On the Dimension and Metric of Space in which a Physical System Can Exist

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Abstract

Based on the movement of a material point along a trajectory in a multi-dimensional space, all possible geometries of such a space are considered.

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It is generally accepted that any physical object can be considered as a set of material points, endowed with a set of physical parameters. Knowing the state of motion of each point of the object is equivalent to the knowing the state of motion of the whole object. Thus, the state of the object is characterized by a number of parameters, and any change in their values is the movement of the object. Parameters may be both discrete (such as spin and charge) and continuous (the coordinates of points, the moments of time). The number of parameters required to describe the physical system is determined by the specific task. For example, Newtonian mechanics requires three coordinates to describe the motion of a particle, while for describing the behavior of a large number of subsystems it is necessary to introduce the concept of configuration and phase spaces.

Here we consider only continuous parameters, called *coordinates*, each of which takes values on the real line. The set of all parameters defines *the coordinate space*, the metric properties of which one needs to know to be able making direct physical measurements that allow determining how the parameters in this physical process change. For example, in the analytical dynamics such parameter space is the space of generalized coordinates. In each physical process, it is possible to determine physical quantities that are functions of parameters, which are determined by indirect measurements and set the state of the object. The set of all possible states determines the *space of states*. Thus, we can summarize that the state of physical objects are realized on the coordinate space. In other words, the coordinate space is a reservoir of physical objects and the arena in which physical processes happen. In this sense it has *the same absolute character as Newton's space*.

Since the coordinate space is a continuous manifold, then we assume that it is a smooth differentiable real *N*-dimensional space, $\mathfrak{D}_{\mathrm{N}}^{\mathbb{R}}$, covered by the curvilinear coordinates (parameters) X^A, $A = 1, 2, ..., N$, *N* is the number of parameters. Changing the state of the system is determined by the motion of a point in this space, so that the point moves along a path, or a smooth curve, the parametric equation of which is $X^A = X^A(s)$. One can choose any parameter as the parameter *s*, but it is most convenient to choose a natural parameter determined by the arc length of the trajectory. In the Special Relativity for subluminal particles, such a parameter is the proper time. Therefore, *the Minkowski space*, just like Euclidean space of Newtonian mechanics, *is also absolute in nature*, rather than the relative one, as it stated in Special Relativity.

The derivative of the coordinates X^A with respect to the parameter *s* at any point of the trajectory defines a tangent vector $V^A = dX^A / ds$. The set of such vectors, which are tangent to all possible trajectories, passing through a given point, forms a tangent *N*-dimensional vector space $\mathbf{V}_{\mathrm{N}}^{\mathbb{R}}$ at this point X^A, which does not coincide with the vector space of the same structure at a neighboring point X'^A of the parameter space. In order to go to a neighboring point it is necessary to make the transformation $X'^A = f^A(X^B)$ in the space $\mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$, which induces a linear transformation $dX'^A = F^A_{\cdot B} dX^B$ in the space $\mathbf{V}^{\mathbb{R}}_{\mathrm{N}}$, where $\hat{\mathbf{F}} = (F^A_{\cdot B}) = (\partial X'^A / \partial X^B)$ is transformation matrix. The transformation in $\mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$ defines the transition from local map $(\mathfrak{U}, \phi)$ to the map $(\mathfrak{U}', \phi')$, where $\mathfrak{U} \subset \mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$ and $\mathfrak{U}' \subset \mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$ are open sets in $\mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$, covering by coordinates $\{X^A\} \in \mathfrak{U}$ and $\{X'^A\} \in \mathfrak{U}'$, respectively. ϕ and ϕ' are homeomorphisms of sets $\mathfrak{U}$ and $\mathfrak{U}'$ onto the open sets $\mathbf{D}$ and $\mathbf{D}'$, respectively, allowing to determine local coordinates $\{\xi^A\} \in \mathbf{D}$ in $\mathfrak{U}$ and $\{\xi'^A\} \in \mathbf{D}'$ in $\mathfrak{U}'$:

$$\phi : \mathfrak{U} \to \mathbf{D}, \text{ or } \xi^A = \phi^A(X^B); \tag{1}$$

$$\phi' : \mathfrak{U}' \to \mathbf{D}', \text{ or } \xi'^A = \phi'^A(X'^B). \tag{2}$$

Introduction of local coordinates allows, first, to determine a local basis at any point of the trajectory and to establish the transformation law of the basis for passing from point to point, and, secondly, to connect local coordinates with tangent vectors by a linear transformation: $d\xi^A = H^A_{\cdot B} dX^B$. Coordinates ξ^A and X^A may be chosen so that the matrix $\hat{\mathbf{H}} = (H^A_{\cdot B})$ will be of canonical form.

From a physical point of view, the measurement process is realized using the local coordinates ξ^A. Then coordinates dX^A will be «naturally measured quantities», as they are called by Einstein,[1] and canonical form of the matrix $\hat{\mathbf{H}}$ determines the metric structure of the tangent space $\mathbf{V}^{\mathbb{R}}_{\mathrm{N}}$. Tangent spaces can be superposed with each other at all points of the space $\mathfrak{D}^{\mathbb{R}}_{\mathrm{N}}$. Therefore, the space $\mathbf{V}^{\mathbb{R}}_{\mathrm{N}}$ with canonical metric can be regarded as a background space, where physical phenomena develop. This agrees with Poincaré's assertion that geometry does not stem from experience and, therefore, "One geometry cannot be more true than another; it can only be *more convenient*".[2]

Determining the canonical form of a matrix $\hat{\mathbf{H}}$ is a completely definite, but rather cumbersome procedure. As is known, its gist consists in the fact that it is necessary to find eigenvalues λ_A of the matrix $\hat{\mathbf{H}}$, and then use the scale transformations to reduce its elements to 0, 1 or –1. λ_A are solutions of characteristic equation

$$\det(\hat{\mathbf{H}} - \lambda \cdot \mathbf{1}) = (-1)^N \mathrm{P}^N(\lambda) = 0, \tag{3}$$

where $\mathrm{P}^N(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + \ldots + a_1\lambda + a_0$ is a polinomial of N-th degree whose coeffecients are expressed in terms of the elements od the matrix $\hat{\mathbf{H}}$. Polinomial $\mathrm{P}^N(\lambda)$ can be represented as

$$\mathrm{P}^N(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_n)^{m_n} \left[\mathrm{P}^2_1(\lambda)\right]^{s_1} \left[\mathrm{P}^2_2(\lambda)\right]^{s_2} \ldots \left[\mathrm{P}^2_z(\lambda)\right]^{s_z}, \tag{4}$$

where m_i ($i = 1, 2, \ldots, n$) are algebraic multiplicity of real roots λ_1, λ_2, ..., λ_n,

s_k ($k = n+1, n+2, \ldots, n+z$) are algebraic multiplicity of complex roots λ_k,

$$\mathrm{P}^2_{k-n}(\lambda) = (\lambda - \lambda_k)(\lambda - \lambda_k^*) = \lambda^2 - \lambda(\lambda_k + \lambda_k^*) + \lambda_k \lambda_k^* \tag{5}$$

is a polinomial of second degree with two complex conjugate roots λ_k and λ_k^*.

Let the real roots λ_1, λ_2, ..., λ_n of the polynomial (4) satisfy conditions

$$\begin{aligned} &\lambda_1 = 0, \ldots, \lambda_r = 0; \\ &\lambda_{r+1} > 0, \ldots, \lambda_{r+p} > 0; \\ &\lambda_{r+p+1} < 0, \ldots, \lambda_{r+p+q} < 0; \end{aligned} \tag{6}$$

[1] "Naturlich gemessene Größen", [1], S. 1058.

[2] [3], p. 50; «Une géométrie ne peut pas être plus vraie qu'une autre; elle peut seulement être *plus commode*» ([2], p. 66-67).

and complex roots λ_{n+1}, λ_{n+2}, ..., λ_{n+z} satisfy conditions

$$\begin{aligned}&\operatorname{Re}\lambda_{n+1}>0,...,\operatorname{Re}\lambda_{n+k}>0\,;\\&\operatorname{Re}\lambda_{n+k+1}<0,...,\operatorname{Re}\lambda_{n+k+l}<0\,;\\&\operatorname{Re}\lambda_{n+k+l+1}=0,...,\operatorname{Re}\lambda_{n+k+l+m}=0\,.\end{aligned}\tag{7}$$

Then $n=r+p+q$, $z=k+l+m$, $N=r+m_1+m_2+...+m_{p+q}+2(s_1+s_2+...+s_z)$, whereas rank of the matrix $\hat{\mathbf{H}}$ is $\operatorname{rank}(\hat{\mathbf{H}})=m_1+m_2+...+m_{p+q}+2(s_1+s_2+...+s_z)=N-r$.

With the help of scrupulous calculations, one can show that the matrix $\hat{\mathbf{H}}$ reduces to the following canonical form

$$\hat{\mathbf{H}}_0=\hat{\mathbf{M}}\hat{\mathbf{H}}\hat{\mathbf{M}}^{-1}=\operatorname{diag}(\mathbf{0}_r,\mathbf{E}_{p+q},\mathbf{A}_{k+l},\mathbf{\Sigma}_m)\,,\tag{8}$$

where

$$\mathbf{E}_{p+q}=\operatorname{diag}(\mathbf{E}_{\mu_1}^{m_1},...,\mathbf{E}_{\mu_p}^{m_p},\mathbf{E}_{\mu_{p+1}}^{m_{p+1}},...,\mathbf{E}_{\mu_{p+q}}^{m_{p+q}})\,,\tag{9}$$

$$\mathbf{A}_{k+l}=\operatorname{diag}(\mathbf{A}_{\sigma_1}^{s_1},...,\mathbf{A}_{\sigma_k}^{s_k},\mathbf{A}_{\sigma_{k+1}}^{s_{k+1}},...,\mathbf{A}_{\sigma_{k+l}}^{s_{k+l}})\,,\tag{10}$$

$$\mathbf{\Sigma}_m=\operatorname{diag}(\mathbf{\Sigma}_{\sigma_{k+l+1}}^{s_{k+l+1}},...,\mathbf{\Sigma}_{\sigma_{k+l+m}}^{s_{k+l+m}})\,,\tag{11}$$

$\hat{\mathbf{M}}$, $\hat{\mathbf{M}}^{-1}$ are degenerate matrices of rank $N-r$ of the transition to the local basis and vice versa, which satisfy the relation $\hat{\mathbf{M}}\hat{\mathbf{M}}^{-1}=\hat{\mathbf{M}}^{-1}\hat{\mathbf{M}}=\operatorname{diag}(\mathbf{0}_r,\mathbf{1}_{N-r})$.

The matrices that make up the diagonal blocks of the matrix (8) have the form

$$\mathbf{E}_{\mu_i}^{m_i}=\underbrace{\begin{pmatrix}+\mathbf{1}_{m_i-\mu_i} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}_{\mu_i}\end{pmatrix}}_{m_i} m_i\,,\ \mathbf{E}_{\mu_i}=\underbrace{\begin{pmatrix}1 & 1 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \cdots & \cdots\\ 0 & 0 & \cdots & 1 & 1\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix}}_{\mu_i}\mu_i\,,\ i=1,2,...,p\,;\tag{12}$$

$$\mathbf{E}_{\mu_i}^{m_i}=\underbrace{\begin{pmatrix}-\mathbf{1}_{m_i-\mu_i} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}_{\mu_i}\end{pmatrix}}_{m_i} m_i\,,\ \mathbf{E}_{\mu_i}=\underbrace{\begin{pmatrix}-1 & 1 & \cdots & 0 & 0\\ 0 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \cdots & \cdots\\ 0 & 0 & \cdots & -1 & 1\\ 0 & 0 & \cdots & 0 & -1\end{pmatrix}}_{\mu_i}\mu_i\,,\ i=p+1,p+2,...,p+q\,;\tag{13}$$

$$\mathbf{A}_{\sigma_a}^{s_a}=\underbrace{\begin{pmatrix}\boldsymbol{\alpha}_{s_a-\sigma_a} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{\sigma_a}\end{pmatrix}}_{2s_a} 2s_a\,,\ \mathbf{A}_{\sigma_a}=\underbrace{\begin{pmatrix}+\boldsymbol{\alpha} & \mathbf{1}_2 & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & +\boldsymbol{\alpha} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \cdots & \cdots\\ \mathbf{0} & \mathbf{0} & \cdots & +\boldsymbol{\alpha} & \mathbf{1}_2\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & +\boldsymbol{\alpha}\end{pmatrix}}_{2\sigma_a}2\sigma_a\,,\ a=1,2,...,k\,;\tag{14}$$

$$\mathbf{A}^{s_a}_{\sigma_a} = \underbrace{\begin{pmatrix} -\boldsymbol{\alpha}_{s_a-\sigma_a} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\sigma_a} \end{pmatrix}}_{2s_a} 2s_a\,,\ \mathbf{A}_{\sigma_a} = \underbrace{\begin{pmatrix} -\boldsymbol{\alpha} & \mathbf{1}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\alpha} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\boldsymbol{\alpha} & \mathbf{1}_2 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\boldsymbol{\alpha} \end{pmatrix}}_{2\sigma_a} 2\sigma_a\,,$$

$$a = k+1, k+2, \dots, k+l\,; \tag{15}$$

$$\boldsymbol{\Sigma}^{s_a}_{\sigma_a} = \underbrace{\begin{pmatrix} \boldsymbol{\sigma}_{s_a-\sigma_a} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\sigma_a} \end{pmatrix}}_{2s_a} 2s_a\,,\ \boldsymbol{\Sigma}_{\sigma_a} = \underbrace{\begin{pmatrix} \boldsymbol{\sigma} & \mathbf{1}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\sigma} & \mathbf{1}_2 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}}_{2\sigma_a} 2\sigma_a\,,$$

$$a = k+l+1, k+l+2, \dots, k+l+m\,; \tag{16}$$

$$\boldsymbol{\alpha}_n = \mathbf{1}_n \times \boldsymbol{\alpha} = \underbrace{\boldsymbol{\alpha} \oplus \boldsymbol{\alpha} \oplus \dots \oplus \boldsymbol{\alpha}}_{n}\,,\ \boldsymbol{\alpha} = \frac{1}{\sqrt{2}}(\mathbf{1}_2 + \boldsymbol{\sigma}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{17}$$

$$\boldsymbol{\sigma}_n = \mathbf{1}_n \times \boldsymbol{\sigma} = \underbrace{\boldsymbol{\sigma} \oplus \boldsymbol{\sigma} \oplus \dots \oplus \boldsymbol{\sigma}}_{n}\,,\ \boldsymbol{\sigma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2\,, \tag{18}$$

$\mathbf{0}_n$, $\mathbf{1}_n$ are zero and unit $n \times n$-matrices, respectively, σ_2 in (18) is the Pauli matrix. At $n = 0$ matrices $\mathbf{0}_n$, $\mathbf{1}_n$, $\mathbf{E}_n$, $\mathbf{A}_n$, $\boldsymbol{\Sigma}_n$ vanish. In formulas (8)-(16) μ_i, σ_i are geometric multiplicities, meaning that the matrix $\hat{\mathbf{H}}$ satisfies the minimal equation

$$(\hat{\mathbf{H}} - \lambda_1)^{\mu_1}(\hat{\mathbf{H}} - \lambda_2)^{\mu_2}\dots(\hat{\mathbf{H}} - \lambda_n)^{\mu_n}\left[\mathrm{P}_1^2(\hat{\mathbf{H}})\right]^{\sigma_1}\left[\mathrm{P}_2^2(\hat{\mathbf{H}})\right]^{\sigma_2}\dots\left[\mathrm{P}_z^2(\hat{\mathbf{H}})\right]^{\sigma_z} = \mathbf{0}\,. \tag{19}$$

The canonical form (8) of the matrix $\hat{\mathbf{H}}$ determines the distance in the space $\mathbf{V}_{\mathrm{N}}^{\mathbb{R}}$, which for infinitely close points on the world line is expressed in terms of local coordinates by means of $\hat{\mathbf{H}}$ in the form

$$dS^2 = (\hat{\mathbf{H}}_0)_{AB}\,dX^A dX^B = \hat{\mathbf{H}}_{AB}\,d\xi^A d\xi^B\,, \tag{20}$$

The representation of the matrix $\hat{\mathbf{H}}$ in the canonical form means that the vector space $\mathbf{V}_{\mathrm{N}}^{\mathbb{R}}$ can be represented as a direct sum of invariant subspaces

$$\mathbf{V}_{\mathrm{N}}^{\mathbb{R}} = \mathbf{V}_r \oplus \mathbf{V}_{\mu_1}^{m_1} \oplus \dots \oplus \mathbf{V}_{\sigma_{k+l+m}}^{s_{k+l+m}}\,, \tag{21}$$

each of which is covered by its own set of local coordinates. It means that line element (20) of the world line can be expressed in the form

$$dS^2 = ds_r^2 + ds_{m_1\mu_1}^2 + \dots + ds_{s_{k+l+m}\sigma_{k+l+m}}^2\,, \tag{22}$$

which contains all possible cases of metrics. For example, if $\hat{\mathbf{H}}_0 = \mathbf{E}_{\mu_1}^{m_1}$, then at $m_1 = 3$, $\mu_1 = 2$ $\hat{\mathbf{H}}$ can be regarded as the metric of a monoclinic crystal with one circular optical axis (see, e. g., [4], p. 97). The metric of an anisotropic space was considered by Edwards in [5], as well as in [6], pp. 82-88, where Einstein coordinates and Edwards coordinates are related by the matrix $\mathbf{E}_{\mu_i}^{m_i}$, (12), at $i = 1$, $m_1 = 4$, $\mu_1 = 2$. The use of metrics of the form (14)-(16) can refer to different superspaces in which the coordinates do not commute with each other.

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