The Paradoxical Collection of Sets Explained

By Jim Rock

Abstract: We explain why there is a collection of sets, which both have and do not have largest elements.

Introduction. For all real numbers \( a \) in the open interval \((0, 1)\)

Let the collection of all \( R_a = \{ y \text{ a real number } | 0 \leq y < a \} \)

Each set in the collection of \( R_a \) has a largest element.

Each \( R_a \) must contain one and only one element \( a' \) that is not an element in the group of all the proper subsets of that \( R_a \), as taken from the collection of all \( R_a \).

Otherwise, since the group of all proper subsets of each \( R_a \) taken from the collection of all \( R_a \) are nested in descending order inside that \( R_a \), and none of the proper subsets contain all the elements of that \( R_a \), each \( R_a \) would be a proper subset of itself.

\( a' \) is the largest element of \( R_a \).

No Set in the collection of \( R_a \) has a largest element.

Suppose there is a largest element \( a' \) in \( R_a \).

\( a' < (a + a')/2 < a \). Let \( b = (a + a')/2 \). Then \( b \) is in \( R_a \) and \( a' < b \).

Note: the question is not whether \( R_a \) actually contains a largest element, but whether or not the two contradictory statements about a largest element are both conclusions of valid logical arguments.

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