol{\phi}| = |\bar{\boldsymbol{\phi}}| = \frac{\phi_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\,. \tag{26}$$

Then using the definition of quaternion multiplication for any two quaternions, $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\begin{cases} \boldsymbol{a}\,\boldsymbol{b} = \left(a_0b_0 - \vec{a}\cdot\vec{b},\, a_0\vec{b} + b_0\vec{a} + \vec{a}\times\vec{b}\right) \\ \boldsymbol{b}\,\boldsymbol{a} = \left(a_0b_0 - \vec{a}\cdot\vec{b},\, a_0\vec{b} + b_0\vec{a} - \vec{a}\times\vec{b}\right)\,, \end{cases} \tag{27}$$

we can define two force fields as derivatives of the potential function,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = -\boldsymbol{\phi}\,\boldsymbol{\nabla} \\ \boldsymbol{\mathcal{F}}^- = -\boldsymbol{\nabla}\boldsymbol{\phi}. \end{cases} \tag{28}$$

Note that the two derivatives are due to non-commutativity of the quaternion multiplication resulting in the left and right derivatives.

Then applying (25) and (27) to (28), we obtain general expressions for quaternion forces,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v,\, -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0 + \vec{\nabla}\times\vec{\phi}_v\right) \\ \boldsymbol{\mathcal{F}}^+ = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v,\, -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0 - \vec{\nabla}\times\vec{\phi}_v\right). \end{cases} \tag{29}$$

Let us look for single-valued functions defining the field components by using commutator and anti-commutator relations,

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \dfrac{1}{2}\left(\boldsymbol{\mathcal{F}}^+ + \boldsymbol{\mathcal{F}}^-\right) \\ \boldsymbol{\mathcal{F}_c} = \dfrac{1}{2}\left(\boldsymbol{\mathcal{F}}^+ - \boldsymbol{\mathcal{F}}^-\right). \end{cases} \tag{30}$$

This results in two types of the generic quaternion field components from (29) and (30),

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v\,,\, -\dfrac{\partial\vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0\right) \\ \boldsymbol{\mathcal{F}_c} = \vec{\mathcal{F}}_c = \vec{\nabla}\times\vec{\phi}_v\,, \end{cases} \tag{31}$$

which can be further expanded using the velocity dependent vector potential, $\phi_v = (\vec{v}/c)\phi$,

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \left(-\dfrac{\partial\phi_0}{c\partial t_0} + \phi\vec{\nabla}\cdot\dfrac{\vec{v}}{c} + (\vec{\nabla}\phi)\cdot\dfrac{\vec{v}}{c}\,,\, -\dfrac{\partial(\vec{v}\phi)}{c^2\partial t_0} - \vec{\nabla}\phi_0\right) \\ \boldsymbol{\mathcal{F}_c} = \vec{\mathcal{F}}_c = \phi\left(\vec{\nabla}\times\dfrac{\vec{v}}{c}\right) + (\vec{\nabla}\phi)\times\dfrac{\vec{v}}{c}\,. \end{cases} \tag{32}$$

The total fields result from (30) and (32),

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}_a} + \boldsymbol{\mathcal{F}_c} \\ \boldsymbol{\mathcal{F}}^- = \boldsymbol{\mathcal{F}_a} - \boldsymbol{\mathcal{F}_c}\,. \end{cases} \tag{33}$$

For the stationary case, when, $\vec{v}\sim 0$, the field components become,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}}^- = \boldsymbol{\mathcal{F}_a} \simeq \left(-\dfrac{\partial\phi_0}{c\partial t_0}\,,\, -\vec{\nabla}\phi_0\right) \\ \boldsymbol{\mathcal{F}_c} = \vec{\mathcal{F}}_c \simeq \vec{0}\,, \end{cases} \tag{34}$$

which further reduce to the classical field expression for the stationary static field, when $\partial\phi_0/\partial t_0 \sim 0$,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}}^- = \vec{\mathcal{F}}_a \simeq -\vec{\nabla}\phi_0\,. \\ \boldsymbol{\mathcal{F}_c} \simeq \vec{\mathcal{F}}_c = \vec{0}\,. \end{cases} \tag{35}$$

Thus, we obtained generic field equations for an arbitrary physical interaction defined by a quaternion potential function, $\boldsymbol{\phi}$. One of the field components, $\boldsymbol{\mathcal{F}_a}$, , is a full quaternion, with both scalar and vector parts, which describe time-dependent fluctuations and linear propagation. The other field component, $\boldsymbol{\mathcal{F}_c}$, is a three-dimensional pure vector, describing rotation in the presence of motion. The field components combine into two expressions for the total quaternion field, $\boldsymbol{\mathcal{F}}^+$ and $\boldsymbol{\mathcal{F}}^-$, which reduce to the classical field expression for the static stationary case. Note that the new field expressions were derived from a velocity dependent quaternion potential and consequently depend on the velocity of the body under investigation relative to the observer measuring apparatus.

VII. QUATERNION ELECTROMAGNETIC FIELDS

Let us consider electromagnetic interaction expressed by a quaternion electromagnetic potential, $\boldsymbol{\phi}$, in the observer reference frame, where we introduce the electric and magnetic fields using (30),

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \boldsymbol{\mathcal{E}} = \left(\mathcal{E}_0\,,\ \vec{\mathcal{E}}\right) \\ \\ \boldsymbol{\mathcal{F}_c} = \boldsymbol{\mathcal{B}} = \left(0,\ \vec{\mathcal{B}}\right). \end{cases} \tag{36}$$

As we can see, the electric field is a full quaternion, with both the scalar and vector components. On the other hand the magnetic field is purely a vector field. We derive full expressions for the fields from (31) and (36),

$$\begin{cases} \mathcal{E}_0 = -\dfrac{\partial \phi_0}{c\partial t_0} + \vec{\nabla}\cdot\vec{\phi}_v \\ \\ \vec{\mathcal{E}} = -\dfrac{\partial \vec{\phi}_v}{c\partial t_0} - \vec{\nabla}\phi_0 \\ \\ \vec{\mathcal{B}} = \vec{\nabla}\times\vec{\phi}_v\,, \end{cases} \tag{37}$$

which reminds of the traditional expressions for the electric and magnetic fields, with the exception of of a scalar component of the electric field, $\mathcal{E}_0$, which is not present in the traditional approach.

Using the velocity dependent vector potential, $\phi_v = (\vec{v}/c)\phi$, we obtain from (32) and (37),

$$\begin{cases} \mathcal{E}_0 = -\dfrac{\partial \phi_0}{c\partial t_0} + \phi\,\vec{\nabla}\cdot\dfrac{\vec{v}}{c} + (\vec{\nabla}\phi)\cdot\dfrac{\vec{v}}{c} \\ \\ \vec{\mathcal{E}} = -\dfrac{\partial(\vec{v}\phi)}{c^2\partial t_0} - \vec{\nabla}\phi_0 \\ \\ \vec{\mathcal{B}} = \phi\left(\vec{\nabla}\times\dfrac{\vec{v}}{c}\right) + (\vec{\nabla}\phi)\times\dfrac{\vec{v}}{c}\,. \end{cases} \tag{38}$$

By applying (36) to the generic definition of the force fields (33), we obtain two quaternion expressions for the total electromagnetic fields,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \left(\mathcal{E}_0\,,\ \vec{\mathcal{E}} + \vec{\mathcal{B}}\right) \\ \\ \boldsymbol{\mathcal{F}}^- = \left(\mathcal{E}_0\,,\ \vec{\mathcal{E}} - \vec{\mathcal{B}}\right), \end{cases} \tag{39}$$

where the field components are given by (38). While the first expression in (39) represents positive electric charges, we suggest that the second expression seem to correspond to negative electric charges.

Next using (39), we obtain two quaternion expressions for the Lorentz electromagnetic force,

$$\begin{cases} \boldsymbol{F}^+ = q\left(\mathcal{E}_0\,,\ \vec{\mathcal{E}} + \vec{\mathcal{B}}\right) \\ \\ \boldsymbol{F}^- = q\left(\mathcal{E}_0\,,\ \vec{\mathcal{E}} - \vec{\mathcal{B}}\right) \end{cases} \tag{40}$$

where q is a positive unit charge and electric and magnetic fields are given by (38).

For the stationary case, $\vec{v} \sim 0$, the electromagnetic fields are,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}}^- = \boldsymbol{\mathcal{E}} \simeq \left(-\dfrac{\partial \phi_0}{c\partial t_0}\,,\ -\vec{\nabla}\phi_0\right) \\ \\ \boldsymbol{\mathcal{F}_c} = \boldsymbol{\mathcal{B}} \simeq 0\,, \end{cases} \tag{41}$$

which further reduces to the classical field expression for the stationary electro-static field, $\partial\phi_0/\partial t_0 \sim 0$,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}}^- = \vec{\mathcal{E}} \simeq -\vec{\nabla}\phi_0\,. \\ \\ \vec{\mathcal{B}} \simeq \vec{0}\,. \end{cases} \tag{42}$$

It seems that the quaternion form of electromagnetic interaction demonstrates a full quaternion electric field and a classical vector magnetic field. In addition, it predicts existence of positive and negative electric charges that behave differently while moving in the magnetic field, as expected from the Hall effect [13].

VIII. QUATERNION GRAVITATIONAL FIELDS

Next, let us assume that the gravitational field can be also described by a potential function in the quaternion form (25). Then, we apply definitions of the quaternion field components (30) in order to derive two components of the gravitational field,

$$\begin{cases} \boldsymbol{\mathcal{F}_a} = \boldsymbol{\varGamma} = \left(\varGamma_0\,,\ \vec{\varGamma}\right) \\ \\ \boldsymbol{\mathcal{F}_c} = \boldsymbol{\varOmega} = \left(0,\ \vec{\varOmega}\right)\,. \end{cases} \tag{43}$$

where the components of the gravitational field include the novel scalar field, $\mathit{\Gamma}_0$, as well as two vector fields, $\vec{\mathit{\Gamma}}$, and $\vec{\mathit{\Omega}}$. Now, we can calculate the field components from the potential using (31),

$$\begin{cases} \mathit{\Gamma}_0 = -\dfrac{\partial \phi_0}{c\partial t_0} + \vec{\nabla} \cdot \vec{\phi}_v \\[2ex] \vec{\mathit{\Gamma}} = -\vec{\nabla}\phi_0 - \dfrac{\partial \vec{\phi}_v}{c\partial t_0} \\[2ex] \vec{\mathit{\Omega}} = \vec{\nabla} \times \vec{\phi}_v \,. \end{cases} \tag{44}$$

where, $\vec{\mathit{\Gamma}}$, and, $\vec{\mathit{\Omega}}$, are gravitational equivalents of the electrical and magnetic fields.

Using the velocity dependent vector potential, $\phi_v = (\vec{v}/c)\phi$, we obtain,

$$\begin{cases} \mathit{\Gamma}_0 = -\dfrac{\partial \phi_0}{c\partial t_0} + \phi\,\vec{\nabla} \cdot \dfrac{\vec{v}}{c} + (\vec{\nabla}\phi) \cdot \dfrac{\vec{v}}{c} \\[2ex] \vec{\mathit{\Gamma}} = -\dfrac{\partial (\vec{v}\phi)}{c^2 \partial t_0} - \vec{\nabla}\phi_0 \\[2ex] \vec{\mathit{\Omega}} = \phi\left(\vec{\nabla} \times \dfrac{\vec{v}}{c}\right) + (\vec{\nabla}\phi) \times \dfrac{\vec{v}}{c}\,. \end{cases} \tag{45}$$

By applying (43) to the generic definition of the force fields (33), we obtain quaternion expressions for the total gravitational fields,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} + \vec{\mathit{\Omega}}\right) \\[2ex] \boldsymbol{\mathcal{F}}^- = \left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} - \vec{\mathit{\Omega}}\right), \end{cases} \tag{46}$$

which are of course equivalent to the general expressions (29). The two expressions for the gravitational field differ by the direction of the torsion field, $\vec{\mathit{\Gamma}}$, similar to the effect of the magnetic field in the electromagnetic force. Therefore, we interpret the two field expressions as representation of two types of particle mass.

Then by using (46), we obtain quaternion expressions for the gravitational forces for negative and positive masses respectively,

$$\begin{cases} \boldsymbol{F}^+ = m\left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} + \vec{\mathit{\Omega}}\right) \\[2ex] \boldsymbol{F}^- = m\left(\mathit{\Gamma}_0\,,\ \vec{\mathit{\Gamma}} - \vec{\mathit{\Omega}}\right), \end{cases} \tag{47}$$

where m is a positive unit mass.

Assuming small variations of the gravitational potential with time, $\partial\phi_0/\partial t_0 \sim 0$, and $\partial\vec{\phi}/\partial t_0 \sim 0$, we obtain the approximate form of the gravitational field,

$$\begin{cases} \mathit{\Gamma}_0 \simeq \phi\,\vec{\nabla} \cdot \dfrac{\vec{v}}{c} + (\vec{\nabla}\phi) \cdot \dfrac{\vec{v}}{c} \\[2ex] \vec{\mathit{\Gamma}} \simeq -\vec{\nabla}\phi_0 \\[2ex] \vec{\mathit{\Omega}} \simeq \phi\left(\vec{\nabla} \times \dfrac{\vec{v}}{c}\right) + (\vec{\nabla}\phi) \times \dfrac{\vec{v}}{c}\,. \end{cases} \tag{48}$$

which is a new form of gravitational field expressions for slow varying fields.

For the stationary case, when, $\vec{v} \sim 0$, the field expressions reduce to,

$$\begin{cases} \boldsymbol{\mathcal{F}}^+ = \boldsymbol{\mathcal{F}}^- = \vec{\mathit{\Gamma}} \simeq -\vec{\nabla}\phi_0\,, \\[2ex] \vec{\mathit{\Omega}} \simeq \vec{0}\,. \end{cases} \tag{49}$$

Thus we obtained quaternion expressions for the gravitational fields and forces similar to the electromagnetic expressions.

IX. CONCLUSIONS

We introduced quaternion space-time and presented a framework for description of physical events using quaternion time intervals. We derived the quaternion form of the Lorentz time transformation and presented an intuitive physical interpretation of the time dilation. Then, we showed that quaternion algebra leads to well behaved quaternion calculus, provided we choose the right derivatives for the observer reference frame. Finally, we proposed a general form of quaternion field expressions, by differentiating a generic quaternion potential function, and applied them to electromagnetic and gravitational interactions. The resulting expressions for fields and forces depend on the particle velocity relative to the observer. The additional novel terms in the field expressions need further study and experimental verification.

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