Null Algebra: The Math of Division by Zero and The Negative Radical

Robert S Miller
Email: rmille4612@hotmail.com

Abstract
An approach to resolution of instances of division by zero, infinities and the negative radical using a primary focus within the mathematical discipline of Algebra via direct substitutions with example problems. It demonstrates the necessity of space-time being a composed of a multiple directions; three space only directions, three time-like space directions, one space-like time direction, and one time-only direction.

The Mathematics of Division by Zero and Solution of $i = \sqrt{-1}$

Algebra has its origins in the ancient history of civilization, even as far back as Babylon. The mathematical discipline of Calculus was discovered by both Isaac Newton and Gottfried Wilhelm Leibniz in 1684 as a way of defining rates of change as well as energy and motion. More than 300 years have passed since that publication. Calculus has advanced and developed along with human society. Though many advancements have been made in mathematics some issues have persisted which seem defy logic. Among those are the idea of dividing by zero and the roots of negative numbers. To resolve these quantities we will return to Algebra and explore their application. Division by zero seems to be impossible to grasp as it implies infinity. No matter how high a number you can imagine infinity is still infinitely far away. Then there is the issue of $\frac{0}{0}$, a figure which can have seeminly multiple values all in the same instance. These values, an apparent contradiction, include infinity, 1 and -1.

These concepts are only confusing and contradictory within the confines of traditional mathematics. In order to make sense of the infinities and multiple value interpretations of $\frac{0}{0}$ requires the inclusion of both extra directions which are present but not seen as well as a new idea of 0, *null math naught*.

When we think of 0 we often think of something we can count, like 0 cookies. Seldom do we think of it as the absence of value but it *can be* this as well. The difficulty is to understand when you're counting cookies and when you're dealing with the void. All of these new concepts require explanation to understand why they are necessary and how to use them in mathematics to resolve infinities and division-by-zero inconsistencies.

In pursuit of this goal it was not intended to tackle the idea of $i = \sqrt{-1}$. However in the process of defining some aspects of division by zero, the apparent solution to the negative radical presented itself. Its recommended you read the entirety of the introduction which provides the explanations and an overview of the Algebra associated with these concepts.

This work has been written in the form of a text book and its assumed the reader has knowledge of several levels of basic Algebra and Trigonometry which are necessary to understand the application of the mathematics. The structure and progression of the chapters is intended to make it easy to understand and use. Take your time working through the material and where available attempt the example problems provided. I appreciate you're taking the time to learn this extension to Algebra.

Robert S. Miller

Introduction

Null Algebra

Intro.i—Subspace:

This is not the Subspace you've learnt about in Linear Algebra classes. Instead of vector spaces it is literally a layer of space in which we all live in. We see it every day without realizing we are looking right at it; it is hidden from us in a sense. For every basis vector in a complete basis there is a subspace axis. For the three directional world defined by the xyz volume there are three corresponding subspaces respectively defined as suw.

These three subspaces, along with a fourth described as the Alternate (A-axis), make up a space-like description of time, denoting the places where the events in the xyz volume have, are and shall occur. Collectively they can be used to describe any and all arrangements of matter and energy as well as their evolution over time. We shall refer to the $suwA$ hyper-volume, as well as their individual component axis, as *dimensional time*.

Dimensional time is different than the idea of *directional time*, the idea of the ticks of the clock, the literal passage of time marked by the increase of entropy, and unfolding of events. Directional time is always positive and denoted by the variable t. Conversion to subspace follows specific rules and reasoning. In order to understand exactly what a subspace is and how to convert to it from a given equation we'll start with an examination of number lines, multiplication and division.

Intro.ii—Understanding Naught Basics:

What exactly is a number line? It's really nothing more than a pictographic representation of counting. We assign values by combining two or more number lines, and then define operations based upon gain or loss within that system. Zero is a transition point along any number line where values flip from negative to positive. Traditional mathematics holds that 0 is neither positive nor negative. This is challenged within the disciplines of Null-Algebra and Null-Calculus. The 0 in a number line is not just between the positive and negative numbers but belongs to both the positive and negative side of any given axis, which means it is simultaneously positive and negative. This is a moot point in traditional mathematics which makes no attempt to resolve infinities and division by zero, but is crucial in null-math.

The 0 in a number line normally holds the idea of none of something; zero cookies. It can also represent the idea of emptiness, the lack of attainable value. In this sense it represents *naught*, denoted by lowercase Greek eta with a zero subscript η_0. This is still zero but not zero as in 0 cookies. It is the absence of value and a result of division by 0. Naught has unique properties.

Note that $\eta_0 = 0$ *but* $\eta_0 \not\equiv 0$. *This requires for the following properties an extension to the traditional order of operations. For instance the properties below will show, for example,* $2 \cdot \eta_0 = 2$. *Although we could use traditional mathematics to easily rewrite this example as* $2 = \frac{2}{\eta_0}$ *it will not yield a correct expression:* $2 \neq \frac{2}{\eta_0} = \eta_0$. *The occurrence of a* η_0 *must be treated as if it and the expression it modifies were wrapped in parenthesis:* $(2 \cdot \eta_0) = 2$. *You must resolve the expression modified by naught first before proceeding with other operations. Note that it would be ok to divide out the entire expression in this example:* $1 = \frac{2}{(2 \cdot \eta_0)}$.

<ins>Naught:</ins>

$\frac{a}{0} = \eta_0 = 0$ Any value *a* divided by 0 will equal *naught*. Special circumstances apply when $a = 0$ which are covered later.

If we are to understand this as how many times *a* cookies be divided into 0 units traditionally this is represented as infinity. The fact that this value is infinitely far away tells us it is unattainable and thereby zero. Not the zero of zero cookies but the rather the zero of non-existence; there are *no ways* in which to divide the up the given objects into zero units. It can still be plotted on the output axis of a graph as 0. The traditional value of zero as in zero cookies is actually attained on the output axis' corresponding subspace. As already mentioned naught is a kind of 0 though the meaning it conveys is slightly different. This will be covered further in the introduction section on subspace conversions.

Naught Additive Identity Property:

To understand naught it is best to compare it to similar operations using 0. The associative property of addition for both 0 and η_0 are identical in structure but not in what they imply.

$a + 0 = 0 + a = a$	Any value plus 0 leaves the value unchanged
$a + \eta_0 = \eta_0 + a = a$	Here there is no concept of value being added to a. The expression at left is nothing but *a* alone. $a = a = a$

Naught Difference Property:

Like the difference property of traditional mathematics subtraction with naught is not commutative. The order of the operation matters and will affect the outcome.

$a - 0 = a \qquad 0 - a = -a$

In the first instance there are *a* cookies and no cookies are removed from this total, leaving you with *a* cookies. In the second instance there are no cookies, and you discover you owe *a* cookies to a friend, leaving you with a deficit of *a* cookies; i.e. *–a.*

$a - \eta_0 = a \qquad \eta_0 - a = -a$

The naught difference property values are again the same as the traditional math counterpart but the reason why is different. In the first instance here you have *a* cookies and no subtraction ever takes place. The value being subtracted is an absence of any value at all and so you always have *a* cookies. In the second instance for the same reasoning you always had a deficit of *a* cookies (*–a*).

Something unique occurs here when $a = 0$.

$\eta_0 - 0 = \eta_0$	Naught, the absence of value, minus 0 is naught. Note this is *positive* naught. The next expression makes this clear. The absence of value minus nothing leaves the absence of value.
$0 - \eta_0 = -\eta_0$	Naught is the absence of value, unattainable as it is at infinity. 0, minus this value at infinity is still naught, still unattainable but was subtracted. This is a way of expressing negative infinity. So the naught is not only negative, it

was always negative. Or by additive inverse, the negative of the absence of value having nothing added to it is still the negative absence of value.

This shows an interesting property of naught. 0 is simultaneously positive and negative. Naught, though an absence of value, is either positive or negative. It carries only the indication of that sign. Positive naught is synonymous with positive infinity. The infinity is unattainable and thereby resolved to naught but maintains its sign as positive. The same is true of negative naught.

<u>Naught Commutative Property:</u>

Traditional mathematics shows that any multiple of 0 is 0 irrespective of the order in which multiplication is applied.

$a \cdot 0 \;=\; 0 \cdot a \;=\; 0$ Any multiple of 0 is 0.

These expressions say, if you have 0 multiples of *a* cookies you have no cookies. Naught changes things up a little. Because naught is the absence of value it tells us there is no concept of a multiple to apply to another value. The expression $a \cdot \eta_0$ says you have *a* cookies and you want to multiply them times, nothing to multiply by, leaving you with just *a* cookies. From this we have, for $a \neq 0$:

$$a \cdot \eta_0 \;=\; \eta_0 \cdot a \;=\; a$$

When $= 0$. Nothing times an absence of value is an absence of value.

$$0 \cdot \eta_0 \;=\; \eta_0 \cdot 0 \;=\; \eta_0$$

If a equals η_0:

$$\eta_0 \cdot \eta_0 \;=\; \eta_0 \cdot \eta_0 \;=\; {\eta_0}^2 \;=\; \eta_0$$

<u>Naught vs. Identity:</u>

Consider the following two examples: $4 = 0$ $4 = \eta_0$

If you see this come up in an equation you most likely immediately check you math for where you went wrong. The first example $4 = 0$ is a contradiction; $4 \neq 0$. Something went wrong during solving or perhaps the equation is unsolvable. However, $4 = \eta_0$ is resolvable. This expression says 4 is equal, and then essentially there is nothing, not even a 0 on the other side of that equals sign. 4 = 0 is nonsense. However 4 = η_0 actually doesn't have anything to the right of the equals sign. It implies, in this example, only 4. So 4 = η_0 really says just 4, or $4 = 4$. Recall that naught is related to infinity and there by this applies to all numbers.

For $a \neq 0$ $\quad a = 0 \quad \longrightarrow \emptyset$ $\qquad$ *only* $a = 0$ *is solution*

For $a = \mathbb{R}$ $\quad a = \eta_0 \quad \longrightarrow a$ $\qquad$ *Solution set is all real numbers*

When $a = 0$ the first equation becomes true but the second equation remains true for all possible values.

Naught Division Property:

Although division is the reverse of multiplication there are of course some differences with naught because of the uniqueness of this value. Again let's begin by considering the properties with 0.

$\frac{0}{a} = 0$ $\qquad$ Traditional mathematics holds 0 divided by any value is except 0, is 0. The only caveat that *a* cannot be 0 is due an inability to resolve the value of $\frac{0}{0}$ within traditional mathematics.

$\frac{a}{0} = \infty = \eta_0$ $\qquad$ This property was discussed above. Traditional mathematics holds the value is infinity but this is not an answer; it means it is increasing without bound. It is for that very reason the value is unattainable and therefore non-existent; naught.

$\frac{\eta_0}{a} = 0$ The logic of this straightforward. The absence of any value, a kind of zero, can be divided into *a* size units how many times? There is no value to divide into *a* size units. So it cannot be done, or rather completed 0 times.

$\frac{a}{\eta_0} = \eta_0$ This is a very similar argument to the one above. Here though you actually begin with a value of some number of units. You have *a* units and want to determine how many times you can spit them up into the absence of value, into non-existence. It cannot be done. Not only can it not be done but 0 times is not a sufficient answer as there is no way to remove something from existence completely. Thus the answer is naught.

There are interesting outcomes when *a* itself equals either 0 or η_0. The application of $a = 0$ in $\frac{0}{a}$ or $\frac{a}{0}$, and application of $a = \eta_0$ in $\frac{\eta_0}{a}$ or $\frac{a}{\eta_0}$ will result in $\frac{0}{0}$ in the first case and $\frac{\eta_0}{\eta_0}$ in the second case. Note that although naught is a kind of zero, $\eta_0 \not\equiv 0$. So a result of $\frac{0}{0}$ and $\frac{\eta_0}{\eta_0}$ will not only have different properties from each other but will also vary in resolution depending on whether it is the result of a constant expression such as $\frac{x}{x}$ or by way of trending, which is explained below.

If either expression is the result of a constant 1 to 1 ratio in an equation, then $\frac{0}{0}$ and $\frac{\eta_0}{\eta_0}$ will equal a constant. $\frac{0}{0}$ will be $\oplus 1$ which will resolve to $+1$ on the originating axis. $\frac{\eta_0}{\eta_0} = \eta_0$ and is sign dependent, being essentially only a positive or negative sign with no actual number value. Despite this if one were to multiply it by a number it would apply its sign in that process similar to, but not identical to 1 or -1 respectively. Values such as $\frac{\eta_0}{a}$, $\frac{a}{\eta_0}$, $\frac{0}{a}$ or $\frac{a}{0}$ will follow the patterns in the above four examples, called trending patterns.

This pattern is called *trending* as the equation or expression itself is already trending toward a specific value at *a* = 0 and $a = \eta_0$. In the 1 to 1 ratio the value of $\frac{0}{0}$ as a constant, $\oplus 1$, is already there. This is best exemplified by $\frac{x}{x}$. When x is equal to any value other than 0 the ratio is equal to 1. The ratio $\frac{x}{x}$ is trending toward 1. When $x = 0$ the ratio $\frac{x}{x}$ is still trending toward 1 and will equal 1.

The occurrence of a $\frac{0}{0}$ in any manner than an expression such as $\frac{x}{x}$ for $x = 0$ the ratio $\frac{0}{0}$ is present as a result of either the numerator or denominator approaching 0 while the other already is 0 and must follow the trending pattern as defined by the equation generating that ratio. For now know that an equation which results in $\frac{0}{0}$ is showing its *feedback value*, which must be resolved via a usage of *naught* in place of the input value. Also note that $\frac{0}{0}$ is referred to as the Ξ operator throughout much of Null-Mathematics. This and more will be covered on the use of $\frac{0}{0}$ and its meanings in a later section.

Additional Naught Properties:

$\eta_0 - \eta_0 = 0$ — The absence of value is removed from the absence of value. The result is no value in the traditional sense of 0. From this the next four properties can be drawn.

$\eta_0 = 0 + \eta_0$ $\qquad$ $\eta_0 - 0 = \eta_0$ $\qquad$ $-\eta_0 = 0 - \eta_0$ $\qquad$ $-\eta_0 + \eta_0 = 0$

$\eta_0 + \eta_0 = \eta_0$ — The absence of value added to the absence of value makes no change.

${\eta_0}^a = \eta_0$ Where $a = \mathbb{R}$ — The value of naught arises for any value divided by 0. So any power of naught as positive or negative integer or fraction will result in naught.

$${\eta_0}^2 = \eta_0 \qquad \sqrt{\eta_0} = \eta_0 \qquad \sqrt[3]{{\eta_0}^2} = \eta_0 \quad \text{etc.}$$

Note that $\eta_0 \not\equiv \frac{0}{0}$. Some of their properties are similar but not identical.

$\eta_0 \sim 0$ however $\eta_0 \not\equiv 0$ (*naught is similar to but not identical to 0*)

$a^0 = 1$ See section on derivatives and limits. Specifically on limits of exponents reaching zero.

<u>Properties from Trending:</u>

$\frac{\eta_0}{\eta_0} = \frac{-\eta_0}{-\eta_0} = \eta_0$ $\frac{-\eta_0}{\eta_0} = \frac{\eta_0}{-\eta_0} = -\eta_0$ $\frac{\eta_0}{0} = \frac{-\eta_0}{0} = 0$ $\frac{0}{\eta_0} = \eta_0$ $\frac{0}{-\eta_0} = -\eta_0$

<u>Naught Exponents and Roots:</u>

$$\left(\frac{a}{b}\right)^{\eta_0} = \begin{cases} \eta_0 \ \ for\ a > b \\ 1 \ \ for\ a = b \\ 0 \ \ or\ a < b \end{cases}$$

The value of naught is obtained as $\frac{n}{0} = \infty$. The value of ratio raised to infinity will depend on the value of numerator and denominator. An infinite result is resolvable to naught.

$$\left(\frac{a}{b}\right)^{-\eta_0} = \left(\frac{b}{a}\right)^{\eta_0} = \begin{cases} \eta_0 \ \text{ for } b > a \\ 1 \ \text{ for } a = b \\ 0 \ \text{ for } b < a \end{cases}$$

$0^0 = 1$ See section on derivatives and limits. Specifically on limits of exponents reaching zero.

$0^{\eta_0} = 0$ The value of naught is obtained as $\frac{n}{0} = \infty$. The value of 0 raised to infinity is itself resolvable to 0.

${\eta_0}^0 = \sqrt[\eta_0]{\eta_0} = \eta_0$ Naught raised to power of 0 is an infinite root of naught and identical to a naught root of naught. This resolves to naught.

${\eta_0}^{\eta_0} = \eta_0$ Naught raised to a power of naught extends to infinity and resolvable to naught.

$\sqrt[\eta_0]{a} = a^0 = 1$ for $a = \mathbb{R}$

$\sqrt[0]{a} = a^{1/0} = a^{\infty} = \infty = \eta_0$

$\sqrt[0]{0} = 0^{1/0} = 0^{\infty} = 0^{\eta_0} = 0$

$\sqrt[0]{\eta_0} = \eta_0{}^{1/0} = \eta_0{}^{\infty} = \eta_0{}^{\eta_0} = \eta_0$

$\sqrt[\eta_0]{\eta_0} = \eta_0$

$$\sqrt[-b]{a} \; = \; a^{-1/b} = \frac{1}{a^{1/b}} = \frac{1}{\sqrt[b]{a}}$$

Intro.iii—Building Toward Subspaces:

Division and multiplication are opposite functions. For any mathematical operation to be determined accurate we must be able to both check that operation by its opposite and be able to use it to accurately portray real world situations. We know that $\frac{10}{5} = 2$ because $5 \cdot 2 = 10$. Just as surely we know that a car traveling at 5mph for 2 hours shall travel a total of 10 miles. The math is verifiable and it can be used for predictive results in the real world. The following chapters will show this can be done for functions which result in division by zero just as easily. Consider the following two examples with the knowledge of 0 and η_0 you gained in section *intro.ii*.

$$\frac{2}{0} = x \qquad \text{and} \qquad 0x = 2$$

If you solve the equation on the right, $0x = 2$, for x you get the equation on the left, $\frac{2}{0} = x$. This approach is the easiest to see the result is η_0. Don't get caught up thinking you can obtain $0x = 2$ by using $x = \frac{2}{0} = 2\left(\frac{1}{0}\right) = 2(\eta_0)$ and try to force it to equal 2. If you try this you get $0\big(2(\eta_0)\big) = 0 \neq 2$. Instead if you use η_0 you get $0(\eta_0) = 2$. Recall from the previous section this will simplify to $\eta_0 = 2$ which is equivalent to saying in this example, just 2. The right side of the equation is alone and best surmised by the mathematical truism $2 = 2$.

$$\eta_0 = \frac{2}{0} = x \qquad 0x = 2 \;\longrightarrow\; 0\left(\frac{2}{0}\right) = 2 \;\longrightarrow\; 0\eta_0 = 2 \;\longrightarrow\; \eta_0 = 2$$

Implies: $2 = 2$

Both $\frac{2}{0} = x$ and $0x = 2$ are mathematically identical and were easily solved to show $x = \eta_0$, a very special kind of zero using the information supplied in section intro.ii. There is another value here which is present but not included in this equation. That value lies on the x-axis subspace, the s-axis. Where this example gave $x = \frac{2}{0} = \eta_0$ you will find $s = 0$.

Using the steps for subspace transforms you'll be able to obtain the values shown in this example for the s-axis when $x = \frac{2}{0}$. Before discussing those transformations we will need to cover notation of points as well as how we arrive at the necessity of the existence of these additional directions.

Intro.iv—Null-Algebra Point Notation:

Let's continue with the example above using given values: $x = \frac{2}{0} = \eta_0$ $\qquad s = 0$

The η_0 element is resolvable to 0 for plotting. The point which we shall call $\mathbb{P}_1$ will show point values just like it would on a Cartesian plane. Since the value of s is a subspace of x it is placed to the right of the real space values separated by a vertical bar.

$$\mathbb{P}_1 = (x|s) = (0|0)$$

This pattern is maintained for all point plotting arrangements. The values are plotted separated by a comma for each degree of freedom present in a real space equation, with their corresponding subspace elements following the vertical divider with comma separation.

Graph Type	Point Arrangement	Actual number of axis
Vertical Line Graph	$(x|s)$	2
Line through a plane	$(x, y|s, u)$	4
Plane within a volume Volumetric Solid	$(x, y, z|s, u, w)$	6

Vertical line with time	$(x\,,t\|s\,,A)$	4
Line through plane with time	$(x\,,y\,,t\|s,u\,,A)$	6
3D evolution with time	$(x\,,y\,,z\,,t\|s\,,u\,,w\,,A)$	8

The values for any or all of the coordinates $x\,,y\,,z\,,s\,,u$ and w may be parametric when time is included. The A coordinate is itself a parametric value with unique properties that will covered more in depth later.

Intro.v—Circle Plus and circumflex signs:

The circle plus operator, ⊕, is used in Null-Mathematics to indicate a number is simultaneously positive and negative. This idea is necessary and will require further discussion of $\frac{0}{0}$, the complex axis and i-multiples. For now know that a number $\oplus a$ is read *plus-and-minus a*. This is NOT the same thing as $\pm a$ which is read *plus-or-minus a*. The value of $\oplus a$ is the result of a negative radical. It is resolved using the positive value of the number on the originating axis whilst its negative value occurs on a subspace plane.

The circumflex, $\hat{a}$ (*read a*-up) or $\breve{a}$ (*read a-down*), are used to keep track of the roots of negative numbers, i-multiples. The upward circumflex represents the positive value of the number while the downward pointing represents the negative of the same value such that $(\oplus a)^2 = \hat{a} \cdot \breve{a} = -(a^2)$.

Intro.vi—Early attempts of division by 0:

The Brahmasphutasiddhanta of Brahmagupta—Intro.vi.1:

Brahmagupta lived in Ujjain, India from 598 to 668 AD and made notable contributions to mathematics and astronomy. In 628 AD he is reported to have written a text called the Brahmasphutasiddhanta, The Opening of the Universe in English, within which is found the earliest known attempt of defining division by zero (Wikipedia: The Brahmasphutasiddhantaof Brahmagupta, *http://en.wikipedia.org/wiki/Brahmasphutasiddhanta*, and *http://www-history.mcs.st-*

andrews.ac.uk/Biographies/Brahmagupta.html). Brahmagupta attempts to define division by zero as equaling zero:

$$\frac{a}{0} = 0 \text{ where } a \text{ is any constant}$$

In a sense this is correct but only if η_0 and subspaces are introduced to the logic. Otherwise you cannot reach this conclusion. The definition given by Brahmagupta appears to be drawn from the idea of defining any function by its opposite. For example we know that $a \cdot 0 = 0$, and $\frac{0}{a} = 0$ are both true statements. It's easy to see how the assumption could be made but it's still unverifiable without inclusion of η_0 and subspaces. Consider the function $y = \frac{1}{x}$. The output value approaches infinity as x approaches 0. Infinity cannot be defined in traditional mathematics as it represents and impossibly huge value—not zero.

If you divide 1 by a very small number you get a very large output. The reasoning is simple logic. The smaller the denominator, the more times a given unit value can be added together to get one whole—the numerator. The smaller that denominator gets, the larger the quotient becomes. When x equals 0 the traditional reasoning says the output has to reach infinity. Again the answer is clearly not zero within the logic of traditional mathematics.

Something has to be happening at this value that traditional reasoning has failed to explain. If we can divide by other values and obtain real results one must exist for this expression as well, it just requires understanding beyond that of Brahmagupta's time. Saying an output is infinity does not actually mean there is no answer if, and only if you include the idea of η_0 and subspaces. y cannot reach the assumed value of infinity. Instead it implies the output does not lie on the y-axis as a traditional number. Instead we resolve this on the y-axis to η_0, represented graphically by numeric 0, though with a very different meaning. The value of y on the adjoining subspace plane will actually be 0.

The expression at the point of division by zero requires understanding of η_0 and an additional output axis which though present was never included in the original example $y = \frac{1}{x}$. These concepts allow the resolution of the infinity and defining the value on the output axis.

Mahavira and Bhaskara II— Intro.vi.2:

Two others followed Brahmagupta in an attempt to correct his definition of division by zero by asserting their own. In 830 A.D. Mahavira wrote a text titled *Ganita Sara Samgraha* in which he claims *A number remains unchanged when divided by zero* (Wikipedia: Division by Zero— *http://en.wikipedia.org/wiki/Division_by_zero*).

$$\frac{a}{0} = a \text{ where } a \text{ is any constant.}$$

Mahavira was undoubtedly brilliant but still made an error. If a number remains unchanged after division, it has either been divided by 1 or not divided at all. With the function $y = \frac{1}{x}$ we clearly see this. One divided by one is one. As *x* becomes larger or smaller than one, no matter how slightly so, *y* will become infinitely small or large respectively. So just avoiding the issue of division at zero and claiming no change takes place won't work.

Bhaskara II, another Indian mathematician who made contributions to calculus, tried his hand at the problem too. He is said to have made the assumption that…*when a finite number is divided by zero, the result is infinity* (http://en.wikipedia.org/wiki/Bhaskara_II*and*http://en.wikipedia.org/wiki/Division_by_zero).

$$\frac{a}{0} = \infty$$

This expression is the foundation of our traditional argument for this operation. It is true, and verifiable in the function $y = \frac{1}{x}$ that dividing by 0 will result in an infinite value for *y* but it doesn't explain what that means. In reality y will reach a limiting value. This occurs when the accuracy of the measuring apparatus we use to define how close we are to 0 on the x-axis can no longer, with certainty, tell the difference between 0 on any other definable non-zero point. When that occurs, though we could mathematically define an ever smaller number for x we could not move that distance without actually saying $x = 0$.

Intro.vii—Developing Subspaces:

There are two concepts of time within Null-Mathematics. We denote directional time by the value *t*. This value is used when we express rates of change with coordinate systems and is synonymous with the idea of the ticks of a clock. Subspaces supply the idea of Dimensional Time. They are related to the idea of time *t,* representing places where events exist in the space of time.

There are as many subspaces in a function as there are degrees of freedom. The three directional world uses four axis to define locations; x, y and z for positions in space and t for time. The subspace directions which correspond to the space axis respectively are s, u and w. The analog of Directional Time t is satisfied by another axis, *The Alternate,* denoted A.

Null-Mathematics makes a distinction between *direction* and *dimension.* The Alternate, although a valid axis, is both a direction and a dimension. The method of calculating the Alternate will be covered in detail later. For now simply know that it is a real axis like the others. The subspaces *s, u*, *w* and *A* define a unique location in the space of time where events, whether having occurred or only as a probability of occurrence, exist independent of the flow of time. In other words any event, in the past, present or future, completed or existing as something that could have occurred, will have a location definable in subspace. Those events continue to exist at that definite location in a subspace domain even after our present has moved past them in our perceived flow of time.

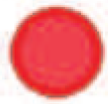
Intro.vii.1—Direction vs Dimension:

Null-Mathematics defines a direction by a number line axis, while a dimension is defined as multiple directions used in combination with one another to plot point positions. When describing the dimensionality of space special notation is used to differentiate between direction, dimensionality and what is perceived from within the 3-drecitonal world. Directions are denoted by

d^{α} where α is the total number of directions in a system. Dimensions are denoted by D^{β} where β is the total number of space dimensions coexisting within that reference frame. R_m will denote space where m is the total number of *perceived* space directions.

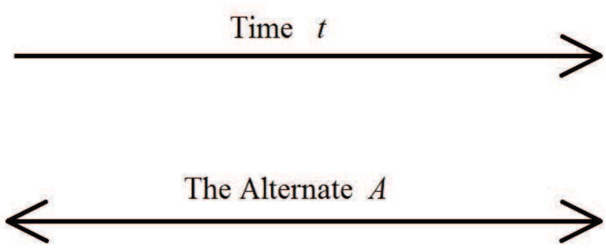
Intro.vii.2—R_0 The Point:

Figure A



To arrive at the presence of subspaces you must begin by considering the absence of space. Zero Space, R_0 means there are no space directions. Figure 1 at right is a singular point; a dot. Though we may draw a point to represent this and even label it, it cannot convey the reality of the point's existence. This is just a thought experiment. To be able to place a point means we have space to place it in regardless of how small it may be. However, if we imagine this point is infinitely small then it need not have space to exist. It's infinitely small. We can't even assign it the 0 vector. The Zero Vector can be said to point in any or all directions simultaneously with no magnitude. Yet since we have no space, the idea of the point being in any direction is meaningless.

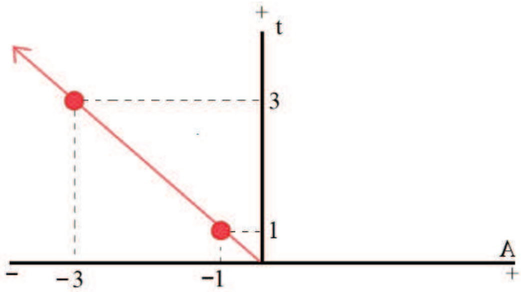
Figure B

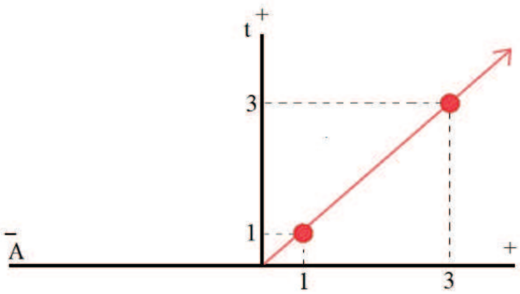


The directional time axis drawn at right alongside the Alternate in Figure B can be used to measure how long the point exits but nothing more. This directional time is important for measuring the passage of time but is more complicated. Physics has shown time to be a direction in its own right. The Null-Mathematics idea of directional time t is always positive and unidirectional. A value of $-t$ implies a time of t units earlier than a point denoted as *now* but not moving backward in time. Yet for it to be a direction this has to be an available possibility, to move either direction on the axis which represents it. Directional time t shows the rate of change of events unfolding about us as entropy increases throughout the universe and for that reason it is unidirectional. To satisfy the need of a real directional axis something other than t is needed. For that we include the Alternate, the A-axis, to represent the actual direction of time.

When space is included in the consideration we find both t and A are actually dimensions whose value must be calculated from interaction of other variables. When considering time alone without space, $A = t$. Both axis are necessary. t measures the passage of time while A is the axis which actually represents position or movement through time. Unlike t, A can have negative values which imply movement backward through time; not just an earlier moment of time. t is a time only direction. The Alternate, A, is a space-like time direction. Figure C shows two of a truly infinite set of possible situations involving both axis.

Figure C:





Intro.vii.3—R_1 The Line:

One-Space, R_1, is obtained by expanding a single infinitely small point out to infinity along one direction. This is best represented by a line; a one directional domain with no up, down, left or right. Only forward and backward exist on this line.

All points are identical and the line infinite, preventing most concepts of measurement. To illustrate consider the line in figure D. A single point on this line can use the passage of directional time as a measurement of how long it takes to move from one point to another on the line. If a second point is added, Figure E, it is possible to measure distance between them as a function of directional time. This second point could be the first point at a later instant of time or simply a second location on the line.

Figure D

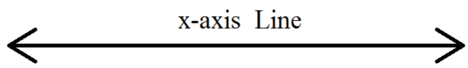
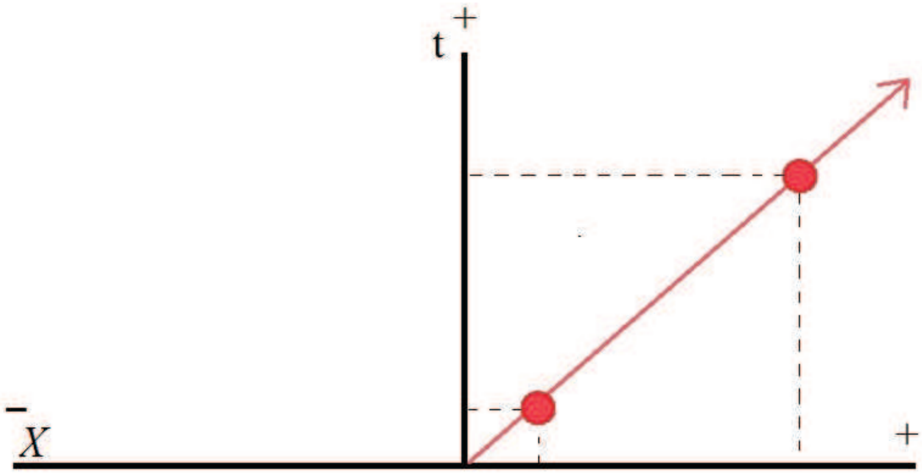


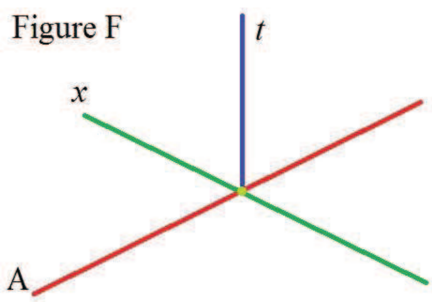
Figure E:



The passage of time marked by the t axis does not represent the actual direction of time due to its unidirectional alignment. This quality is included by the dimensional time axis of the Alternate. It's inclusion with the x and t-axis shows two things. 1) That time is a direction which can be moved along like any other and 2) that although we experience a passage of time positively from one moment to the next, all time, represented by the Alternate, is simultaneously present all the time. By acknowledging and representing both aspects of time together, it's perceived unidirectional passage, and its bidirectional A-axis direction, then every point on x for any and all time t is simultaneously represented by the xtA hypervolume. These collective points may be the present position of the point on x, or in its past or future. Regardless values of $(x\ ,A)$ points plotted at a given value of t persist; they exist irrespective of the current value of t.

Using the x-axis as our first example of one-space consider how equations for x are formed. First, equations on the xt-Plane take the form of $x = f(t)$. We can include the Alternate as $x = f(t, A)$. Figure F here shows one possible way of visualizing this arrangement.

Figure F



The x and A-axis are both directions in the same sense. The x-axis, used to label a direction of the three-directional world is experienced easily. A and t are related. A represents a directional axis where events experienced along the passage of t occur in the space of time, denoted on this space-like time axis. This same consideration must be given for x. Where t, a time-only axis was initially paired with the Alternate, a space-like time axis, so too must the space only x-axis be paired with a time-like space axis.

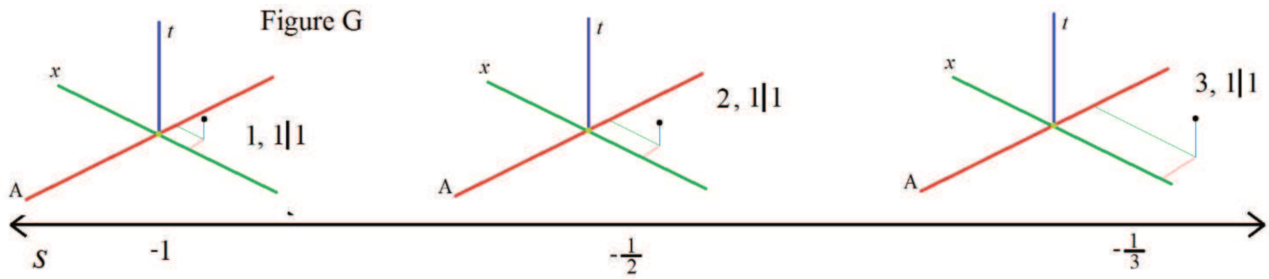
We can see that all time (*the Alternate*) is present all the time (*The t axis*). Similarly all points on x are present at once for the placement of an object on the x-axis for any given instant in time. Consider this thought experiment: Within the one directional world of the line you are standing at $x = 1$. However you could have chosen to begin at any other point on the x-axis at that same instant of time. It all of these available options simultaneously coexist, why then do you not see an infinite set of copies of yourself standing at all other values of x simultaneously, stretching on toward infinity both in front and behind you on the line? The reason is these other positions of the same you at the same instant of time are themselves separated out on yet another axis turned orthogonal to the one on which you stand.

Just as the passage of time as t required the presence of the Alternate to be a true direction so too does the presence of any space axis require another axis to represent the place in time where a given event on that axis occurs separate from all others. For the x-axis we label this as the s-axis and refer to s as a subspace of x. For any given instance of time t, there is a point which corresponds to position in time on the Alternate, while all possible values of x at that same instant of time are themselves separated from one another along the s-axis.

If it seems difficult to rationalize this argument the next section on subspace transformations will clarify it considerably. You will examine the presence of infinity arising within simple equations, what that implies and why it can be resolved with the presence of these directions.

For now though we have the following, a one-directional line, represented by the x-axis which has been shown to actually be a four-directional point system of the form $(x\ , t|s\ , A)$. This is rather difficult to graph out as we are running out of ways to show easily comprehendible projections of higher space directionality onto a 2D paper or screen surface. Nonetheless here in

Figure-G is an attempt using s as an orthogonal axis beneath several possible arrangements of the remaining $x, t|A$ points.



Note the values for A in Figure G have been rounded off. The points from left to right are:

$$(x, t|s, A)$$

$$(1\,,1|-1\,,1.414) \qquad \left(2\,,1|-\tfrac{1}{2}\,,1.118\right) \qquad \left(3\,,1|-\tfrac{1}{3}\,,1.054\right)$$

It's clear A no longer equals t. This is a result of what the s-axis is. s is a time-like space axis. It must be so because it separates the infinite possible values of x at any instant of time. This means the s-axis subspace must be incorporated into the space of time just like A is in respect to t. This is why we experience only one possible value for a point placed on x at a given point in time. It also means that the value of time t must be adjusted to the distance travelled across the collective expanse of the subspaces. For the line this is:

$$dt = \sqrt{(ds)^2 + (dA)^2}$$

What though is the value of A? This equation can be solved directly for A giving:

$$dA = \sqrt{(dt)^2 - (ds)^2}$$

Observe then what happens when you plug in the value of dA in the equation for dt.

$$dt = \sqrt{(ds)^2 + (dA)^2} = \sqrt{(ds)^2 + \left(\sqrt{(dt)^2 - (ds)^2}\right)^2} = dt$$

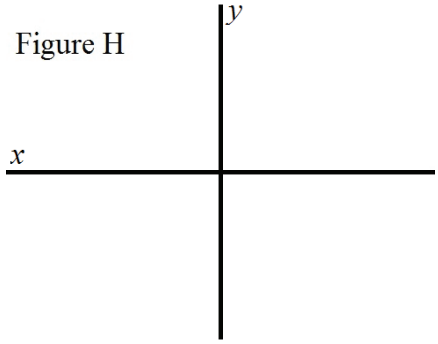
The values dA, ds, and dt are used here to indicate these are a change in value for that axis. However the equations can be simplified by assuming the distance is taken from the origin at which point the given values for each can be rewritten as A, s and t.

For the line, dimensional time is comprised of the distance across the sA-Hyperplane even though directional time remains unchanged, represented by t. Note that if the point on the x-axis never moves, it will never move on the s-axis either. These two points are bound to each other as negative reciprocates and will be covered in detail in the transformations section of this chapter. However, time is still passing and so the position along the Alternate also continues to advance. Though there are four directional axis present this is still only a line in one space direction. The four total coordinates create a unique arrangement for every possible position to place a point on the line at any given value of time, for all values of time. Also note that we can choose to ignore the inclusion of t as its own axis. If this is done you obtain graphs for all possible points of x for given equation with A being equal to s. This is the full subspace equivalent to a graph of a point on the line x in one space without time as a parameter. The difference being on the full subspace graph you could plot multiple or all choices for x simultaneously and yet they remain separate from each other.

In summary for one space direction, R_1: 1) t is present as a perceived direction but being unidirectional is a time-direction only. 2) There are d^3 total space directions present; one space-only direction x, one time-like space direction s and one space-like time direction A. 3) The inclusion of the s and A directions are inseparable from x and t even though traditional mathematics may choose to not represent all of them. This provides there are D^4 total dimensions present: the XS-hyperplane, XA-hyperplane, SA-hyperplane and XSA-hypervolume for any instant in time t. We can further conclude as these properties result from expansion of an infinitely small point into a line that space is an emergent property of time.

Intro.vii.4—R_2 The Plane:

Figure H



The plane is created by expanding the line in a direction orthogonal to its orientation. We will label this new direction the y-axis. Figure H here shows this orientation for space directions only and should be easily recognized as the standard Cartesian Plane.

The same considerations which applied to the x-axis apply to the y-axis. It will have a subspace axis which corresponds to it as a time-like space direction. All possible values of y for any instance of time exist simultaneously but are separated from each other along the y subspace, the u-axis. The inclusion of the u-axis will require modification of the definition of the value of time across what is now a subspace volume.

$$dt = \sqrt{(dA^2) + (ds^2) + (du^2)} \qquad dA = \sqrt{(dt^2) - (ds^2) - (du^2)}$$

Again notice the value for the perceived change in time is unaffected by the inclusion of the additional directional.

$$dt = \sqrt{\left(\sqrt{(dt^2) - (ds^2) - (du^2)}\right)^2 + (ds^2) + (du^2)} = \sqrt{(dt^2) - (ds^2) - (du^2) + (ds^2) + (du^2)} = dt$$

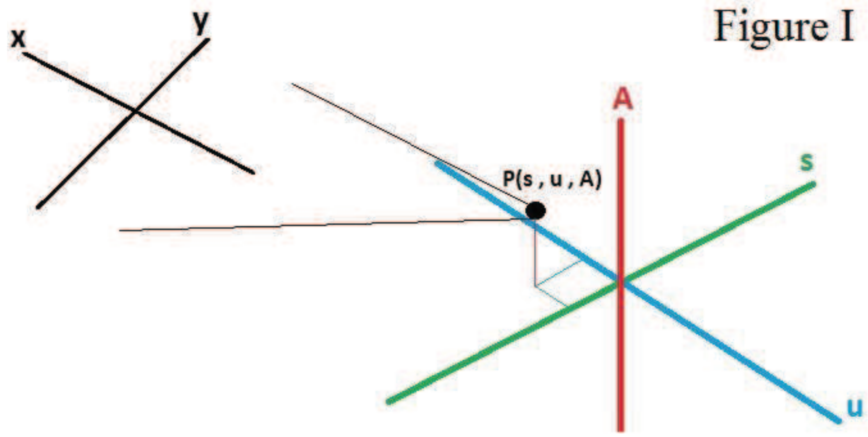
Equations within the Cartesian Plane usually take the form of $y = f(x)$. They could also be written to include time directly as a second input in the form of $y = f(x\,, t)$ or even parameterized in two separate equations to plot points: $(x\,, y) = \big(x = f(t)\,, y = g(t)\big)$. Regardless of the format used these graphs will include three additional directions, all of which are orthogonal to each other and define points $(x\,, y\,, t|s\,, u\,, A)$. Again it is possible to ignore t and simply plot the full subspace equivalent to $y = f(x)$, a line through the Cartesian Plane which is not dependent on time t. It will show the path taken of the same line through space and subspace irrespective of the passage of time with $dA = \sqrt{-(ds^2) - (du^2)}$ instead of $dA = \sqrt{(dt^2) - (ds^2) - (du^2)}$.

In summary for two space directions, R_2: 1) t is present as a perceived direction but being unidirectional is a time-direction only. 2) There are d^5 total space like directions; two of space only (x and y), two time-like space directions (s and u) and one space-like time direction (A). There are D^{26} total dimensions present as listed below. Though many exist these are all still a two directional plane. Those only in the first row represent environments which can be considered a real plane using the remaining directions as subspaces. All others represent various considerable combinations of real and subspace directions.

Two Directional Planes and Hyperplanes:							D^{10} *total*		
XY	XS	XU	XA	YS	YU	YA	SU	SA	UA
Three Directional Volumes and Hypervolumes:							D^{10} *total*		
XYS	XYU	XYA	XSU	XSA	XUA	YSU	YSA	YUA	SUA
Four Directional Hypervolumes:							D^5 *total*		
$XYSU$	$XSUA$	$XYUA$	$XYSA$	$YSUA$					
Five Directional Hypervolume:							D^1 *total*		
$XYSUA$									

Figure I shows one possible way of depicting the five-directional XYSUA Hyperspace.

Figure I



Each of the five total directional axes are orthogonal to each other. They have to be. If we take the dot product of any two corresponding vectors they will equal zero, making them orthogonal. This is seen most easily with the x and y axes.

$$x \cdot y = \langle x_n , 0\rangle \cdot \langle 0 , y_n\rangle \qquad 0 = [(x_n \cdot 0) + (0 \cdot y_n)]$$

In other words both the x and y axis are turned 90° to each other. This can be expanded to include all five axes.

$x \cdot y = \langle x_n , 0 , 0 , 0 , 0\rangle \cdot \langle 0 \cdot y_n \cdot 0 \cdot 0 \cdot 0\rangle = 0$

$x \cdot s = \langle x_n , 0 , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , s_n , 0 , 0\rangle = 0$

$x \cdot u = \langle x_n , 0 , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , u_n , 0\rangle = 0$

$x \cdot A = \langle x_n , 0 , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , 0 , A_n\rangle = 0$

$y \cdot s = \langle 0 , y_n , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , s_n , 0 , 0\rangle = 0$

$y \cdot u = \langle 0 , y_n , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , u_n , 0\rangle = 0$

$y \cdot A = \langle 0 , y_n , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , 0 , A_n\rangle = 0$

$s \cdot u = \langle 0 ,0 , s_n , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , u_n , 0\rangle = 0$

$s \cdot A = \langle 0 , 0 , s_n , 0 , 0\rangle \cdot \langle 0 , 0 , 0 , 0 , A_n\rangle = 0$

$u \cdot A = \langle 0 ,0 , 0 , u_n , 0\rangle \cdot \langle 0 , 0 , 0 , 0 , A_n\rangle = 0$

Choosing any of the axis to input values will result in 0 indicating they are in fact each orthogonal to all others. This works for directional time t as well expanding to the full six available directions. The dot product for $x \cdot t$ is shown below.

$$x \cdot t = \langle x_n , 0 , 0 , 0 , 0 , 0\rangle \cdot \langle 0 , 0 , t_n , 0 , 0 , 0\rangle = 0$$

Intro.vii.5—R_3 The Volume:

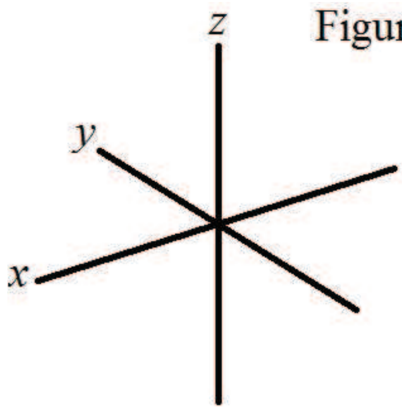


Figure J

Volumes have three perceivable space directions and are a direct expansion of a plane orthogonal to its two perceived directions. Figure J depicts the standard 3-space volume. Equations usually take the form of $z = f(x, y)$ defining a plane within a given volume with two inputs, x and y. Time can be included as a third input as $z = f(x, y, t)$ or as a parameter which defines points in the space as $(x , y , z) = \big(x = f(t) \ , \ y = g(t) , z = h(t)\big)$.

For the same reasons all possible placements on x and y are spearated out into their own subspace direciton z must also be given this same consideration with the w-axis subsapce. The presence of this additional axis will alter the definition for directional and dimensional time to the following:

$dt = \sqrt{(dA^2) + (ds^2) + (du^2) + (dw^2)}$ where $dA = \sqrt{(dt^2) - (ds^2) - (du^2) - (dw^2)}$

Observe the newly included subspace w still results in no change for the perceived passage of directional time.

$$dt = \sqrt{\left(\sqrt{(dt^2) - (ds^2) - (du^2) - (dw^2)}\right)^2 + (ds^2) + (du^2) + (dw^2)}$$

$$dt = \sqrt{(dt^2) - (ds^2) - (du^2) - (dw^2) + (ds^2) + (du^2) + (dw^2)}$$

$$dt = \sqrt{dt^2} = dt$$

Points for this system are plotted in the format $(x , y , z , t | s , u , w , A)$. Again it is possible to ignore t and simply plot the full subspace equivalent to $z = f(x, y)$, a plane through a given volume of space which is not dependent on time. It will show the expanse of the plane through space and subspace irrespective of the passage of time with $dA = \sqrt{-(ds^2) - (du^2) - (dw^2)}$ instead of $dA = \sqrt{(dt^2) - (ds^2) - (du^2) - (dw^2)}$.

In summary for three space directions, R_3: 1) t is present as a perceived direction but being unidirectional is a time-direction only. 2) There are d^7 total space like directions; three of space only (x, y and z), three time-like space directions (s, u, and w), and one space-like time direction (A). There are D^{89} total dimensions present as listed below. Though many exist these all remain a 3-directional volume. Only arrangements of three directional groups can be considered a perceivable space volume like that lived in by the human species and populated with planets, stars and galaxies, etc. These chosen three-directional groups are accompanied by the remaining four directions which serve as subspaces and include dimensional time A.

Three Directional Volumes and Hyper-volumes: D^{35} Total:

xyz	syz	xsu	xuA	xsz	ysu	yuA
xys	zsu	zuA	uyz	xsw	xwA	xuz
ysw	ywa	xyu	zsw	zwA	wyz	xsA
suw	xwz	ysA	suA	xyw	zsA	swA
Ayz	xuw	uwA	xAz	yuw	xyA	zuw

Four Directional Hyper-volumes: D^{35} Total:

$suwA$	$xuwA$	$yzwA$	$yzsw$	$yuwA$	$xzwA$	$xzsw$
$zuwA$	$xywA$	$xysw$	$xswA$	$yzuA$	$yzsu$	$yswA$
$xzuA$	$xzsu$	$zswA$	$xyuA$	$xysu$	$xsuA$	$yzuw$
$xyzA$	$ysuA$	$xzuw$	$xyzw$	$zsuA$	$xyuw$	$xyzu$
$xsuw$	$yzsA$	$xyzs$	$ysuw$	$xzsA$	$zsuw$	$xysA$

Five Directional Hyper-volumes: D^{11} Total:

$xyzsu$	$wyzsu$	$xwzsu$	$xywsu$	$xyzwu$	$xyzsw$
$Ayzsu$	$xAzsu$	$xyAsu$	$xyzAu$	$xyzsA$	

Six Directional Hyper-volumes: D^{7} Total:

$xyzsuw$	$Ayzsuw$	$xAzsuw$	$xyAsuw$	$xyzAuw$	$xyzsAw$
$xyzsuA$					

Seven Directional Hyper-volume: D^{1} Total:

$xyzsuwA$

Intro.viii.—Subspace Transformations:

Chapter 2 will discuss properties and applications of $\frac{0}{0}$. This value is labeled in traditional mathematics as the indeterminate form. Without the principles discussed in Null-Algebra and Null-Calculus this is a reasonable label. Unless you can apply reasonable rules for deciphering the nature of the expression, depending upon how you approach the value, it can be interpreted to mean both signed and unsigned infinity as well as 1 and -1.

Within Null-Calculus the expression is labeled by two unique operators. The Xi Operator, Ξ, represents the value of $\frac{0}{0}$ as ∞. The second is the subspace transform operator $\frac{\mathit{\Xi}}{\varsigma n}$ where n is the letter used to represent the subspace value resulting from the transformation. The $\mathit{\Xi}$ in the numerator indicates the new variable is infinite in its representation of a number line and the ς in the denominator indicates the new variable is a subspace.

A simple explanation of the natural need for subspaces can be seen with the equation $y = \frac{1}{x}$. As the value of x approaches 0, the value of y will approach $+\infty$ or $-\infty$ depending on whether approaching from the positive or negative side of the x-axis respectively. At $x = 0$ the entire expanse of the y-axis seems be implied by the result which is best represented as the unsigned infinity, ∞.

The first thing to realize is what the presence of this ∞ means. We use infinite items all the time in mathematics. The x-axis can be symbolically represented in an equation by simply writing x. Although we only look at a portion of the graph represented by that equation, or may be interested only in some values of the equation, that x is infinite, representing all values of the x-axis at once. A number line, represented by a coordinate axis is infinite. The output value $y = \infty$ in the example of $y = \lim_{x \to 0} \frac{1}{x}$ implies the value exists on a coordinate axis which, though present, is not included in the equation and thereby not literally on the y-axis unless resolved to 0 as naught. The value on y is best represented by η_0, which can be resolved to 0 but means the absence of value rather than no count of given item.

For a deeper understanding of the following information read chapter 2 dealing with $\frac{0}{0}$. The value of 0 in a number line belongs to both the positive and negative side of the number line meaning 0 is simultaneously positive and negative. Normally this doesn't cause any issues. Whether you have positive-no-cookies or negative-no-cookies doesn't matter; you still have no cookies. It's for this reason traditional mathematics argues 0 is neither positive nor negative. As we develop this idea we shall see 0 is truly positive and negative. This means we can interpret the value of $\frac{0}{0}$ as 1 and as -1. Notice I say *and* not *or*. Another feature though is that a set of 0 can be divided into 0 units an infinite number of times and expressed as $+\infty$ and $-\infty$.

These qualities are important because it means in one single expression we have a value which relates infinites to real value numbers. Consider again the equation $y = \frac{1}{x}$. At $x = 0$ the output value for y is infinite. We can use the Ξ and $\frac{\Xi}{\varsigma n}$ operators to resolve what the infinite output means. The infinity occurs on y so this side of the equation must be related to a real number value. This is done by distributing $\frac{0}{0}$ against the y using the Ξ operator and resolving it to positive-and-negative one (*represented as* $\oplus 1$). The opposite side of the equation represents the input values and needs related to an infinite value. This is done using the Surface Operator, $\frac{\Xi}{\varsigma n}$, which corresponds to $\frac{0}{0} = \infty = n$ where n is the subspace axis of the original output value. In this example we are dealing with y so the subspace is u. On the input function side of the equation the value of the infinite is distributed across as the new variable of the subspace input. Even though one side of the equation is being multiplied by an infinite, as a new variable, it is still the result of a constant 1 to 1 ratio. The value of $\frac{0}{0}$ has taken unique forms on each side of the equation but remains a ratio of 1. Thus despite changing the form of the equation it has not altered its value; only provided what value it also has on a subspace axis.

$$\Xi y = \frac{\Xi}{\varsigma u}\,\frac{1}{x} \qquad \longrightarrow \qquad \oplus 1 = \frac{u}{x}$$

Now we need to solve for u. Because this is a transformation, dependent upon both real values of $\frac{0}{0}$ being considered both signs of the $\oplus 1$ will interact with the sign of x when solving this example and will effectively change its sign from positive to negative or negative to positive.

$$u = x \ \cdot \oplus 1$$
$$u = x \ \cdot (+\ -1)$$
$$u = -x$$

Notice the value of u is the negative reciprocate of y. This is always the case with the transformations against the output side of an equation. Had y been an input to an equation or simply a standalone value, for example $y = 3$, we could just as easily take the transform and arrive at the subspace value of u.

As an undefined input variable: $y = y$

$\Xi y = \frac{\Xi}{\varsigma u} y \qquad \oplus 1 = yu \qquad u = -\frac{1}{y}$

Notice this exactly what was received with the above example of $y = \frac{1}{x}$.

$$u = -\frac{1}{y} = -\frac{1}{\frac{1}{x}} = -x$$

As a stand-alone value: $y = 3$

$\Xi y = \frac{\Xi}{\varsigma u} 3 \qquad \oplus 1 = 3u \qquad u = -\frac{1}{3}$

You can do a reverse-transform by swapping the Ξ and $\frac{\Xi}{\varsigma n}$ operators. This is not *reverse* as in transforming from a subspace back to a real space. To do that you simply repeat the transform on the subspace exactly as you did to obtain it. Consider this with the equation $u = -x$.

$$\Xi u = -\frac{\Xi}{\varsigma y} x \qquad \oplus 1 = -xy \qquad y = \frac{1}{x}$$

The reverse transform for the example of $y = \frac{1}{x}$ would look like this:

$$\frac{\Xi}{\varsigma \mathbf{n}} y = \Xi \frac{1}{x}$$

Notice the term indicating the subspace is bolded and not indicated; it is left as n. This is to prove a point. We can solve for the subspace of an input. For x this is:

$$x = x \qquad \Xi x = \frac{\Xi}{\varsigma s} x \qquad \oplus 1 = xs \qquad s = -\frac{1}{x}$$

This is because in this example y is a function of x. In this instance $y = \frac{1}{x}$. But the value of the input is x, not $\frac{1}{x}$. The s-axis subspace caries values which are negative reciprocates of the x-axis. So, if, the reverse-transform were solving for is the subspace of the input, then $\frac{\Xi}{\varsigma n} y = \Xi \frac{1}{x}$

should provide that $s = -\frac{1}{x}$. See below this is not the case:

$$\frac{\Xi}{\varsigma s} y = \Xi \frac{1}{x} \qquad sy = \oplus 1 \qquad s = -\frac{1}{y}$$

Where y is known to equal $\frac{1}{x}$.

$$s = -\frac{1}{y} = -\frac{1}{\frac{1}{x}} = -x$$

$$s \neq -x$$

This does not provide the correct value for s. We can see this if we substitute the value of s in for x in the original example equation $y = \frac{1}{x}$.

Given $s = -\frac{1}{x}$ $(NOT - x)$ We know that $x = -\frac{1}{s}$ by simple algebra. Thus:

$$y = \frac{1}{x} = \ y = \frac{1}{-\frac{1}{s}} = -s$$

Where $s = -\frac{1}{x}$ and thus we regain the original equation:

$$y = -s = -\left(-\frac{1}{x}\right) = \frac{1}{x}$$

So the arrangement of, $\frac{\Xi}{\varsigma n} y = \Xi \frac{1}{x}$ cannot be solving for the x-axis subspace. The Ξ operator is used to relate a real number value to an infinite arising in a single variable. There is no issue with any infinite on the x-axis, so although it will still resolve to $\oplus 1$ nothing is removed since the operator's value of $\frac{0}{0}$ need not be distributed through an infinite. Likewise the $\frac{\Xi}{\varsigma n}$ operator can no longer be used to relate an infinite value as a new variable as the subspace of the original output. It's on the wrong side of the equation to do that. So instead it will resolve to a new variable represented as **T** but again nothing is removed.

$$\frac{\Xi}{\varsigma\mathbf{T}}y = \Xi\frac{1}{x} \qquad \mathbf{T}y = \oplus 1 \cdot \frac{1}{x} \qquad \mathbf{T} = \frac{-\frac{1}{x}}{y} \qquad \mathbf{T} = \frac{-\frac{1}{x}}{\frac{1}{x}} \qquad \mathbf{T} = -1$$

This successful reverse-transform has provided a value for what would be considered a subspace except it doesn't appear to relate to anything; all of the data representing the original equation is gone. Since it represents a subspace the real-space value will require another application of a reverse-transform to obtain it. You'll see that all this does is make the value positive. So whether you apply a reverse-transform or standard transform to $\mathbf{T} = -1$ you'll get the same answer. What follows is the second application of the reverse-transform to return this to what represents the real-space value of this system.

$$\frac{\Xi}{\varsigma\mathbf{T}}\mathbf{T} = -\Xi 1 \qquad \mathbf{T}^2 = -1 \cdot \oplus 1 \qquad \mathbf{T} = \sqrt{1} = 1$$

You have to use $\frac{\Xi}{\varsigma\mathbf{T}}$ because there is nothing else in the equation to indicate what a real space axis could be possibly be for the second application of the reverse-transform. So what is this mysterious value **T**? Attempting to continue this process again, a third application of the process will result in the square root of a negative number. This third application of the reverse-transform returns you again to what should be the subspace of the system. Chapter 2 will show that this is a positive-and-negative number which represents an i-multiple, a value which corresponds to multiples of 1 and -1 of $\frac{0}{0}$ on what traditional mathematics calls the complex plane. Chapter 2 shows these values of 1 and -1 relate to that complex axis via $\frac{0}{0}$. In short **T** can be positive or negative 1 which represents the simultaneously positive-and-negative value of $\frac{0}{0}$, the value $\oplus 1$. This is because $\frac{0}{0}$ lies on its own axis and is its own subspace. This becomes important when dealing with some values for the Alternate which will be covered in a later chapter.

Below follows a summary of subspace transformations:

<u>Intro.viii.1—Transform 1:</u>

An expression of the form $y = \frac{a}{x}$

Where a equals any non-zero constant, and x is a variable.

Intro.viii.1a—Method 1: Direct Conversion

Output Value: $\Xi y = \frac{\Xi}{\varsigma u}\frac{a}{x} \quad \longrightarrow \quad u = -\frac{x}{a}$

Input Value: $\Xi x = \frac{\Xi}{\varsigma s}x \quad \longrightarrow \quad s = -\frac{1}{x}$

Intro.viii.1b—Method 2: Trigonometric Conversion

a. Trigonometric descriptions of points in the Cartesian plane are governed by polar coordinates

b. These are: $x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r^2 = x^2 + y^2$

Method 2:

This method directly relates the circular rotation of angle Theta and the real space coordinate points (x, y) to values attained on the *u* and *s* axes.

1. Use the equations $x = r\cos\theta$, $y = r\sin\theta$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and $r^2 = x^2 + y^2$ to determine the values for r and θ.

2. With these values known apply the Surface Value and Xi Operators to both the x and y equations. The x value has an actual equation in this instance and will be take the transform in the same fashion as the y equation.

$$\Xi x = \frac{\Xi}{\varsigma s} r\cos\theta \qquad \Xi y = \frac{\Xi}{\varsigma u} r\sin\theta$$

$$\oplus 1 = (r\cos\theta)s \qquad \oplus 1 = (r\sin\theta)u$$

$$s = -\frac{1}{r}\sec\theta \qquad u = -\frac{1}{r}\csc\theta$$

Both method 1 and method 2 will result in the same values. Consider the following example. Evaluate the equation $y = \frac{1}{x}$ for the values $x = \{2\,,3\}$. Show the subspace equations and solve using both methods. *Note that decimals in the trigonometric operations have been heavily truncated to save space. If you perform the operations without rounding the values you will receive the answer shown. Otherwise you will receive an approximation accurate to the decimal value you chose to include.*

Method I:

$x = 2$

$y = \frac{1}{x} = \frac{1}{(2)}$

Method II:

$x = 2$

$r = \sqrt{x^2 + y^2} = \sqrt{4.25} \approx 2.062$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(.25) \approx 0.245\ rads$

For the Y-subspace *u*:

Method I:

$\Xi y = \frac{\Xi}{\varsigma u}\frac{1}{2}$

$\oplus 1 = \left(\frac{1}{2}\right)u$

$u = -2$

Method II:

$\Xi y = \frac{\Xi}{\varsigma u}(2.062)\sin(0.245)$

$\oplus 1 = \big((2.062)\sin(0.245)\big)u$

$u = -\frac{1}{2.062}\csc(0.245) = -2$

For the X-subspace *s*:

Method I:

$\Xi 2 = \frac{\Xi}{\varsigma s}2$

$s = -\frac{1}{2}$

Method II:

$\Xi x = \frac{\Xi}{\varsigma s}(2.062)\cos(0.245)$

$s = -\frac{1}{2.062}\sec(0.245) = -\frac{1}{2}$

If we perform the subspace transforms again we return to the original x and y values.

By Method I:

$\Xi u = -\frac{\Xi}{\varsigma y} 2$

$\oplus 1 = (-2)y$

$y = \frac{1}{2}$

$\Xi s = -\frac{\Xi}{\varsigma s}\frac{1}{2}$

$\oplus 1 = \left(-\frac{1}{2}\right)x$

$x = 2$

The same will be achieved using Method II replacing the equations which define s and u to define x and y. You may attempt this on you own.

The same processes are used to obtain the subspace values when $x = 3$. Repeating them will undo the operation and again provide the original values.

By Method I:

$u = -3$

$s = -\frac{1}{3}$

Method II:

$u = -\frac{1}{3.018} csc(0.1107)\ rads$

$u = -\frac{9.052\ldots}{3.018\ldots}$

$u = -3$

$s = -\frac{1}{3.018}\sec(0.1107) = -\frac{1}{3}$

Intro.viii.1c—Reverse Transform:

Method I—Transform to Subspace:

$\frac{\Xi}{\varsigma \mathbf{T}} y = \frac{a}{x}\ \Xi \qquad \mathbf{T}y = \frac{a}{x}(\oplus 1)$

$T = \frac{-\frac{a}{x}}{y} \qquad T = -\frac{\frac{a}{x}}{\frac{a}{x}} \qquad T = -\frac{a}{x}\left(\frac{x}{a}\right) \quad T = -1$

Method I—Return Transform to Real Space:

$\frac{\Xi}{\varsigma \mathbf{T}} T = -\Xi 1 \qquad \mathbf{T} = 1$

Method 2—Trigonometric Reverse Transformation:

$\frac{\Xi}{\varsigma\mathbf{T}} x = r\cos\theta \ \Xi$ $\qquad$ $\frac{\Xi}{\varsigma\mathbf{T}} y = r\sin\theta \ \Xi$

$Tx = \oplus 1(r\cos\theta)$ $\qquad$ $Ty = \oplus 1(r\sin\theta)$

$Tx = -r\cos\theta$ $\qquad$ $Ty = -r\sin\theta$

$T = -\frac{r\cos\theta}{x}$ $\qquad$ $T = -\frac{r\sin\theta}{y}$

$T = -\frac{r\cos\theta}{r\cos\theta}$ $\qquad$ $T = -\frac{r\ \sin\theta}{r\sin\theta}$

$T = -1$ $\qquad$ $T = -1$

Repeating the operation again provides $T = 1$. There are three other general formats for a given equation upon which subspace transformations may be applied. Method 1 or 2 may be used to make the transformation. They are shown here, completed by Method 1. You may try method 2 at your leisure.

Intro.viii.2—Transform 2

An expression of the form $y = \frac{a}{x}$ where a equals 0, and x is a variable.

$$\Xi y = \frac{\Xi}{\varsigma u}\left(\frac{0}{x}\right)$$

$$\oplus 1 = \left(\frac{0}{x}\right)\infty$$

$\oplus 1 = \left(\frac{0}{x}\right)u$ This equation is in terms of *u*. So we want to keep *u.*

Multiply by *x* and then divide by zero.

$$-\frac{x}{0} = u$$

$u = -\infty = -\eta_0$ negative Naught

Look at the first three steps to this equation very carefully:

$\Xi y = \frac{\Xi}{\varsigma u}\left(\frac{0}{x}\right)$ $\qquad$ $\oplus 1 = \left(\frac{0}{x}\right)\infty$ $\qquad$ $\oplus 1 = \left(\frac{0}{x}\right)u$

Whenever a subspace transformation is conducted you are linking a new variable value through use of an infinite to the output which was removed from the equation. In this example that new variable is u. If we repeat the subspace transform on $-\frac{x}{0} = u$ we get back the original expression $y = \left(\frac{0}{x}\right)$. So an infinite arising in $y = f(x)$ may be replaced by the u value at the same x-input point which originally resulted in the infinite on y. Likewise an infinite occurring in $u = f(x)$ may be replaced by y value at the same x input which originally resulted in the infinite on the u axis. They will have the same value numerically but different meanings; one 0 and the other η_0.

When an input causes an infinite output we can evaluate the expression. Continuing with the example shown here, we can evaluate the expression $u = -\infty$ such that $-\infty = y$ without changing the nature of what it implies. In other words, in $u = f(x) = -\frac{x}{0}$, we can evaluate this as $u = y = 0$ for all x in the $u(x)$ example. u will equal η_0 for all x, while in the corresponding y equation will equal 0 for all values of x. The relation to infinity via the subspace transform shows us we can relate each of the u output values to the corresponding value on y for the same value of x making $u = 0$ for all x. We may also resolve the η_0 which results directly from the u equation in this example to 0 as the absence of value it implies.

Note that this zero on u, though resolved by linking it to the value on y at the same value of the input x is useful they do have separate meanings. The y axis output of 0 in this example mean 0 as in zero cookies. The u axis value of 0 is naught, which though graphable as 0 remains the absence of value.

In this example if $x = 0$ we get $\frac{0}{0}$ on both the y and u equations. Trending in this example dictates the value for y is 0 and for u is $-\eta_0$ when $x = 0$ given x as an input representing all values of x in the respective equations. For more on why see Chapter 3, section 1 on graphs of $\frac{y}{x}$ as either value approaches 0.

Intro.viii.3—Transform 3: Reverse transform of transform 2:

The reverse of Transform 2 results in the temporal constant just like in transform 1.

$$\frac{\mathit{\Xi}}{\varsigma\mathbf{T}}y = \left(\frac{0}{x}\right)\Xi \qquad T\cdot y = \oplus\, 1\left(\frac{0}{x}\right) \qquad T = \frac{-\frac{0}{x}}{y}\,T = -\frac{\frac{0}{x}}{\frac{0}{x}} \qquad T = -\frac{0}{x}\left(\frac{x}{0}\right)$$

$$T = -\left(\frac{0}{0}\right) = -(\oplus\, 1) = -(1) = -1$$

This final step requires the resolution of $-(\oplus\, 1)$. The $\oplus\, 1$ naturally arises in the equation and will resolve to $+1$ on the axis where it occurs. Normally, because this is a transformation both signs interact simultaneously and would result in a 1. However since the $\frac{0}{0}$ arose naturally in the equation it is resolved first to $+1$ and then multiplied by the negative sign giving -1. This represents the subspace value for T. Likewise the $+1$ for the resolved value of $\frac{0}{0}$ resulting from this subspace transform implies the corresponding 1 for what will be its real space equivalent. That real space equivalent will be found here for T by repeating the reverse transform. For further descriptions see Chapter 2 on $\frac{0}{0}$ as i multiples and Chapter 3 on graphs of $\frac{y}{x}$ as either value approaches 0.

A repeat of the reverse transform will return you to the value which corresponds to what represents a real space value for this equation. It shows that via the two transformations together T $=\oplus$ 1 just as before.

$$\frac{\mathit{\Xi}}{\varsigma\mathbf{T}}T = -\Xi 1 \qquad \mathbf{T}^2 = -1\cdot\oplus 1 \qquad \mathbf{T} = \sqrt{1} = 1$$

Intro.viii.4—Transform 4:

An expression of the form $y = 0x$.

$$\Xi y = \frac{\mathit{\Xi}}{\varsigma u}(0x) \qquad \oplus\, 1 = (0x)u \qquad \oplus\, 1 = (0)u \qquad u = -\frac{1}{0} = -\infty = -\eta_0$$

We find u is $-\eta_0$ for all values of x. A repeat transform will return you to the original y equation.

$$\Xi\text{u} = -\frac{\Xi}{\varsigma y}(\eta_0) \qquad \oplus 1 = -\eta_0 y$$

Normally Naught multiplied by a given value will result in that value ($\eta_0 \cdot a = a$ *See Naught Properties above*). This however is a transformation and cannot be interrupted. This means you must first divide $\oplus 1$ by naught. The Naught itself is the presence of some value divided by 0. That is acknowledged and substituted as such in this process.

$$y = \frac{\oplus 1}{-\eta_0} = \frac{\oplus 1}{-\frac{a}{0}} = \frac{0}{a} = 0$$

The final form looks different than the original equation $y = 0x$ but as you can see results in the same thing, y is 0 for all values of x. If we chose not to evaluate the naught as above we could say $y = \frac{\oplus 1}{-\eta_0} = \eta_0 = 0$. The ultimate result is the same but in this case would represent the absence of value. You will know which to use base upon what the naught was originally generated by. In this case the original $y = f(x)$ equaled 0 for all x so we know it must be as in $y = \frac{\oplus 1}{-\eta_0} = \frac{\oplus 1}{-\frac{a}{0}} = \frac{0}{a} = 0$ for this example.

Intro.viii.5—Transform 5, reverse transform of Transform 4:

$$\frac{\Xi}{\varsigma \mathbf{T}} y = \Xi 0x \qquad T \cdot y = (0) \oplus 1 \qquad T \cdot y = -0 \quad T = -\left(\frac{0}{0}\right) = -(\oplus 1) = -1$$

Recall that 0 is itself simultaneously positive-and-negative. In the step $T \cdot y = (0) \oplus 1$ we get $T \cdot y = -0$. In traditional mathematics this would just be 0. However within Null Algebra subspace transformations we must keep track of the sign here. Next we divide by y which we know from the original equation equals 0 for all instances of x. The negative is pulled out of the expression which is then evaluated as $\frac{0}{0} = \oplus 1$ before being resolved $+1$ where it arises. Lastly the negative is distributed.

$T = -\left(\frac{0}{0}\right) = -(\oplus 1) = -1$

Incidentally had we left the sign in place without pulling it out we still get -1 we would get the following:

$$T = \left(\frac{-0}{0}\right) = \frac{-0}{+0} = -1$$

A repeat of the reverse transform will give $T = 1$.

$$\frac{\Xi}{\varsigma\mathbf{T}}T = -\Xi 1 \qquad \mathbf{T}^2 = -1 \cdot \oplus 1 \qquad \mathbf{T} = \sqrt{1} = 1$$

For further description on interpreting $\frac{0}{0}$ see chapter 2 on $\frac{0}{0}$ as i multiples and Chapter 3 on graphs of $\frac{y}{x}$ as either value approaches 0.

<u>Intro.viii.6—Transform 6:</u>

An expression of the form $y = a \cdot x$, where a is any non-zero constant.

$$\Xi y = \frac{\Xi}{\varsigma u}ax \qquad \oplus 1 = (ax)u \qquad u = -\frac{1}{ax}$$

<u>Intro.viii.7—Transform 7, reverse transform of transform 6:</u>

$\frac{\Xi}{\varsigma\mathbf{T}}y = \Xi ax \qquad T \cdot y = \oplus 1(ax) \qquad T = -\frac{ax}{y} \qquad T = -\frac{ax}{ax} \qquad T = -1$

A repeat of the reverse transform will give $T = 1$. For further description on interpreting $\frac{0}{0}$ see chapter 2 on $\frac{0}{0}$ as i multiples and Chapter 3 on graphs of $\frac{y}{x}$ as either value approaches 0.

Intro.ix—Null Algebra:

Intro.ix.1—Plotting Points:

Figure K below shows two graphs. The equations of both are fairly straight forward and don't require a lot of visualization.

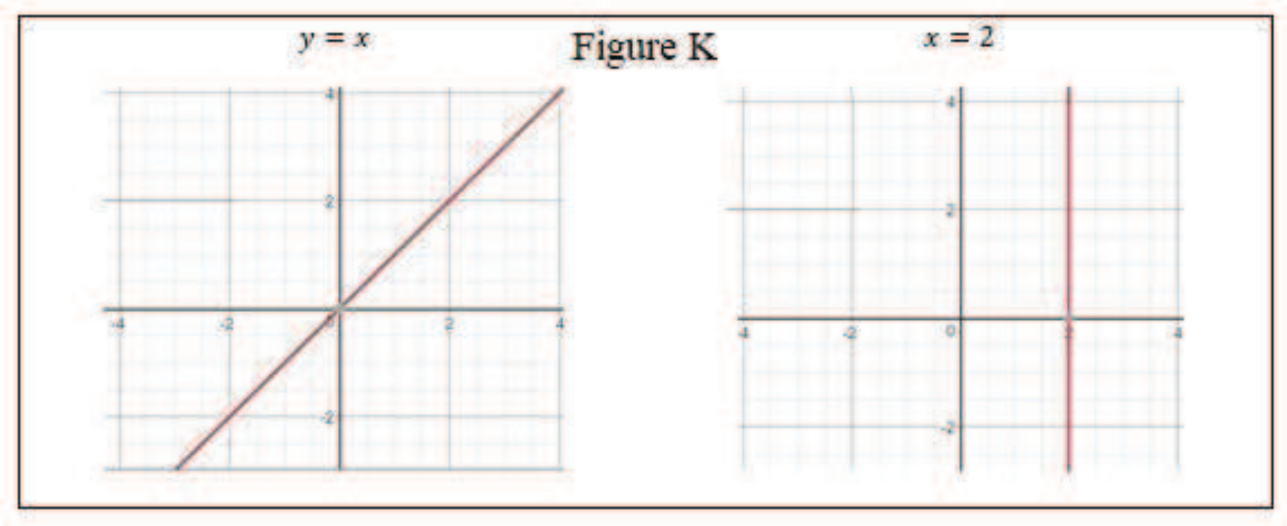
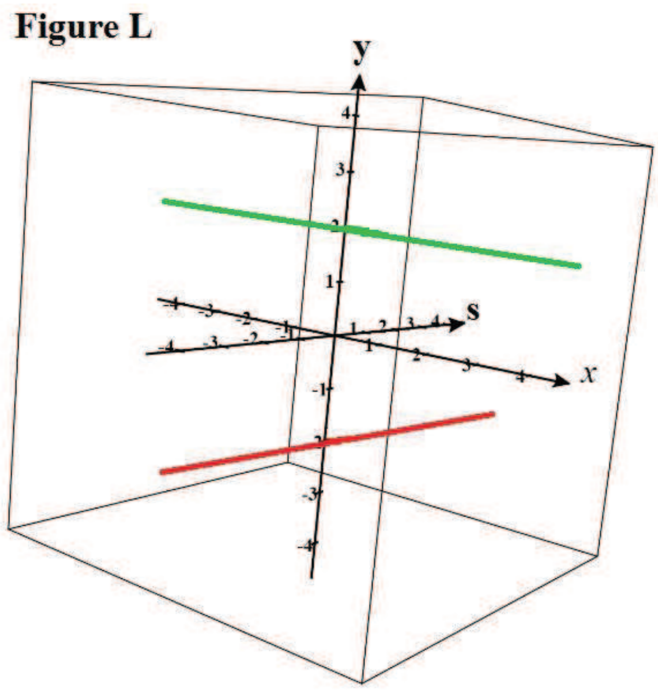


Figure K

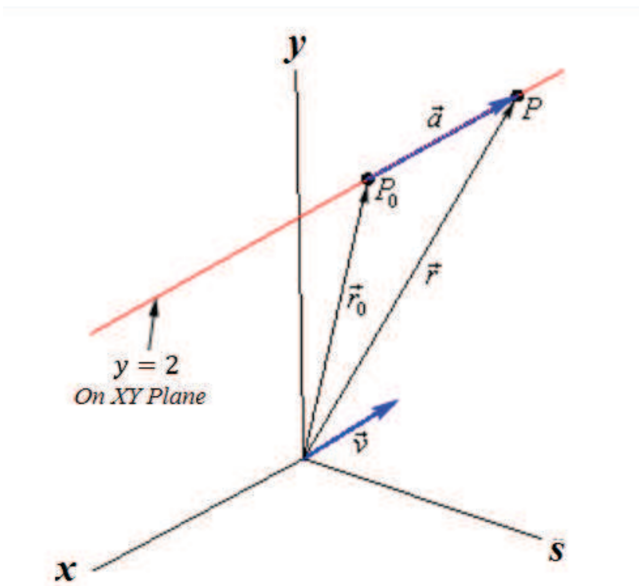
What about an equation of the form $y = \oplus 2$? Chapter 2 discusses the nature of $\oplus$ numbers which are ultimately i multiplies, that is magnitude-multiples of $\frac{0}{0}$ and $\oplus 1$. The graph of $y = \oplus 2$ is not $y = 2$. Neither is it $y = -2$. The value $\oplus 2$ indicates simultaneously plus and minus, telling us this is a graph of $y = 2$ but a subspace axis is included which results in $y = -2$. Examine Chapter 2 to see how this obtained. The graph $y = 2$ is still a collection of points in the XY Plane of the form $P(x, y) = (\mathbb{R}, 2)$. The positive half of the graph of $y = \oplus 2$ is identical to $y = 2$. Its location depends on the Plane you're focusing on for the instance of the $\oplus 2$. Assuming we are focused on real space equations of the form $y = f(x)$ this is indeed on the XY Plane, graphed as a horizontal line parallel to the x-axis at $y = 2$. The negative half, $y = -2$, is on the YS Plane. Together these are a complete graph of $y = \oplus 2$ for a plus-and-negative number originating on the XY plane. See Figure L.

Figure L:



You can obtain graphs of lines through three directional XYS space which are identical to these.

Figure L.i here shows the basics for plotting out an equation of a line through a three directional plane.



We shall define the following points which are known to exist on the line.

Point $P_0 = \langle x , s , y \rangle = \langle 0 , 0 , 2 \rangle = \overrightarrow{r_0}$ and Point $P = \langle x , s , y \rangle = \langle 2 , 0 , 2 \rangle = \vec{r}$

Both points lay on the line parallel to the x-axis at height of $y = 2$. The vector $\vec{a}$ between the two points equals $\vec{r} - \overrightarrow{r_0}$. Additional The vector $\vec{v}$ is held to be parallel to vector $\vec{a}$ such that $\vec{a} = t\vec{v}$. The x-

axis in this example is parallel to vector $\vec{a}$. So we may chose one such value to be $\vec{v} = \langle x\,, s\,, y\rangle = \langle 2\,, 0\,, 0\rangle$.

Finally we write the equation for the line through XSY 3-Space as:

$$\vec{r} = \overrightarrow{r_0} + t\vec{v} \;=\; \langle 0\,, 0\,, 2\rangle + t\langle 2\,, 0\,, 0\rangle$$

$$\vec{r} = \langle x\,, s\,, y\rangle$$

$$x = 0 + 2t$$

$$s = 0 + 0t$$

$$y = 2 + 0t$$

We can do the same things for $y = -2$ in the YS Plane.

$$\vec{r} = \overrightarrow{r_0} + t\vec{v} \;=\; \langle 0\,, 0\,, -2\rangle + t\langle 0\,, -2\,, 0\rangle$$

$$\vec{r} = \langle x\,, s\,, y\rangle$$

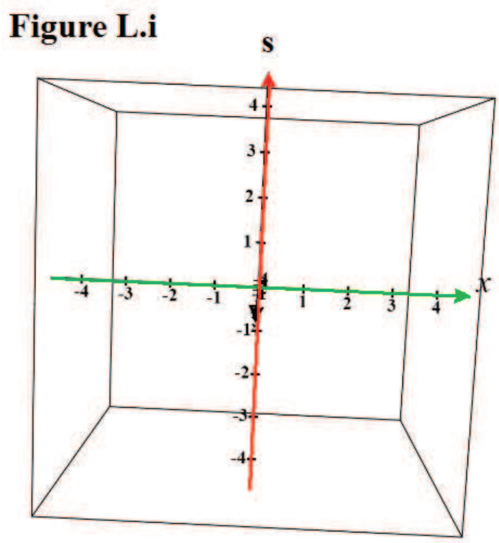
$$x = 0 + 0t$$

$$s = 0 - 2t$$

$$y = -2 + 0t$$

Although both of these vector-line equations in the XSY 3-Space are identical respectively to the graphs of $y = 2$ in the XY Plane and $y = -2$ in the XS Plane, only $y = \oplus 2$ originating on the YX place describes their simultaneous placement. If the graph of $y = \oplus 2$ had originated on the YS Plane we would still have two simultaneous graphs but their placement would be reversed with $y = 2$ occurring in the SY Plane and $y = -2$ occurring in the XY Plane. Note if you look down the y-axis you see the XS Plane. In the XS plane the same graphs have equivalents to $x = 0$ and $s = 0$ respectively. Yet they remain in reality the simultaneous placement of two graphs indicated by $y = \oplus 2$ on the XY Plane. See Figure L.ii:-

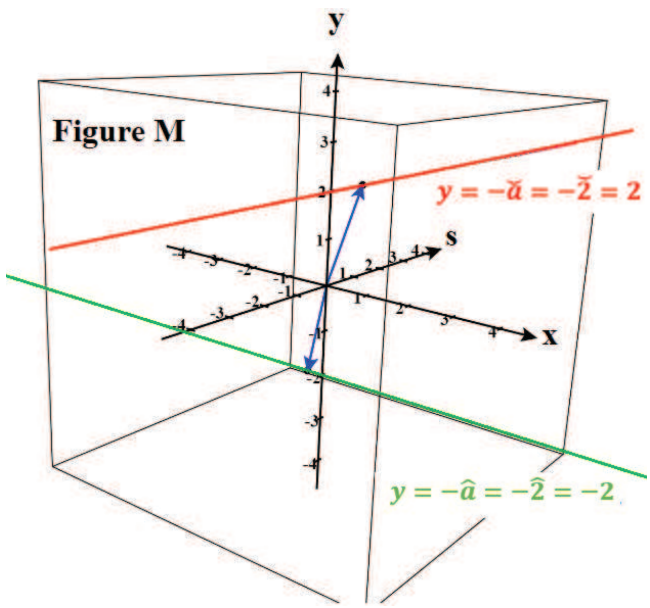
Figure L.ii



The Chapter 2 section on $\frac{0}{0}$, i-multiples and negative-radicals will expand on this idea.

Intro.ix.2—Negative Points

Is it possible to have a negative subspace point? A number $\oplus a$ is already representative of *positive-and*-negative value. If we have $y = -2$ we can just as easily write $-y = 2$. This can be done for subspace numbers as well in the form $-y = \oplus 2$. Using methods in Chapter 2 a number $\oplus a$ will resolve to the positive value where it originates as $y = \hat{a}$ and the negative value on a paired subspace axis $y = \breve{a}$. For a negative subspace number of the form $-y = \oplus a = \oplus 2$ it will resole to $y = -\hat{a} = -\hat{2} = -2$ where it originates and $y = -\breve{a} = -\breve{2} = 2$ on a paired subspace.



Intro.ix.3—Point Plotting Notation in Real Space and Subspace:

For any point on the Cartesian plane in real space $P_n = (a\,, b)$, where a corresponds to a value on the x-axis *and* b to a value on the y-axis, there shall exist a point in subspace $P_s = (c\,, d)$ where c corresponds to the s-axis definable as $c = -\frac{1}{a}$, and d to the u-axis definable by $d = -\frac{1}{b}$.

The Alternate, The A-axis is definable as $\sqrt{t^2 - (c^2) - (d^2)}$ or $\sqrt{-(c^2) - (d^2)}$ if equations are not time dependent. The Alternate has no subspace. On the A-axis $\sqrt{-a^2} = -a$ and $\sqrt{a^2} = a$. The reason becomes clear when you observe how values are resolved. Absent a given value of t in a time dependent equation large enough at least exactly cancel the $-(c^2) - (d^2)$, the Alternate will result in a negative radical; $\sqrt{-(c^2) - (d^2)}$. These components are squared but they are subtracted from the *time* component t.

The Chapter 2 section detailing the nature of i multiples shows the root of negative numbers is a positive-and-negative number. This same section details that for positive-and-negative numbers the value represents the positive part of the number on the axis which it occurs and the negative component on the corresponding subspace. This is a little different with the Alternate.

This axis has no subspace. The root of a negative number on the Alternate axis will result in a positive-and-negative value but it is resolved differently than such an occurrence on any other axis. Having no subspace, the only transform which can be conducted against the Alternate is a reverse transform. For example consider $A = \sqrt{-4} = \oplus\, 2$. The value $\oplus\, 2$, the result of the negative radical, is used for the complete reverse transform; one application of the reverse transform represents the subspace value of the Alternate if it existed. A second application of the reverse transform will return to actual value held on the Alternate axis.

$$\frac{\mathit{\Xi}}{\varsigma\mathbf{T}}A = \oplus\, \Xi 2 \qquad \longrightarrow \qquad A\mathbf{T} = \oplus\, 2 \cdot \oplus\, 1$$

From the section on the transforms we know that $\mathbf{T} = \oplus\, 1$. So the equaiton is being multiplied on both sides $\oplus 1$. Although this will change the form of the equaiton it does not change its value.

$$A\mathbf{T} = \oplus 2 \cdot \oplus 1$$
$$A\mathbf{T} = -2$$

Because A has no subspace and we know from above examples that **T** also equals $\oplus 1$ we must determine what value **T** holds at this stage in the transform. So first solve for **T.**

$$\mathbf{T} = \frac{-2}{A}$$

The value of A has not yet been resolved as the transforms are not complete. You must use the value for A as a result of the negative radical $A = \sqrt{-4} = \oplus 2$.

$$\mathbf{T} = \frac{-2}{A} = \frac{-2}{\oplus 2} = 1$$

Using this value for **T** in the first reverse transform we get:

$$A\mathbf{T} = \oplus 2 \cdot \oplus 1$$
$$A\mathbf{T} = -2$$
$$A = -2$$

Since A has no subspace you must take another reverse transform to obtain its actual value. Again we must mention that **T** equals $\oplus 1$. Its postivie value was used for the first reverse transform. Thus we must use its negative value for the second reverse transform which will give us the value for the negative radical on the Alternate.

$$\frac{\Xi}{\varsigma\mathbf{T}} A = -\Xi 2$$
$$A\mathbf{T} = -2 \cdot \oplus 1$$
$$A(-1) = 2$$
$$A = -2$$

We see from this example that negative radicals on the alternate will resolve to the negative of the absolute value of the root. Roots of positive numbers will have positive-or-negative solutions. For the Alternate, which represents a time direction, to remain consistent with the idea of the continuity of time we make the rule that positive radicals will equal the positive value of the root on the Alternate axis

$$A = \sqrt{-4} = -2 \qquad A = \sqrt{4} = 2$$

Intro.ix.4—Adding and subtracting subspace values:

Addition and subtraction are not difficult concepts. Expressions such as $y = x + 1$ and $y = x^2 - 3$ are straightforward and require little thought to understand. What about an the expression $y = x + (\oplus 2)$ and $y = x - (\oplus 2)$?

Chapter 2 shows that $\oplus$ numbers are i multiples which resolve to their positive magnitude component on the axis which they occur and their negative magnitude on a paired corresponding subspace for all axis except the Alternate. In the instance, an equation such as $y = x + (\oplus 2)$ the plus-and-minus number is not an axis output. Neither is it the input axis variable(s). Instead it's a constant being, in this example added, to an axis input in order to get the corresponding value for the output. It is a constant Δx added to the input value x.

Recall from the previous section that, assuming a $y = f(x)$ expression of the form $y = \oplus n$ where n is any number, results in $y = n$ on the XY-Plane and $y = -n$ on the adjoining SY-Plane. In the instance of the example equations $y = x + (\oplus 2)$ and $y = x - (\oplus 2)$ we must again account for both the positive and negative values of the constant. In the example of form $y = \oplus n$ the $\oplus n$ is an n valued multiple of $\oplus 1$ which resolves to the $+n$ multiple on the place of occurance, on for the value of y; the XY-Plane. The subspace value is the $-n$ multiple for y but applies to the subspace in the SY-Plane. It has to because y equals both values and the input receiving the plus-and-minus number, given you originally using the XY-Plane, can only relate to extension across the x-axis. This is because, even though we are saying y equals a number the line graph it generates is parallel to the entiriety of the x-axis in the XY-Plane.

With an equation of the form $y = x + (\oplus 2)$ you have a $\Delta\, x = \oplus n$ on the x-axis on the XY-Plane. The $\Delta\, x$ must account for both the positive and negative magnitudes of the value on the x-axis. The $\oplus n$ value is an n multiple of $\oplus 1$. The $+n$ is used on the x-axis at place of occurrence on the XY-plane. The $-n$ multiple will occur on the x-axis as well but corresponds to a subspace. The only place it can be located here is on the *co-adjoining* subspace of the XU-Plane.

Thus we have the following resolutions:

$y = x + (\oplus 2) \quad \longrightarrow \quad y = x + 2$	$y = x - (\oplus 2) \quad \longrightarrow \quad y = x - 2$
$u = -\frac{1}{x-2}$	$u = -\frac{1}{x+2}$

To further illustrate consider that $y = x + (\oplus 2) \;\; \equiv \;\; y = x + \sqrt{-4}$. If instead we used an example equation of the form $y = x + \sqrt{x}$, for the lone instance of $x = -4$ we get the same value in the from the radical as we do in the original example equation of $y = x + (\oplus 2)$ used above.

The equation $y = x + \sqrt{x}$ is a more general instance where we must consider such exchanges of positive or negative values and what axis they apply to. For positive values of x you will resolve the equation to $y = x \pm \sqrt{x}$. With negative values of x we resolve the equation to with a particular input of $x = -4$ as a model of how to resolve $y = x + (\oplus 2)$ using the same $x =$ $y = x \pm \oplus\sqrt{|x|}$. Note this is because roots still produce $\pm$ values. Its just that in this instance value is an i-multiple, a plus-and-minus number.

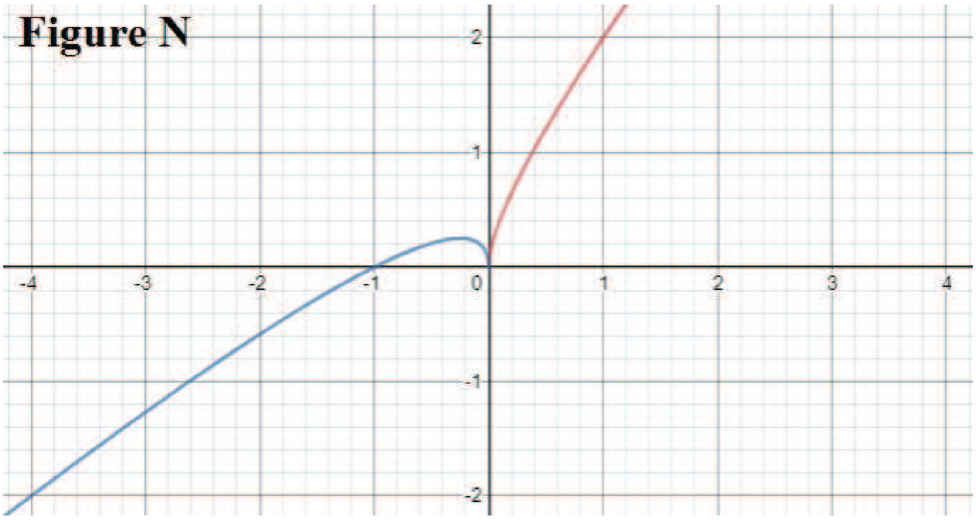
If we use $y = x + \sqrt{x}$ with $x = -9$ we get the following:

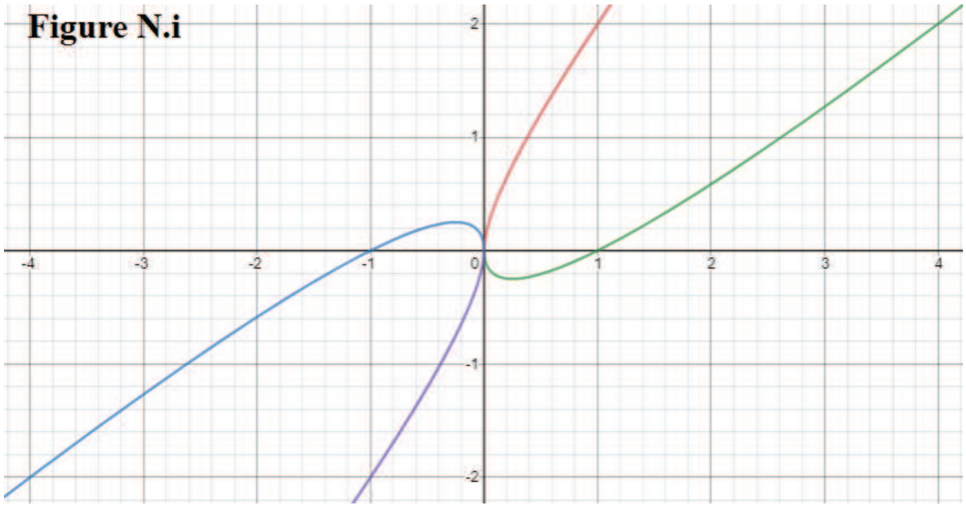
Remember the result of a radical is $\pm$.

$$y = -9 \pm \oplus\sqrt{|-9|} \quad \longrightarrow \quad y = -9 \pm \oplus 3 \quad \longrightarrow \quad y = -9 \pm \hat{3} = -6 \text{ or } -12$$

$$u = -\frac{1}{-9 \pm \check{3}} = -\frac{1}{-9 \pm (-3)} = \frac{1}{12} \text{ or } \frac{1}{6}$$

Figure N below shows the graph of the example equation $y = x \pm \sqrt{x}$. Note this graph only shows the positive values from the root. Roots provide a $\pm$ solution. Figure N.i shows the complete graph with the negative components of the root as well.





The graph of Figure N for the domain $x \geq 0$, and $a = \sqrt{x}$ is denoted by the red curve:

$$y = x + \sqrt{x} \;=\; x + a$$

Since the radical is $\pm$ the green cruve shows $y = x - \sqrt{x} \;=\; x - a$

Likewise for the domain of $x < 0$ we have $\hat{a} = \sqrt{x}$ denoted by the blue curve:

$$y = x + \sqrt{x} \;=\; x + \hat{a}$$

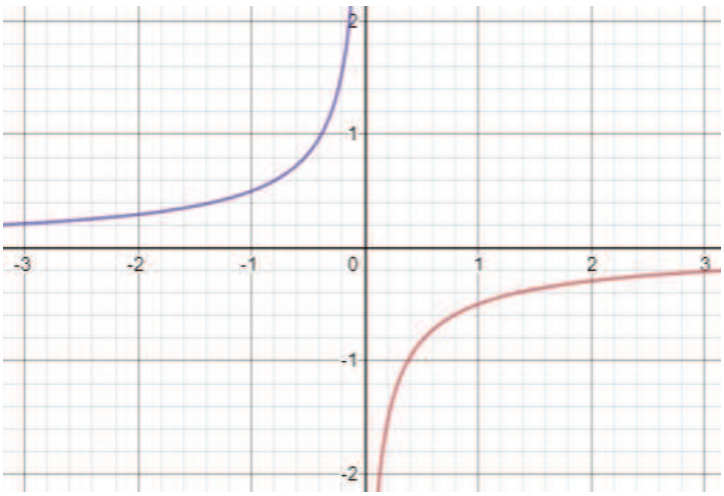
Finally the because of the radical's $\pm$ we have the purple curve shows

$$y = x - \sqrt{x} \;=\; x - \hat{a}$$

This is graphable as a set of four equations for most graphing utilities: The upper curves of $y = x \pm \sqrt{x}$ are defined by $y = x + (x)^{0.5}$ (*the red curve*) and $y = x + (-x)^{0.5}$ (*The blue curve*). The lower curves correspond to the subtraction of the radical and are defined by $y = x - (x)^{0.5}$ (*The green curve*) and $y = x - (-x)^{0.5}$ (*The purple curve*). This method is necessary as calculators and computers at the time of this document's publishing are not programmed to understand roots with a negative argument.

The paired components resulting from the plus-and-minus values will be on the co-adjoining XU-Plane. Beginning with the equation $y = x \pm \sqrt{x}$ you can use a subspace transformation to obtain $u = -\frac{1}{x \pm \sqrt{x}}$. For the positive radical we use the equation $u = -\frac{1}{x+\sqrt{x}}$. With positive value of x you get the red line graph below in the lower right quadrant. The negative values of x produce $u = -\frac{1}{x + \oplus\sqrt{|x|}}$. Since this is the co-adjoining subspace of the original equation containing the $\oplus$ number, it will resolve to its *negative* component. Giving $u = -\frac{1}{x-\sqrt{|x|}}$ for negative values of x the purple line graph below.

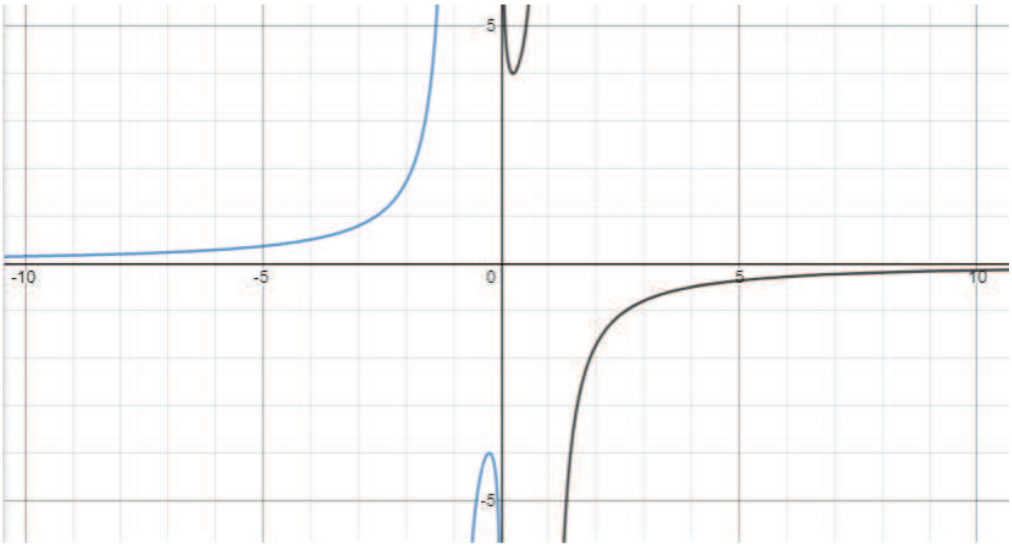
Figure N.i.1:



The Second set of equations come from subtracting the radical out. From $u = -\frac{1}{x \pm \sqrt{x}}$ we use the equation $u = -\frac{1}{x-\sqrt{x}}$. For the positive values of x this gives the graph below in black. The negative values of x will produce $u = -\frac{1}{x - \oplus\sqrt{|x|}}$. Again though the value plus-and-minus number is occurring in the co-adjoining subspace of its original occurrence and will take its negative value.

This gives $u = -\frac{1}{x+\sqrt{|x|}}$ for negative values of x show in in the blue line graph.

Figure N.i.2



Again note that graphing utilities at the time of this paper publishing are not yet programmed to understand radicals with negative arguments. Thus to graphed the output use the following equations which produce the same result on the screen. For the four $u(x)$ equations shown above, the red graph is given by $u = -\frac{1}{x+(x)^{0.5}}$ and the purple graph by $u = -\frac{1}{x-(-x)^{0.5}}$. The black is given by $u = -\frac{1}{x-(x)^{0.5}}$ and the blue graph by $u = -\frac{1}{x+(-x)^{0.5}}$.

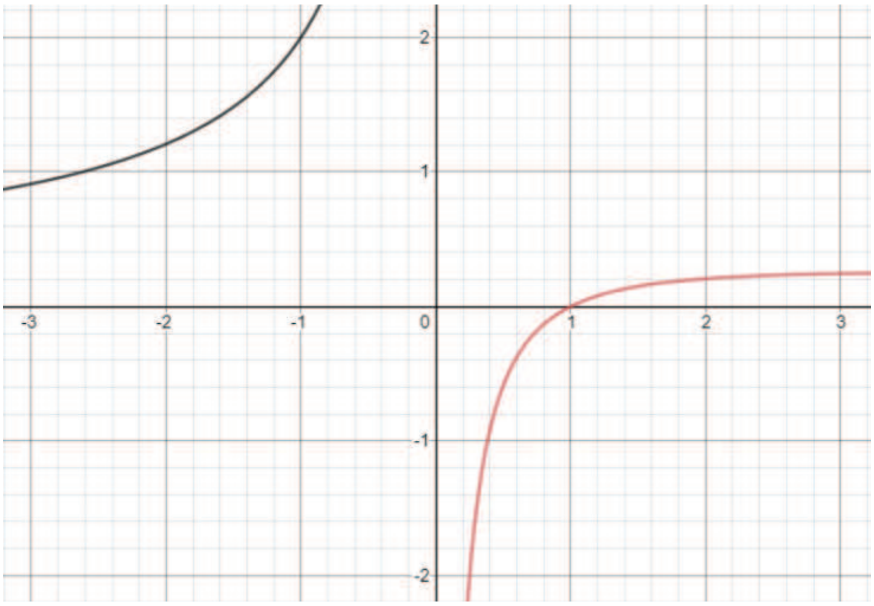
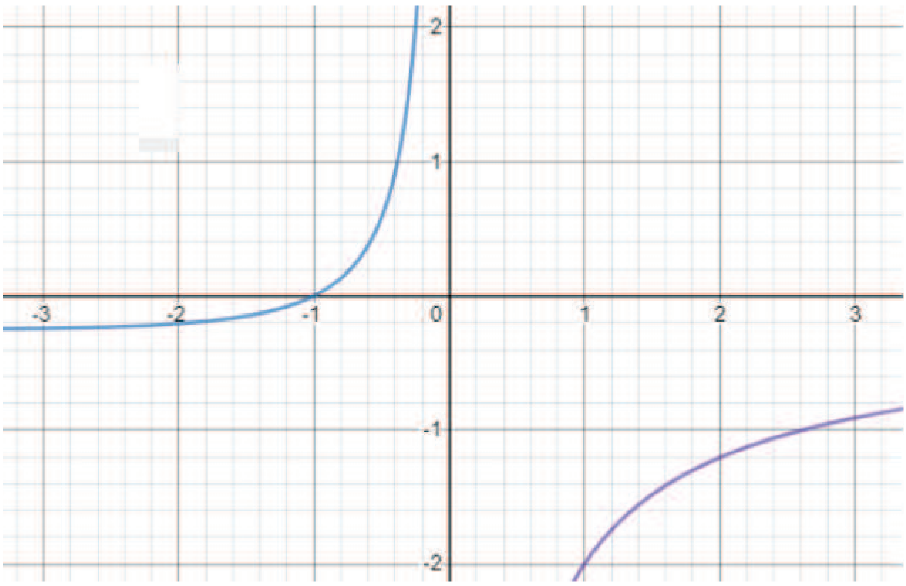
The equations of the SY-Plane and SU-Plane can easily be generated by performing and transformation on the x variable inputs such that $x = -\frac{1}{s}$.

$y = f(s)$	$u = f(s)$
$y = x \pm \sqrt{x} \quad \longrightarrow \quad y = -\frac{1}{s} \pm \sqrt{-\frac{1}{s}}$	$u = -\frac{1}{x \pm \sqrt{x}} \quad \longrightarrow \quad u = -\frac{1}{-\frac{1}{s} \pm \sqrt{-\frac{1}{s}}}$

Each will again have four total components. There will be two equations corresponding to each half of the s-axis inputs; one for the radical having a positive argument and one for it having negative argument. Each of those equations will have a form for which the radical is added and for which it is subtracted accounting for the radical's $\pm$ values. Although the inputs are those of the subspace s-axis, we are ausming to begin with the equation in the SY-Plane. This is the adjoining subspace of the XY-Plane. Despite switching to the subspace input the negative radical arguments are still producing the plus-and-minus number outputs in a way which applies them as a Δs on the input side of the equation. The positive component will exist on the SY-Plane and the negative on the co-adjoining SU-Plane.

$y = f(s)$

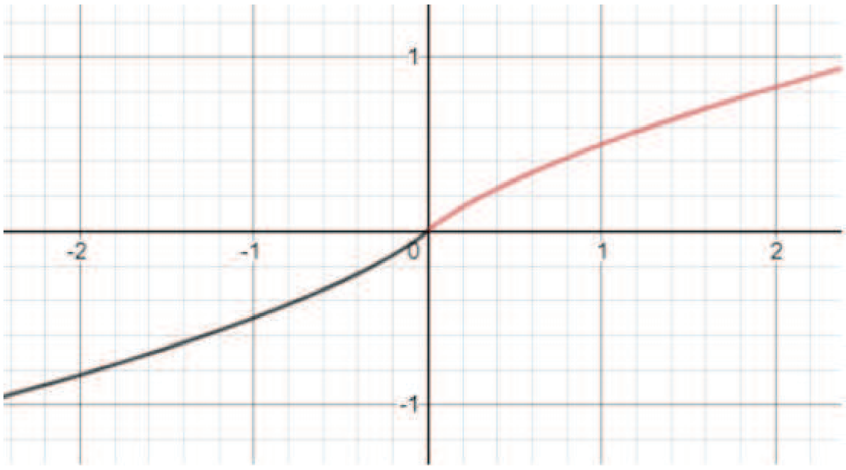
Equation	Graphable As:
Positive s / radical added	*Red Graph*
$y = -\frac{1}{s} + \oplus \sqrt{\left\|-\frac{1}{s}\right\|}$	$y = -\frac{1}{s} + \left(\frac{1}{s}\right)^{0.5}$
$y = -\frac{1}{s} + \sqrt{\left\|-\frac{1}{s}\right\|}$	
	Black Graph
Negative s / radical added	$y = -\frac{1}{s} + \left(-\frac{1}{s}\right)^{0.5}$
$y = -\frac{1}{s} + \sqrt{-\frac{1}{s}}$	
Positive s / radical subtracted	Purple Graph
$y = -\frac{1}{s} - \oplus \sqrt{\left\|-\frac{1}{s}\right\|}$	$y = -\frac{1}{s} - \left(\frac{1}{s}\right)^{0.5}$
$y = -\frac{1}{s} - \sqrt{\left\|-\frac{1}{s}\right\|}$	

<table>
<tr><td>Negative s / radical subtracted
$$y=-\frac{1}{s}-\sqrt{-\frac{1}{s}}$$</td><td><u>Blue Graph</u>
$$y=-\frac{1}{s}-\left(-\frac{1}{s}\right)^{0.5}$$</td></tr>
<tr><td colspan="2">N.i.3

N.i.3.a
</td></tr>
</table>

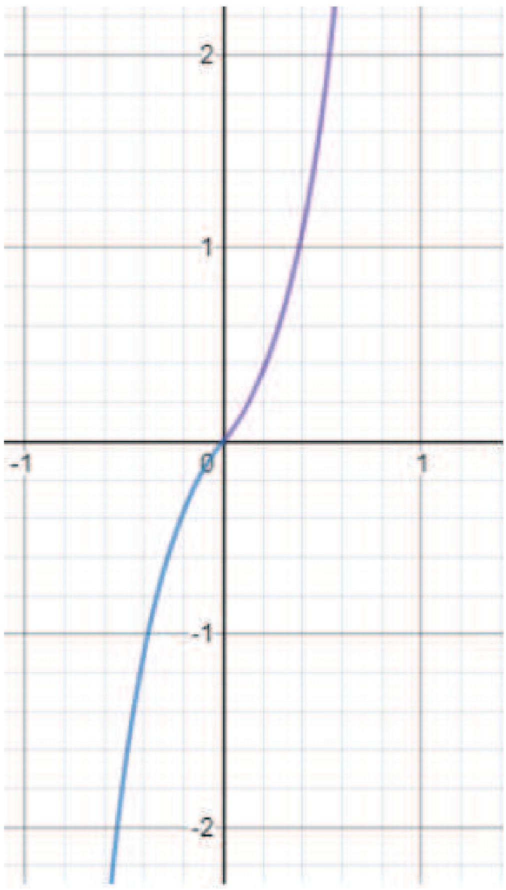
$u = f(s)$

Equation	Graphable As:
Positive s / radical added takes neg component	*Red Graph*
$u = -\frac{1}{-\frac{1}{s}+\oplus\sqrt{\left\|-\frac{1}{s}\right\|}}$ $u = -\frac{1}{-\frac{1}{s}-\sqrt{\left\|-\frac{1}{s}\right\|}}$	$u = -\frac{1}{-\frac{1}{s}-\left(\frac{1}{s}\right)^{0.5}}$
Negative s / radical added	*Black Graph*
$u = -\frac{1}{-\frac{1}{s}+\sqrt{-\frac{1}{s}}}$	$u = -\frac{1}{-\frac{1}{s}+\left(-\frac{1}{s}\right)^{0.5}}$
Positive s / radical subtracted takes neg component	Purple Graph
$u = -\frac{1}{-\frac{1}{s}-\oplus\sqrt{\left\|-\frac{1}{s}\right\|}}$ $u = -\frac{1}{-\frac{1}{s}+\sqrt{\left\|-\frac{1}{s}\right\|}}$	$u = -\frac{1}{-\frac{1}{s}+\left(\frac{1}{s}\right)^{0.5}}$
Positive s / radical subtracted	Blue Graph
$u = -\frac{1}{-\frac{1}{s}-\sqrt{-\frac{1}{s}}}$	$u = -\frac{1}{-\frac{1}{s}-\left(-\frac{1}{s}\right)^{0.5}}$

N.i.4



N.i.4.a



Intro.ix.5—Multiplication and Division of subspace numbers with real space numbers:

The duel sign of subspace numbers cannot directly interact with other numbers except within a valid subspace transformation where both must be simultaneously applied. So equations like $y = x(\oplus 2)$ and $y = \frac{x}{\oplus 2}$ must have their subspace values resolved before the operations can be completed. Here again the $\oplus$ pertains to the x-axis as a Δx to produce the output y in these examples. So the positive magnitude will be used on the x-axis of the XY-Plane whilst the negative magnitude corresponds to the x-axis on the co-adjoining XU-Plane subspace.

$y = (\oplus 2)x \quad \equiv \quad \oplus 2x \quad \equiv \quad y = 2x$ $\qquad\qquad$ $y = \frac{x}{\oplus 2} \quad \equiv \oplus \frac{x}{2} \quad \equiv \; y = \frac{x}{2}$

The resolution of the $\oplus$ sign on the XY-Plane will take the positive magnitude of $2x$ in the first example and $\frac{x}{2}$. In the second example.

$y = 2x$ $\qquad\qquad$ $y = \frac{x}{2}$

After transforming to the co-adjoining subspace XU-Plane equation the $\oplus$ sign will resolve to the negative magnitude of $2x$ in the first example and $\frac{x}{2}$. In the second example.

$\Xi y = \frac{\Xi}{\varsigma u}(\oplus 2x)$ $\qquad\qquad$ $\Xi y = \frac{\Xi}{\varsigma u}\left(\oplus \frac{x}{2}\right)$

$u = -\frac{1}{(\oplus 2)x} = \frac{1}{2x}$ $\qquad\qquad$ $u = -\frac{1}{\oplus \frac{x}{2}} = \frac{2}{x}$

Intro.ix.5.a—Addition / Subtraction and Multiplication / Division of multiple subspace terms:

Equations containing multiple subspace terms can be grouped together if the form of the equations allows it without changing its form. Addition and subtraction will operations will change the magnitude number modifying the term but leave it with a plus-and-minus sign. Multiplication and division will both result in the essential squaring of the plus-and-minus sign itself; the $\frac{0}{0} = \oplus 1$ which is the base of all i-multiples. This will provide a negative magnitude. Examples given below assume original equations are given in $y = f(x)$ format on the XY-Plane.

Addition

$y = \sqrt{-9} + \sqrt{-4}$

$y = \oplus 3 + \oplus 2 = \oplus 5$

Subtraction

$y = \sqrt{-9} - \sqrt{-4}$

$y = \oplus 3 - \oplus 2 = \oplus 1$

Output y equals a subspace i-multiple. Positive magnitude exists in the XY-Plane with the negative magnitude is the SY-Plane.

XY-Plane: $y = 5$ SY-Plane: $y = -5$

XY-Plane: $y = 1$ SY-Plane: $y = -1$

Multiplication:

$y = \sqrt{-16} \cdot \sqrt{-9}$

$y = \oplus 4 \cdot \oplus 3 = -12$

Division:

$y = \frac{\sqrt{-16}}{\sqrt{-9}}$

$y = \frac{\oplus 4}{\oplus 3} = -\frac{4}{3}$

In both instances the output y equals a real number. The subspace transformations no longer include subspace values in the inputs and will transform normally.

$u = \frac{1}{12}$

$u = \frac{3}{4}$

Intro.ix.6—Squares and Radicals of Subspace Numbers:

The square of a subspace constant will equal a negative number. For a further description on this process see Chapter 2 on i multiples.

$$n^2 = \hat{n}^2 = (\oplus\, n)^2 = [n^2]\,[\oplus\, 1^2] = -n^2$$

$$2^2 = \hat{2}^2 = (\oplus\, 2)^2 = [2^2]\,[\oplus\, 1^2] = -4$$

The paired component to the $\hat{n}$ is the $\breve{n}$. Where $\hat{n}$ resolves to positive at place of occurrence, the $\breve{n}$ component resolves to the negative component. Whether squaring $\hat{n}$ or $\breve{n}$ the answer is identical. It is as if you multiplied it with its opposite signed counterpart. You are squaring the magnitude of the plus and minus number, multiplied by the square of the i-multiple base; $\frac{0}{0} = \oplus\, 1$. It's important to understand this and will be clearer in a moment when examining higher powers of plus-and-minus numbers. Consider the same example used above but with $\check{2}$ in place of the originally used $\hat{2}$.

$$-2^2 = \check{2}^2 = [2^2][\oplus 1^2] = -4$$

Also note $n = -\check{n} = \hat{n}$ and $-n = \check{n} = -\hat{n}$. In the examples above we begin by squaring $n = 2$ and $n = -2$. Respectively the first steps in the examples show $2^2 = \hat{2}^2$ and $-2^2 = \check{2}^2$. Though true statements $2^2 \not\equiv \hat{2}^2$ and $-2^2 \not\equiv \check{2}^2$. These exchanges are made either by keeping track of subspace numbers, likely working backwards through a resolution, or from other factors which indicate you need to make the exchanges. One such instance of this occurs with Completing the Negative Square, a subspace version of the quadratic formula to complete the square when factoring.

The subspace up and down components which themselves possess a negative sign are called cross terms. In the paragraph above these are $n = -\check{n}$ and $-n = -\hat{n}$. These values are not only equal but are also identical to each other. Resolving them will result in the same answer. Both $-\check{n}$ and $-\hat{n}$ represent a negative magnitude.

$$-2^2 = -\hat{2}^2 = [(-2^2)(\oplus 1^2)] = -4$$

$$2^2 = -\check{2}^2 = [(-2^2)(\oplus 1^2)] = -4$$

The square root of positive number is a $\pm$ value. The square root of a negative number is a $\oplus$ number. The square root of a subspace number, a $\oplus$ value, will ultimately be a $\pm$ value. These roots are the product of the square root of the magnitude of the argument and the square root of the i-multiple base, ; $\frac{0}{0} = \oplus 1$. The square root of $\oplus 1$ is identical to its square. For further information on roots of $\oplus$ numbers see chapter 2 on i-multiples.

$$\sqrt{\oplus n^2} = \sqrt{\hat{n}^2} = \sqrt{n^2} \cdot \sqrt{\oplus 1} = \pm n \cdot -1 = \pm n$$

$$\sqrt{\oplus 4} = \sqrt{\hat{4}} = \sqrt{4} \cdot \sqrt{\oplus 1} = \pm 2 \cdot -1 = \pm 2$$

Here the cross value terms have an impact on the sign. The root of the magnitude has a negative argument. It will result in a positive-and-negative number which will resolve to positive at place of occurrence on the originating plane. Note again that since $\sqrt{-\hat{n}} \equiv \sqrt{\check{n}}$ use $\sqrt{\check{n}}$ in its place to avoid confusion or going in circles.

$$\sqrt{-\hat{4}} \;=\; \left[\sqrt{\hat{4}} \cdot \sqrt{-1}\right] \;=\; \left[\left(\sqrt{4} \cdot\!\oplus 1^2\right) \cdot\!\oplus 1\right] \quad \left[(\pm 2 \cdot -1) \cdot \hat{1}\right] \quad = \; \pm 2$$

Observe this is identical to $\sqrt{\check{4}}$

$$\sqrt{\check{4}} \;=\; \sqrt{4} \cdot \sqrt{\oplus 1} \quad = \quad [\pm 2 \cdot -1] \quad = \quad \pm 2$$

$$\sqrt{-\check{4}} = \left[\sqrt{\check{4}} \cdot \sqrt{-1}\right] \;=\; \left[\left(\sqrt{4} \cdot\!\oplus 1^2\right) \cdot\!\oplus 1\right] \quad \left[(\pm 2 \cdot -1) \cdot \hat{1}\right] \quad = \; \pm 2$$

Observe this is identical to $\sqrt{\hat{4}}$

Intro.ix.7—Higher Powers and Roots of Subspace Numbers:

Cubes and higher powers of subspace numbers are identical to raising the magnitude of the subspace number to the given power, multiplied by ⊕1 raised to the same power. For a further description on this process see Chapter 2 on i multiples.

Where $n \geq 2$ and n is even

$$(\oplus a)^n \quad = \quad -(a^n) \qquad \text{for even } n$$

$$2^4 \;=\; \hat{2}^4 = (\oplus 2)^4 \;=\; [2^4][\oplus 1^4] \quad = \quad 16 \cdot -1 \quad = \quad -16$$

This will work for the down components too:

$$-2^4 = \check{2}^4 = (\oplus 2)^4 \;=\; [2^4][\oplus 1^4] \quad = \quad 16 \cdot -1 \quad = \quad -16$$

Remember the up and down components are resolved halves of the same plus-and-minus number.

Where $m \geq 3$ and m is odd

$$(\oplus a)^m \quad = \quad a^m$$

$$2^5 = \hat{2}^5 = (\oplus 2)^5 \;=\; [2^5][\oplus 1^5] \quad = \quad 32 \cdot 1 \quad = \quad 32$$

The down components result in an identical answer:

$$-2^5 = \check{2}^5 = (\oplus 2)^5 \;=\; [2^5][\oplus 1^5] \quad = \quad 32 \cdot 1 \quad = \quad 32$$

The crossed-sign terms will cause a sign change when dealing with powers of $m \geq 3$ and m is odd

Here is $n \geq 2$ and n is even with cross signed terms.

$$-2^4 = -\hat{2}^4 = \left[\hat{2}^4 \cdot (-1)^4\right] = [(2^4) \cdot (\oplus 1^4)\,] \cdot 1 \;=\; [16 \cdot -1] \;\cdot 1 \;=\; -16$$

With the corresponding down component:

$$2^4 = -\check{2}^4 = \left[\check{2}^4 \cdot (-1)^4\right] = [(2^4) \cdot (\oplus 1^4)\,] \cdot 1 \;=\; [16 \cdot -1] \;\cdot 1 \;=\; -16$$

Where $m \geq 3$ and m is odd

$$-2^5 = -\hat{2}^5 = \left(\hat{2}^5 \cdot (-1)^5\right) = [(2^5) \cdot (\oplus 1^5)\,] \cdot -1 = [(32 \cdot 1)] \cdot -1 = -32$$

With the corresponding down component:

$$2^5 = -\check{2}^5 = \left(\check{2}^5 \cdot (-1)^5\right) = \;[(2^5) \cdot (\oplus 1^5)\,] \cdot -1 \;= [32 \cdot 1] \cdot -1 = -32$$

Note again that $2^5 \not\equiv -\check{2}^5$*. If you see* 2^5 *you'll very likely simplify this expression as* 32*. Though not identical,* 2^5 *does equal* $-\check{2}^5$*. When you resolve the down component you obtain* 2^5*. Whether you keeping track of the up and down components and working backwards from the resolved value, or some other factor ques you need to switch to a subspace component (see Completing the Negative Square) you know to make the exchange. This same consideration must be given to any value.*

Cube roots and higher roots of a subspace numbers follow directly from the application of higher powers. The result is the positive or negative value of the n^th^-root of the magnitude of the subspace term multiplied by ⊕1 raised to the same *power*. For ⊕1, the i- multiple, its roots are identical to its powers. See chapter 2 on i-multiples for more on this. Again watch out for the crossed-sign terms as the negative-up and negative-down terms will have a sign change for odd powered-roots.

Where $n \geq 2$ and n is even with $b = \sqrt[n]{a}$.

$$\sqrt[n]{\oplus a} \;=\; \sqrt[n]{\hat{a}} \;=\; \sqrt[n]{a} \cdot (\oplus 1)^n = \pm b \cdot -1 = \pm b$$

$$\sqrt[4]{\oplus 16} = \sqrt[4]{\widehat{16}} \;=\; \sqrt[4]{16} \cdot (\oplus 1)^4 \;=\; \pm 2 \;\cdot\; -1 \;= \pm 2$$

Will get the same values for the down components:

$$\sqrt[4]{\widetilde{16}} \;=\; \sqrt[4]{16} \cdot (\oplus 1)^4 \;=\; \pm 2 \cdot -1 \;=\; \pm 2$$

Where $m \geq 3$ and m is odd with $b = \sqrt[m]{a}$

$$\sqrt[m]{\oplus a} \;=\; \sqrt[m]{\hat{a}} \;=\; \sqrt[m]{a} \cdot \oplus 1^m = b \cdot 1 = b$$

$$\sqrt[3]{\oplus 8} = \sqrt[3]{\hat{8}} = \sqrt[3]{8} \cdot \oplus 1^3 = 2 \cdot 1 = 2$$

For the corresponding down values:

$$\sqrt[3]{\breve{8}} = \sqrt[3]{8} \cdot \oplus 1^3 = 2 \cdot 1 = 2$$

The crossed-signed values with results are shown here below:

Where $n \geq 2$ and n is even and $b = \sqrt[n]{a}$

$$\sqrt[n]{-\hat{a}} = \left(\sqrt[n]{\hat{a}} \cdot \oplus 1\right) = \left[\left(\sqrt[n]{a}\right) \cdot (\oplus 1^n)\right] \cdot \left(\hat{1}\right) = \pm b \cdot -1 \cdot 1 = \pm b$$

$$\sqrt[4]{-\widehat{16}} = \sqrt[4]{\widehat{16}} \cdot \oplus 1 = \left[\left(\sqrt[4]{16}\right) \cdot (\oplus 1^4)\right] \cdot \left(\hat{1}\right) = [\pm 2 \cdot -1 \cdot 1] = \pm 2$$

For the negative down crossed-sign component:

$$\sqrt[4]{-\widetilde{16}} = \left(\sqrt[4]{\widetilde{16}} \cdot \oplus 1\right) = \left[\left(\sqrt[4]{16}\right) \cdot (\oplus 1^4)\right] \cdot \left(\hat{1}\right) = [\pm 2 \cdot -1 \cdot 1] = \pm 2$$

Where $m \geq 3$ and m is odd with $b = \sqrt[m]{a}$

$$\sqrt[m]{-\hat{a}} = \left(\sqrt[m]{\hat{a}} \cdot -1\right) = \left[\sqrt[m]{a} \cdot \oplus 1^m\right] \cdot -1 = \pm b \cdot 1 \cdot -1 = \pm b$$

$$\sqrt[3]{-\hat{8}} = \sqrt[3]{\hat{8}} \cdot -1 = \left[\sqrt[3]{8} \cdot \oplus 1^3\right] \cdot -1 = \pm 2 \cdot 1 \cdot -1 = \pm 2$$

For the corresponding negative down crossed-sign value:

$$\sqrt[3]{-\breve{8}} = \sqrt[3]{\breve{8}} \cdot -1 = \left[\sqrt[3]{8} \cdot \oplus 1^3\right] \cdot -1 = \pm 2 \cdot 1 \cdot -1 = \pm 2$$

$\sqrt[n]{-1} = \oplus 1$ $\sqrt[m]{-1} = -1$ Where n is even and m is odd

For further information on roots and powers of subspace numbers see Chapter 2 on powers and root of $\frac{0}{0}$.

Chapter 1

Algebra Basics Review

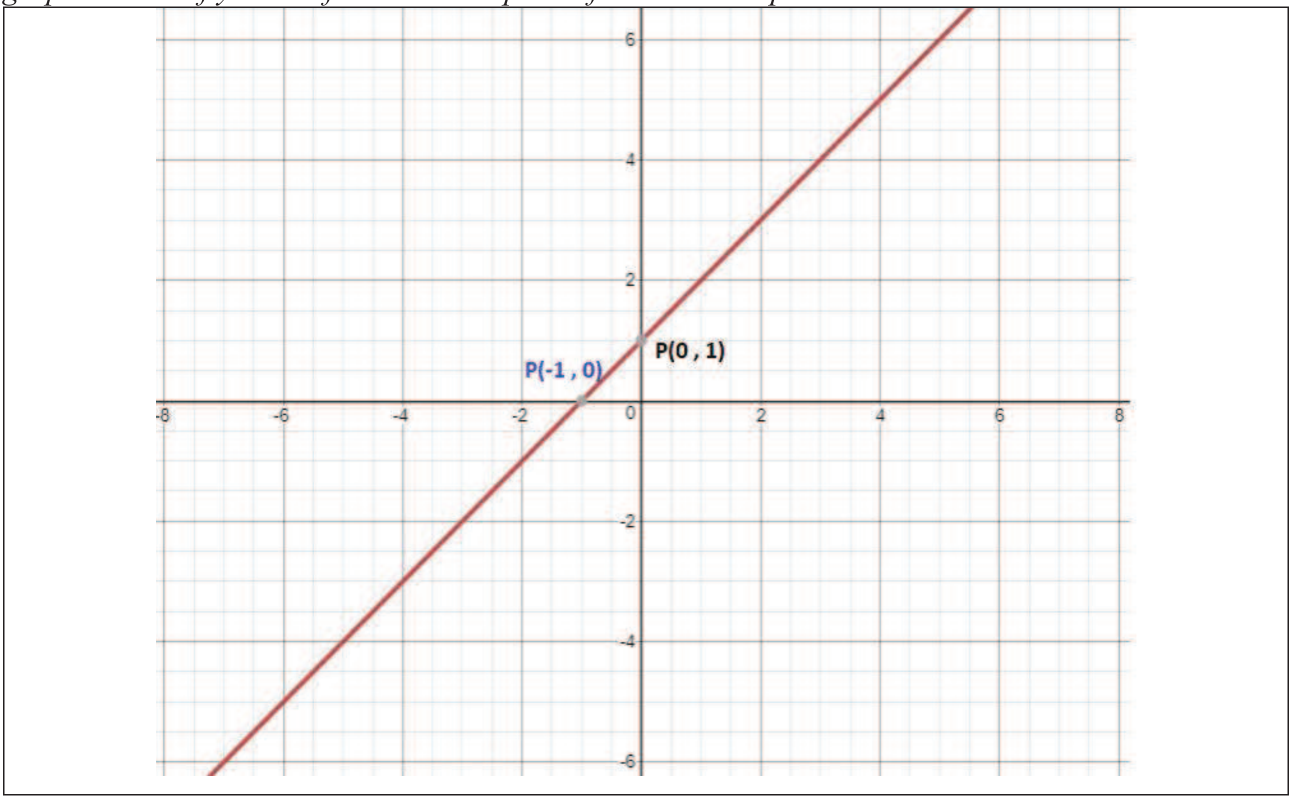
Before jumping into the more complicated math of Null Calculus we should first revisit some basic principles. They will form a basis for understanding those more complicated concepts. The first stop then is the equation for lines in the Cartesian plane and their implied counterparts, those in the corresponding subspace plane.

<u>1.a—Lines in a Plane:</u>

Lines in a Cartesian plane are usually written in one of three forms:

Standard Form:	$Ax + By = C$
Slope-Intercept:	$y = mx + b$
Point-Slope	$y - y_1 = m(x - x_1)$

Figure 1:
The graph below represents: $-x + y = 1$, $y = x + 1$ *and* $y - 2 = (1)[x - 1]$. *All three equations are different ways of describing the same set of coordinates. The x and y intercepts described in section 1.a.1 are marked on this graph. Use section 1.a.1 and the equation of this graph and see if you can find the same points for the intercepts.*



<u>1.a.1—Standard equation for a line on the Cartesian plane:</u>

The standard form equation has both x and y intercept points. These are found by setting either x or y to 0 and then solving for the other. For the XY-Plane we have:

X-Intercept	Y-Intercept
$Ax + B(0) = C$ $x = \frac{C}{A}$	$A(0) + By = C$ $y = \frac{C}{B}$

From Null Algebra we find each of these intercept points is definable in a corresponding subspace by way of a subspace transformation. Note, without further resolution of values as defined within null mathematics, these values are more than likely *not* the values of the S-intercept or U-intercept in the $u(s)$ equation which corresponds to a given $y(x)$ equation. These are provided to show what their corresponding values are in comparison to their real-space partners. This will be used in a moment to show that any real space equation can be expressed in terms of its subspace variable equivalents and vice versa. It will also provide your first example of division by 0 in calculating the slopes of real-space lines using subspace term equivalencies.

S-Value at X-Intercept	U-Value at Y-Intercept
$\Xi x = \frac{\Xi}{\varsigma s}(\frac{C}{A}) \quad s = -\frac{A}{C}$	$\Xi y = \frac{\Xi}{\varsigma u}(\frac{C}{B}) \quad u = -\frac{B}{C}$

1.a.2—The subspace equation of the Standard Form:

The actual subspace intercepts must be found by solving for them from the subspace equations themselves. Consider the transformation of the Standard Form line equations into those of their subspace equivalents:

Observe the transformation upon the standard form of the line requires seeing the variables each as an input while the slope-intercept form is a standard transform. Note that even though we can calculate the value of s at the x-intercept via a simple transformation (*shown just previously as* $s = -\frac{A}{C}$) the value of the s-intercept itself will almost certainly not equal this value obtained from that transformation. You must determine the value of the s and u intercepts from the $u(s)$ equation. In transforming from Ax + By = C to it subspace equivalent we obtain the equations below. Two methods will be shown of solving for the s and u intercepts. Method Alpha will ignore the rule of operations change and directly resolve the division by 0 without first obtaining same denominators. It will be shown later in this section that you can use null mathematics to resolve these values to obtain a real value via this method but it will still be a different point than the traditional intercepts defined by the $u(s)$ equation. Method Beta will follow the rule.

Note that an instance of n_0 in an equation though equivalent to some value divided by 0, if appearing as a single figure instead of in a fraction would not require this consideration.

Method α:

$$C = A\frac{\Xi}{\varsigma s}x + B\frac{\Xi}{\varsigma u}y \qquad\qquad C = -\frac{A}{s} - \frac{B}{u}$$

The S-intercept is found by setting u to 0 and solving for s

$$C = -\frac{A}{s} - \frac{\text{B}}{0}$$

The next step here is important. Again, there is a modification required to rules for order of operations. If you are adding or subtracting something to or from a term divided by 0 you must first set terms to have the same denominator. Only then can you resolve the value being divided by 0. This is seen below in some of the examples. However we could argue this equation's two components are each their own equation and require isolation within parentheses. From here the resolution seems straight forward but will be different than using same denominators.

$$C = \left(-\frac{A}{s}\right) - \left(\frac{\text{B}}{0}\right) \qquad C = -\frac{A}{s} - \eta_0 \qquad \frac{1}{C} = -\frac{s}{A} \qquad s = -\frac{A}{C}$$

If instead we were to follow the rule and set the denominators equal, we still need to at least first isolate the s.

<u>Method β:</u>

$$C = -\frac{A}{s} - \frac{\text{B}}{0} \qquad \frac{A}{s} = -C - \frac{\text{B}}{0} \qquad \frac{A}{s} = -\frac{0}{0} - \frac{\text{B}}{0}$$

$$\frac{A}{s} = -\frac{\text{B}}{0} \qquad \frac{A}{s} = -\eta_0 \qquad \frac{s}{A} = -\frac{1}{\eta_0} \qquad s = \eta_0$$

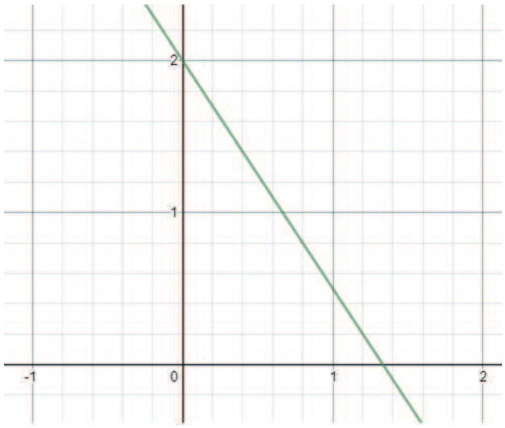
Resolves to $s = 0$

Are these two ideas identical? Are they representing the same thing? The subspace transform of the X-intercept and solving for the S-intercept by Method Alpha above show $s = -\frac{A}{C}$. Method Beta on the other hand shows naught, resolvable to 0. $-\frac{A}{C}$ could be 0 but could also be an infinite number of other values. Note the value of *s* obtained through method Beta is resolved to zero, and thereby graphable as zero. This value is accurate but does not imply exactly the same thing as $-\frac{A}{C}$ obtained by method Alpha. To illustrate lets examine an actual equation.

$$3x + 2y = 4 \quad \equiv \quad y = 2 - \frac{3}{2}x$$

From these equations we can easily see the *y*-intercept is at point $(0\,,2)$ and the *x*-intercept is at point $\left(\frac{4}{3}\,,0\right)$.

<u>Figure Alpha:</u>



From these equations we can solve for the $u(s)$ equation. There are a couple of ways to do this depending on the form of the equation you start with.

Direct conversion of variable to subspace from Ax + By = C format

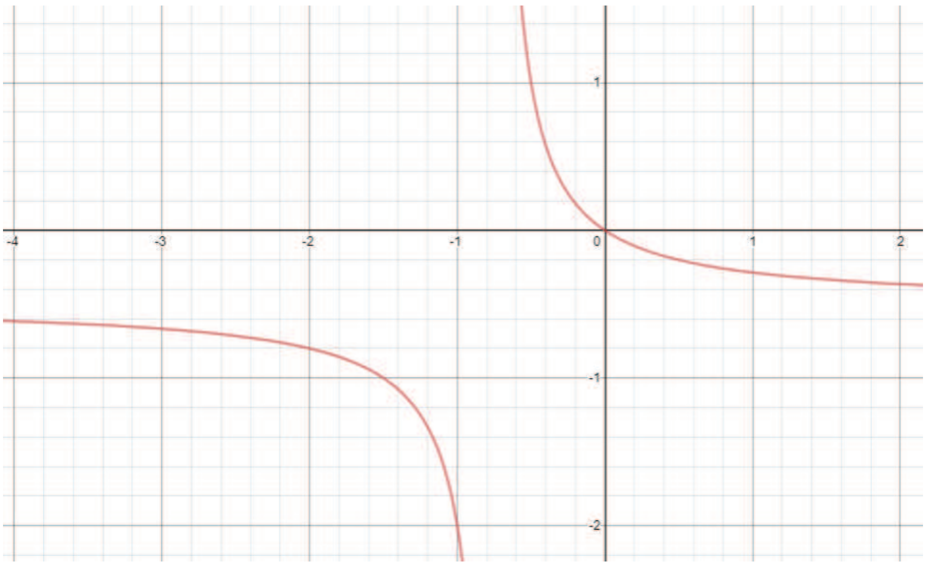
$$3x + 2y = 4 \quad \longrightarrow \quad -\frac{3}{s} - \frac{2}{u} = 4$$

Subspace transform against y and transformation of x-variable to s.

$$y = 2 - \frac{3}{2}x \quad \longrightarrow \quad u = -\frac{1}{2+\frac{3}{2s}} \quad \longrightarrow \quad u = \frac{2s}{4s+3}$$

Regardless you'll get the graph of shown in Figure Beta:

Figure Beta:



If we use Methods α and β to determine the s and u intercepts we can then compare the answers they provide. We will use the As + Bu = C form of the equation for simplicity. Note from here forward in notes ≐ normally used for *approaches limit* shall instead be used to denote *resolved to*.

$-\frac{3}{s}-\frac{2}{u}=4$	
Method Alpha	Method Beta
S-Intercept: $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{s}-\frac{2}{0}=4$ $-\frac{3}{s}-\eta_0=4 \quad\longrightarrow\quad \frac{1}{s}=-\frac{4}{3}$ $s=-\frac{3}{4}$	S-Intercept: $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{s}-\frac{2}{0}=4$ $-\frac{2}{0}-\frac{4}{1}=\frac{3}{s} \quad\longrightarrow\quad -\frac{2}{0}-\frac{0}{0}=\frac{3}{s}$ $-\eta_0=\frac{3}{s} \quad\longrightarrow\quad -\frac{\eta_0}{3}=\frac{1}{s}$ $s=-\frac{3}{\eta_0}=-\eta_0 \doteq 0$
U-Intercept: $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{0}-\frac{2}{u}=4$ $-\eta_0-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{1}{u}=2$ $u=-\frac{1}{2}$	U-Intercept: $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{0}-\frac{2}{u}=4$ $-\frac{3}{0}-\frac{4}{1}=\frac{2}{u} \quad\longrightarrow\quad -\frac{3}{0}-\frac{0}{0}=\frac{2}{u}$ $-\eta_0=\frac{2}{u} \quad\longrightarrow\quad -\frac{\eta_0}{2}=\frac{1}{u}$ $u=-\frac{2}{\eta_0}=-\eta_0 \doteq 0$

The solution here is the realization that the values obtained by Method Alpha lay on asymptotic lines. The S-Intercept value obtained from the special consideration in Method Alpha shows the value for the intercept to be at $s=-\frac{3}{4}$. When $s=-\frac{3}{4}$ the there is an asymptotic line, meaning the value is actually infinitely far away when considered within traditional mathematics. The infinity shows up in this example as $-\eta_0$, resolvable within Null Mathematics as 0. Further, if we were to insert this value into the equation and then solve for u we will again get $-\eta_0 \doteq 0$.

$$-\frac{3}{s}-\frac{2}{u}=4 \qquad\longrightarrow\qquad -\frac{3}{-\frac{3}{4}}-\frac{2}{u}=4$$

$$4-\frac{2}{u}=4 \qquad\longrightarrow\qquad -\frac{2}{u}=0$$

$$-\frac{u}{2}=\frac{1}{0} \qquad\longrightarrow\qquad u=-\frac{2}{0}$$

$$u=-\eta_0 \doteq 0$$

Likewise the U-Intercept value obtained from Method Alpha is $u = -\frac{1}{2}$. Again at $u = -\frac{1}{2}$ there is an asymptotic line. At $u = -\frac{1}{2}$ the value is again actually at infinity, just this time on the s-axis. The infinity is resolvable to 0 as shown in Method B. Again if we use the value of $u = -\frac{1}{2}$ and solve for s will find $-\eta_0 \doteq 0$.

$$-\frac{3}{s} - \frac{2}{u} = 4 \qquad \longrightarrow \qquad -\frac{3}{s} - \frac{2}{-\frac{1}{2}} = 4$$

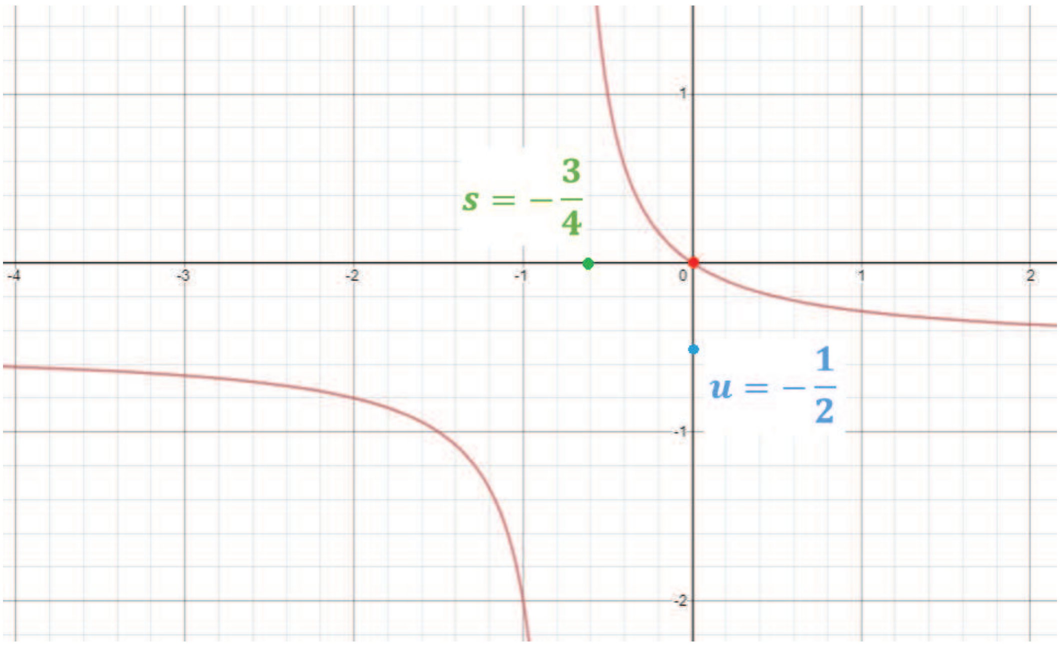
$$-\frac{3}{s} + 4 = 4 \qquad \longrightarrow \qquad -\frac{3}{s} = 0$$

$$-\frac{s}{3} = \frac{1}{0} \qquad \longrightarrow \qquad s = -\frac{3}{0}$$

$$s = -\eta_0 \doteq 0$$

Thus we need to add in two additional intercepts which are representative of values at infinity on the s and u axis. These points the s intercept at point $s = -\frac{3}{4}$ and the u intercept at $u = -\frac{1}{2}$. These are shown below in figure Beta-1

Figure Beta-1:



These intercepts exist as a result of what is happening on the original equation, $3x + 2y = 4$. When $= -\frac{3}{4}$, $x = \frac{4}{3}$ and the y values reaches 0. Though both the x and y axis are orthogonal to the US-Plane, $y = 0$ directly intersects with it and does so when, in this example, $s = -\frac{3}{4}$. Likewise when $= -\frac{1}{2}$, $y = 2$ and the x values reaches 0. When $x = 0$ it intersects with the US-Plane and in this example, that occurs when $u = -\frac{1}{2}$. There will be more on this later. There are two ways to consider what this means, none of which is any more correct than the other. One is that the points the intercepts shown by Method Alpha for this example exist on the XY-Plane and do not actually affect the US-Plane even though the exist at intersection with it and we instead see them as asymptotic curves whose infinities are resolvable to 0 but imply absence of meaning rather then 0 of something countable. The other is, were you actually able to reach the point at infinity paralleling either axis in this example, whether you chose positive or negative infinity, you'd find the intercept point marked above which corresponds to it.

By Method Beta we can solve for the values of the intercepts which exist on the US-Plane proper.

Method Beta	
S-Intercept	U-Intercept
$-\frac{3}{s}-\frac{2}{u}=4 \quad \longrightarrow \quad -\frac{3}{s}-\frac{2}{0}=4$	$-\frac{3}{s}-\frac{2}{u}=4 \quad \longrightarrow \quad -\frac{3}{0}-\frac{2}{u}=4$
$-\frac{2}{0}-4=\frac{3}{s} \quad \longrightarrow \quad -\frac{2}{0}-\frac{0}{0}=\frac{3}{s}$	$-\frac{3}{0}-4=\frac{2}{u} \quad \longrightarrow \quad -\frac{3}{0}-\frac{0}{0}=\frac{2}{u}$
$-\frac{2}{0}=\frac{3}{s} \quad \longrightarrow \quad -\frac{0}{2}=\frac{s}{3}=0$	$-\frac{3}{0}=\frac{2}{u} \quad \longrightarrow \quad -\frac{0}{3}=\frac{u}{2}$
$s=0$	$u=0$

Using the rule of operations addition to require same denominators when adding or subtracting something to or from a value divided by 0 we keep the values confined within the plane defined by the equation, in this example the US-Plane. We obtain the origin for both the s and u intercept. This is the point shown in red in figure Beta-1.

One last method of solving for the intercepts we shall call method Gamma. You may instead use relation that although $\frac{n}{0} \not\equiv \eta_0 \not\equiv \infty$ it remains true that $\frac{n}{0} = \eta_0 = \infty$ for all $n \neq 0$. When $n = 0$ there are other considerations at play which will be described in detail later. By using the infinity was can again directly solve for the intercepts of this example and will receive the same values found in Method Bata as this process keeps a values confined to the plane you are in. Though you are using an infinity you have not resolved it and thereby remain in the same plane. resolve instances of division by 0.

Method Gamma	
S-Intercept $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{s}-\frac{2}{0}=4$ $-\frac{3}{s}-\infty=4 \quad\longrightarrow\quad -\infty-4=\frac{3}{s}$ $-\infty=\frac{3}{s} \quad\longrightarrow\quad -\frac{1}{\infty}=\frac{s}{3}$ $-0=\frac{s}{3} \quad\longrightarrow\quad s=0$	U-Intercept $-\frac{3}{s}-\frac{2}{u}=4 \quad\longrightarrow\quad -\frac{3}{0}-\frac{2}{u}=4$ $-\infty-\frac{2}{u}=4 \quad\longrightarrow\quad -\infty-4=\frac{2}{u}$ $-\infty=\frac{2}{u} \quad\longrightarrow\quad -\frac{1}{\infty}=\frac{u}{2}$ $-0=\frac{u}{2} \quad\longrightarrow\quad u=0$

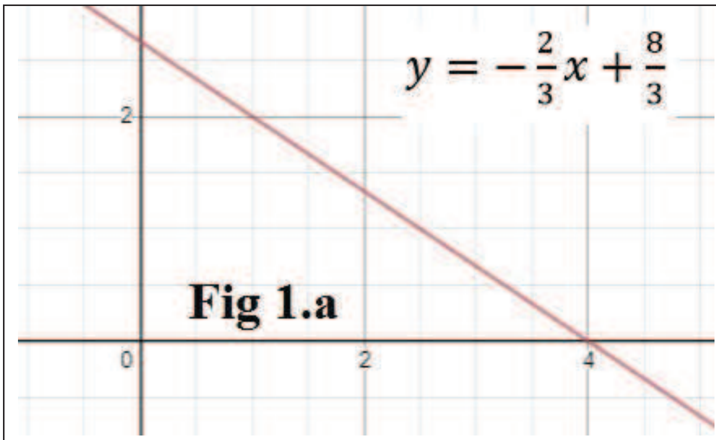
1.a.3—S and U intercepts mixed planes:

The intercepts for the UX and YS planes are found just like discussed above. See the examples below.

$u(x)$-Equation		$y(s)$-Equation	
$Ax+By=C$		$Ax+By=C$	
$Ax-\frac{B}{u}=C$		$-\frac{A}{s}+By=C$	
u-intercept	x-intercept	s-intercept	y-intercept
$A0-\frac{B}{u}=C$	$Ax-\frac{B}{0}=C$	$-\frac{A}{s}+B0=C$	$-\frac{A}{0}+By=C$
$u=-\frac{B}{c}$	$Ax=C+\frac{B}{0}$	$s=-\frac{A}{c}$	$By=C+\frac{A}{0}$
	$Ax=\frac{0}{0}+\frac{B}{0}$		$By=\frac{0}{0}+\frac{A}{0}$
	$x=\frac{\eta_0}{A}\doteq 0$		$y=\frac{\eta_0}{B}\doteq 0$

An example equation is $y=-\frac{2}{3}x+\frac{8}{3}$. In the Standard Form this is identical to $8=2x+3y$. If you make the transformations on the various terms you will get these four equations. By using Method Beta again terms involving division by 0 you will remain on the given plane and obtain the intercepts. If you use instead method Alpha you'll find the intercept which occurs as the adjoin subspace term approaches 0. That same intercept is found by resolving the infinity on the given plane. Observer below.

$$8 = 2x + 3y$$

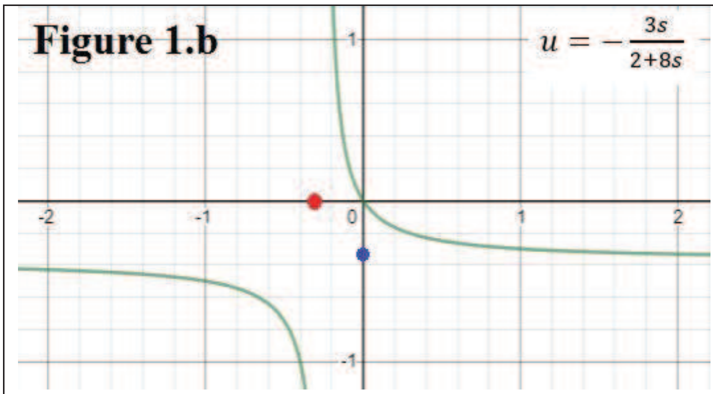


XY-Plane: $y = -\frac{2}{3}x + \frac{8}{3}$

X-Intercept: 4

Y-Intercept: $\frac{8}{3}$

$$8 = -\frac{2}{s} - \frac{3}{u}$$



US-Plane: $u = -\frac{3s}{2+8s}$

<u>S-Intercept :</u>

By Alpha— $s = -\frac{1}{4}$

By Beta— $s = -\eta_0 \doteq 0$

<u>U-Intercept:</u>

By Alpha— $u = -\frac{3}{8}$

By Beta— $u = -\eta_0 \doteq 0$

$$8 = 2x - \frac{3}{u}$$

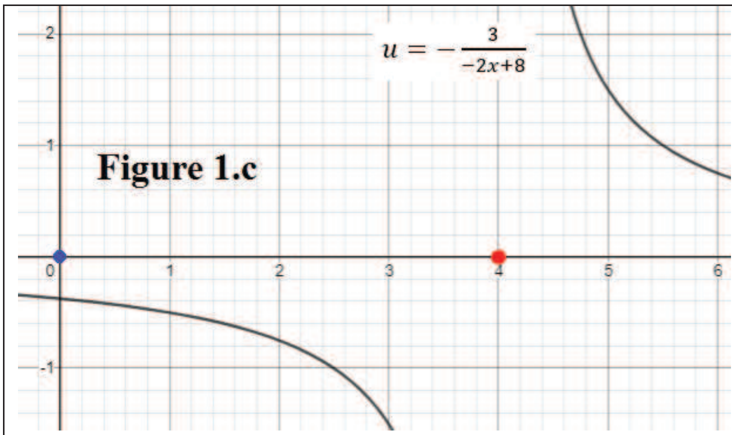


Figure 1.c

UX-Plane: $u = -\frac{3}{-2x+8}$

U-Intercept: $-\frac{3}{8}$

X-Intercept:

Alpha— $x = 4$

Result of resolution of output u-axis value to 0 when at infinity along vertical asymptote at x = 4.

Beta— $x = 0$

Result of resolution of input x-axis value at infinity to 0.

$$8 = -\frac{2}{s} + 3y$$

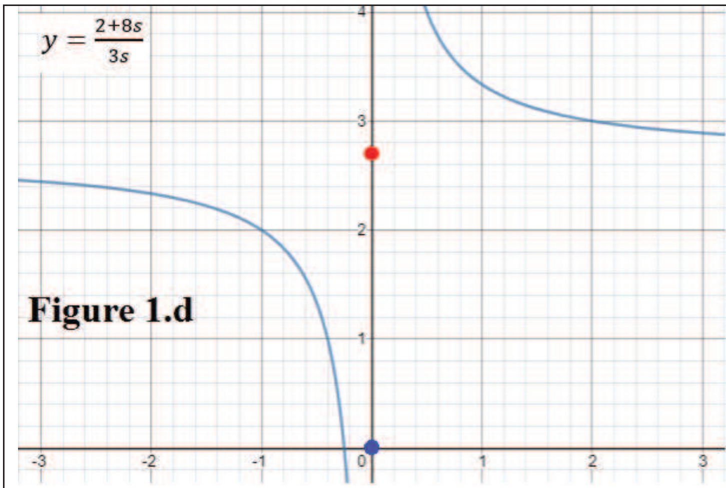


Figure 1.d

YS-Plane: $y = \frac{2+8s}{3s}$

S-Intercept: $-\frac{1}{4}$

Y-Intercept:

Alpha— $y = \frac{8}{3}$

Result of resolution of input s-axis value at infinity to 0.

Beta— $y = 0$

Result of resolution of output y-axis value at infinity to 0.

1.a.4—Slope Intercept Line Equation:

An equation which is already in the form of $y = f(x)$ like the Slope-Intercept equation can be directly transformed into its point slope equation.

$$y = mx + b$$

$$\Xi y = \frac{\Xi}{\varsigma u}(mx + b)$$

$$u = -\frac{1}{mx + b}$$

From these two equations we easily obtain: $y = -\frac{m}{s} + b$ $u = -\frac{s}{-m+bs}$

Using again the same example of $y = -\frac{2}{3}x + \frac{8}{3}$, which above was shown to have the Standard Form equation $8 = 2x + 3y$. Using $m = -\frac{2}{3}$ and $b = \frac{8}{3}$, all one must do is insert these values into the four Slope intercept equation forms which arise from this example and you will find exactly the same equations and axis intercepts shown above.

XY-Plane: $y = -\frac{2}{3}x + \frac{8}{3}$

US-Plane: $u = -\frac{3s}{2+8s}$

UX-Plane: $u = -\frac{3}{-2x+8}$

YS-Plane: $y = \frac{2+8s}{3s}$

1.b—Point Slope Line Equation:

The point slope of a line is found by examining two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ to calculate the slope of a line and an equation of that line. On the XY-Cartesian Plane slope is identified by $m = \frac{y_2 - y_1}{x_2 - x_1}$. The equation for the line which passes through the point $P_1(x_1, y_1)$ will follow the form of

$$y - y_1 = m(x - x_1)$$

$$y = m(x - x_1) + y_1$$

These values will have correspondences on the subspace axes. Using subspace transformations we find

$x_1 = x_1$	$x_2 = x_2$	$y_1 = y_1$	$y_2 = y_2$
$\Xi x_1 = \frac{\Xi}{\varsigma s} x_1$	$\Xi x_2 = \frac{\Xi}{\varsigma s} x_2$	$\Xi y_1 = \frac{\Xi}{\varsigma u} y_1$	$\Xi y_2 = \frac{\Xi}{\varsigma u} y_2$

$s_1 = -\frac{1}{x_1}$	$s_2 = -\frac{1}{x_2}$	$u_1 = -\frac{1}{y_1}$	$u_2 = -\frac{1}{y_2}$

From these equations we see the values of the corresponding subspace points are:

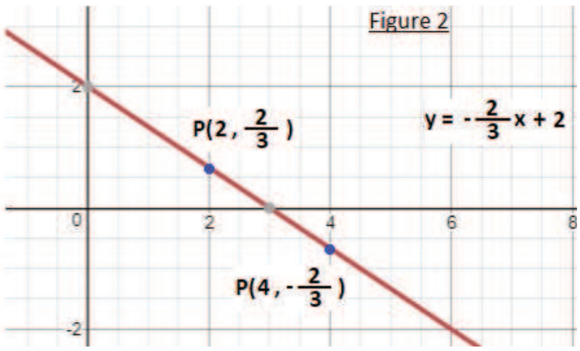
$$\mathbf{P}_{su_1}(s_1\,, u_1) = (-\tfrac{1}{x_1}, -\tfrac{1}{y_1}) \quad \text{and} \quad \mathbf{P}_{su_2}(s_2\,, u_2) = (-\tfrac{1}{x_2}, -\tfrac{1}{y_2})$$

The slope of the y = *f*(x) line can be expressed in terms of *u* and *s* without altering its value.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \frac{-\frac{1}{u_2} + \frac{1}{u_1}}{-\frac{1}{s_2} + \frac{1}{s_1}} \quad m = \frac{-\frac{u_1}{u_1 u_2} + \frac{u_2}{u_2 u_1}}{-\frac{s_1}{s_1 s_2} + \frac{s_2}{s_2 s_1}} \qquad m = \frac{\frac{u_2 - u_1}{u_1 u_2}}{\frac{s_2 - s_1}{s_1 s_2}}$$

$$m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1}$$

To see that this is so consider the following equation in the form of y = mx +b



Example 1:

Consider Figure 2 at right. Given points $P_1(2, \frac{2}{3})$ and $P_2(4, -\frac{2}{3})$ which both lie on the line defined by $y = -\frac{2}{3}x + 2$ determine the slope of the line. Then show the slope of the line can be expressed in terms of subspace values s and u.

The slope *m* for this example is calculable as $m = \frac{y_2 - y_1}{x_2 - x_1}$ and as $m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1}$. Thus, by assigning the appropriate values both calculations for *m* should result in $-\frac{2}{3}$. The subspace points which correspond to those used in the above *y*-equation for this example will be $\mathbf{P}_{su_1} = (-\frac{1}{2}, -\frac{3}{2})$ and $\mathbf{P}_{su_2} = \left(-\frac{1}{4}, \frac{3}{2}\right)$. Note that the order of point 1 and point 2 matters. They are arranged from most negative (least value) to least negative (most value) for their *s*-values.

$$-\frac{1}{2} < -\frac{1}{4}$$

$m = \frac{y_2 - y_1}{x_2 - x_1}$	$m = \frac{\left(-\frac{2}{3}\right) - \frac{2}{3}}{4 - 2}$	$m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1}$
$m = \frac{-\frac{4}{3}}{2}$	$m = -\frac{2}{3}$	$m = \frac{\left[\left(-\frac{1}{4}\right) - \frac{1}{2}\right]\left[\frac{3}{2} + \frac{3}{2}\right]}{\left[\frac{3}{2}\left(-\frac{3}{2}\right)\right]\left[-\frac{1}{4} - \left(-\frac{1}{2}\right)\right]}$

	$m = \frac{\frac{6}{16}}{-\frac{9}{16}}$ $m = -\frac{6}{9} = -\frac{2}{3}$

Note an interesting situation will occur if you were to have chosen $P_2(3\,,0)$ instead of $P_2(4,-\frac{2}{3})$. Both points exist on the same line defined by $y = -\frac{2}{3}x + 2$ but you'll encounter the presence of a non-zero values divided by 0. Observe below.

Given points $P_1(2,\frac{2}{3})$ and $P_2(3\,,0)$ which both lie on the line defined by $y = -\frac{2}{3}x + 2$ determine the slope of the line. Then show the slope of the line can be expressed in terms of subspace values *s* and *u*.

<u>Solution:</u>

The slope *m* for this example is calculable as $m = \frac{y_2-y_1}{x_2-x_1}$ and as $m = \frac{(s_1s_2)u_2-u_1}{(u_1u_2)s_2-s_1}$. Thus, by assigning the appropriate values both calculations for *m* should result in $-\frac{2}{3}$. The subspace points will be $\mathbf{P}_{su_1} = (-\frac{1}{2},-\frac{3}{2})$ and $\mathbf{P}_{su_2} = \left(-\frac{1}{3},\frac{1}{0}\right)$. Note these points are arranged with the *s*-value in increased value from most to least negative. Or in others words from least to greatest value. *As* you'll see the $\frac{1}{0}$ elements end up being removed in a way.

$m = \frac{(s_1s_2)u_2-u_1}{(u_1u_2)s_1-s_2}$	$m = \frac{\left[-\frac{1}{2}(-\frac{1}{3})\right]\left[\frac{1}{0}-\left(-\frac{3}{2}\right)\right]}{\left[-\frac{3}{2}(\frac{1}{0})\right]\left[-\frac{1}{3}-\left(-\frac{1}{2}\right)\right]}$

A subspace conversion on the $\frac{1}{0}$ value will show how to resolve division by 0 trending in the form a non-zero value divided by 0. This value is not synonymous with a subspace number and will resolve to 0. The meaning of this 0 requires some explanation which will be covered here below. Recall subspace numbers are $(+\,-)$, plus-and-minus numbers. In this book subspace numbers are indicated by either the circle-plus sign $\oplus$ or via the circumflex marks, $\hat{a}$ (*a-up*) and $\check{a}$ (*a-down*). The applications of the transform are listed in the table here below.

The transform will provide a result of 0. This process sets $\frac{1}{0} = \infty$. *We will use the dummy axis* h *to represent the primary axis and* g *to represent the subspace axis.*

$h = \frac{1}{0}$	Initial value set to dummy axis
$\Xi h = \frac{\Xi}{\varsigma g}\infty$	Xi operator subspace transform

$$\oplus 1 = \infty(g) \quad \longrightarrow \quad g = \frac{\oplus 1}{+\infty} \quad \longrightarrow \quad g = \frac{-1}{\infty}$$

$g = \frac{-1}{\infty} = 0$	The subspace value is a small number divided by infinity and thereby 0.

The return transform to the h-axis will result in division by 0 and infinity once again. This process is resolved by relating the infinity in the transformation to the axis being exchanged. Thus the infinite output in the h-equation *can be exchanged for the value on* g *at the same input.*

$h = \frac{1}{0} = \infty = g = 0$	This value of 0 is representative of the unattainability of infinity but also literally shows that the tip of the infinite spire (*as well as the bottom of the infinite valley which accompanies it*) described by $h = \frac{1}{0}$ is equal to h = 0.

For the purpose of graphing, the value on the h-*axis will be 0. However its meaning is not the value of 0 objects, but rather 0 value. Therefore it still represents a point of discontinuity in the graph. In Null Algebra and Null Calculus this idea of 0 is called naught. For Null-Math we will use the symbol below to indicate this value when it interacts with other numbers.*

Naught

η_0

Now we may return to the slope equation, and using the properties of Naught provided in the introduction, solve the equations given. Recall we are using the example equation $y = -\frac{2}{3}x + 2$ with points $P_1\left(2, \frac{2}{3}\right)$ and $P_2(3, 0)$. The slope of the equation has been verified to be $-\frac{2}{3}$. The subspace equivalents of these points are:

$$\mathbf{P}_{su_1}(-\tfrac{1}{2},-\tfrac{3}{2}) \qquad \text{and} \qquad \mathbf{P}_{su_2}(-\tfrac{1}{3},\tfrac{1}{0})$$

Using subspace equivalents for x and y we calculated the slope as listed below.

$$m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} \qquad m = \frac{\left[-\frac{1}{2}(-\frac{1}{3})\right]\left[\frac{1}{0}-\left(-\frac{3}{2}\right)\right]}{\left[-\frac{3}{2}(\frac{1}{0})\right]\left[-\frac{1}{3}-\left(-\frac{1}{2}\right)\right]}$$

Because Naught is 0 as the absence of value it is ultimately removed. In the denominator, $\left[-\frac{3}{2}(\frac{1}{0})\right]$, this is applied immediately before multiplication in order to resolve the infinite to the real value it implies in the equation. In the numerator, $\left[\frac{1}{0}-\left(-\frac{3}{2}\right)\right]$, the addition operation requires an extension to order of operations rules; it must be handled first. $\frac{1}{0}-\left(-\frac{3}{2}\right)$ is a single value by which $\left[-\frac{1}{2}(-\frac{1}{3})\right]$ will be multiplied. $\frac{1}{0}$ is a unique fraction and you must add the correct ratio to it in the correct format. Therefore the correct sum of $\frac{1}{0}-\left(-\frac{3}{2}\right)$ must be obtained before resolving the value of the number divided by 0. By setting both fractions to like denominators the addition can be completed. In this situation $\frac{0}{0}$ will show up but is not substituted for its plus-and-minus value as there is no need; whether added or subtracted it won't change value. You will see both fractions have 0 as their denominator and addition is then handled directly.

$$m = \frac{\frac{1}{6}\left[\frac{2}{0}+\frac{0}{0}\right]}{\left[\eta_0\left(-\frac{3}{2}\right)\right]\left[-\frac{2}{6}+\left(\frac{3}{6}\right)\right]} \qquad m = \frac{\frac{1}{6}\left[\frac{2}{0}\right]}{\left[\eta_0\left(-\frac{3}{2}\right)\right]\left[-\frac{2}{6}+\left(\frac{3}{6}\right)\right]} \qquad m = \frac{\frac{1}{6}[\eta_0]}{\left[\eta_0\left(-\frac{3}{2}\right)\right]\frac{1}{6}} \qquad m = \frac{\frac{1}{6}}{\left(-\frac{3}{2}\right)\frac{1}{6}}$$

$$m = \frac{1}{-\frac{3}{2}} \qquad m = -\frac{2}{3}$$

Consider the following example below with a different equation to further illustrate this situation.

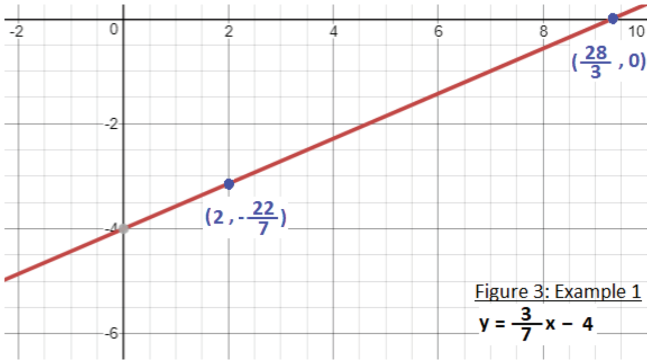


Figure 3: Example 1
$y = \frac{3}{7}x - 4$

Example 1:

Consider figure 3 at right. Using the equation $y = \frac{3}{7}x - 4$ and the points $P_1(2, -\frac{22}{7})$ and $P_2(\frac{28}{3}, 0)$ determine the corresponding subspace points. Show the slope remains unchanged whether expressed in terms of x and y or s and u.

Solution:

The slope of the equation given the two points is defined as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - \left(-\frac{22}{7}\right)}{\frac{28}{3} - 2} = \frac{\frac{22}{7}}{\frac{22}{3}} = \frac{3}{7}$$

The subspace points are those using the corresponding equivalents of values for x and y in terms of s and u. They are:

$$\mathbf{P}_{su_1}(-\frac{1}{2}, \frac{7}{22}) \qquad \mathbf{P}_{su_2}(-\frac{3}{28}, \frac{1}{0})$$

These values may be directly plugged into the subspace *m* equivalence equation.

$$m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} \qquad m = \frac{\left[-\frac{1}{2}\left(-\frac{3}{28}\right)\right]\left[\frac{1}{0} - \left(\frac{7}{22}\right)\right]}{\left[\frac{1}{0}\left(\frac{7}{22}\right)\right]\left[-\frac{3}{28} - \left(-\frac{1}{2}\right)\right]}$$

$$m = \frac{\frac{3}{56}\left[\frac{7}{0} - \frac{0}{0}\right]}{\left[\eta_0\left(\frac{7}{22}\right)\right]\left[-\frac{3}{28} + \frac{14}{28}\right]} \qquad m = \frac{\frac{3}{56}[\eta_0]}{\left[\eta_0\left(\frac{7}{22}\right)\right]\left(\frac{11}{28}\right)} \qquad m = \frac{\frac{3}{56}}{\left(\frac{7}{22}\right)\frac{11}{28}}$$

$$m = \frac{\frac{3}{56}}{\frac{77}{616}} \qquad m = \frac{3}{7}$$

1.c The line equation in subspace:

The equation of a line has a corresponding equation in subspace.

$$\Xi y = \frac{\Xi}{\varsigma u}(mx + b) \qquad u = -\frac{1}{mx + b} \qquad u = -\frac{1}{m\left(-\frac{1}{s}\right) + b}$$

It is tempting to think you could rewrite the value of *m* itself as the slope of the subspace equation by simply applying a transform like show below:

$$\Xi m = \frac{\Xi}{\varsigma M}\left[\frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$M_{su} = -\frac{x_2 - x_1}{y_2 - y_1} = -\frac{u_2 - u_1}{s_2 - s_1}$$

This is certainly a negative-reciprocate of *m* in terms of *s* and *u* but its incorrect. The slope *m*, whether in terms of *x* and *y* or *s* and *u*, as represented in the above examples, is the slope of a line defined by a linear equation. The *u*(*s*) equation is the subspace of the *y*(*x*) equation. Though the *y*(*x*) equation of the previous examples were linear equations the *u*(*s*) subspace equation which corresponds to it most likely isn't. The slope on this equation is changing at each point. It must be calculated as an instantaneous rate of change by way of the derivative of the equation.

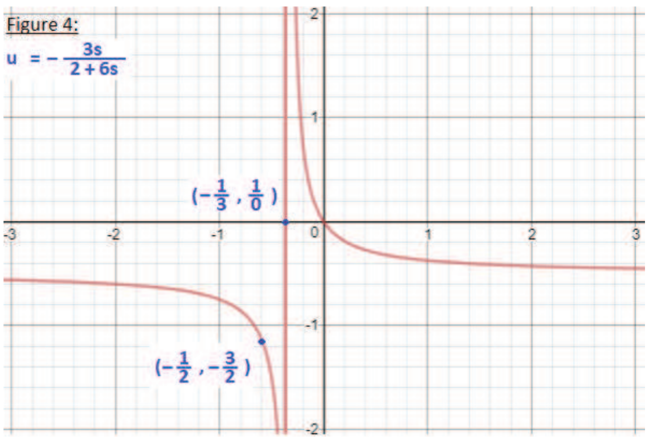
Consider the application of this with the example equation $y = -\frac{2}{3}x + 2$. Whether you use two points to calculate rise over run or take the derivative of this equation you will find the same answer; the slope of this $y(x)$ equation is a constant $-\frac{2}{3}$, and it is linear. This will not be the case for its corresponding subspace equation.

Converted to subspace we have

$$u = -\frac{1}{-\frac{2}{3}x+2} \qquad u = -\frac{1}{\left[-\frac{2}{3}\left(-\frac{1}{s}\right)\right]+2}$$

$$u = -\frac{1}{\frac{2}{3s}+2} \qquad u = -\frac{1}{\frac{2+6s}{3s}}$$

$$u = -\frac{3s}{2+6s}$$



In the equation $y = -\frac{2}{3}x + 2$ if we enter the value of x = 2, the function returns two-thirds; the point defined in an earlier example with this equation as $P_2(2, \frac{2}{3})$. The subspace equation which corresponds to the real space equation of $y = -\frac{2}{3}x + 2$, whether in terms of *x* as $u = \frac{3}{2x-6}$ or in terms of *s* as $u = -\frac{3s}{2+6s}$ the value of u should match the same values obtained if the points are directly transformed as $P_{su_2}(-\frac{1}{2}, -\frac{3}{2})$.

Using *x*-value of 2	Using *s*-value of -(1/2)
$u = \frac{3}{2(2) - 6}$ $u = \frac{3}{4 - 6}$ $u = -\frac{3}{2}$	$u = -\frac{3s}{2 + 6s}$ $u = -\frac{3\left(-\frac{1}{2}\right)}{2 + 6\left(-\frac{1}{2}\right)}$ $u = -\frac{-\frac{3}{2}}{2 + \left(-\frac{6}{2}\right)}$ $u = -\frac{-\frac{3}{2}}{-1}$ $u = -\frac{3}{2}$

Likewise the you will receive $\frac{1}{0}$ whether expressing *u* in terms of *x* or *s*: the point $P_{su_1}(-\frac{1}{3}, \frac{1}{0})$.

Using *x*-value of 3	Using *s*-value of -(1/3)
$u = -\frac{3}{-2(3) + 6}$ $u = -\frac{3}{-6 + 6}$ $u = -\frac{3}{0} \equiv \frac{1}{0} \equiv \eta_0$	$u = -\frac{3s}{2 + 6s}$ $u = -\frac{3\left(-\frac{1}{3}\right)}{2 + 6\left(-\frac{1}{3}\right)}$ $u = -\frac{-\frac{3}{3}}{2 + \left(-\frac{6}{3}\right)}$ $u = -\frac{-1}{0}$ $u = \frac{1}{0} \equiv \eta_0$
Using *x*-value of 4	Using *s*-value of -(1/4)

$u = -\frac{3}{-2(4)+6} = \frac{3}{2}$	$u = -\frac{3\left(-\frac{1}{4}\right)}{2+6\left(-\frac{1}{4}\right)} = \frac{3}{2}$

If the *u*-equation were linear like the *y*-equation in this example, $= -\frac{2}{3}x + 2$, we would expect a transform on it to provide a constant slope. The slope *m* in the *y*-equation was $-\frac{2}{3}$. A transform suggests the slope of the corresponding subspace equation to be $\mathrm{M} = \frac{3}{2}$ if the corresponding subspace equation were linear. Let's try this with two points from the above example equation.

Using the points $P_1(2, \frac{2}{3})$ and $P_2(4, -\frac{2}{3})$ which both lie on the line defined by $y = -\frac{2}{3}x + 2$ determine the corresponding subspace equation *u*(*s*), and the corresponding subspace points for point 1 and 2. Next show that the determined values for *s* generate the expected outputs for *u*. Attempt to determine the slope of the *u*-equation in the fashion used for the *y*-equation. Show that $\mathrm{M} \neq \frac{\Xi}{\varsigma M} m$.

<u>Solution:</u>

The first step is to calculate the slope of the *y*-equation.

$$m = \frac{-\frac{2}{3} - \frac{2}{3}}{4-2} = -\frac{2}{3}$$

Although the below is not correct the transform implies the following would describe the subspace equation's slope:

$$\Xi m = \frac{\Xi}{\varsigma M}\left[\frac{y_2 - y_1}{x_2 - x_1}\right] \qquad M_{su} = -\frac{1}{\frac{y_2-y_1}{x_2-x_1}} = -\frac{1}{\frac{-\frac{1}{y_2}+\frac{1}{y_1}}{-\frac{1}{x_2}+\frac{1}{x_1}}} = -\frac{x_1-x_2}{y_1-y_2} = -\frac{\frac{1}{s_2}-\frac{1}{s_1}}{\frac{1}{u_2}-\frac{1}{u_1}} = -\frac{\frac{s_1-s_2}{s_1 s_2}}{\frac{u_1-u_2}{u_1 u_2}} = -\frac{(u_1 u_2)s_2 - s_1}{(s_1 s_2)u_2 - u_1}$$

You should easily recognize this final step. $M_{su} = -\frac{(u_1 u_2)s_2 - s_1}{(s_1 s_2)u_2 - u_1}$ is the negative reciprocate of the value for *m* written in terms of *u* and *s*. In other words, if the $u(s)$-equation were linear, this would give the value for the slope on that line. However, the goal here is to show it is not the actual value of the slope. A glance at the graph of $u = -\frac{3s}{2+6s}$ will give no doubt the equation is not linear. Yet another test is to examine the value of the first derivative of *u*.

We will examine whether M_{su}, the subspace conversion of the slope of $y(x)$, is identical to $u'(s)$. Begin by calculating the derivative of the *u*-equation via the quotient rule for differentiation.

Let: $h(s) = -\frac{3s}{2+6s}$ Then: $f(s) = -3s$ $\acute{f}(s) = -3$

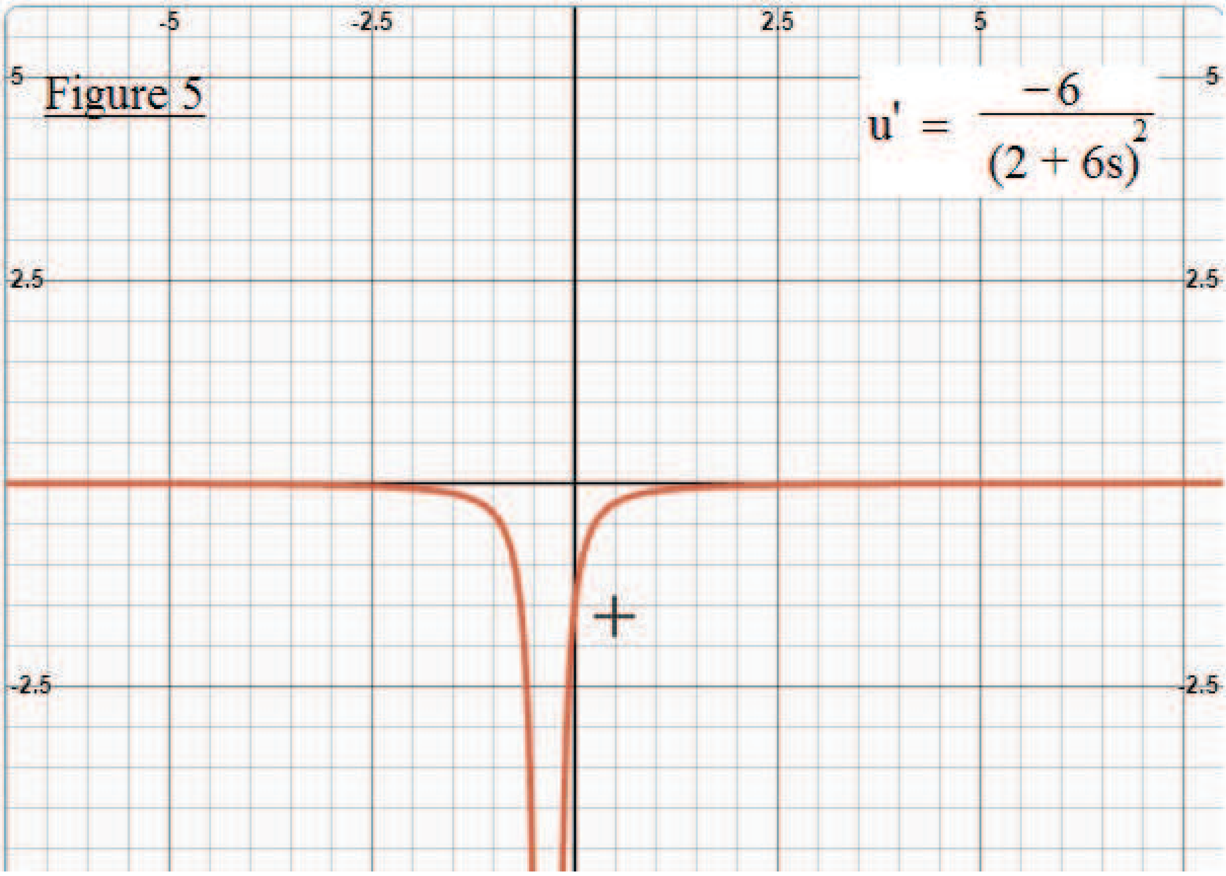
$g(s) = 2 + 6s$ $\acute{g}(s) = 6$

$$\acute{h}(s) = \frac{\acute{f}(s)g(s)-f(s)\acute{g}(s)}{\left(g(s)\right)^2} = \frac{[(-3)(2+6s)]-[(-3s)(6)]}{36s^2+24s+4} = -\frac{6}{(6s+2)^2}$$

If we use the points $P_1\left(2,\frac{2}{3}\right)$ and $P_2\left(4,-\frac{2}{3}\right)$ on $y = -\frac{2}{3}x + 2$ to define the points $\mathrm{P}_{su_1}\left(-\frac{1}{2},-\frac{3}{2}\right)$ and $\mathrm{P}_{su_2}\left(-\frac{1}{4},\frac{3}{2}\right)$ we will get the following for M_{su}:

$$M_{su} = -\frac{(u_1u_2)s_2-s_1}{(s_1s_2)u_2-u_1} = -\frac{\left[\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right)\right]\left[\left(-\frac{1}{4}\right)-\left(-\frac{1}{2}\right)\right]}{\left[\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)\right]\left[\left(\frac{3}{2}\right)-\left(-\frac{3}{2}\right)\right]} = -\frac{\left[-\frac{9}{4}\right]\left[\frac{1}{4}\right]}{\left[\frac{1}{8}\right]\left[\frac{6}{2}\right]} = -\frac{-\frac{9}{16}}{\frac{6}{16}} = \frac{9}{16}\left[\frac{16}{6}\right] = \frac{3}{2}$$

It does return positive three-halves, the negative reciprocate of the slope of the original example $y(x)$ equation, in terms of u and s. However this is not the slope of $u = -\frac{3s}{2+6s}$ equation derived from the supplied example $y = -\frac{2}{3}x + 2$. $M_{su} = -\frac{(u_1u_2)s_2-s_1}{(s_1s_2)u_2-u_1}$ is describing the slope of a completely different equation, an equation which is neither the original $y(x)$ equation nor the $u(s)$ equation derived from it. Consider below the graph of $\acute{u} = -\frac{6}{(6s+2)^2}$ in Figure 5. Compare this to the chart for some of the inputs. Not only is the slope at that point not positive three-halves but it continues to vary along the curve.



s	$\acute{u} = -\frac{6}{(2+6s)^2}$
0	$-\frac{3}{2}$
-1	$-\frac{3}{8}$
$-\frac{1}{2}$	-6
$-\frac{1}{3}$	η_0
$-\frac{1}{4}$	-24
$-\frac{1}{5}$	$-\frac{75}{8}$

Chapter 1 Example Problems:

Take your time with the example problems listed below. There aren't many so don't get overwhelmed. Try to reach the solutions yourself. The solutions are given at the end of the section.

Instructions:

For each of the problems below do the following:

1.) Determine the value for which y will equal 0; $0 = f(x)$.
2.) Determine the value for $y = f(2)$.
3.) Using these two points
    a. Determine the slope of $y = f(x)$.
    b. Determine the corresponding subspace points based on the $f(x)$ points. Hint: examine the value of the *s*-values of the two subspace points. Be certain to assign them in order from least value to highest value as point s_1 and s_2.
    c. Show the slope m found in step *3a* is still reachable using the subspace points determined in step *3b*.
4.) Finally determine the corresponding *u*(*s*) equation.
    a. Show that the *u*-value for *u*(*s*) corresponding to the value of step 2 above matches the *u*-value of the point determined in step *3b*.
5.) Feel free to try additional points and equations on your own.

Example 1: $y = 5x + \frac{2}{3}$

Example 2: $y = 7x - 3$

Example 3: $y = 4x + \frac{1}{8}$

Example 4: $y = \frac{3}{4}x + 2$

Solution Example 1:

$$y = 5x + \frac{2}{3}$$

1. $f\left(-\frac{2}{15}\right) = 5\left(-\frac{2}{15}\right) + \frac{2}{3} = 0$ $\qquad$ $P_1\left(-\frac{2}{15}, 0\right)$

2. $f(2) = 5(2) + \frac{2}{3} = \frac{32}{3}$ $\qquad$ $P_2\left(2, \frac{32}{3}\right)$

3a. $m = \frac{\Delta y}{\Delta x} = \frac{\frac{32}{3} - 0}{2 - \left(-\frac{2}{15}\right)} = \frac{\frac{32}{3}}{\frac{30}{15} + \frac{2}{15}} = \frac{32}{3} \cdot \frac{15}{32} = 5$

3b. $\boldsymbol{P}_{su_1}\left(-\frac{1}{2}, -\frac{3}{32}\right)$ $\qquad$ $\boldsymbol{P}_{su_2}\left(\frac{15}{2}, \frac{1}{0}\right)$ $\qquad$ Ordered by: $-\frac{1}{2} < \frac{15}{2}$

3c. $m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} = \frac{\left[\left(-\frac{1}{2}\right)\left(\frac{15}{2}\right)\right]\left[\frac{1}{0} - \left(-\frac{3}{32}\right)\right]}{\left[\left(-\frac{3}{32}\right)\left(\frac{1}{0}\right)\right]\left[\frac{15}{2} - \left(-\frac{1}{2}\right)\right]} = \frac{\left[-\frac{15}{4}\right]\left[\frac{32}{0} + \frac{0}{0}\right]}{\left[\left(-\frac{3}{32}\right)\left(\frac{1}{0}\right)\right]\left[\frac{16}{2}\right]} = \frac{\left[-\frac{15}{4}\right][\eta_0]}{\left[\left(-\frac{3}{32}\right)(\eta_0)\right]\left[\frac{16}{2}\right]} = \frac{-\frac{15}{4}}{-\frac{48}{64}} = 5$

4a. $\Xi y = \frac{\Xi}{\varsigma u}\left(5x + \frac{2}{3}\right) \longrightarrow \oplus 1 = \left(5x + \frac{2}{3}\right)u \longrightarrow u = -\left(\frac{1}{5x + \frac{2}{3}}\right)$

$u = -\left(\frac{1}{-\frac{5}{s} + \frac{2}{3}}\right) = -\left(\frac{1}{-\frac{15}{3s} + \frac{2s}{3s}}\right) = -\left(\frac{1}{\frac{-15+2s}{3s}}\right) = -\left(\frac{3s}{-15+2s}\right)$

$u = -\left(\frac{3s}{-15+2s}\right)$

4b. $u\left(-\frac{1}{2}\right) = -\left(\frac{3\left(-\frac{1}{2}\right)}{-15 + 2\left(-\frac{1}{2}\right)}\right) = -\left(\frac{-\frac{3}{2}}{-15 - \frac{2}{2}}\right) = -\left(\frac{-\frac{3}{2}}{-16}\right) = -\left(-\frac{3}{2} \cdot -\frac{1}{16}\right) = -\frac{3}{32}$

$u\left(\frac{15}{2}\right) = -\left(\frac{3\left(\frac{15}{2}\right)}{-15 + 2\left(\frac{15}{2}\right)}\right) = -\left(\frac{\frac{45}{2}}{-15+15}\right) = -\left(\frac{\frac{45}{2}}{0}\right) = -\left(\frac{45}{2} \cdot \frac{1}{0}\right) = -\frac{45}{0} \equiv -\frac{1}{0} = -\eta_0$

Solution Example 2:

$$y = 7x - 3$$

1. $f\left(\frac{3}{7}\right) = 7\left(\frac{3}{7}\right) - 3 = 0$ $\qquad$ $P_1\left(\frac{3}{7}, 0\right)$

2. $f(2) = 14 - 3 = 11$ $\qquad$ $P_2(2, 11)$

3a. $m = \frac{\Delta y}{\Delta x} = \frac{11 - 0}{2 - \frac{3}{7}} = \frac{11}{\frac{14}{7} - \frac{3}{7}} = \frac{11}{1} \cdot \frac{7}{11} = 7$

3b. $\boldsymbol{P}_{su_1}\left(-\frac{7}{3}, \frac{1}{0}\right)$ $\qquad$ $\boldsymbol{P}_{su_2}\left(-\frac{1}{2}, -\frac{1}{11}\right)$ $\qquad$ Ordered by: $-\frac{7}{3} < -\frac{1}{2}$

3c. $m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} = \frac{\left[\left(-\frac{7}{3}\right)\left(-\frac{1}{2}\right)\right]\left[-\frac{1}{11} - \left(\frac{1}{0}\right)\right]}{\left[\left(\frac{1}{0}\right)\left(-\frac{1}{11}\right)\right]\left[-\frac{1}{2} - \left(-\frac{7}{3}\right)\right]} = \frac{\left[\frac{7}{6}\right]\left[\frac{0}{0} - \frac{11}{0}\right]}{\left[\left(-\frac{1}{11}\right)\left(\frac{1}{0}\right)\right]\left[\frac{11}{6}\right]} = \frac{\left[\frac{7}{6}\right][-\eta_0]}{\left[\left(-\frac{1}{11}\right)(\eta_0)\right]\left[\frac{11}{6}\right]} = \frac{-\frac{7}{6}}{-\frac{1}{6}} = 7$

4a. $\Xi y = \frac{\Xi}{\varsigma s}(7x - 3) \longrightarrow \oplus 1 = (7x - 3)u \longrightarrow u = -\left(\frac{1}{7x-3}\right)$

$u = -\left(\frac{s}{-7-3s}\right)$

4b. $u\left(-\frac{7}{3}\right) = -\left(\frac{-\frac{7}{3}}{-7-3\left(-\frac{7}{3}\right)}\right) = -\left(\frac{-\frac{7}{3}}{-7+7}\right) = -\left(\frac{-\frac{7}{3}}{0}\right) = -(-\eta_0) \equiv \frac{1}{0}$

$u\left(-\frac{1}{2}\right) = -\left(\frac{-\frac{1}{2}}{-7-3\left(-\frac{1}{2}\right)}\right) = -\left(\frac{-\frac{1}{2}}{-7+\frac{3}{2}}\right) = -\left(\frac{-\frac{1}{2}}{-\frac{11}{2}}\right) = -\left(\frac{1}{11}\right) = -\frac{1}{11}$

Solutions Example 3:

$y = 4x + \frac{1}{8}$

1. $f\left(-\frac{1}{32}\right) = 4\left(-\frac{1}{32}\right) + \frac{1}{8} = 0$ $\quad P_1\left(-\frac{1}{32}, 0\right)$

2. $f(2) = 4(2) + \frac{1}{8} = \frac{65}{8}$ $\quad P_2\left(2, \frac{65}{8}\right)$

3a. $m = \frac{\Delta y}{\Delta x} = \frac{\frac{65}{8} - 0}{2-\left(-\frac{1}{32}\right)} = \frac{\frac{65}{8}}{\frac{64}{32}+\frac{1}{32}} = \frac{65}{8} \cdot \frac{32}{65} = 4$

3b. $\boldsymbol{P}_{su_1}\left(-\frac{1}{2}, -\frac{8}{65}\right)$ $\quad \boldsymbol{P}_{su_2}\left(32, \frac{1}{0}\right)$ Ordered by: $-\frac{1}{2} < 32$

3c. $m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} = \frac{\left[\left(-\frac{1}{2}\right)(32)\right]\left[\frac{1}{0} - \left(-\frac{8}{65}\right)\right]}{\left[\left(-\frac{8}{65}\right)\left(\frac{1}{0}\right)\right]\left[32 - \left(-\frac{1}{2}\right)\right]} = \frac{[-16]\left[\frac{65}{0} + \frac{0}{0}\right]}{\left[\left(-\frac{8}{65}\right)\left(\frac{1}{0}\right)\right]\left[\frac{65}{2}\right]} = \frac{[-16][\eta_0]}{\left[\left(-\frac{8}{65}\right)(\eta_0)\right]\left[\frac{65}{2}\right]} = \frac{-16}{-4} = 4$

4a. $\Xi y = \frac{\Xi}{\varsigma s}\left(4x + \frac{1}{8}\right) \longrightarrow \oplus 1 = \left(4x + \frac{1}{8}\right)u \longrightarrow u = -\left(\frac{1}{4x+\frac{1}{8}}\right)$

$u = -\left(\frac{1}{-\frac{4}{s}+\frac{1}{8}}\right) = -\left(\frac{1}{-\frac{32}{8s}+\frac{s}{8s}}\right) = -\left(\frac{1}{\frac{-32+s}{8s}}\right) = -\left(\frac{8s}{-32+s}\right)$

$u = -\left(\frac{8s}{-32+s}\right)$

4b. $u\left(-\frac{1}{2}\right) = -\left(\frac{8\left(-\frac{1}{2}\right)}{-32+\left(-\frac{1}{2}\right)}\right) = -\left(\frac{-4}{-32-\frac{1}{2}}\right) = -\left(\frac{-4}{-\frac{65}{2}}\right) = -\left(-4 \cdot -\frac{2}{65}\right) = -\frac{8}{65}$

$u(32) = -\left(\frac{8(32)}{-32+(32)}\right) = -\left(\frac{256}{0}\right) = -(\eta_0) = -\frac{1}{0} \equiv \frac{1}{0}$

The negative sign on naught here is identical to $\frac{1}{0}$. *Its no longer interacting with others values and is itself the output of the equation, the final value arrived at. Whether naught is positive or negative,* $\frac{1}{0}$ *or* $-\frac{1}{0}$, *standing alone it is still equates to positive or negative infinity. That value is only resolved by a transform which shows the infinite relates to the subspace's value of 0. As it is not adding or multiplying any other number it is taken as this value of naught…. zero. Thereby* $-(\eta_0) = -\frac{1}{0} \equiv \frac{1}{0}$ *in this example.*

Solution Example 4:

$$y = \frac{3}{4}x + 2$$

1. $f\left(-\frac{8}{3}\right) = \frac{3}{4}\left(-\frac{8}{3}\right) + 2 = 0$ $\quad$ $P_1\left(-\frac{8}{3}, 0\right)$

2. $f(2) = \frac{3}{4}(2) + 2 = \frac{7}{2}$ $\quad$ $P_2\left(2, \frac{7}{2}\right)$

3a. $m = \frac{\Delta y}{\Delta x} = \frac{\frac{7}{2} - 0}{2 - \left(-\frac{8}{3}\right)} = \frac{\frac{7}{2}}{\frac{6}{3} + \frac{8}{3}} = \frac{7}{2} \cdot \frac{3}{14} = \frac{3}{4}$

3b. $\boldsymbol{P}_{su_1}\left(-\frac{1}{2}, -\frac{2}{7}\right)$ $\quad$ $\boldsymbol{P}_{su_2}\left(\frac{3}{8}, \frac{1}{0}\right)$ $\quad$ Ordered by: $-\frac{1}{2} < \frac{3}{8}$

3c. $m = \frac{(s_1 s_2)u_2 - u_1}{(u_1 u_2)s_2 - s_1} = \frac{\left[\left(-\frac{1}{2}\right)\left(\frac{3}{8}\right)\right]\left[\frac{1}{0} - \left(-\frac{2}{7}\right)\right]}{\left[\left(-\frac{2}{7}\right)\left(\frac{1}{0}\right)\right]\left[\frac{3}{8} - \left(-\frac{1}{2}\right)\right]} = \frac{\left[-\frac{3}{16}\right]\left[\frac{7}{0} + \frac{0}{0}\right]}{\left[\left(-\frac{2}{7}\right)\left(\frac{1}{0}\right)\right]\left[\frac{7}{8}\right]} = \frac{\left[-\frac{3}{16}\right][\eta_0]}{\left[\left(-\frac{2}{7}\right)(\eta_0)\right]\left[\frac{7}{8}\right]} = \frac{-\frac{3}{16}}{-\frac{14}{56}} = \frac{3}{4}$

4a. $\Xi y = \frac{\Xi}{\varsigma s}\left(\frac{3}{4}x + 2\right) \longrightarrow \oplus 1 = \left(\frac{3}{4}x + 2\right)u \longrightarrow u = -\left(\frac{1}{\frac{3}{4}x + 2}\right)$

$$u = -\left(\frac{1}{-\frac{3}{4s} + 2}\right) = -\left(\frac{1}{-\frac{3}{4s} + \frac{8s}{4s}}\right) = -\left(\frac{1}{\frac{-3+8s}{4s}}\right) = -\left(\frac{4s}{-3+8s}\right)$$

$$u = -\left(\frac{4s}{-3+8s}\right)$$

4b. $u\left(-\frac{1}{2}\right) = -\left(\frac{4\left(-\frac{1}{2}\right)}{-3+8\left(-\frac{1}{2}\right)}\right) = -\left(\frac{-2}{-3-4}\right) = -\frac{2}{7}$

$$u\left(\frac{3}{8}\right) = -\left(\frac{4\left(\frac{3}{8}\right)}{-3+8\left(\frac{3}{8}\right)}\right) = -\left(\frac{\frac{12}{8}}{-3+\frac{24}{8}}\right) = -\left(\frac{\frac{12}{8}}{0}\right) = -\left(\frac{12}{8} \cdot \frac{1}{0}\right) = -\frac{12}{0} \equiv -\frac{1}{0} =$$

$$-\eta_0 \equiv \frac{1}{0}$$

Chapter 2

Review of Algebraic Structures

2.a—Equations, Relations and Functions:

2.a.1—How functions and relations are linked

Any Function in mathematics is a Relation. A relation is a general term used to mean a way of describing how numbers and variables in an equation are related. Relations do not have to be equations but commonly are. Subspace mathematics is no different other than subspace numbers and infinitals are included in the possible values of numbers.

Examples of Relations:

Sets of Ordered Pairs: $\left\{(2\,,5)\,,\left(-1\,,\frac{1}{3}\right)\,,\left(\oplus 2\,,\frac{7}{0}\right)\,,\left(-\frac{1}{2}\,,\oplus 4\right)\right\}$

Expressions: $2+x$ $\quad x-\frac{x}{\oplus 2}$

Functions: $y=2+x$ $\quad u=-\left[\frac{s}{2s-1}\right]$

2.a.2—What defines a function:

Functions are unique as relations because every input will have exactly one matching output. Input values are part of the function's domain and output values are those within its range.

Functions: $f(x)=\frac{1}{4}x+5$ Domain and Range $\mathbb{R}$

$g(s)=-\left[\frac{4s}{-1+20s}\right]$ Domain and Range $\mathbb{R}$

Since there is only one output for every input this relation is a function. Therefore, we say the relation *f*, is a function of x, and the relation *g* is a function of *s*. These are then functions *f* of *x* and *g* of *s*.

2.a.3—Composite functions:

Composite functions have one function within another function such that $h(x)=f\big(g(x)\big)$. The function is solved by completing operations within the inner function first and working your way out.

$h(x)=\sqrt{2+x^2}$ $\qquad h(x)=f\big(g(x)\big)$ where $\mathrm{f}(x)=\sqrt{x}$ and $g(x)=2+x^2$

$h(s)=\frac{\left(\sqrt{s+3}\right)^2}{s}$ $\qquad h(s)=\frac{f\left(g(s)\right)}{k(s)}$ where $f(s)=s^2$, $g(s)=\sqrt{s+3}$ and $k(s)=s$

In each example start with the *g*-equation and evaluate for a given input. The output value from that equation is then used as the input for *f*-equation. The result is the value of *h* at the given input.

Be aware this can become very complicated. There is no reason to expect an imbedded equation will be composed of only two equations. There could be several.

2.a.4—Inverse functions

Inverse functions cancel each other out. There are two ways to understand the concept. A function $y = f(x)$ has an inverse function $x = f(y)$. If you have $y = x^2$ solving for *x* will provide the inverse equation. In this example the inverse of $y = x^2$ is $x = \sqrt{y}$. Note it is a convention to swap *x* and *y* in the inverse equation so that its still in terms of x as the input.

Applying the convention we have $f(x) = x^2$ and $g(x) = \sqrt{x}$. We know these are inverse of each other because chaining them leaves only the input value selected.

$$f\big(g(x)\big) = g\big(f(x)\big) = x$$

$$\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x$$

Another way of denoting an inverse function is with the negative exponent, where $f(x) = x^2$ and $f^{-1}(x) = g(x) = \sqrt{x}$. This is read *f-inverse of x*.

2.a.5—Parametric Equations and Rectangular Form:

There are situations where it is beneficial to represent both the input(s) and output values of an equation in terms of another variable; a parameter. This value is usually represented as *t* which makes sense as the real-world application is usually with the introduction of *time.* However for graphing purposes, on a Cartesian Plane, the two parametric equations must be resolved and combined into the rectangular form of the Cartesian Plane.

A Cartesian plane involving 2-directional coordinates can be in terms of *x* and *y, s* and *u* or any other pairings of axis. Examples of parametric equations involving the same are shown here. Note there are two sets of examples. The values for s and u are only shown as examples of parametric equation in this section, not the subspace values of the x and y in the other parametric example.

Parametric example in terms of x and y:

$$x = 2 - t \qquad\qquad y = t + 5$$

Parametric example in terms of s and u:

$$s = t - 1 \qquad\qquad u = -t + 3$$

The method of converting to rectangular coordinates is by solving the parameter against the value which is used as the input value in a rectangular equation. In the examples above this will be $t = -x + 2$ and $t = s + 1$. All else that's needed is to substitute the value of t into the respective y and u equations.

$$y = t + 5 \quad = \quad -x + 2 + 5 \quad = \quad 7 - x$$

$$u = -t + 3 \quad = \quad -(s + 1) + 3 \quad = \quad 2 - s$$

2.b—The Negative Radical:

The laws of exponents have some unique properties when dealing with subspace numbers. The logic however extends simply from laws of exponents of standard Algebra and Calculus. Before discussing exponents you'll need to understand what happens when a $\oplus$ sign is squared. That itself requires discussion of the negative radical.

In Algebra and Calculus there arise situations where a squared value must equal -1. Consider the sum of perfect squares. If we have a value defined as $a^2 + b^2$ how can we factor this? Unless you accept the square of the b value to be negative this is not possible as you will otherwise always have cross-term values in the answer. In Algebra and Calculus this value is defined as $i = \sqrt{-1}$.

$$a^2 - b^2 \quad = \quad (a + b)(a - b) \quad = \quad a^2 - ab + ab - b^2$$

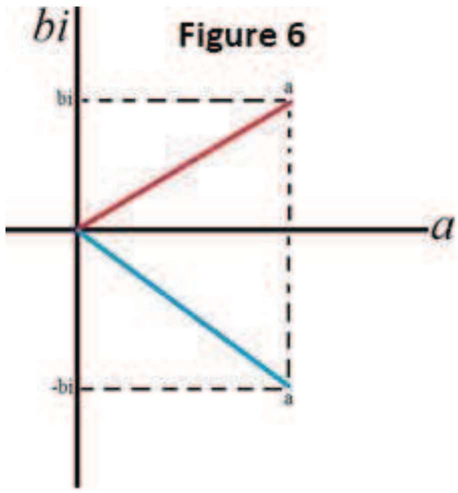
$$a^2 + b^2 \quad \neq \quad (a + b)(a + b) \quad = \quad a^2 + 2ab + b^2$$

$$a^2 + b^2 \quad \neq \quad (a - b)(a - b) \quad = \quad a^2 - 2ab + b^2$$

$$a^2 + b^2 \quad = \quad (a + bi)(a - bi) \quad = \quad a^2 + abi - abi - (bi^2) \quad = \quad a^2 + b^2$$

The value of i is identical to $\frac{0}{0}$. Treating it as the root of negative one, i has a number of its own properties. To get to this conclusion first examine the concept of the complex plane. Complex numbers have the form of $a \pm bi$. The a component is called the *real* component. This value represents a value in real space which you could count, like five coins, or two apples, etc. The bi value is called the *imaginary* component. This is not to say this value is not itself real. It has to be or values like $a^2 + b^2$ and trigonometric properties like $\sin^2 x + \cos^2 x = 1$ could not exist. It's imaginary because visualizing this number cannot be done with an example of something we could count.

Therefore complex numbers are given their own Cartesian style plane. Real values of a are placed on the horizontal axis and imaginary bi values are on the vertical axis. Figure 6 shows a typical layout of complex values on the complex plane.



The complex numbers have two parts. They are arranged in a way that each component is mirrored about the a-axis. This mirroring of these value is the crucial element in the squaring of a complex number, the event that ultimately results in a negative value after squaring. It is because complex numbers are both $a + bi$ and $a - bi$; they possess a positive and negative quality just like the subspace numbers introduced above.

In reality a can be positive or negative. For this example we will assume it is positive. We then label numbers in the upper quadrant $z = a + bi$ whilst those in the lower quadrant are $z^* = a - bi$. These are called complex conjugates and together represent the positive and negative qualities of complex numbers. To represent the squared value, z^2, both parts of a complex number must be represented. In doing so we find z^2 is always real and positive.

$$z^2 = z^* z = (a - bi)(a + bi) = a^2 + b^2$$

For this to be so, whatever the value of i is, it must equal -1 when squared. We cannot set i equal to either 1 or -1 without changing the outcome of the squaring of these complex numbers.

To see what's happening and to really understand where the -1 comes from let's first consider the nature of the complex plane. What's complex about the plane, is that the imaginary part exists along a direction that cannot be directly experienced. The bi axis represents what $\frac{0}{0}$ does when used to relate any and all real numbers to their subspaces. This bi axis is separate from all other axis and numbers, sitting outside of their countable range yet is capable to relating them to one another.

The bi axis represents multiples of $\frac{0}{0}$, the Ξ operator used to transform between values and their corresponding subspace values. We want to examine only the number i. In place of i we substitute $\frac{0}{0}$. Because $\frac{0}{0}$ is $\oplus 1$ we will examine how this develops from only one possible arrangement for a complex number; $a + bi$.

Note that 0 is unique. It sits between an infinite set of numbers to its positive and negative side. Although this is true of any number, 0 has an infinite number of positive numbers only to one side and an infinite number of negative values only to its other side. No other number possesses this specific property. Traditionally 0 is held to be neither positive nor negative. However these infinite amounts of positive and negative values to either side do not cancel out the value of 0 itself. This is a transition point between positive and negative and assigning either sign to 0 does not change its meaning. It is also shared by both sides of the number line as an origin point. Because of this it must be both positive and negative. Thus $\frac{0}{0}$ is really $\frac{+0}{+0} = \frac{-0}{-0} = 1$ and simultaneously $\frac{-0}{+0} = \frac{+0}{-0} = -1$.

Applying this to $a + bi$ with the above specifications we get:

For $a = 0$ and $b = 1$ $\quad a + bi \quad \longrightarrow \quad 0 + (1)i \quad \longrightarrow \quad i \quad$ Such that $i = \frac{0}{0} = \oplus 1$.

If you now acknowledge that a and b may be any value and substitute $i = \oplus 1$ we instead arrive at

$$a + b(\oplus 1) \quad = \quad \begin{cases} a + b \\ a - b \end{cases}$$

These two values are representative of the values which require a real component and an imaginary component. They still exist on the complex plane and is why we are able to simultaneously multiply the b magnitude by the positive and negative value of $\frac{0}{0}$. If we were not on the complex plane, a place mating a real axis to an axis of $\frac{0}{0}$ itself we would instead have been forced to resolve its value to its positive component at the point of origin since this is not itself a subspace transformation. We have simply resolved the given b magnitude into its positive and negative multiples of $\frac{0}{0}$ on the imaginary bi-axis. This shows that any such complex-plane value has two parts which simultaneously exist; complex numbers of the form:

$$z = a + bi$$

$$z^* = a - bi$$

In the case shown here the values of i are shown unresolved. The use of i in the expressions requires us to maintain the idea that this usage of $\frac{0}{0}$ exists as the root of -1. With $\frac{0}{0}$ being a value on its own bi-axis, one which simultaneously represents $+1$ and -1 then squaring it means multiplying both of its possible values on the complex plane, the positive and the negative component, together ($b \cdot -b = -(b^2)$) This representation depicts the actual act of squaring a complex number. See figure 7 at right:

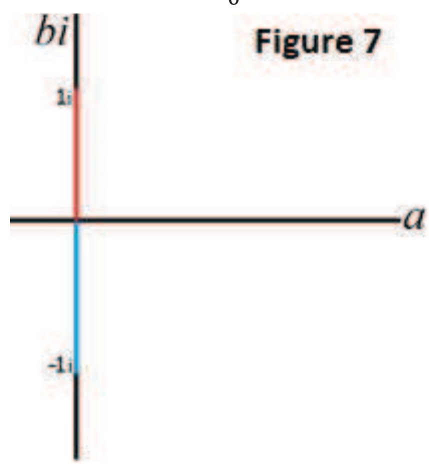


Figure 7

For $a = 0$ and $b = 1$

$$z = a + bi \quad \longrightarrow \quad 0 + 1(\oplus 1) \quad \longrightarrow \quad \begin{matrix} z = 1 \\ \\ z^* = -1 \end{matrix}$$

This property of $\frac{0}{0}$ exists because it is its own subspace. We see that the value of i is itself both positive and negative. Consider that $\left(\frac{0}{0}\right)^2 = \frac{0}{0}$. The value has not been changed by squaring and as was already shown $i = \frac{0}{0} = \oplus 1$ and that squaring it actually requires multiplying both its values together, represented as $1 \cdot -1 = -1$.

We can feasibly express any value on any axis as $a + bi$ by entering the given values of a, b and resolving i in a specific manner which will be covered momentarily. Important is holding the value of $\frac{0}{0}$ to be a root of -1 rather than a value which can be resolved via a subspace transformation when dealing with the idea of the complex plane. The $\oplus$ numbers are the roots of negative numbers. This is different from when a number like $\frac{0}{0}$ shows up on another axis as a result of inputs and outputs. In an equation like $y = \frac{1-\frac{x}{3}}{x-3}$ the result of $\frac{0}{0}$ comes from the equation itself when $x = 3$. This is called a *feedback* value, the result of *naught* values arising within the equation which must be resolved by another means. In the equation $y = \frac{1-\frac{x}{3}}{x-3}$ when $x = 3$, $y = -\frac{1}{3}$. This can be found by direct substitution and will be discussed later.

The root of a negative number is a multiple of $\frac{0}{0}$, bound to the bi-axis. Although this number is simultaneously positive and negative, it is a root of the number you intend to represent and bound to the $\frac{0}{0}$, bi-axis. To resolve the value as well as determine the value on a respective subspace of the equation it arises in requires you square the term. The application of this is shown below in a moment.

Values possessing this quality of the form $a + bi$ can have any value a and b but must represent both values of i. Thus we get the usual form for complex numbers.

$$z = a + bi$$

$$z^* = a - bi$$

Consider that $2i$ equals 2 and -2 on the $\frac{0}{0}$ axis; $\oplus$ 2. A value $\oplus$ 2 on a real space axis like the y-axis represents a numeric magnitude of length, with the $\oplus$ sign indicating the positive value of the magnitude exists on the given axis and the negative on the adjoining subspace. In other words a plus-and-minus number on a real space axis, which cannot express simultaneously positive and negative values, must be resolved via a transform against the bi axis. The plus and minus value on the real space axis is resolved via the transform with the $\frac{0}{0}$, Ξ operator:

$$y = \oplus 2$$

The magnitude is 2. The $\oplus$ component is the $\oplus$ 1 of the bi-axis. The value of $\oplus$ 2 on the y-axis here is the result of a negative root, $y = \sqrt{-4} = \oplus 2$.

Notes the any radical will produce and $\pm$ result. The information presented here assumes we are discussing only the + value of the root. In which case $y = \sqrt{-4}$ will produce $y = \oplus 2$ rather than $y = \pm(\oplus 2)$. In both instances you must resolve the plus-and-minus value as shown below.

Even though we've set y in this example to a constant, $\oplus$ 2, this constant sits in place of the input side of the equation. If we had $y = \sqrt{x}$ the value of $x = -4$ will generate $y = \oplus 2$ but its only one point on a line which is filled out by all values of x. Regardless whether we show $y = \oplus 2$ as the result of a function of x or simply a declared constant you have a real space axis which appears to have two values.

In fact this point is both $y = 2$ and simultaneously $y = -2$, but the points as you will see are not in the same planes. Assume $y = \oplus 2$ is originally declared in the XY Plane and is declared as a constant, not as a function of x. This was shown in chapter 1 in the section dealing with plotting subspace numbers. You see that $y = \oplus 2$ will resolve to $y = 2$ on the XY plane (*assuming this is the plane you are using when plotting this $\oplus$ value*). That resolved expression $y = 2$ is a line parallel to the x-axis at a height of $y = 2$. The x inputs aren't gone. Rather $y = 2$ for all values of x. The value of $y = -2$ lies then in SY-Plane.

A transformation could of course be done on the y equation to see what its corresponding u equation is. Whether you first resolve the value of y from the plus-and-minus value assigned to it or perform the transform first you will end up with the same values.

Lets first examine resolving the plus-and-minus value in $y = \oplus\, 2$. This will involve examining the $\oplus\, 2$ as if it were in the form of a complex number such that

$$\oplus\, 2 = \oplus\, b = bi = \begin{cases} 0 + b \\ 0 - b \end{cases} = \begin{cases} 0 + b(1) \\ 0 + b(-1) \end{cases}$$

2 is the magnitude of the values when represented on complex plane, of which the complex portion is $i = \oplus 1 = \frac{0}{0}$. The transform here against $\oplus 1$ of the complex plane i-axis. Since both the positive and negative value is on the y-axis we use the operator $\frac{\Xi}{\varsigma y}$ to obtain $\check{y}$ representing the negative half of the i-axis. The operator $\frac{\Xi}{\varsigma y}$ is used to obtain $\hat{y}$ representing the positive half of the i-axis. This provides the signs the values take on the y-axis. These values are each multiplied by the magnitude of the original i-multple. A transform is lastly used to show which value applies to which dimensional plane involving the y-axis. The magnitude itself does not require a transform.

$$\Xi\mathrm{y} = \frac{\Xi}{\varsigma y} \oplus 1 \quad \longrightarrow \quad y = \check{1} = -1$$

$$\Xi\mathrm{y} = -\frac{\Xi}{\varsigma y}\, 1 \quad \longrightarrow \quad y = \hat{1} = 1$$

$y = \hat{1}$ corresponds to the sign of the magnitude on the originating axis while $y = \check{1}$ is the sign on the of the magnitude on the adjoining subspace. This means given $y = \oplus\, 2$ on the XY-Plane, we know that y must equal simultaneously 2 and -2 and that this value is given to y by an assigned input value. In the case of the XY-Plane the equation y equals some value for all values of x after resolving $y = \oplus\, 2$. Thus the pairing between the real XY-Plane must be with the subspace of the input value of that plane in order to allow y to have both values. This is seen in an subspace transform against x; see <u>Same Axis Transformations</u> below. The positive value will apply to the given magnitude on y in the XY-Plane (*the plane chosen for this example where* $y = \oplus\, 2$

originates). Simultaneously the SY-Plane will hold the negative value of the given magnitude for y. For $y = \oplus\, 2$ we have $y = 2$ on the XY-Plane and $y = -2$ on the SY-Plane. The below section marked <u>Same Axis Transformations</u> explains why this necessitates the plus-and-minus value on y be defined by the adjoining subspace planes XY and SY for the example equation $y = \oplus\, 2$.

<u>Same Axis Transformations—</u>

When the complex plane bi-axis is respresented as multiples of $\frac{0}{0}$, it is used for relating two values together. Complex numbers already represent this in that they have two parts; $a + bi$ and $a - bi$. The Ξ operator uses $\frac{0}{0}$ to resolve infinities with subspace axis. Additionally roots of negative numbers result in $\oplus$ numbes which are i-multiples, or rather multiples of $\frac{0}{0}$. Being a multiple of $\frac{0}{0}$ any $\oplus$ number occurring on an axis must represent the positive and negative pairs of the magnitude of the number shown in a fashion representative of the positive and negative pairs of the bi-axis. Using the XY plane as an example, a value of $y = \oplus\, n$ will have two values simultaneously present: $y = \hat{n} = n$ and $y = \check{n} = -n$. Because the $\oplus$ value is assigned to the y-axis the Ξ operator is used against the other axis of the dimension to obtain the correct assignment of the positive and negative halves of the value within the correct adjoining planes by locating the proper subspace axis.

On the XY-Plane the x-axis will take the role of the transformation to determine *which dimensional plane the transform against the plus-and-minus number applies to.* The value held by the plus-and minus number assigned to the given axis, the y-axis in this example, is obtained by a separate transformation using a same-axis. This is because both the positive and negative value lies on the y-axis and this step is used only to resolve the correct sign within the plane.

Given: $y = \oplus\, 2$ on the XY-Plane

<u>Step 1:</u>
Set the magnitude of the plus-and-minus value aside as a variable. Perform a transformation against the x-axis to dertermine the adjoining subspace axis which pairs with y as a dimensional plane and will hold the *down* component of the plus and minus value. Also perform transformation against the plus-and-minus value, $\oplus\, 1$. This is the base of the plus-and-minus magnitude assigned to y using a same-axis transformation to determine its sign in the subspace adjoining plane. This transformation resolves one half of the value which is representative of the bi-axis as multiples of $\frac{0}{0}$.

Magnitude: $M = 2$ $\Xi y = \frac{\Xi}{\varsigma y} \oplus 1$ $y \;=\; M[-1] \;=\; -2 \;=\; \check{2}$	$\Xi x = \frac{\Xi}{\varsigma s}\, x$ $s = -\frac{1}{x}$ Substituting $x = -\frac{1}{s}$ in the above equation we get $s = s$ Will be using the SY-Plane for this value
$y = -2$ on the SY-Plane	

Step 2:
Repeat the transformation again on both sides after assigning the magnitude to a variable and setting it aside. The same-axis transformation will provide the sign of the magnitude which corresponds to the *up* component on the original plus-and-minus value and resolve the remaining half of the value which is representative of the bi-axis as multiples of $\frac{0}{0}$. The transform against the subspace axis, s in this example, will provide the axis which pairs with that value on the dimension plane.

Magnitude: $M = 2$ $\Xi y = -\frac{\Xi}{\varsigma y}\ 1$ $y \ = \ M[1] \ = \ 2 \ = \ \hat{2}$	$\Xi s = \frac{\Xi}{\varsigma x}\ s$ $x = -\frac{1}{s}$ Substituting $s = -\frac{1}{x}$ in the above equation we get $x = x$ Will be using the XY-Plane for this value
$y = 2$ on the XY-Plane	

In instances of higher dimensionality, such as XYZ plane, and instant of a single axis being assigned a value of a plus-and-minus number, for example $x = \oplus 3$, will have additional instances of the down value being shared with adjoining subspace dimensional planes.

Now consider this same situation but resulting from a function instead of a constant, using the equation $y = \sqrt{x}$. y will equal the $\pm$ root of x. For the positive values of x traditional mathematics has no issues. The negative values will be resolved as described. If $a = \sqrt{x}$ then for negative values of x, we have $y = \pm\sqrt{x} = \pm(\oplus\ a)$. Ignoring the $\pm$ we will focus on only the positive values of the root. This leaves us with $y = \sqrt{x} = (\oplus\ a)$ for negative arguments of x. Note there are corresponding subspace equations. We will consider all of these in the following example.

XY-Plane:	$y = \sqrt{x}$	SY-Plane:	$y = \sqrt{-\frac{1}{s}}$
XU-Plane:	$u = -\frac{1}{\sqrt{x}}$	SU-Plane:	$u = -\frac{1}{\sqrt{-\frac{1}{s}}}$

For the value $x = -9$ we receive $\oplus$ values in both equations using x as an input. Whether you chose to resolve the $\oplus$ number before or after the transformations is irrelevant as you will still receive the same values for the various equations.

Resolved before transform	Resolved after transform
$y = \sqrt{x}$ For $x = -9$ $y = \sqrt{-9} = \oplus 3$ Resolving provides— XY: $y = 3$ SY: $y = -3$ Transform to UX: $u = -\frac{1}{\sqrt{x}}$ For $x = -9$ $u = -\frac{1}{\sqrt{-9}} = -\frac{1}{\oplus 3}$ Resolving provides— XU: $u = -\frac{1}{3}$ SU: $u = \frac{1}{3}$ Using the resolved value we have the following direct transformations: XY-Plane: $y = \sqrt{x} = \sqrt{-9} = \oplus 3 = 3$ $\Xi y = \frac{\Xi}{\varsigma u}3 \longrightarrow$ XU-Plane: $u = -\frac{1}{3}$ SY-Plane: $y = \sqrt{-\frac{1}{s}} = \sqrt{-\frac{1}{\frac{1}{9}}} = \oplus 3 = -3$ $\Xi y = -\frac{\Xi}{\varsigma u}3 \longrightarrow$ SU-Plane : $u = \frac{1}{3}$	$y = \sqrt{x}$ For $x = -9$ $y = \sqrt{-9} = \oplus 3$ Resolving provides— XY: $y = 3$ SY: $y = -3$ The SY equation is $y = \sqrt{-\frac{1}{s}}$. The original $\oplus 3$ value is generated when $x = -9$. This is equivalent to the point when $s = \frac{1}{9}$. SY-Plane: $y = \sqrt{-\frac{1}{\frac{1}{9}}} = \oplus 3$ Resolving this value is done just like when you started in the XY plane except we know this is the subspace of adjoining plane of the place where the $\oplus$ value originated so you solve for the *up* value on the first transform and the *down* value on eh second. Magnitude $= 3$ SY-Plane $\Xi y = \frac{\Xi}{\varsigma y}\oplus 1 \longrightarrow$ XY-Plane $y = M\hat{1} = \hat{3} = 3$ XY-Plane $\Xi y = \frac{\Xi}{\varsigma y}1 \longrightarrow$ SY-Plane $y = M\check{1} = \check{3} = -3$ These values match those in the left column and these above. Transform to XU: $u = -\frac{1}{\sqrt{-9}} = -\frac{1}{\oplus 3}$ Resolving provides— XU: $u = -\frac{1}{3}$ SU: $u = \frac{1}{3}$ The SU equation is $u = -\frac{1}{\sqrt{-\frac{1}{s}}}$. The original $\oplus 3$ value occurs on UX at $x = -9$. Though the original equation is in fact the $y = f(x)$ equation of the XY plane the transform to the co-adjoining UX plane equation preserves the output of a $\oplus 3$ from the same x-axis input. The corresponding s-axis value to $x = -9$ is at $s = \frac{1}{9}$.

	$u = -\frac{1}{\sqrt{-\frac{1}{\frac{1}{9}}}} = -\frac{1}{\oplus 3}$ Resolving this value is done just like when you started in the XU plane except we know this is the subspace of co-adjoining plane of the place where the ⊕ value originated so you solve for the *up* value on the first transform and the *down* value on eh second. Magnitude $= 3$ SU-Plane $u = -\frac{1}{\oplus 3}$ Using *h* as a dummy axis to represent the lone value $\Xi\mathrm{h} = -\frac{\Xi}{\varsigma h}1 \longrightarrow \; M\hat{1} = \hat{3} = 3$ XU-Plane $u = -\frac{1}{\oplus 3} = -\frac{1}{3}$ Using *h* again to return the magnitude back to value on SU-Plane $\Xi\mathrm{h} = \frac{\Xi}{\varsigma h}\oplus 1 \longrightarrow \; M\check{1} = \check{3} = -3$ SU-Plane $u = -\frac{1}{\oplus 3} = \frac{1}{3}$ These values match those in the left column and these above.

The up and down values are necessary to keep track at this stage that these are in fact i-multiples. Note also that this was done by considering only the positive values of the root. In actually the radical still requires we consider the root as $\pm$. So the final answers would include this to be complete:

$y = \sqrt{x} = \sqrt{-9} = \oplus\, 3$	
XY-Plane $y = \sqrt{-9} = \pm\hat{3} = \pm 3$ XU-Plane:	SY-Plane $y = \sqrt{-\frac{1}{\frac{1}{9}}} = \pm\check{3} = \mp\check{3}$ SU-Plane:

$u = -\frac{1}{\sqrt{-9}} = -\frac{1}{\pm\check{3}} = \mp\frac{1}{3}$	$u = -\frac{1}{\sqrt{-\frac{1}{\frac{1}{9}}}} = -\frac{1}{\pm\hat{3}} = \pm\frac{1}{3}$

The $\mp$ was used to denote values which originate from resolved *down* components. The overall meaning is the same as $\pm$.

Consider below a twist on this same example in the XY-Plane with the value assigned to x as a constant. As mentioned above it does not matter whether you resolve the plus-and-minus value before or after making a transformation to a subspace equation.

XY-Plane equation	$x =\oplus 2$
Magnitude: $M = 2$ $\Xi x = \frac{\Xi}{\varsigma x} \oplus 1$ $x = M[-1] = -2 = \check{2}$	$\Xi y = \frac{\Xi}{\varsigma u}\, y$ $u = -\frac{1}{y}$ Substituting $y = -\frac{1}{u}$ in the above equation we get $u = u$ Will be using the XU-Plane for this value
Magnitude: $M = 2$ $\Xi x = -\frac{\Xi}{\varsigma x}\, 1$ $x = M[1] = 2 = \hat{2}$	$\Xi \mathrm{u} = \frac{\Xi}{\varsigma y}\, u$ $y = -\frac{1}{u}$ Substituting $u = -\frac{1}{y}$ in the above equation we get $y = y$ Will be using the XY-Plane for this value

Notice this not the same thing solving for the subspace of an equation which generates a positive-or-negative value during the transformation process.

$$x = 2 \qquad \Xi x = \frac{\Xi}{\varsigma s} 2 \quad \longrightarrow \quad s = -\frac{1}{2}$$

$$x = -2 \qquad \Xi x = -\frac{\Xi}{\varsigma s} 2 \quad \longrightarrow \quad s = \frac{1}{2}$$

The $\oplus$ sign for a number on a real space axis indicates the sign of the value is positive on the axis and plane (*dimension*) upon which the number occurs and negative on the adjoining subspace plane formed from the same axis which was originally assigned the plus-and-minus value, and the subspace of the original input axis.

Another way of arriving at the same value when resolving a $\oplus$ number on the axis which it originates is to use i directly. You must square i on the input side and yet not change the value of the equation. This is done by multiplying both sides of the equation by $\frac{0}{0} =\oplus 1 = i$. Recall that these numbers are multiplies of $\frac{0}{0}$. The squaring will occur only on the $\frac{0}{0} = i$ portion. The reaming $\oplus$ 1 is simply divided out applying another application of i.

$$x =\oplus 2 \quad \longrightarrow \quad x = 2i$$

$$xi = 2i^2 \quad \longrightarrow \quad x = 2i^3$$

$$x = 2$$

Believe it or not i divided by i is identical to i times i and additionally, that i^3 equals positive 1. This will be covered in depth in a moment with properties of i. This second method will allow direct resolution of a plus-and-minus value set to a given axis but ignores the adjoining subspace plane and assumes you are dealing with the originating location for the occurrence of the plus-and-minus value.

Consider the following example which shows another way of applying this same method:

$$y = \sqrt{x}$$

For the value of $x = -4$ we get $y =\oplus 2 = 2i$.

<table>
<tr><td colspan="2">

$$y = \sqrt{x} \text{ for } x = -4$$

$$y = \sqrt{-4} = \oplus 2$$

The two examples below show two ways of handling this second method of directly resolving any $\oplus$ value. The left column shows usage of i and the right column shows use of $\oplus$ numbers. In essence they are identical.

</td></tr>
<tr><td>

$$(i)y = 2i(i)$$

$$(i)y = 2i^2$$

$$y = 2\frac{i^2}{i}$$

$$y = 2i^3 = 2$$

</td><td>

$$(\oplus 1)y = \oplus 2(\oplus 1)$$

$$(\oplus 1)y = 2[(\oplus 1)(\oplus 1)]$$

$$y = 2\left(\frac{[(\oplus 1)(\oplus 1)]}{(\oplus 1)}\right)$$

$$y = 2$$

Because are using $\frac{0}{0}$ as $\oplus$ 1 on both sides of the equation this is essentially a transformation which stays on the same plane without making exchanges. It permits both signs of each instance of $\oplus$ 1 to interact directly.

</td></tr>
</table>

This will always provide the value for the subspace number on the axis which it originates. Despite this method ignoring the adjoining subspace it is nonetheless present. A subspace number appearing on a given axis represents two values on that axis. It is a given magnitude, multiplied by $\frac{0}{0} = i = 1$ in the originating dimension and $\frac{0}{0} = i = -1$ in the implied adjoining subspace dimension. This makes the $\oplus$ number on any given axis an i-multiple. The $\oplus$ number is bound to the axis on which it occurs, resolved by a transforms as defined above. This will provide the sign of the magnitude of the original value on originating and adjoined dimensional planes.

In summary for $y = \sqrt{-4} = 2i$ the i as $\oplus$ 1 originates on the y-axis; y is simultaneously 2 and -2. The value of 2 is the magnitude of the subspace number which is associated with y. The input axis on the originating plane takes a transform to define the axis which form the adjoin subspace dimensional plane on which the *down* value is located . The transform against base of the plus-and-minus number, $\oplus$ 1, multiplied by the magnitude of the number provides the sign of the value on the given axis it the adjoining subspace dimensional plane. A repeat application of these transforms will proved the sign of the magnitude of the number as the originating real space dimensional plane of its occurrence.

From this we may now explore the laws of exponents for all numbers; positive, negative and subspace. A number as some value bi, can be expressed as $\oplus\, b$, a b magnitude multiple of 1 and -1 on the $\frac{0}{0}$ Ξ-axis. It may be resolved by way of a transform against i alone as the respective originating input axis or by multiplying both sides of the equation by i. Even in the form $z^*z = (a-bi)(a+bi) = a^2 + b^2$ the i multiple magnitude resolves to the positive component at location of occurrence and to its negative value on the paired subspace.

2.b.1—Theorem 1: The Negative Radical

$i = \sqrt{-1}$

Given: $\frac{0}{0} = \frac{\pm 0}{\pm 0} = \oplus 1$ with a = 0 and b = 1

$a + bi$ becomes $0 + 1(\oplus\, 1)$

Such that both values of i (1 and -1)are represented on the same axis as

$$z = 1 \qquad \text{and} \qquad z^* = -1$$

The square of such a number must incorporate both values such that:

$$z^2 = z^*z = (-1)(1) = -1 = i^2$$

Thereby we conclude that:

$$\frac{0}{0} = \oplus 1 = \begin{cases} z = 1 \\ z^* = -1 \end{cases}$$

Where: $\left(\frac{0}{0}\right)^2 = z^2 = z^*z = [-1][+1] = -1$

If $\frac{0}{0} = \oplus 1 = \begin{cases} z = 1 \\ z^* = -1 \end{cases}$, and $(\oplus 1)^2 = z^*z = i^2 = -1$ then $i = \frac{0}{0}$.

$$\sqrt{-1} = \frac{0}{0} = i = \oplus 1$$

$$\left(\frac{0}{0}\right)^2 = \sqrt{\frac{0}{0}} = i^2 = \sqrt{i} = -1$$

2.c—Laws of Exponents:

2.c.1—Squares and Roots:

Squares and roots are inverse functions. A square is a value multiplied by itself whilst a root provides the value which when squared returns the original input. For a number found on standard number lines will provide a positive, real number whenever squared.

Positive number squared:	$n \cdot n = n^2$	$2^2 = 4$
Negative number squared:	$-n \cdot -n = (-n)^2$	$(-2)^2 = 4$

The squaring of subspace numbers is different. For the reasons above in section 2b squared subspace values will result in negative value.

$$(\oplus n)^2 = [n^2][(\oplus 1)^2] = [n^2][(1) \cdot (-1)] = -(n^2)$$

$$(\oplus 2)^2 = [2^2][(\oplus 1)^2] = [2^2][(1) \cdot (-1)] = -4$$

This can also be defined as the positive magnitude of $\oplus 2$ *on the originating axis multiplied by the negative magnitude of* $\oplus 2$ *on the corresponding subspace plane.*

$$(\oplus 2)^2 = (2)(-2) = -4$$

For numbers on standard number lines the root of a number can have two possible answers. One answer is positive whilst the other is negative. Regardless the squaring of either value will result in the original value. Roots for these values can only be taken from positive numbers because squaring of positive and negative numbers from standard number lines both result in a positive value.

Root of a positive number: $\sqrt{n^2} = \pm n$ $\sqrt{4} = \begin{cases} 2 \\ -2 \end{cases}$

Subspace numbers are the result of roots of negative numbers from any number line. They are resolved via usage of i or the transform upon i.

Root of a negative number: $y = \sqrt{-4} = \oplus\, 2 = \begin{cases} 2i \\ y = 2 \;\; XY\ plane \\ y = -2 \;\; SY\ plane \end{cases}$

The root of a subspace number. Subspace numbers are themselves positive ordered roots of negative numbers. So a square root of a plus-and-minus number cannot be resolved to its positive and negative components before taking the root. Instead use the identity that a plus-and-minus number $\oplus\, n \equiv ni$. The solution then is the root of the magnitude n multiplied by the root of i which is identical to its square. See below:

Root of a subspace number: $y = \sqrt{\oplus\, 4}$

The subspace number is inside a radical so the value which will be assigned to the output, y, will be resolved by taking the root of the plus-and-minus value as an i-multiple directly. Assuming the value here to be an x input, as $x = \oplus\, 4 = 4i$. The root will equal the root of the magnitude of the plus-and-minus number multiplied by the root of i to the same power.

The section dealing with the negative radical will show that roots of i are identical to powers of i of to the same power as the order of the root. This is resolved as the root of the magnitude of the number multiplied i raised to the same power as the root.

$$y = \sqrt{\oplus\, 4} = 2 \cdot i^2 = -2$$

Note that roots always take a positive and negative value. So this final answer is $y = \mp 2$.

Consider further an example equation $y = 2 + \sqrt{x}$. The negative values of x will provide a $\oplus$ value which is then added to 2. The negative root occurs on the x-axis along with the positive 2 value added to it to obtain the output value which is ultimately set to y for each input. So the positive input will be applied to the x-axis and the negative component to the x-axis inputs on the co-adjoining subspace plane. If we assign $n = \sqrt{x}$ then for negative values of x in the example equation $y = 2 + \sqrt{x}$ we obtain $y = 2 \pm (\oplus\, n)$. For simplicity we assume we are only dealing with the positive component of the root which then provides $y = 2 + (\oplus\, n)$.

Both the *up* and *down* component must both be represented. The resolved components on must both exist as a change in x on the x-axis. The positive *up* component will be used on the XY-Plane. The negative *down* component will be used on the co-adjoining subspace XU-Plane. Note it is a convention in many graphing utilities will show on the positive component of roots.

For $x = -4$ in $y = 2 + \sqrt{x}$ we have the following:

<u>Using only the Positive value from the root:</u>

$$y = 2 + \sqrt{x} \quad \longrightarrow \quad 2 + \sqrt{-4} \quad = \quad 2 + (\oplus\, 2) \quad = \quad 2 + \left(\hat{2}\right) \quad = \quad 4$$

<u>Using the plus or minus value from the root:</u>

$$y = 2 + \sqrt{x} \quad \longrightarrow \quad 2 + \sqrt{-4} \quad = \quad 2 + \left(\pm(\oplus\, 2)\right) \quad = \quad 2 + \left(\pm\left(\hat{2}\right)\right) \quad = \quad \begin{cases} 2 + 2 = 4 \\ 2 - 2 = 0 \end{cases}$$

In the co-adjoining subspace XU-Plane the equation is $u = -\frac{1}{2+\sqrt{x}}$. When $x = -4$ it again creates a plus-and-minus number. In this instance, on the co-adjoining plane, it will take the negative *down* component. However the plus or minus from the root still applies.

<u>Using only the Positive value from the root:</u>

$$u = -\frac{1}{2+\sqrt{x}} \quad \longrightarrow \quad u = -\frac{1}{2+\sqrt{-4}} = -\frac{1}{2+(\oplus 2)} = -\frac{1}{2+(\check{2})} = -\frac{1}{2+(\check{2})} = -\frac{1}{0} = -\eta_0 \doteq 0$$

Using the plus or minus value from the root:

$$u = -\frac{1}{2+\sqrt{x}} \quad \longrightarrow \quad u = -\frac{1}{2+\sqrt{-4}} \;=\; -\frac{1}{2+(\pm(\oplus 2))} \;=\; -\frac{1}{2+\left(\pm(\check{2})\right)} \;=\; \begin{cases} -\frac{1}{2-2} \doteq 0 \\ -\frac{1}{2+2} = -\frac{1}{4} \end{cases}$$

Similar situations occur in the adjoining SY Subspace Plain and its co-adjoining US Subspace Plane. The equation for SY-Plane is $y = 2 + \sqrt{-\frac{1}{s}}$. In this equation negative roots occur when s is positive. For the instance in which $x = -4$ the corresponding value on the s-axis is $s = \frac{1}{4}$.

Using only the Positive value from the root:

$$y = 2 + \sqrt{-\frac{1}{s}} \quad \longrightarrow \quad 2 + \sqrt{-\frac{1}{\frac{1}{4}}} \;=\; 2 + \oplus\, 2 \;=\; 2 + (\hat{2}) \;=\; 4$$

Using the plus or minus value from the root:

$$y = 2 + \sqrt{-\frac{1}{s}} \quad \longrightarrow \quad 2 + \sqrt{-\frac{1}{\frac{1}{4}}} \;=\; 2 + (\pm(\oplus\, 2)) \;=\; 2 + \left(\pm(\hat{2})\right) \;=\; \begin{cases} 2 + 2 = 4 \\ 2 - 2 = 0 \end{cases}$$

The equation for the co-adjoining US Subspace Plane is $u = -\frac{1}{2+\sqrt{-\frac{1}{s}}}$. Again using the $x = -4$ correspondance for the s-axis, $s = \frac{1}{4}$, we get the following:

Using only the Positive value from the root:

$$u = -\frac{1}{2+\sqrt{-\frac{1}{s}}} \quad \longrightarrow \quad = -\frac{1}{2+\sqrt{-\frac{1}{\frac{1}{4}}}} \;=\; -\frac{1}{2+\oplus 2} \;=\; -\frac{1}{2+(\check{2})} \;=\; -\frac{1}{0} \;=\; -\eta_0 \doteq 0$$

Using the plus or minus value from the root:

$$u = -\frac{1}{2+\sqrt{-\frac{1}{s}}} \quad \longrightarrow \quad = -\frac{1}{2+\sqrt{-\frac{1}{\frac{1}{4}}}} \;=\; -\frac{1}{2+(\pm(\oplus\, 2))} \;=\; -\frac{1}{2+\left(\pm(\check{2})\right)} \;=\; \begin{cases} -\frac{1}{2+(\check{2})} \doteq 0 \\ -\frac{1}{2-(\check{2})} = -\frac{1}{4} \end{cases}$$

2.c.2—Exponents and Roots of Higher Powers:

The higher powers and roots of numbers on the standard number lines are straight forward. The powers represent repeated instances of the given value multiplied by itself and work for positive numbers as well as negative numbers.

Powers of Positive numbers: *Where m is even and n is odd—*	$x^m = x^m$	$x^n = x^n$
	$2^2 = 4$	$2^3 = 8$
Powers of Negative number: *Where m is even and n is odd—*	$-2^2 = 4$	$-2^3 = -8$

The higher powers of subspace numbers will work in a similar fashion. A number bearing the $\oplus$ represents a magnitude multiple of $\frac{0}{0}$ or i. The value of the power is obtained by raising the magnitude of the subspace number to the power indicated, and the multiplying it by i raised to the same power.

Consider the following three values:

$$(\oplus 2)^1 \qquad (\oplus 2)^2 \qquad (\oplus 2)^3$$

Each of these three values are synonymous with multiples of the magnitude of $\frac{0}{0}$ or i. They therefore can be written as:

$$(2i)^1 \qquad (2i)^2 \qquad (2i)^3$$

The value of $(2i)^2$ comes right out of traditional algebra and calculus as -4. The odd powers of i don't have a place in these traditional mathematics. Traditionally $(2i)^3$ would be seen as $-4i$. The solution is to note that $i = \frac{0}{0} = \oplus 1$. Thus the number $\oplus 2$ is twice the magnitude of $\frac{0}{0}$.

A transform is conducted against the value of the i component, the $\oplus$ 1, representative of $\frac{0}{0}$, *bi*-axis. A dummy axis is used to determine the sign values will take at the instance of occurrence and the corresponding subspace. With the dummy axis this can be done even with a constant and no actual indication of which axis is being used.

Conversion to dummy-subspace axis: $\Xi h = \frac{\Xi}{\varsigma h} \oplus 1 \quad \longrightarrow \quad \check{h} = -1$

Conversion to real space dummy axis: $\Xi \check{h} = -\frac{\Xi}{\varsigma h} 1 \quad \longrightarrow \quad \hat{h} = 1$

From this the value of i on the originating axis is 1 and will be -1 on the corresponding subspace. For $(\oplus 2)^1$ we evaluate this as $(\oplus 2)^1 = \oplus 2 = 2 \cdot (\oplus 1) = 2$ on the originating axis and -2 on the corresponding subspace.

The squared value, $(\oplus 2)^2$ will equal -4. The reasons for why the squaring of the $\oplus$ results in a negative value were given above in section 2.b.

The $(2i)^3$ will equal positive 8. The reasoning begins with separating out the magnitude component from the value of i. Thus $(2i)^3$ is a magnitude of 2^3 times that of i^3. Begin by examining for the i component it is identical to saying:

$$i^3 = \left[\frac{0}{0} \cdot \frac{0}{0} \cdot \frac{0}{0}\right]$$

The first two terms combine through multiplication squaring their value. To identify what these result in on the real number line axis we resolve them first to $\oplus 1$. So the squaring, the multiplication of the first two instances of is the same as saying $\oplus 1 \cdot \oplus 1$. It results in a negative value but leaves behind $\frac{0}{0} \cdot \frac{0}{0}$, meaning it is being essentially squared a second time in completing the cube of i.

$$i^3 = \left[\frac{0}{0} \cdot \frac{0}{0} \cdot \frac{0}{0}\right] = -\left[\left[\frac{0}{0}\right] \cdot \frac{0}{0}\right] = --1 = 1$$

The two negative signs will cancel leaving a value of positive 1 for the cubed i component. Finally the magnitude of the term is multiplied by this value. Thus:

If $x = (2i)^3$ $\qquad x = 8$

If $x = (2i)^4$ $\qquad x = -16$

Thus for any subspace number where m is even and n is odd:

$$(\oplus x)^m = -(x^m) \qquad\qquad (\oplus x)^n = x^n$$

This can be better understood with the following synopsis using y values of the XY-Plane and its co-adjoining XU-Plane Subspace. The magnitude in each instant here below is 1 as we are only raising i to various powers. As is shown below, because each instance of i exists as two halves across two adjoined planes, each successive power of i is but another instance of these two halves multiplied by each other.

i	The square root of -1	$\left\{\frac{0}{0}, \oplus 1\right.$	$y = \sqrt{-1}$ $y_{xy} = 1$ $\qquad y_{xu} = -1$
i^2	The square of i	$\left\{\left(\frac{0}{0}\right)^2, (\oplus 1)^2\right.$	$y^2 = i^2 = y_{\mathrm{xy}} \cdot y_{xu} = -1$

Because $\left(\frac{0}{0}\right)^2 = \frac{0}{0}$ from here forward every higher additional application of multiplication by $\frac{0}{0}$ is identical to squaring $\frac{0}{0}$. The power of 1 represents the existence of the separate values with opposite signs in space and a respective subspace. Squaring raises them at power of 2.

To further illustrate this consider the differences and similarities between an equation like $y = a + \sqrt{b^2}$ for $a = 0$ and $b^2 < 0$ and the actual complex plane numbers, $a + bi$ and $a - bi$.

If we hold $a = 0$ we are left with: $y = \pm\sqrt{b^2}$ $bi = \hat{b}$ and $-bi = \check{b}$

Should we use $b^2 = -4$ (*and focus on the positive result of the radical*) for $y = \sqrt{b^2} = \sqrt{-4} = \oplus 2$. For reasons explained earlier this is resolved to $y = 2$ on the XY-Plane, and $y = -2$ on the XU-Plane. The squaring of this number can mean different things depending on what starting information you have. Had you not known you started in the XY-Plane, $y = 2$ is actually $y = \hat{2}$ you would attribute $y^2 = 4$. However, again for reason already explained, $y^2 = \hat{2}^2 = -4$. From the perspective of the XY-Plane $y = \hat{2}$ is only half of a number and squaring it requires we square its magnitude and the value of $i = \frac{0}{0} = \oplus 1$, a value which indicates not only where the other half of the number is but also that squaring involves both of these values' signs despite their being on planes which have no direct contact with each other. We can write this several ways:

y is the squared magnitude X squared i $y^2 = \hat{2}^2 = 2i^2 = 2^2 \cdot i^2 - 4$
Its equivalent as a complex number: $\left(0 + \hat{2}\right)\left(0 + \check{2}\right) = -4$

This second example may not look like a complex number format but it is. The complex plane bi-axis is complex because it is the mashing together of a positive portion of an axis from one plane and the negative portion from its adjoining subspace. Resolving the values we have:

$$\left(0 + \hat{2}\right)\left(0 + \check{2}\right) = -4$$
$$(0 + 2)(0 - 2) = -4$$

This second set looks much closer to $(a + bi)(a - bi)$ for $a = 0$ and $b = 2$. If we use the bi-axis itself to repreent these numbers its positive half represent the $y = \hat{n}$ where n is the positive component of an i-multiple. Likewise the bottom, negative half of the bi-axis will represent the $y = \check{n}$ component of the same i-multiple in the adjoining subspace. The difference here is we have compressed the values which are in those separate planes into a single axis. In the instance of the bi-axis the value of i is not resolved; rather it is always a part of the axis even though it still implies the idea of simultaneous positive and negative values. It is kept included without resolution to maintain the fact that this bi-axis is hybrid across two planes which adjoin each other through subspace.

This is the point which earlier, we set $a = 0$ and $b = 1$ to find that whatever i is. It had to allow $(0 + i)(0 - i) = i^2 = -1$, meaing that i itself had to be 1 and -1 at the same time if were are to hold true that $-1 = i^2 = i \cdot i$. This value was shown to be $\frac{0}{0} = i = \oplus 1$. On the complex plane the i component is part of the axis, indicated as just mentioned, the axis is a hybrid, composed of portions from two planes. Yet values marked on it must follow the pattern of complex numbers for which this plane was originally suited. The i-mutiples positive and negative components are simultaneously plotted. Using again $y = \sqrt{b^2} = \sqrt{-4} = \oplus 2$ as an example we plot this out as:

XY-Plane: $y = \oplus 2 = 2i = \hat{2} = 2$

SY-Plane: $y = \oplus 2 = 2i = \check{2} = -2$

<u>aBi-Plane:</u>

XY-Plane	$y = b = \oplus 2$	Such that for $a = 0$	$(a + bi)$ $(0 + \oplus 2)$ $(0 + 2i)$
SY-Plane	$y = b = \oplus 2$	Such that for $a = 0$	$(a - bi)$ $(0 - \oplus 2)$ $(0 - 2i)$

The values in the XY-Plane and SY-Plane above show resolved values. Those on the complex plane are not resolved and remain in the from $a \pm bi$. The signs of the $\oplus 2$ when used as its equivalent $2i$ doesn't show the dual signs which exist on the resolved XY and SY planes of the example. This is because whether using $\oplus 2$ or $2i$ the value remains un-resolved. The plus and minus signs shown in the complex plane components are those of the complex plane itself.

Note a major difference. On the bi-axis version of this we get 4, where as the squaring of either $y = \hat{2}$ or $y = \check{2}$ will result in -4. Squaring is the idea of Area, a base times a height of a given region whose base and height are of identical length. The length of the base and height in $y^2 = \hat{2}^2$ gives an area representative of the resolved value of $y = \oplus 2$ in the XY Plane, which when squared, provides an area based upon a value which is really half of a number. The value of -4 is the value which is synonymous with an area whose value is seen from the perspective of the XY-Plane only even though it extends into the SY-Plane, a part of the space which though present cannot be directly interacted with from the perspective of the XY-Plane.. Hence it's a negative area as it in a sense takes something away from the expression. The same thing comes from $y^2 = \check{2}^2 = -4$ in the SY-Plane. If we say instead $y = (0 + 2i)(0 - 2i) = 4$ we are again squaring a number which could feasibly been seen as an area. Yet from this perspective we are providing the value which represents the idea of area shared between the two adjoined subspace planes. This means seeing both the $\hat{2}$ and $\check{2}$ component of $\oplus 2$ as a single number on both adjoining subspace planes as if they were a single entity; the true nature of such a number not just the way we see from the space we can directly experience.

This value is positive, and in the sense that the squaring represents an area, a real area which contributes to rather than removes from the calculation. This is still present when dealing with values of $a \pm bi$ for $a \neq 0$. It can be a little more difficult to see and you will again derive different answers depending upon not only what starting information you have available but also on what it is you intend to express in squaring the number.

To see this consider the complex plane values: $3 \pm 5i$

As a complex number squaring will provide: $(3+5i)(3-5i) = 9 + 25 = 34$

Yet if we assume this represents a number on the XY-Plane it resolves as: $y = 3 + (\oplus 5)$
Resolved we are left with: XY-Plane: $y = 3 + \hat{5} = 3 + 5 = 8$
XU-Plane: $u = 3 + \check{5} = 3 - 5 = -2$

If we had no idea this value originated as a sum which included a resolved i-multiple we could choose to square either on its own place and obtain an answer to which we are all accustomed to from traditional mathematics

Squared value on XY-Plane only: $y^2 = 8^2 = 64$
Squared value on XU-Plane only: $u^2 = -2^2 = 4$

If we did have the ability to keep track of the i-multiples we are left with the following numbers:

XY-Plane: $y = 3 + \hat{5} \; = \; 3 + 5$
XUY-Plane: $u = 3 + \check{5} \; = \; 3 - 5$

In either instant the whole portion of the right side of the equation must be treated as a single number. This means you must use the complete value of the i-multiple. It is tempting to think that squaring the term would be done as follows:

XY-Plane: $y^2 = \left(3 + \hat{5}\right)\left(3 + \hat{5}\right) = 9 + 15 + 15 - 25 = 14$
XU-Plane: $u^2 = \left(3 + \check{5}\right)\left(3 + \check{5}\right) = 9 + 15 + 15 - 25 = 14$

These values are incorrect. Its only when they are combined into the value of the complex plane in which we can determine the value of square from the perspective of both planes together. If ware squaring y as used in this example we are squaring an expression which contains a resolved i-multiple. That expression in this example has its paired component in the co-adjoining subspace XU-Plane. It is that expression which must multiply it. The use of a single dot over top of a number is used to denote it is a fully resolved term. When multiplying the resolved terms with a real number they multiply as expected in traditional mathematics. Yet the right hand number is actually an i-multiple and will provide a negative number just as on the complex plane.

If $y = 3 + \hat{5}$ in the XY-Plane the square of the $\hat{5}$ will necessitate multiplication by its negative counterpart which lies, in this example, in the co-adjoining XU-Plane.

Square in Real Space	Square on Complex Plane
$y = 3 + \dot{5}$ $y^2 = \left(3 + \dot{5}\right)^2$ $y^2 = y \cdot u = \left(3 + \dot{5}\right)\left(3 - \dot{5}\right)$	$y^2 = (a + bi)(a - bi)$ $y^2 = \left(3 + \hat{5}\right)\left(3 - \check{5}\right)$ $y^2 = (3 + 5i)(3 - 5i)$ $y^2 = 9 + 25 = 34$

$y^2 = 9 - 15 + 15 + [\hat{5} \cdot \check{5}]$ $y^2 = 34$	

Power three and higher repeat this and continue to multiply additional negative signs. Observe below in the cube the is a negative sign added from the first application of $\frac{0}{0} \cdot \frac{0}{0}$. The cube of i is:

$$i^3 = \frac{0}{0} \cdot \frac{0}{0} \cdot \frac{0}{0} \quad \longrightarrow \quad i^3 = -\left(\frac{0}{0} \cdot \frac{0}{0}\right) = -(i^2) \quad \longrightarrow \quad i^3 = -(-1) \quad \longrightarrow \quad i^3 = 1$$

i^3 The cube of i $\left\{ -\left(\frac{0}{0}\right)^2, -(\oplus 1)^2, 1 \right.$

$y_{xy} = 1$ $y_{sy} = -1$ $y^3 = i^3 = -(y_{\text{xy}} \cdot y_{\text{sy}}) = -(-1) = 1$

The trend is shown continued here:

i^4 Power of 4 $\left\{ --\left(\frac{0}{0}\right)^2, --(\oplus 1)^2 \right.$ = $--(-1) = -1$

i^5 Power of 5 $\left\{ ---\left(\frac{0}{0}\right)^2, ---(\oplus 1)^2 \right.$ $---(-1) = 1$

Etc.

The value of positive or negative one obtained from repetitions of the powers of i are applied to the magnitude of the original multiple, the magnitude of the original plus-and-minus number which was raised to a power. The resolution of a single value such as $\oplus 2$, or $2i$ was shown to be magnitude of 2 times 1 on the y-axis of the XY-Plane and magnitude of 2 times -1 on the y-axis of the SY-Plane (*or vice versa if the plus-and-minus number originated on the SY-Plane*). The square is the multiplication of these values with each other via the value of i on the complex plane which relates them to each other. Each repeat power applied to a value such as $(2i)^n$ or $(\oplus 2)^n$ where n is any non-zero number represents the coefficient (*in this example a 2*) raised to that power then multiplied by the sign which results from raising i to the same power.

Examples:

$y = 2i$ $i = \oplus 1 \longrightarrow$ XY-Plane $y = 2$ SY-Plane $y = -2$

$y = (2i)^2$ $y = 2^2[(-)^0(\oplus 1)^2] = -4$

$y = 4[(1)(-1)] = -4$

$y = 2(i^2)$ $y = 2[(-)^0(\oplus\,1)^2]$

$y = 2[-1] = -2$

$y = (2i)^3$ $y = 2^3[(-)^1(\oplus\,1)^2]$
$y = 2^3[-(-1)] = 8$

$y = 2(i^3)$ $y = 2[(-)^1(\oplus\,1)^2]$
$y = 2[-(-1)]$
$y = 2[1] = 2$

$y = (2i)^4$ $y = 2^4[(-)^2(\oplus\,1)^2]$
$y = 16[1(-1\,)] = \ -16$

$y = 2(i^4)$ $y = 2[-1] = -2$

$y = (2i)^5$ $y = 2^5[(-)^3(\oplus\,1)^2]$
$y = 32[-1(-1)] = 32$

$y = 2(i^5)$ $y = 2[1] = 2$

Etc.

This swapping of positive and negative values of powers of subspace numbers is seen with roots of i as well.

<u>Roots of Subspace Numbers:</u>

Consider the following root: $y = \sqrt{-16}$.

From section 2.b and 2.c above we know this is $y = \sqrt{-16} = 4i = \oplus 4$. We also know $4i = \oplus 4$ is resolved by a transform against the value of i such that $y = 4$ in the XY-Plane and $y = -4$ on the SY-Plane. Note this is only when considering the positive value of the root, as a root will have $\pm$ answer.

The next root, the cube-root can allow negative arguments. Consider $y = \sqrt[3]{8}$ and $y = \sqrt[3]{-8}$. The cube root of each easily solvable:

$$y = \sqrt[3]{8} = 2$$
$$y = \sqrt[3]{-8} = -2$$

However if there exists an i-multiple as the argument you must now take the root of the magnitude of the number given to the power of the root, and multiply it by the same power of root of i. The roots of i are identical to the matching powers of i. Consider $y = \sqrt[3]{8i}$ and $y = \sqrt[3]{-8i}$.

$$y = \sqrt[3]{8i} \qquad y = \sqrt[3]{8} \cdot \sqrt[3]{i} \qquad y = 2 \cdot \left(\frac{0}{0} \cdot \frac{0}{0} \cdot \frac{0}{0}\right) \qquad y = 2 \cdot \left[-1\left(\frac{0}{0} \cdot \frac{0}{0}\right)\right]$$

$$y = 2 \cdot [-1(-1)] \qquad y = 2$$

$$y = \sqrt[3]{-8i} \qquad y = \sqrt[3]{-8} \cdot \sqrt[3]{i} \qquad y = -2 \cdot \left(\frac{0}{0} \cdot \frac{0}{0} \cdot \frac{0}{0}\right) \qquad y = -2 \cdot \left[-1\left(\frac{0}{0} \cdot \frac{0}{0}\right)\right]$$

$$y = -2 \cdot [-1(-1)] \qquad y = -2$$

<u>Multiple Roots of i:</u>

i	The square root of -1	$\left\{\frac{0}{0}, \oplus 1\right.$	XY-Plane $y = 1$	SY-Plane $y = -1$
$\sqrt{i}$	The square root of i	$\left\{\sqrt{\frac{0}{0}} = \left(\frac{0}{0}\right)^2 = -1\right.$		

$$y = \sqrt{i} = (i)^2 = \left(\frac{0}{0}\right)^2 = -1$$

$$y = \sqrt{-i} = \left[\sqrt{-1} \cdot \sqrt{i}\right] = \left[\oplus 1 \cdot \left(\frac{0}{0}\right)^2\right] = [\hat{1} \cdot -1] = -\hat{1}$$

Don't get tripped up thinking that $\left[\oplus 1 \cdot \left(\frac{0}{0}\right)^2\right]$ *would be* $\left(\frac{0}{0}\right)^3$. *The expression* $\sqrt{-i}$ *was separated into the root of its magnitude* $\sqrt{-1}$ *and root of the* i*-multiple* $\sqrt{i}$. *These are taken separately and although they are multiplied together,* $\oplus 1 \cdot \left(\frac{0}{0}\right)^2$, *the exponent will be handled first leaving* -1. *Then only after resolving the plus-and-minus number to* $\hat{1}$ *can the multiplication be completed.*

$\sqrt[3]{i}$	Cube root of i	$= (-)^1 \left(\frac{0}{0}\right)^2 = 1$	$-(y_{\mathrm{xy}} \cdot y_{\mathrm{sy}}) = 1$
$\sqrt[4]{i}$	Forth root of i	$= (-)^2 \left(\frac{0}{0}\right)^2 = -1$	$--(y_{\mathrm{xy}} \cdot y_{\mathrm{sy}}) = -1$
$\sqrt[5]{i}$	Fifth root of i	$= (-)^3 \left(\frac{0}{0}\right)^2 = 1$	$---(y_{\mathrm{xy}} \cdot y_{\mathrm{sy}}) = 1$

Roots of a given number:

Focus is upon the positive value of the radical.

$$y = \sqrt{25} \qquad y = 5$$
$$y = \sqrt{-25} \qquad y_{\mathrm{xy}} = 5i = \hat{5}$$
$$y = \sqrt{25i} \qquad y = 5 \cdot \sqrt{i} = -5$$
$$y = \sqrt{-25i} \qquad y = 5i \cdot \sqrt{i} = -\hat{5}$$

Don't get tripped up in the last example and think to use $5i^3$. The expression $\sqrt{-25i} = 5i \cdot \sqrt{i} \neq 5i^3$. *The i in the argument of the root, has a magnitude of -25. The magnitude will resolve first and then you may multiply the $\sqrt{i}$.*

With the cube and any other odd roots:

Both positive and negative values can be used in odd powered roots of three and higher without resulting in a subspace number. The subspace in this situation is not bound to the bi-axis and is found by way of a transform upon the given equation.

$$y = \sqrt[3]{25} \qquad y = 2.924$$

$$y = \sqrt[3]{-25} \qquad y = -2.924$$

$$y = \sqrt[3]{25i} \qquad y = 2.924 \cdot i^3 \quad \longrightarrow \quad y = 2.924$$

$$y = \sqrt[3]{-25i} \qquad y = -2.924 \cdot i^3 \quad \longrightarrow \quad y = -2.924$$

$$y = 1 + \sqrt[3]{25i} \qquad y = 1 + 2.924 = 3.924$$

$$y = 1 + \sqrt[3]{-25i} \qquad y = 1 + (-2.924 \cdot i^3) = 1 - 2.924 \qquad y = -1.924$$

Higher roots:

Pattern will continue to follow that of 3rd roots for odd roots and 4th root for even roots.

$$y = \sqrt[4]{25} \qquad y = 2.236$$
$$y = \sqrt[4]{-25} \qquad y = 2.236i = \hat{2}.236$$
$$y = \sqrt[4]{25i} \qquad y = 2.236 \cdot i^4 = -2.236$$
$$y = \sqrt[4]{-25i} \qquad y = 2.236i \cdot i^4 = -\hat{2}.236$$

$y = \sqrt[5]{25}$ $\quad y = 1.904$
$y = \sqrt[5]{-25}$ $\quad y = -1.904$
$y = \sqrt[5]{25i}$ $\quad y = 1.904 \cdot i^5 = 1.904$
$y = \sqrt[5]{-25i}$ $\quad y = -1.904 \cdot i^5 = -1.904$

2.c.3—Multiplying and Dividing same base raised to powers:

When two values of the same base x are multiplied together and raised to powers, the base is kept and the powers added together.

Adding Exponents: $\quad x^a \cdot x^b = x^{a+b}$

$$x^2 \cdot x^3 = x^5$$

Likewise when two values of the same base x are divided and raised to powers, the base is kept and the power of the denominator subtracted from the power of the numerator.

Subtracting Exponents: $\quad \frac{x^a}{x^b} = x^{a-b}$

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2}$$

These same properties can be extended to subspace numbers:

Adding Exponents on subspace numbers:

$$\oplus\, 3^2 \cdot \oplus\, 3^3 = \oplus\, 3^5 \qquad \longrightarrow \qquad 243$$

Note you must use this property to combine the powers of the i multiples. If you attempt to first raise these values to their powers separately and then multiply them together you get the wrong value:

$$\oplus\, 3^2 \cdot \oplus\, 3^3 \quad \longrightarrow \quad (3i)^2 \cdot (3i)^3 = -9 \cdot 27 = -243$$

This would only be correct if for some reason you had applied parentheses around the separate instances of the exponents requiring they be handled apart from one another first:

$$\oplus\, 3^2 \cdot \oplus\, 3^3 = \oplus\, 3^5 \qquad \longrightarrow \qquad 243$$

$$(\oplus\, 3^2) \cdot (\oplus\, 3^3) \quad \longrightarrow \quad ((3i)^2) \cdot\ ((3i)^3)\ =\ -9 \cdot 27\ =\ -243$$

Thus we see that grouping order will matter when subspace numbers are included. Although $x^a \cdot x^b = x^{a+b}$ will hold true for the numeric value (*i.e.*$3^5 = 243$ and $-3^5 = -243$) the sign will vary for $\oplus$ 3 depending upon whether you chose to group its stages and raise it to the power indicated. See below:

$$(3i)^2 \cdot\ (3i)^3\ =\ -9 \cdot 27\ =\ -243 \qquad \not\equiv \qquad (3i)^5 = 243$$

This is because the value of i and its powers will determine whether the value on the given axis is positive or negative. Thus when subspace values such as $\oplus\, x^a \cdot \oplus\, x^b$ you cannot consider these as separate values. You must recognize that the number of times the value of $\oplus\, x$ is being raised to a power is the value of $a + b$. Thus, unless parentheses are being used to indicate the values are to be evaluated separately the term must be evaluated as $\oplus\, x^{a+b}$. Only in the final form $\oplus\, x^{a+b}$ can you obtain the correct magnitude to raise to the power $a + b$ which must then be multipled by i raised to the same power.

Thus:

$$\oplus\, 3^2 \cdot \oplus\, 3^3 = \oplus\, 3^5\ = [3^5 \cdot i^5] =\ 243$$

Likewise the same rules apply to the division of a subspace values of the same base raised to different powers.

Subtracting Exponents on subspace number:

$$\frac{\oplus x^a}{\oplus x^b} = \oplus\, x^{a-b} \qquad\qquad \frac{\oplus x^3}{\oplus x^5} = \oplus\, x^{3-5} = \oplus\, x^{-2}$$

$$\frac{\oplus\, 2^3}{\oplus\, 2^5} =\ 2i^{3-5} = 2i^{-2} = -\frac{1}{4}$$

2.c.4—Negative Exponents:

Negative exponents imply the value must be reciprocated before evaluation

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ $\left(-\frac{1}{2}\right)^{-3} = (-2)^3 = -8$ $\left(\oplus\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}i\right)^2 = -\frac{16}{9}$

2.c.5—Fractional Exponents:

When raising a value to a fraction, the power is raised to value of the numerator and then to the root-power of the denominator. The order can be reverse, first taking the root of the denominator and then raising to the power of the numerator.

$$x^{a/b} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

$3^{5/3} = \sqrt[3]{3^5} = 6.240$ $2^{4/5} = -2^{4/5} = \sqrt[5]{2^4} = \sqrt[5]{-2^4} = 1.741$

$-2^{3/4} = \sqrt[4]{-2^3} = 1.681i = \hat{1}.681$

$-2^{3/4} = \left(\sqrt[4]{-2}\right)^3 = \left(\hat{1}.189\right)^3 = \bar{1}.681$

$-5^{4/3} = \sqrt[3]{-5^4} = 8.54$

$-5^{4/3} = \left(\sqrt[3]{-5}\right)^4 = 8.54$

2.c.6—Factoring:

Factoring is the processes of unmultiplying an equation or expression and is quite useful in mathematics. This at times requires massaging the data given into a form which can be factored by use of Completing the Square or the Conjugate Method. Lastly the Quadratic equation will provide an exact decimal value for the factors sought.

Some forms of factorable values were covered in an earlier in section 2.b. Two common types of values encountered are *binomials* (*of the form* $(a+b)$) and *trinomials* (*of the forms* $Ax^2 + Bx + C$). Some forms of these numbers have known patterns for factoring which will not require polynomial long division to determine factors.

Difference of Perfect Squares $a^2 - b^2 = (a+b)(a-b)$

$x^2 - 9 = (x+3)(x-3)$

Difference of Squares	$a^2 - 2ab + b^2$	$=$	$(a-b)(a-b)$
	$s^2 - 4s + 4$	$=$	$(s-2)(s-2)$
Difference of Perfect Cubes	$a^3 - b^3$	$=$	$(a-b)(a^2+ab+b^2)$
	$x^3 - 8$	$=$	$(x-2)(x^2+2x+4)$
Sum of Perfect Cubes	$a^3 + b^3$	$=$	$(a+b)(a^2-ab+b^2)$
	$x^3 + 8$	$=$	$(x+2)(x^2-2x+4)$
Sum of Perfect Squares	$a^2 + b^2$	$=$	$(a+bi)(a-bi)$

$$x^2 + 9 = (x+3i)(x-3i) = (x+\hat{3})(x+\check{3})$$

$$= (x+\hat{3})(x-\check{3})$$

A reminder here on the use of the accent notation. The subspace values which arise from factoring the sum of perfect squares, $\pm 3i$, are resolved via subspace transform against i. Assuming the example above of x^2+9 is on the XY-Plane, the as the originating location for the expression the positive component resolved $\oplus b$ portion represents the part bound to the XY-Plane. The squaring implies multiplying it by the negative magnitude of the same value. This negative counterpart is on the coadjoining subspace XU-Plane. When the squaring of this term occurs the values on these two orthogonal axis interact and result in a negative. Thus it becomes necessary when dealing with the factored terms to keep track values which are resolved from i-multiples. The upward pointing circumflex is used to represent the resolved i multiple on the originating plane which resolves to positive, whilst the downward circumflex represents the value on the corresponding subspace plane, which resolves to negative. Don't get caught up thinking this will always be $x = \hat{b}$ on the XY-Plane , $x = \check{b}$ on the XU-Plane. This particular arrangement would only apply to i multiples originating on the x-axis in equations on the XY-Plane. If the i multiple originates on the x-axis of the XU-Plane the values are reversed; $x = \check{b}$ on XY-Plane and $x = \hat{b}$ on

XU-Plane.

The application of this will provide the below values. Note that the cross terms use the sign associated with the resolved terms only as this is the only sign they can see; the result of a resolved subspace term multiplied by a real-space number. This is the portion which for this example exists on the x-axis and the part of the value which is modified when multiplying with x. The cross terms will cancel as expected. The b^2 term is a square of an i multiple. It takes the value of the square of the *magnitude of the* i *multiple*, times the square of i itself times the square of the signs of the terms are seen from real space (*the plus and minus sign from the two separate binomial terms*).

Consider the following:

$x^2 + 9$	The given binomial term is a sum of perfect squares. There are no cross terms in the binomial so it must have a factored form matching $(x+?)(x-?)$. Traditional math requires the squaring of the second value to generate an additional negative sign to math the original plus sing. Traditionally this is held to be for the form ni where n is the root of the second term.
$(x + 3i)(x - 3i)$	This factored expression has two separate occurrences of an i-multiple; $3i$. The plus and minus signs are a result of the factoring of the original binomial in order to remove the cross terms which result from multiplication of the two factored binomials. There are not from the resolved values of the i-multpiles. Since both i-multiples originate out of the factored expression $x^2 + 9$, both will resolve to $\hat{3}$.
$(x + \hat{3})(x - \hat{3})$	The cross terms will cancel out in this expression as expected. The $\hat{3}$ is easily handled. First recall that this merely the resolved form of $\oplus 3 \equiv 3i$. The final step in multiplication of these two binomials is $(+\hat{3}) \cdot (-\hat{3})$. The plus sign multiplied by a negative sing is negative. The numbers

	themselves, $\hat{3} \cdot \hat{3} = \hat{3}^2$. Recall that squaring either the up or the down component of a plus-and-minus number is identical to squaring the magnitude of the number and then multiplying this by i raised to the same power. For squares, this example will generate a -9. This is identical to $\hat{3} \cdot \check{3}$.
$(x + \dot{3})(x - \dot{3})$	Lastly, we can fully resolve the values to the dotted form. This is how they are seen from the perspective of the plane in which these numbers originate as they interact with other real numbers there. The positive component of each plus-and-minus number is shown, modified by the sign of the binominal in which they exist. From this perspective it no longer matters whether the component was an up or a down component. Multiplying the two binomial halves together results in the squaring of a resolved i-multiple half, in this example $\dot{3}$ such that $\dot{3}^2 = -9$.

$$(x + \dot{3})(x - \dot{3})$$

$$x^2 - \bar{3}x + \bar{3}x - (\hat{3} \cdot \check{3})$$

$$x^2 - \dot{3}^2 \quad \equiv \quad x^2 - (3i)^2 \quad \equiv \quad x^2 - (\oplus 3)^2 \quad \equiv \quad x^2 - (\hat{3} \cdot \check{3})$$

The are several subspace equations which correspond to $y = x^2 + 9$:

XU-Plane: $u = -\frac{1}{x^2+9}$ SY-Plane: $y = \frac{9s^2+1}{s^2}$

SU-Plane: $u = -\frac{s^2}{9s^2+1}$

We begin with the x-axis inputs of the given equation on the XY-Plane, $y = x^2 + 9$. The values being added to the x inputs in the factored equation are $\hat{3}$. The same resolved value is added to x inputs in the XU-Plane:

$$u = -\frac{1}{x^2 + 9} = -\frac{1}{x^2 - (\hat{3} \cdot \check{3})} = -\frac{1}{(x + \dot{3})(x - \dot{3})}$$

In both of the equations using s-axis the values added in are the down counterparts to those added to the x inputs.

YS-Plane: $y = \frac{9s^2+1}{s^2} \quad = \quad \frac{(1+\hat{3}s)(1-\hat{3}s)}{s^2} \quad \equiv \quad \left(\frac{1}{s} + \hat{3}\right)\left(\frac{1}{s} - \hat{3}\right)$

US-Plane: $u = -\frac{s^2}{9s^2+1} \quad = \quad -\frac{s^2}{(1+\breve{3}s)(1-\breve{3}s)} \quad \equiv \quad -\frac{1}{\left(\frac{1}{s}+\breve{3}\right)\left(\frac{1}{s}-\breve{3}\right)}$

If you resolve the values on any of the up or down components you still end up with the same equations. You only have to keep track of the fact that these values now resolved are still i-multiples.

Consider the $u(s)$ equation $u = -\frac{s^2}{9s^2+1}$ and the $u(x)$ equation $u = -\frac{1}{x^2+9}$. The two equations will have corresponding values in x and s which result in the same values for u. For $x = -\frac{1}{s}$ and thereby $= -\frac{1}{x}$, $u = -\frac{1}{x^2+9}$ and $u = -\frac{s^2}{9s^2+1}$ will have equivalent values for u. Both equations have a component which can clearly be factored. Since we are discussing values on the x-axis we'll begin by considering $-\frac{1}{x^2+9}$ instead of with *s*.

Here in the coadjoining subspace XU-Plane (*the originating plane is XY with* $y = x^2 + 9$) the equation $u(x)$ will have factored values which resolve to the down-values. Factored, $u(x)$ provides:

$$u = -\frac{1}{x^2+9} \quad = \quad -\frac{1}{(x+\breve{3})(x-\breve{3})}$$

If you instead begin with $y = -\frac{1}{x^2+9}$ as the originating equation instead of $y = x^2 + 9$ you still use the *up* correspondences of the x value inputs XY-Plane and the *down* correspondences of the *x* value inputs on the XU adjoining subspace plane because the use of the i-multiples are attached to the x-axis inputs as defined by the originating equation, from the perspective of the XY-Plane as the plane of origin.

The SY and SU planes will factor the same way. The SY plane uses inputs assigned to the s-axis as variables. Though having converted to s-axis inputs the factoring still causes plus-and-minus numbers to originate on this adjoining subspace SY-plane. The SY-Plane will use resolved *up* value components while the coadjoining SU-Plane will use the resolved *down* components.

YS-Plane: $y = \frac{9s^2+1}{s^2} \quad = \quad \frac{(1+\hat{3}s)(1-\hat{3}s)}{s^2} \quad \equiv \quad \left(\frac{1}{s}+\hat{3}\right)\left(\frac{1}{s}-\hat{3}\right)$

US-Plane: $u = -\frac{s^2}{9s^2+1} \quad = \quad -\frac{s^2}{(1+\breve{3}s)(1-\breve{3}s)} \quad \equiv \quad -\frac{1}{\left(\frac{1}{s}+\breve{3}\right)\left(\frac{1}{s}-\breve{3}\right)}$

If you begin XY-Plane as the plane of origin, $y = \frac{9s^2+1}{s^2}$ will use the positive *up* components of the i multiples. Though the SY-Plane is an adjoining subspace the plus-and-minus terms will originate here from factoring just as they do in the XY-Plane holding the original given equation. The equation, $u = -\frac{s^2}{9s^2+1}$ of the SU-Plane, a coadjoining subspace plane will use the negative, *down,* components of the i-multiples.

If you convert from $y = x^2 + 9$ as the originating equation into $u = -\frac{s^2}{9s^2+1}$ with changes for the up and down components you will see the conversion goes forward without issue.

$y = x^2 + 9 \quad = \quad (x+\hat{3})(x-\hat{3})$ Where $\hat{3}$ is the positive $3i$ multiple components.

$u = -\frac{s^2}{9s^2+1} \quad = \quad -\frac{s^2}{(1+\breve{3}s)(1-\breve{3}s)}$ Where negative i multiple components are used with *s* input equivalent of *x*.

Converting either *s* to *x* or vice versa will result in the expected i multiple.

$$-\frac{s^2}{(1+\breve{3}s)(1-\breve{3}s)} \quad = \quad -\frac{\left(-\frac{1}{x}\right)^2}{\left(1+\breve{3}\left(-\frac{1}{x}\right)\right)\left(1-\breve{3}\left(-\frac{1}{x}\right)\right)} \quad = - \frac{\frac{1}{x^2}}{\left(1-\frac{\hat{3}}{x}\right)\left(1+\frac{\hat{3}}{x}\right)}$$

$$= -\frac{\frac{1}{x^2}}{1+\frac{\hat{3}}{x}-\frac{\hat{3}}{x}-\left(-\frac{9}{x^2}\right)} \quad = \quad -\frac{\frac{1}{x^2}}{\left(1+\frac{9}{x^2}\right)} \quad = \quad -\frac{\frac{1}{x^2}}{\left(\frac{x^2+9}{x^2}\right)}$$

$u = -\frac{1}{x^2+9} \quad u = \quad -\frac{1}{(x+\hat{3})(x-\hat{3})}$

The subspace conversion to $y(x)$ provides: $y = x^2 + 9 \ = \ (x+\hat{3})(x-\hat{3})$

Values which fit any of these binomial or trinomial patterns can be immediately factored. Where factoring becomes important is when you need to determine the zeros of an expression and, in dealing with fractions, when you get $\frac{0}{0}$ as an output. Consider the following polynomial examples and identities:

<u>Binomial and Trinomial Identities of Subspace Numbers:</u>

$$(a+\dot{b})^2 = (a+\dot{b})(a+\dot{b}) = a^2 + \overline{2ab} - b^2 \qquad (x+\dot{3})^2 = x^2 + \bar{6}x - 9$$

$$(a-\dot{b})^2 = (a-\dot{b})(a-\dot{b}) = a^2 - \overline{2ab} - b^2 \qquad (x-\dot{3})^2 = x^2 - \bar{6}x - 9$$

$$(\dot{a}+b)^2 = (\dot{a}+b)(\dot{a}+b) = -a^2 + \overline{2ab} + b^2 \qquad (\dot{x}+3)^2 = -x^2 + \bar{6}x + 9$$

$$(\dot{a}-b)^2 = (\dot{a}-b)(\dot{a}-b) = -a^2 - \overline{2ab} + b^2 \qquad (\dot{x}-3)^2 = -x^2 - \bar{6}x + 9$$

$$(\dot{a}+\dot{b})^2 = (\dot{a}+\dot{b})(\dot{a}+\dot{b}) = -a^2 - 2ab - b^2 \qquad (\dot{x}+\dot{3})^2 = -x^2 - 6x - 9$$

$$(\dot{a}-\dot{b})^2 = (\dot{a}-\dot{b})(\dot{a}-\dot{b}) = -a^2 + 2ab - b^2 \qquad (\dot{x}-\dot{3})^2 = -x^2 + 6x - 9$$

$$(a+\dot{b})(a-\dot{b}) = a^2 - a\bar{b} + a\bar{b} + b^2 = a^2 + b^2 \qquad (x+\dot{3})(x-\dot{3}) = x^2 + 9$$

$$(\dot{a}+b)(\dot{a}-b) = -a^2 - \bar{a}b + \bar{a}b - b^2 = -a^2 - b^2 \qquad (\dot{x}+3)(\dot{x}-3) = -x^2 - 9$$

$$(\dot{a}+\dot{b})(\dot{a}-\dot{b}) = -a^2 + \bar{a}b - \bar{a}b + b^2 = -a^2 + b^2 \qquad (\dot{x}+\dot{3})(\dot{x}-\dot{3}) = -x^2 + 9$$

$$(a+\dot{b})(\dot{a}-b) = \bar{a}^2 - ab - ab - \bar{b}^2 = \bar{a}^2 - 2ab - \bar{b}^2$$

$$(x+\dot{3})(\dot{x}-3) = \bar{x}^2 - 6x - \bar{9}^2$$

$$(\dot{a}+b)(a-\dot{b}) = \bar{a}^2 + ab + ab - \bar{b}^2 = \bar{a}^2 + 2ab - \bar{b}^2$$

$$(\dot{x}+3)(x-\dot{3}) = \bar{x}^2 + 6x - \bar{9}^2$$

For all a-components as resolved value $\dot{a}$

$(\dot{a}+b)\left(-a^2-\overline{ab}+b^2\right)$

$(\dot{a}+b)\left((\dot{a})^2-\left((\dot{a})(b)\right)+b^2\right)$

$a^3+a^2b+\bar{a}b^2-a^2b-\bar{a}b^2+b^3$

a^3+b^3

$(\dot{a}-b)\left(-a^2+\overline{ab}+b^2\right)$

$(\dot{a}-b)\left((\dot{a})^2+\left((\dot{a})(b)\right)+b^2\right)$

$a^3-a^2b+\bar{a}b^2+a^2b-\bar{a}b^2-b^3$

a^3-b^3

$(\dot{x}+3)\left((\dot{x})^2-\left((\dot{x})(3)\right)+(3)^2\right)$

x^3+27

$(\dot{x}-3)\left((\dot{x})^2+\left((\dot{x})(3)\right)+(3)^2\right)$

x^3-27

Note that the first form of each expression is factored out into it's resolved plus-and-minus components before performing further multiplication which ultimately results in sums and differences of cubes. If instead you either didn't know the a components in the $-a^2-\overline{ab}+b^2$ and $-a^2+\overline{ab}+b^2$ were resulting from plus-and-minus number, or were simply dealing with expression using real trinomials of this form you would get different answers:

$(\dot{a}+b)(-a^2-ab+b^2)$

$-\bar{a}^3-2\bar{a}b+b^3$

$(\dot{a}-b)(-a^2+ab+b^2)$

$-\bar{a}^3+2\bar{a}b-b^3$

$(\dot{x}+3)(-x^2-3x+9)$

$-x^3-6x^2+27$

$(\dot{x}-3)(-x^2+3x+9)$

$-x^3+6x^2-27$

For all b-components as resolved value $\dot{b}$

$\left(a+\dot{b}\right)\left(a^2-\overline{ab}-b^2\right)$

$\left(a+\dot{b}\right)\left(a^2-\left((a)(\dot{b})\right)+(\dot{b})^2\right)$

$a^3-a^2\bar{b}-ab^2+a^2\bar{b}+ab^2+b^3$

a^3+b^3

$\left(a-\dot{b}\right)\left(a^2+\overline{ab}-b^2\right)$

$\left(a-\dot{b}\right)\left(a^2+\left((a)(\dot{b})\right)+(\dot{b})^2\right)$

$a^3+a^2\bar{b}-ab^2-a^2\bar{b}+ab^2-b^3$

a^3-b^3

$\left(x+\dot{3}\right)\left(x^2-\left((x)(\dot{3})\right)+(\dot{3})^2\right)$

x^3+27

$\left(x-\dot{3}\right)\left(x^2+\left((x)(\dot{3})\right)+(\dot{3})^2\right)$

x^3-27

Again though, the first form of each expression is factored out into it's resolved plus-and-minus components before performing further multiplication which ultimately results in sums and differences of cubes. If instead you either didn't know the b components in the $a^2-\overline{ab}-b^2$ and $a^2+\overline{ab}-b^2$ were resulting from plus-and-minus number, or were simply dealing with expression using real trinomials of this form you would get different answers:

$(a+\dot{b})(a^2-ab-b^2)$ $a^3-2ab^2-b^3$

$(a-\dot{b})(a^2+ab-b^2)$ $a^3-2ab^2+b^3$

$(x+\dot{3})(x^2-3x-9)$ $x^3-18x-27$

$(x-\dot{3})(x^2+3x-9)$ $x^3-18x+27$

For all a-components and b-components as resolved as values $\dot{a}$ and $\dot{b}$

$(\dot{a}+\dot{b})(-a^2+ab-b^2)$

$(\dot{a}+\dot{b})\left((\dot{a})^2-\dot{a}\dot{b}+(\dot{b})^2\right)$

$a^3-a^2b+ab^2+a^2b-ab^2+b^3$

a^3+b^3

$(\dot{a}-\dot{b})(-a^2-ab-b^2)$

$(\dot{a}-\dot{b})\left((\dot{a})^2+\dot{a}\dot{b}+(\dot{b})^2\right)$

$a^3+a^2b+ab^2-a^2b-ab^2-b^3$

a^3-b^3

$(\dot{x}+\dot{3})\left((\dot{x})^2-\left((\dot{x})(\dot{3})\right)+(\dot{3})^2\right)$

x^3+27

$(\dot{x}-\dot{3})\left((\dot{x})^2+\left((\dot{x})(\dot{3})\right)+(\dot{3})^2\right)$

x^3-27

Again the first form of each expression is factored out into it's resolved plus-and-minus components before performing further multiplication which ultimately results in sums and differences of cubes. If instead you either didn't know the a and b components in the $-a^2+ab-b^2$ and $-a^2-ab-b^2$ were resulting from plus-and-minus numbers, or were simply dealing with expression using real trinomials of this form you would get different answers:

$(\dot{x}+\dot{3})(-x^2+3x-9)$

$-x^3-27$

$(\dot{x}-\dot{3})(-x^2-3x-9)$

$-x^3+27$

<u>Examples with Factoring:</u>

<u>Example 1:</u>

$y=x^2-16$

The Equation is a difference of perfect squares: $y=(x+4)(x-4)$

The zeros exist at: $x=\begin{cases}4\\-4\end{cases}$

Example 2:

$y = \frac{x^2-16}{x^2+x-12}$

Factoring the polynomials provides: $y = \frac{(x+4)(x-4)}{(x+4)(x-3)}$

Traditional Algebra and Calculus dictate a valid zero exists at $x = 4$. However the same disciplines show a vertical asymptote at $x = 3$ and a hole in the graph at $x = -4$. From the positive side of $x = 3$ the value of y approaches ∞. From the negative side of $x = 3$ it approaches $-\infty$. This is seen clearly at $f(3) = -\frac{7}{0}$. This value at $x = 3$ is simply naught, η_0, and may be resolved to 0 thereby (*see section 1.b for review of this transformation*).

At $x = -4$ traditional mathematics shows this value as $\frac{0}{0}$ and suggests nothing can be done except to cancel out the $(x + 4)$ factors which appear in the numerator and denominator of the example function. Although it still requires factoring to see there are two terms which can cancel out there is no issue with evaluating the function at $x = -4$ if you understand what is occurring.

When dealing with polynomial fractions the output of $\frac{0}{0}$ is a *feedback solution*. This is not a $\frac{0}{0}$ arising within a single equation. Instead it's the result of an input being used when it is no longer required resulting in *naught* as an output. This results in an unintended zero clobbering out the rest of the meaningful information via multiplication. Without necessary information of what to do when this presents itself you'd be left with a hole in the graph. Once factoring out components you can identify the corresponding parts of the equation which result in a 0 in the numerator and denominator for the same value. At the instance of the trouble-value input, *naught* is used as the input in the components which would otherwise generate the simultaneous zeros in the numerator and denominator. The actual input value is used for everything else.

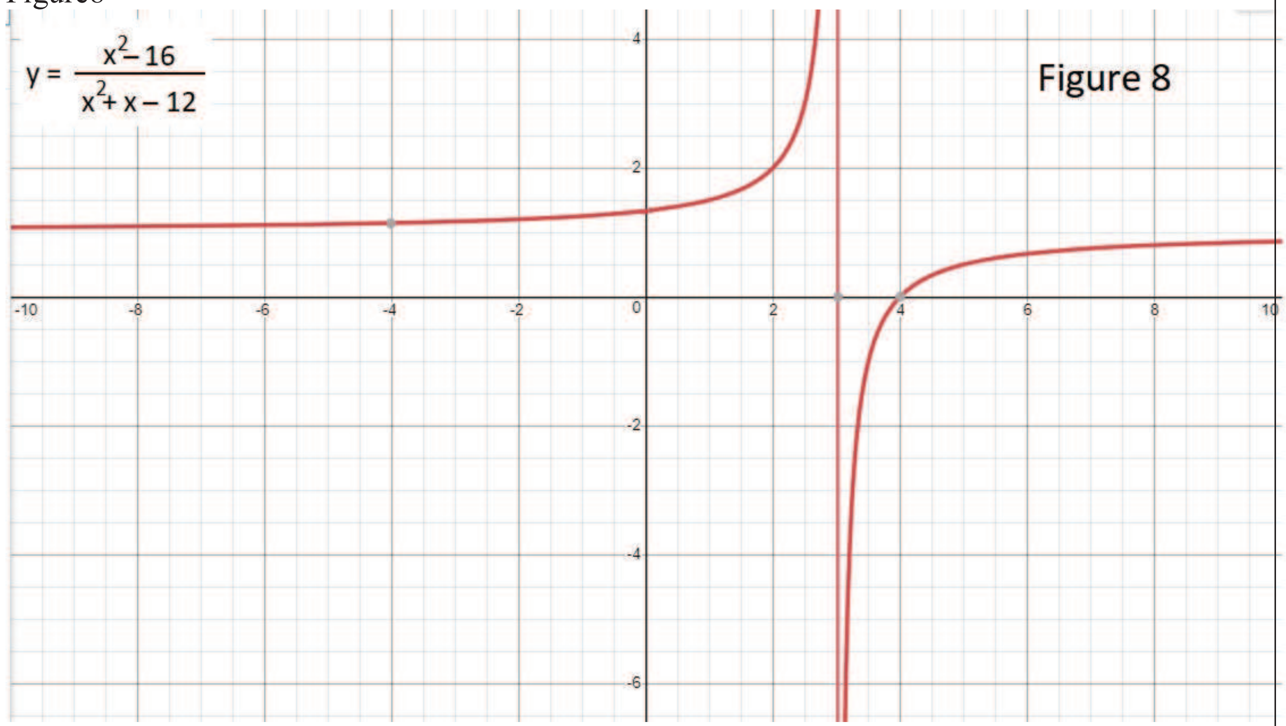
There is a unique rule of operation change to consider depending upon whether or not you end up with a lone η_0 in the numerator or denominator in the last step of such an equation; This will be seen in Example 11. The mathematics used here will resolve the issues with the components which are causing the $\frac{0}{0}$ output. The use of naught is necessary since this is what their zero implies in the $\frac{0}{0}$ output. In the example below when $x = -4$ the $(x + 4)$ components will generate 0. This isn't zero as in zero cookies. Instead it implies they no longer contribute to the overall value of the function; zero as in η_0. Using naught in place of the input in those components therefore provides the correct output from the remainder of the equation.

Doing this with the example equation we get:

$y = \frac{(x+4)(x-4)}{(x+4)(x-3)}$ $\qquad f(-4) = \frac{(\eta_0+4)(x-4)}{(\eta_0+4)(x-3)} = \frac{(4)(-8)}{(4)(-7)} = \frac{8}{7}$

The output value doesn't just approach $\frac{8}{7}$, it actually is this value. See Figure 8:

Figure8



It's easy to dismiss this with the thinking given that copies of both factors exist in the numerator and denominator and thereby will always cancel out due to their persisting 1 to 1 ratio. After all, this is what we see with the $(x + 4)$ factors in the above example. Whether using the whole factor or 4 alone, they both cancel out. What then if there is nothing to cancel, but we still get a $\frac{0}{0}$ output? Consider the following example.

$$y = \frac{\frac{x}{3} - 1}{x^2 - 7x + 12} = \frac{\frac{x}{3} - 1}{(x-3)(x-4)}$$

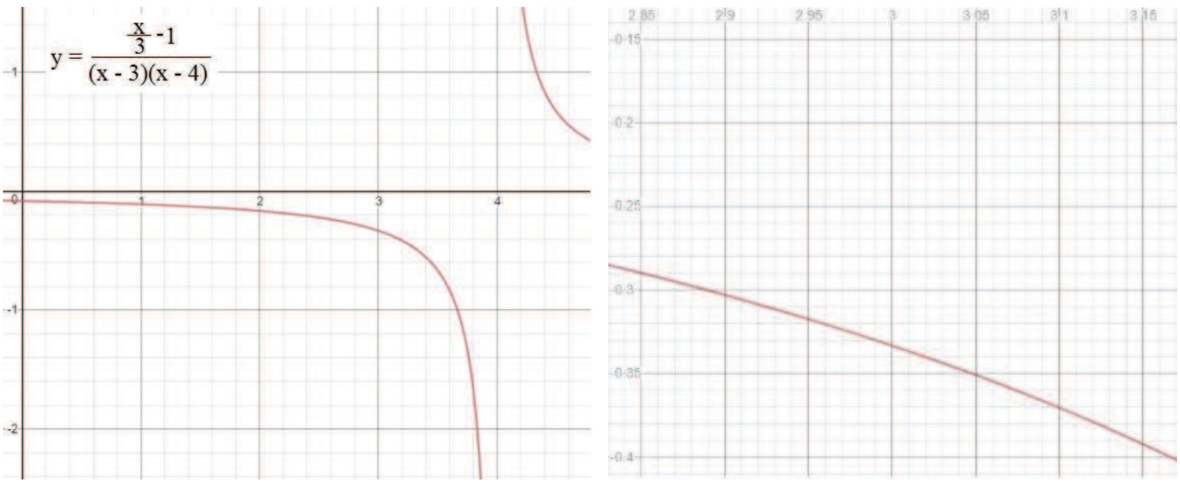
There's nothing else to factor here and nothing cancels. There is an asymptote which resolves to $y = \eta_0 = 0$ at $x = 4$. If you input $x = 3$ the output will be $\frac{0}{0}$. Resolve this using the implied value of *naught* from the *feedback value / solution* in the components of the expression producing it.

In the numerator you will get $\frac{\eta_0}{3} - 1$. The *naught* is a kind of 0. Naught divided by any number will result in 0. So your left with $0 - 1$ in the numerator. In the denominator we get $(\eta_0 - 3)(3 - 4) = (-3)(-1) = 3$. Putting this together we get:

$$y = \frac{\frac{\eta_0}{3} - 1}{(\eta_0 - 3)(3 - 4)} = \frac{-1}{(-3)(-1)} = -\frac{1}{3}$$

Examine figure 9 which shows a graph of this equation. You can clearly see the value of y is $-\frac{1}{3}$ when $x = 3$. The blown-up section on the right makes it easily visible.

Figure 9:



Consider the following additional examples:

Example 3:

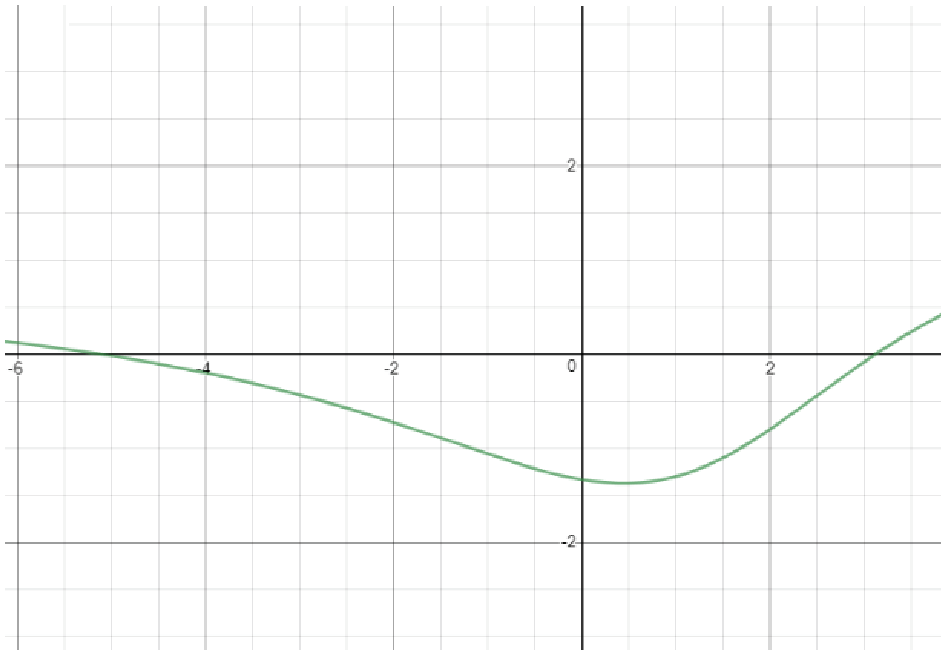
A non-factorable polynomial fraction. The example here below is one such expression. The numerator and denominator are both polynomials. It has values which at a glance might even look factorable but it's not.

$$\frac{x^2 + 2x - 16}{x^2 - 3x + 12}$$

The graph shows: Zeros at $x = \begin{cases} -5.123 \\ 3.123 \end{cases}$

There are no asymptotic values as this graph's denominator never equals 0 at any value of x.

Figure 10:



Example 4:

A factorable polynomial fraction with 1 to 1 ratio imbedded in the expression. Factoring is possible and its clear parts of the expression will cancel out. It is nonetheless possible to solve the value with naught.

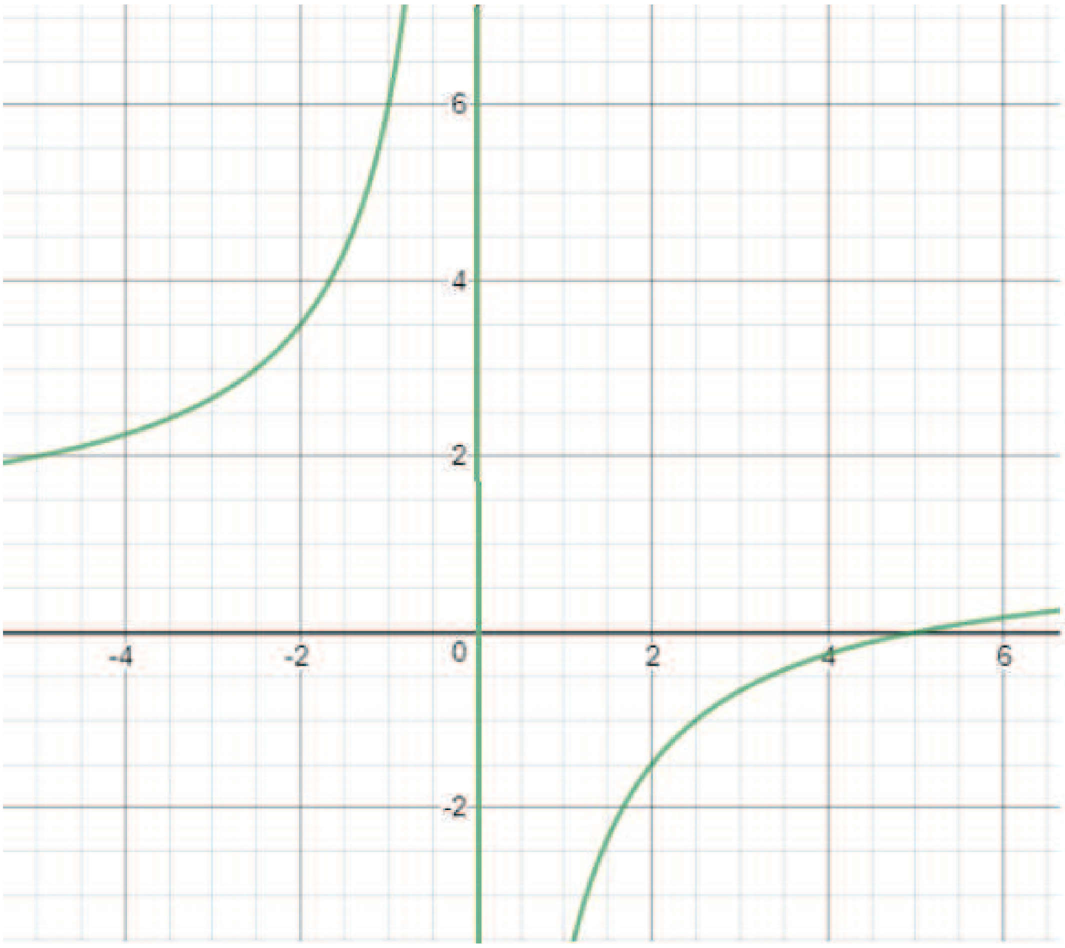
$$\frac{x^2 - 2x - 15}{x^2 + 3x} = \frac{(x+3)(x-5)}{(x+3)x}$$

Examining the graph you'll find the following properties.

Zero at $x = 5$. The vertical asymptote exists at $x = 0$ and resolves to $y = \frac{-5}{0} = -\eta_0 \doteq 0$. Finally there is the feedback solution at $x = -3$ giving $y = \frac{0}{0}$. Whether you chose to cancel the values out directly or enter naught for x in the $(x+3)$ components you'll still get the same answer; $f(-3) = \frac{8}{3}$. This is shown below with use of naught. The graph is shown in figure 11.

$$f(-3) = \frac{(\eta_0 + 3)(-3-5)}{(\eta_0 + 3)(-3)} = \frac{(3)(-8)}{(3)(-3)} = \frac{8}{3}$$

<u>Figure 11</u>



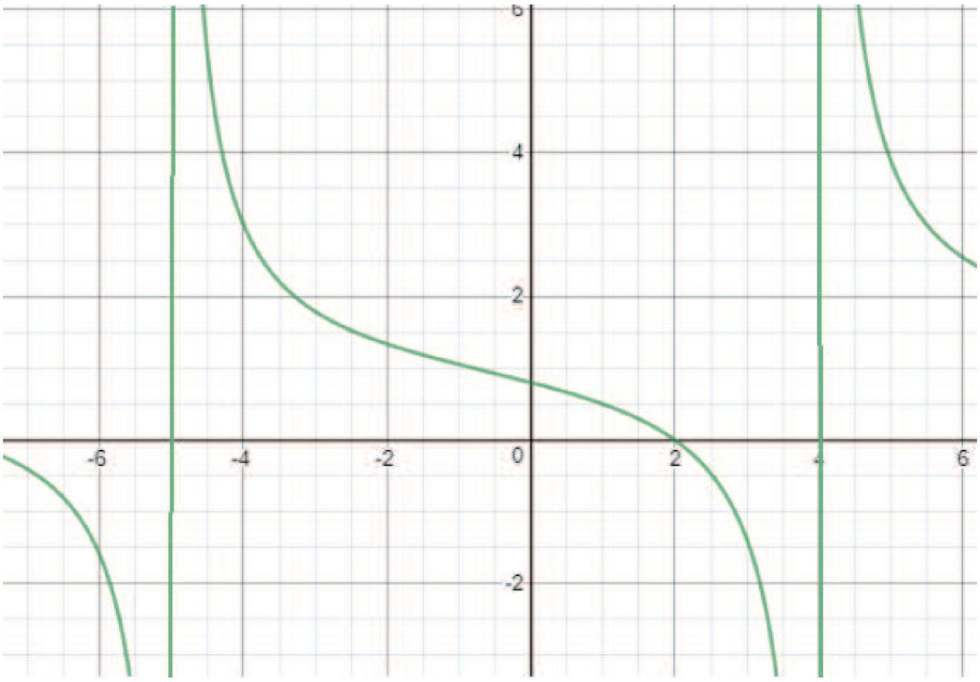
<u>Example 5:</u>

A factorable polynomial fraction but no cancelling terms.

$$\frac{x^2 + 6x - 16}{x^2 + x - 20} = \frac{(x+8)(x-2)}{(x-4)(x+5)}$$

The graph clearly shows there are zeros at $x = -8$ and 2. There exist asymptotes at two values; $x = -5$ and 4. Respectively these are resolved by $f(-5) = -\eta_0 \doteq 0$ and $f(4) = \eta_0 \doteq 0$. Though there are zeros and incidents of naught arising in the output from division by zero there are no instances of the feedback value. This is because no arrangement of the factors will result in a 0 output simultaneously in both the numerator and denominator for any given value of x. Figure 12 below shows the graph of this example function.

Figure 12:



Example 6:

Factoring Polynomial fraction with a feedback value, no canceling terms, and division by zero in a factor fraction. This type of example is uncommon but possible. It involves the presence of a fraction in one of the factors which goes to 1 at the same time the value of another factor opposite it goes to 0. When 1 is subtracted from the fraction containing factor you get the feedback solution as an output. Consider the example:

$$\frac{\frac{-8-x^2}{x}+6}{x^2-5x+4} = \frac{\left(\frac{4}{x}-1\right)(x-2)}{(x-4)(x-1)}$$

If you didn't know $\frac{-8-x^2}{x}+6$ factored out to $\left(\frac{4}{x}-1\right)(x-2)$ you would likely simplify it to $-\frac{8}{x}-x+6$ and believe you cannot go further. This pattern is still useful and will be covered in a moment. The graph of equations shows the following properties:

Zero at $x=2$. Vertical Asymptote at $x=1$ which resolves to $y=-\eta_0 \doteq 0$, and another at $x=0$ which is likewise resolvable $y=-\frac{8}{0}=-\eta_0 \doteq 0$. This second asymptote isn't immediately visible in the math and is a result of a missing factor of x in both numerator and denominator. If you do not attempt to replace the instance of division by 0 with a *naught*, and instead just use the 0 directly you can see it. The use of naught and its resolution would only apply if this resulted in a feedback value solution. It doesn't so we use direct substitution.

$$f(0) = \frac{\left(\frac{4}{0} - 1\right)(0-2)}{(0-4)(0-1)} = \frac{(\infty)(-2)}{4} = \infty$$

Infinity can be divided into four parts an infinite amount of times. The Unsigned infinity implies both positive-and-negative infinity. This means at the value of $x = 0$ we have $y = \infty$. This solution is then finally resolvable via rules of *naught*. $y = \infty \doteq \eta_0 \doteq 0.$

Feedback solution at $x = 4$ giving $y = \frac{0}{0}$, resolvable by use of *naught*.

$$f(4) = \frac{\left(\frac{4}{\eta_0} - 1\right)(4-2)}{(\eta_0 - 4)(4-1)}$$

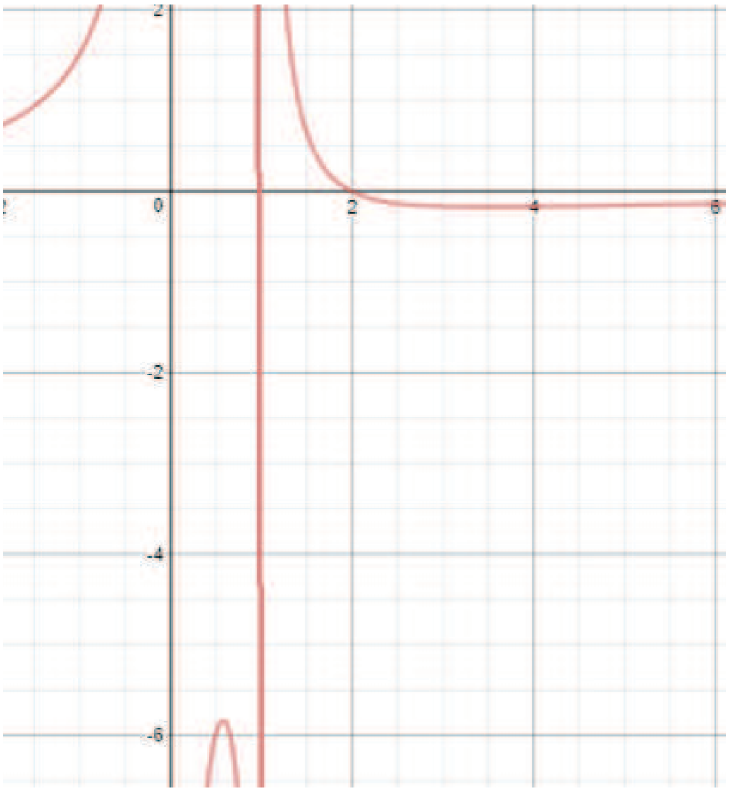
The $\left(\frac{4}{\eta_0} - 1\right)$ in the numerator contains a fraction which goes to 1 at the feedback value. The naught is in the denominator of this factor. In this instance we are asking how many times the numerator value can be divided by a naught resolved to zero. Yet there's another concern here too. The 1 cannot be subtracted until its set to same denominator. This must be handled before resolving the division by naught. After subtracted out we are left with $\frac{4}{0}$ which again resolves to 0 as η_0. In the $(\eta_0 - 4)$ factor the naught is simply resolved to 0. Thus we have:

$$f(4) = \frac{\left(\frac{4}{\eta_0} - 1\right)(4-2)}{(\eta_0 - 4)(4-1)} = \frac{\left(\frac{4}{0} - \frac{0}{0}\right)(4-2)}{(0-4)(4-1)} = \frac{(\eta_0)(4-2)}{(-4)(4-1)} = \frac{(2)}{(-4)(3)}$$

$$= -\frac{2}{12} = -\frac{1}{6}$$

Observe Figure 13 below:

Figure 13:



Return now again to the form of the numerator used above as $-\frac{8}{x} - x + 6$. I mentioned this pattern is still useful. If you examine the fully factored example expression, $\frac{\left(\frac{4}{x}-1\right)(x-2)}{(x-4)(x-1)}$, you'll see this is easy arrived at in multiplying the terms in the numerator.

$$\left(\frac{4}{x} - 1\right)(x-2) \;=\; 4 - \frac{8}{x} - x + 2 \;=\; -\frac{8}{x} - x + 6$$

If you multiply this expression by x it will have a form which is easily identifiable and factorable. This means multiplying the entire expression $\frac{-\frac{8}{x}-x+6}{(x-4)(x-1)}$ by $\frac{x}{x}$.

$$\frac{x}{x} \cdot \frac{-\frac{8}{x} - x + 6}{(x-4)(x-1)} \;=\; \frac{-8 - x^2 + 6x}{x(x-4)(x-1)}$$

We can rewrite the numerator into a more familiar form as $-x^2 + 6x - 8$. You can also see the values for zeros and the asymptotic occurrences, including the one where $x = 0$.

$$\frac{-x^2 + 6x - 8}{x(x-4)(x-1)}$$

This version of the numerator is factorable by two different ways. The fist is to utilize the resolved plus-and-minus values they imply.

$$\frac{-x^2 + 6x - 8}{x(x-4)(x-1)} = \frac{(\dot{x} - \dot{2})(\dot{x} - \dot{4})}{x(x-4)(x-1)} \doteq -\frac{(\dot{x} - \dot{2})}{x(x-1)}$$

The final step shows a sign change. The canceling of the $(x-4)$ terms occurs because they maintain a 1 to 1 ratio in the numerator and denominator. However, the numerator version, contains resolved i-multiples $(\dot{x} - \dot{4})$. For the purposes of canceling out with the $(x-4)$ of the denominator they are identical. However in the multiplication of with the other numerator $(\dot{x} - \dot{2})$ components they result in a negative. With $(\dot{x} - \dot{4})$ cancelled out we loose this and thus must supply back the negative sign.

The other way of factoring $-x^2 + 6x - 8$ is to simply remove a negative sign from the expression and then factor.

$$\frac{-x^2 + 6x - 8}{x(x-4)(x-1)} = \frac{(-1)(x^2 - 6x + 8)}{x(x-4)(x-1)} = \frac{(-1)(x-2)(x-4)}{x(x-4)(x-1)} = -\frac{(x-2)}{x(x-1)}$$

If you check the values for $f(2)$ you'll get a zero for the graph. Likewise $f(0)$ and $f(1)$ produce vertical asymptotes which resolve to 0. Finally $f(4) = -\frac{1}{6}$.

<u>Example 7:</u>

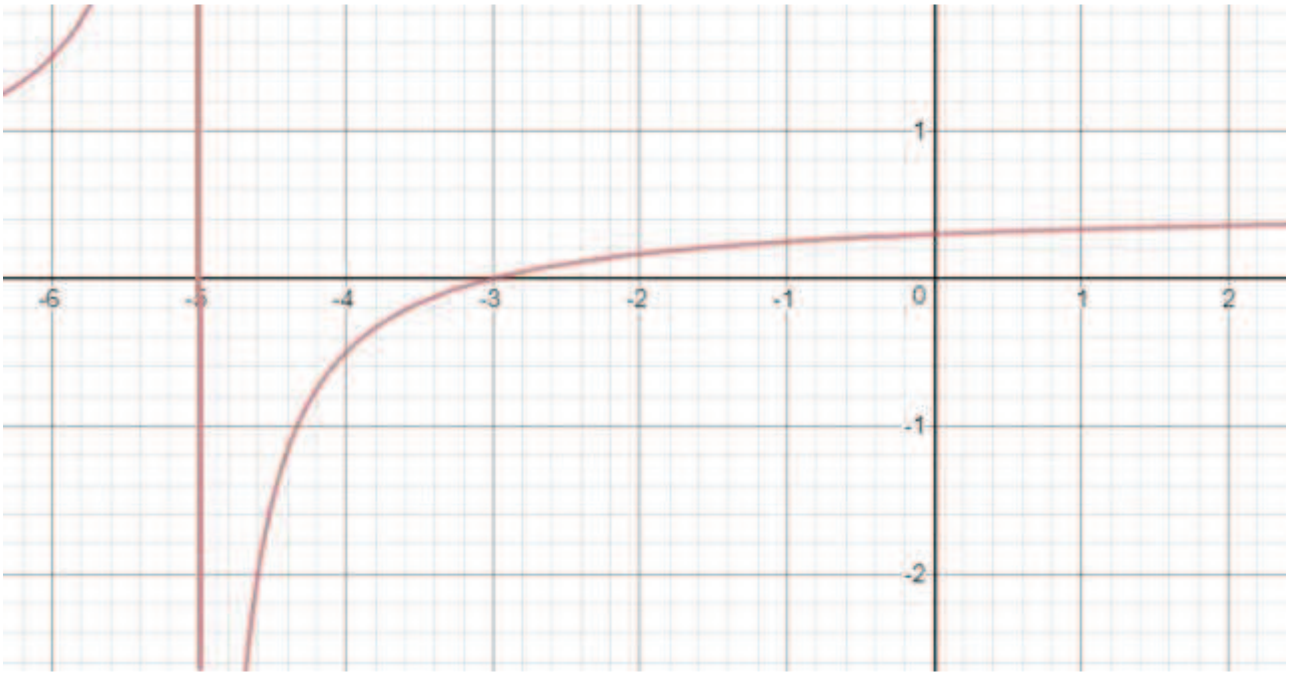
Factoring polynomial fraction, with: a feedback solution, no canceling terms and a fraction containing factor not having division by *naught*.

$$\frac{\frac{x^2 + x}{2} - 3}{x^2 + 3x - 10} = \frac{\left(\frac{x}{2} - 1\right)(x+3)}{(x-2)(x+5)}$$

Again if you didn't know $\frac{x^2+x}{2} - 3$ factored to $\left(\frac{x}{2} - 1\right)(x + 3)$ you may not do more with it than write it out as $\frac{x^2}{2} + \frac{x}{2} - 3$. Again it's still useful and will be covered moment. There is a zero at $x = -3$ and an asymptote at $x = -5$ giving $y = -\eta_0 \doteq 0$. The feedback value occurs at $x = 2$ giving $y = \frac{0}{0}$ resolvable by naught. In the $\left(\frac{x}{2} - 1\right)$ component the naught gives 0 divided by 2 which gives leaves with $0 - 1$. In the denominator the component $(x - 2)$ resolves with naught to $\eta_0 - 2 = -2$. Figure 14 shows the graph.

$$\frac{\left(\frac{\eta_0}{2} - 1\right)(2 + 3)}{(\eta_0 - 2)(2 + 5)} = \frac{(0 - 1)(5)}{(-2)(7)} = \frac{5}{14} = 0.357$$

Figure 14:



Like was done in example 6 the numerator $\frac{x^2+x}{2} - 3$ can be changed into a more common and easily factorable form by multiplying it by 2. To keep the ratio unchanged the denominator must also be multiplied by 2.

$$\frac{2}{2} \cdot \frac{\frac{x^2 + x}{2} - 3}{x^2 + 3x - 10} = \frac{x^2 + x - 6}{2(x - 2)(x + 5)} = \frac{(x + 3)(x - 2)}{2(x - 2)(x + 5)} = \frac{(x + 3)}{2(x + 5)}$$

The zero still exists at $f(-3)$. There is a vertical asymptote at $f(-5)$ and the $f(2) = 0.357$.

Example 8:

A factorable polynomial in the denominator of a fraction with a single non-factorable, fractional component in the numerator.

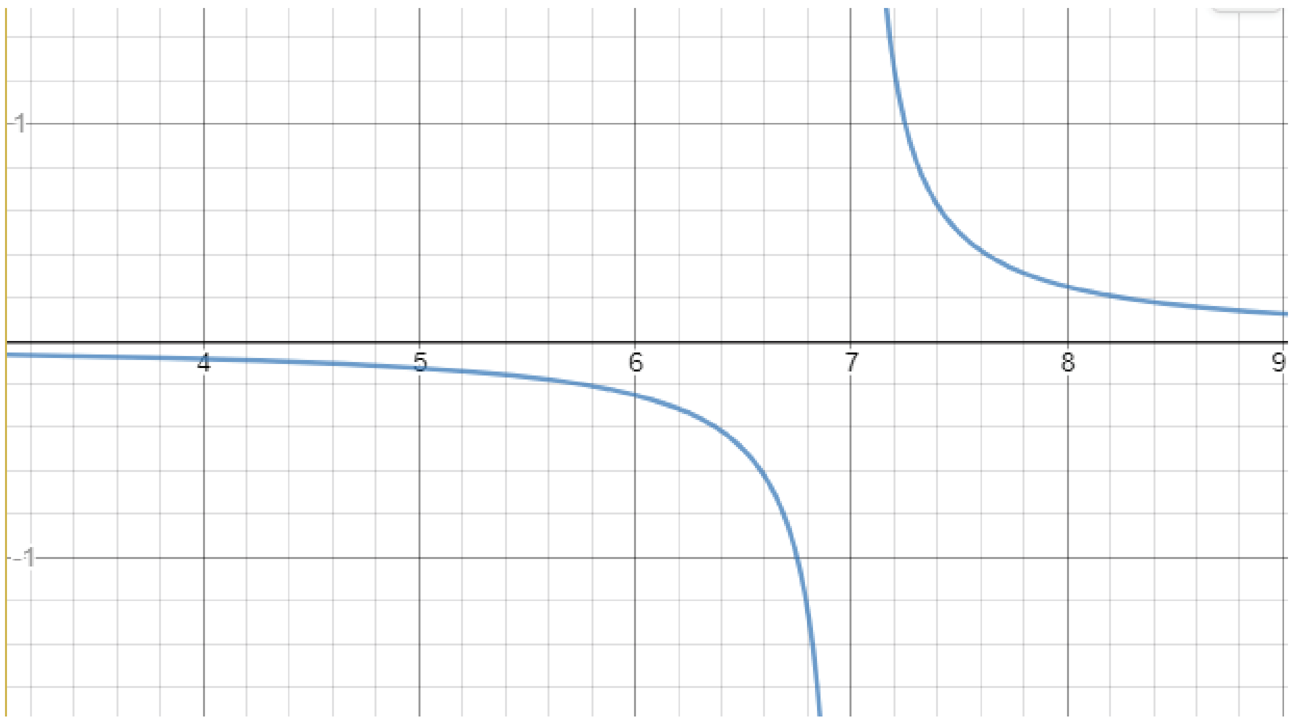
$$\frac{\frac{x}{4}-1}{x^2-11x+28} = \frac{\frac{x}{4}-1}{(x-4)(x-7)}$$

By now this should be getting fairly comfortable to work with. There are actually no zeros for this example. Instead there is a horizontal asymptote at $y = 0$. There is also a vertical asymptote at $x = 7$ giving $y = \eta_0 = 0$. The feedback value occurs at $x = 4$ and is resolved with naught shown below.

$$f(4) = \frac{\frac{\eta_0}{4}-1}{(\eta_0-4)(4-7)} = \frac{-1}{(-4)(-3)} = -\frac{1}{12} = -0.08\overline{3}$$

Figure 15 shows the graph.

Figure 15:



Example 9:

A fraction with factorable polynomials and a factor containing a fraction which will cause division by 0.

$$\frac{x^2-2x-3}{\frac{-x^2+24}{x}-5} = \frac{(x+1)(x-3)}{\left(\frac{3}{x}-1\right)(x+8)}$$

Zero at $x=-1$. Vertical asymptote at $x=-8$ giving $y=-\eta_0=0$. The feedback value results at $x=3$.

$$\frac{(3+1)(\eta_0-3)}{\left(\frac{3}{\eta_0}-1\right)(3+8)} = \frac{(4)(-3)}{\left(\frac{3}{0}-\frac{0}{0}\right)(11)} = \frac{-12}{(\eta_0)(11)} = -\frac{12}{11}$$

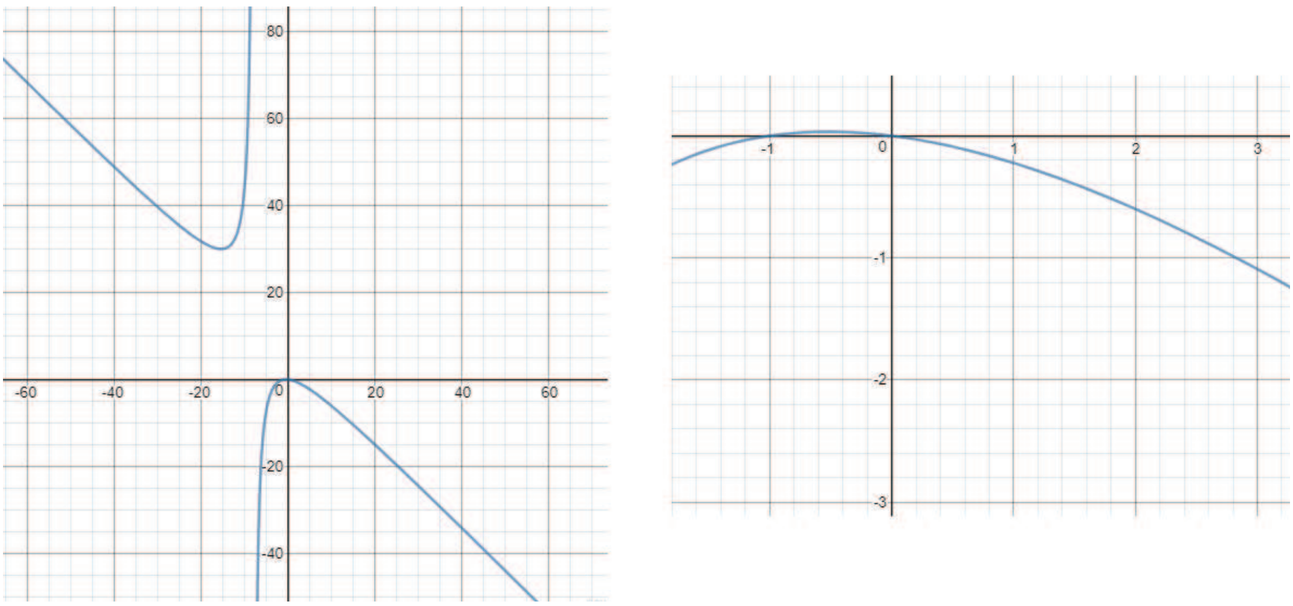
As before you can multiply the numerator and denominator by $\frac{x}{x}$ and receive a more easily usable form.

$$\frac{x}{x}\cdot\frac{x^2-2x-3}{\frac{-x^2+24}{x}-5} = \frac{x(x+1)(x-3)}{-x^2-5x+24} = \frac{x(x+1)(x-3)}{(\dot{x}-\dot{3})(\dot{x}+\dot{8})} = -\frac{x(x+1)}{(\dot{x}+\dot{8})}$$

You can also remove a negative one from the denominator $-x^2-5x+24$ and factor directly.

$$\frac{x}{x}\cdot\frac{x^2-2x-3}{\frac{-x^2+24}{x}-5} = \frac{x(x+1)(x-3)}{-x^2-5x+24} = \frac{x(x+1)(x-3)}{(-1)(x-3)(x+8)} = -\frac{x(x+1)}{(x+8)}$$

Figure 16:



Example 10:

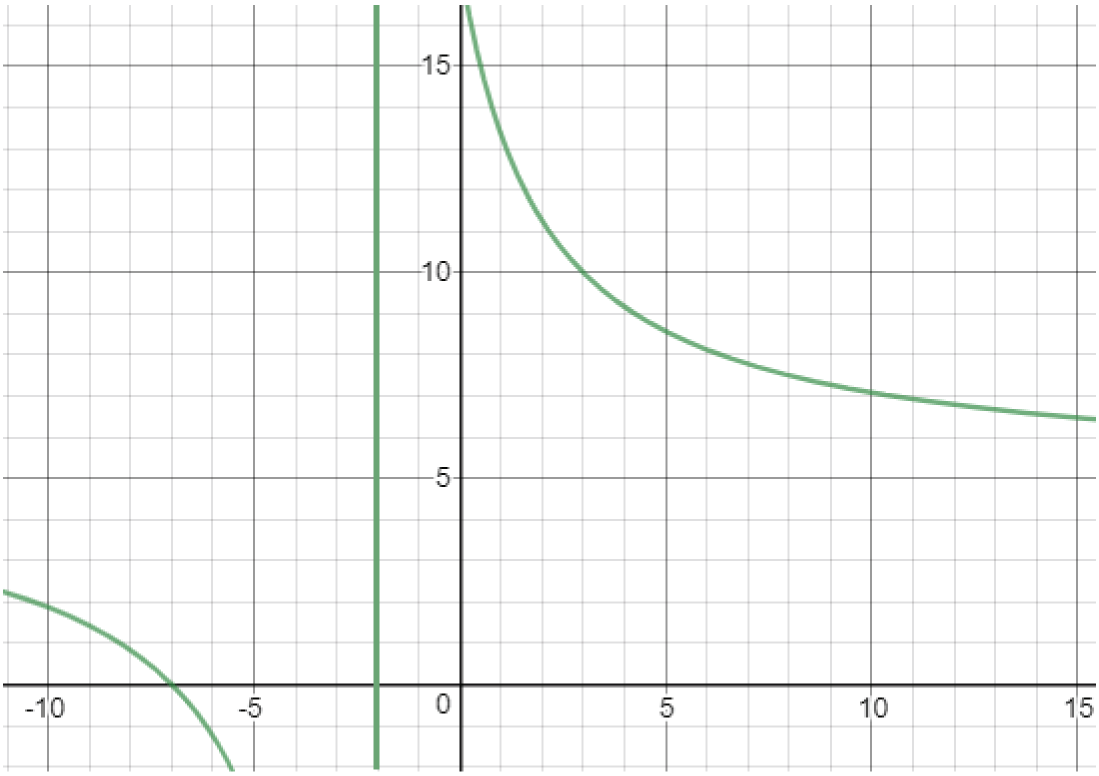
$$\frac{x^2+2x-35}{\frac{x^2-3x}{5}-2} = \frac{(x-5)(x+7)}{\left(\frac{x}{5}-1\right)(x+2)}$$

Zero at $x = -7$. Vertical asymptote at $x = -2$ giving $y = \eta_0 \doteq 0$. The feedback value is at $x = 5$ giving $y = \frac{0}{0}$ resolvable by naught.

$$\frac{(\eta_0-5)(5+7)}{\left(\frac{\eta_0}{5}-1\right)(5+2)} = \frac{(-5)(12)}{(0-1)(7)} = \frac{60}{7} = 8.571$$

Graph is shown in Figure 17.

Figure 17:



Example 11:

Earlier it was mentioned there is a rule of operations addition which must be used in the last step of solving an equation which produces a feedback value.

If that equation is:

- A fraction
- Produces a feedback value

- The final step results in a lone η_0
    - in its numerator
    - In its denominator
    - In both numerator and denominator

The final step must be calculated as multiplication. The use of η_0 to remove the issue of the components which no longer contribute to the solution at the problem input value is required. Although η_0 is not identical to 0 it is resolvable to it. The meaning of the result is different depending upon whether a value is multiplied by or divided by η_0. In order to ensure the use of naught is applied properly and information not accidently destroyed the final stage of solving in this situation must be conducted via multiplication only.

Consider the following three examples:

Ex. 11a $\dfrac{\frac{x}{2}-1}{(x-2)(x+3)}$ Ex. 11b $\dfrac{\frac{2}{x}-1}{(x-2)(x+3)}$ Ex. 11c $\dfrac{(x-2)(x+3)}{\frac{2}{x}-1}$

Ex. 11d $\dfrac{(x-2)(x+3)}{\frac{x}{2}-1}$ Ex. 11e $\dfrac{\frac{x}{2}-1}{\frac{x}{2}-1}$ Ex. 11f $\dfrac{\frac{2}{x}-1}{\frac{2}{x}-1}$

Ex. 11g $\dfrac{\frac{x}{2}-1}{\frac{2}{x}-1}$ Ex. 11h $\dfrac{\frac{2}{x}-1}{\frac{x}{2}-1}$ Ex. 11i $\dfrac{\frac{2}{x}-\frac{2}{x}}{(x-2)}$ Ex. 11j $\dfrac{\frac{2}{x}+\frac{2}{x}}{(x-4)}$

Each of these examples illustrates the differences and issue in the final step of solving with usage of η_0. First consider Examples 11a and 11b.

<u>Example 11a and 11b:</u>

Ex. 11a $\dfrac{\frac{x}{2}-1}{(x-2)(x+3)}$ Ex. 11b $\dfrac{\frac{2}{x}-1}{(x-2)(x+3)}$

These two equations are very similar, with their numerator components only varying by the inverted fraction contained in the numerator.

Example 11a:

$$f(2) = \frac{\frac{(\eta_0)}{2} - 1}{(\eta_0 - 2)((2)+3)}$$

$$= \frac{-1}{(-2)(5)}$$

$$= \frac{1}{10}$$

Example 11b:

$$f(2) \quad = \frac{\frac{2}{(\eta_0)} - 1}{((\eta_0) - 2)((2)+3)}$$

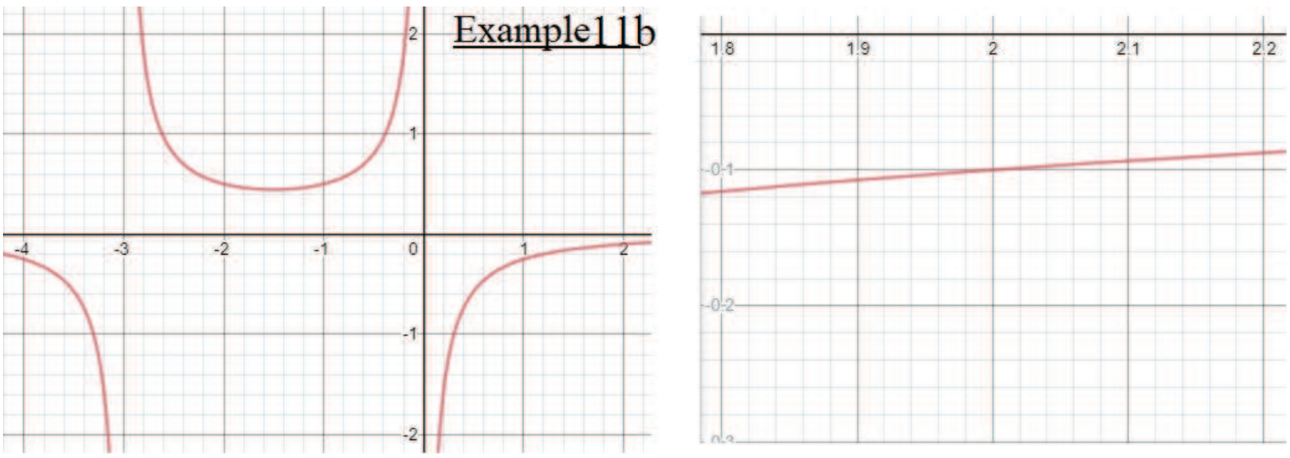
$$= \frac{\frac{2}{(\eta_0)} - \frac{0}{0}}{(-2)(5)}$$

$$= \frac{\eta_0}{-10} \equiv \eta_0 \cdot \frac{1}{-10} = -\frac{1}{10}$$

Example 11a is straight forward. However 11b ends up with a lone naught in the numerator. An examination of their respective graphs shows these values are correct.

Graph of 11a:

Example 11b:



Note that for Example 11b the zeros when $x = \{-3\,, 0$. These result from the standard application of naught at these values:

$$f(-3) = \frac{\frac{2}{-3}-1}{(-3-2)(-3+3)} = \frac{-\frac{5}{3}}{0} = \eta_0 = 0$$

$$f(0) \quad = \frac{\frac{2}{0}-1}{(0-2)(0+3)} = \frac{\frac{2}{0}-\frac{0}{0}}{-6} = \frac{\eta_0}{-6} = 0$$

<u>Example 11c and 11d:</u>

Ex. 11c $\dfrac{(x-2)(x+3)}{\frac{2}{x}-1}$ Ex. 11d $\dfrac{(x-2)(x+3)}{\frac{x}{2}-1}$

These two examples are simply reciprocates of the 11a and 11 but will illustrate what happens when the same situation occurs in the denominator rather than in the numerator.

<u>Example 11c:</u>

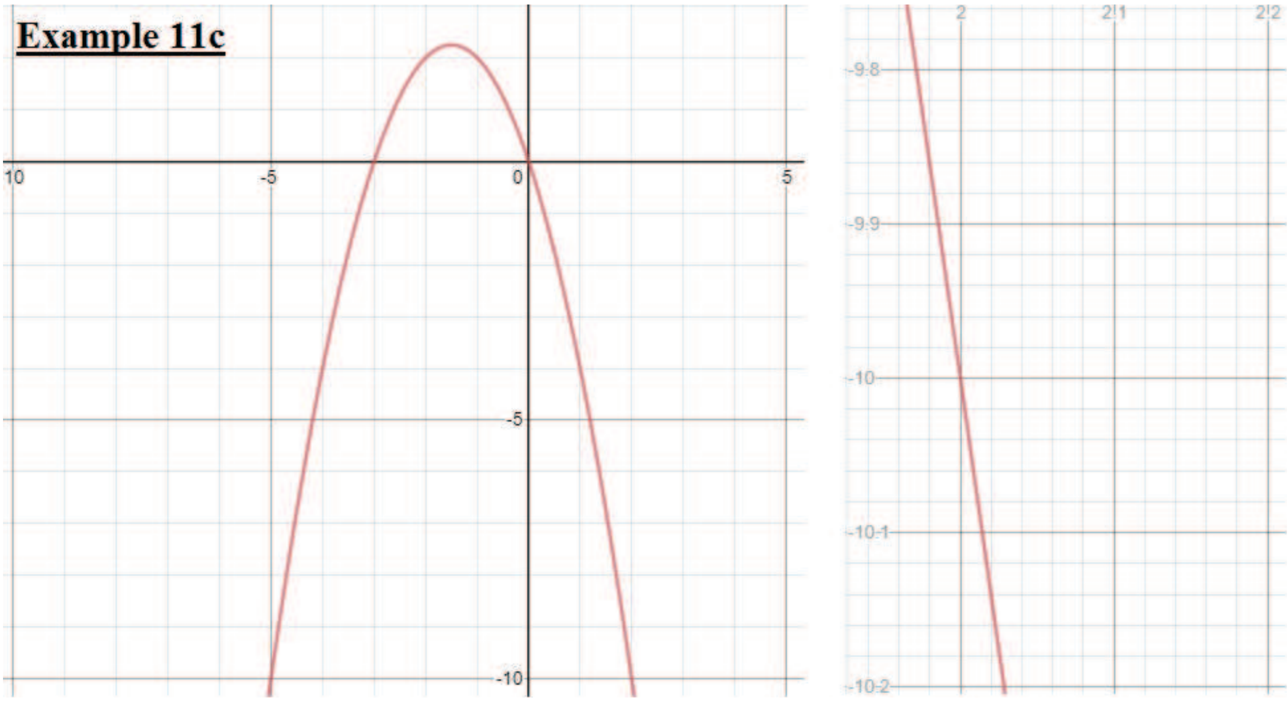
$$f(2) \ = \frac{(\eta_0-2)(2+3)}{\frac{2}{\eta_0}-1} = \frac{(-2)(5)}{\frac{2}{\eta_0}-\frac{0}{0}} = \frac{-10}{\eta_0} \quad \equiv \quad -10 \cdot \left(\frac{1}{\eta_0}\right) = -10 \cdot \eta_0 = -10$$

<u>Example 11d:</u>

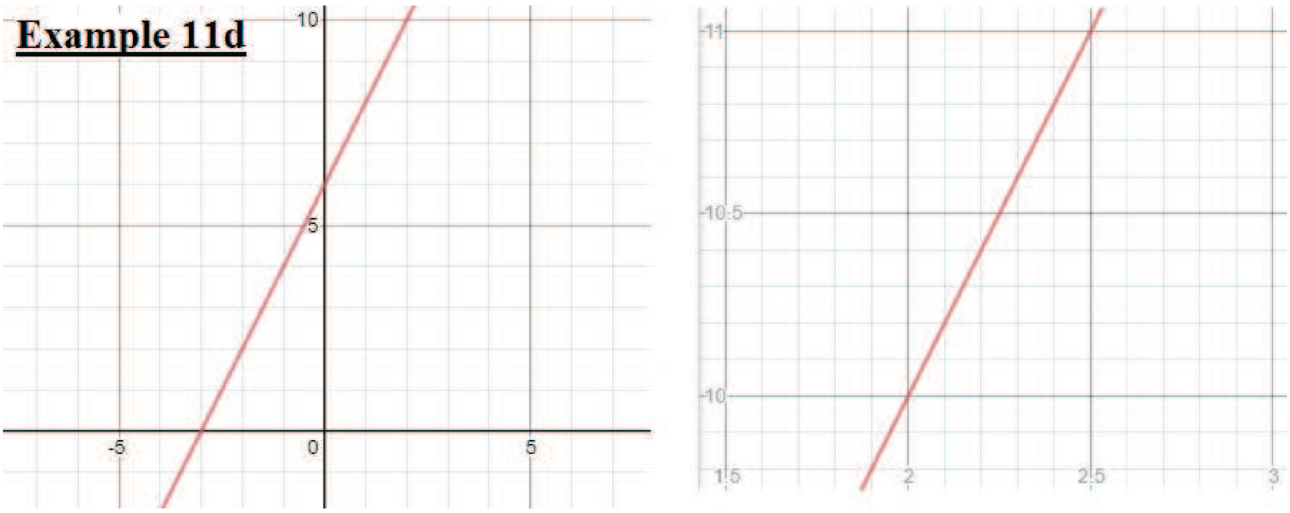
$$f(2) \ = \frac{(\eta_0-2)(2+3)}{\frac{\eta_0}{2}-1} = \frac{(-2)(5)}{0-1} = 10$$

Example 11c results in a lone naught in the denominator. It is resolved by treating it as multiplication just as before. Example 11d has no lone naught occurring in either numerator or denominator at final step and is resolved with standard usage of naught. Again the graphs of each equation will show these in fact are the values the graph reaches.

Graph of 11c:



Graph 11d:



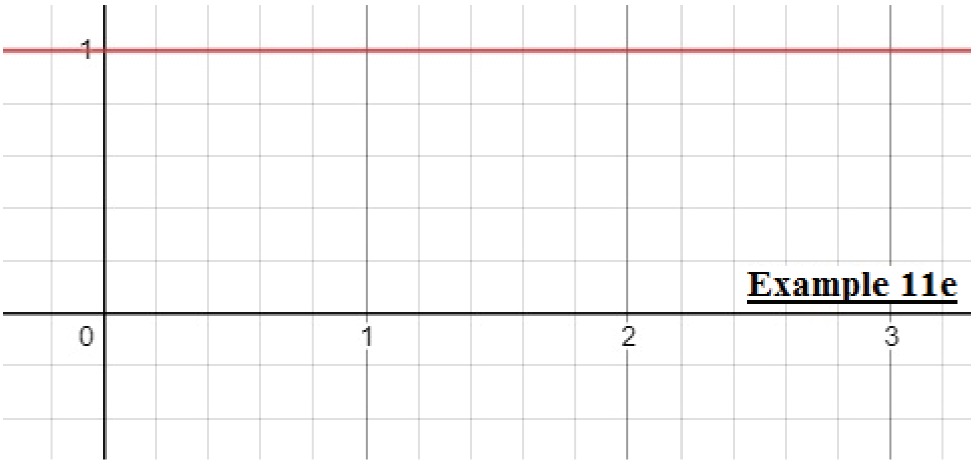
Example 11e:

$$\frac{\frac{x}{2}-1}{\frac{x}{2}-1}$$

Here again at $f(2)$ we get the feedback value $\frac{0}{0}$. In this instance the numerator and denominator are identical and so exist in a 1 to 1 ratio. The value of the expression should equal 1. In solving via naught we find it is indeed 1.

$$f(2) \ = \frac{\frac{\eta_0}{2}-1}{\frac{\eta_0}{2}-1} = \frac{0-1}{0-1} = 1$$

Graph 11e:



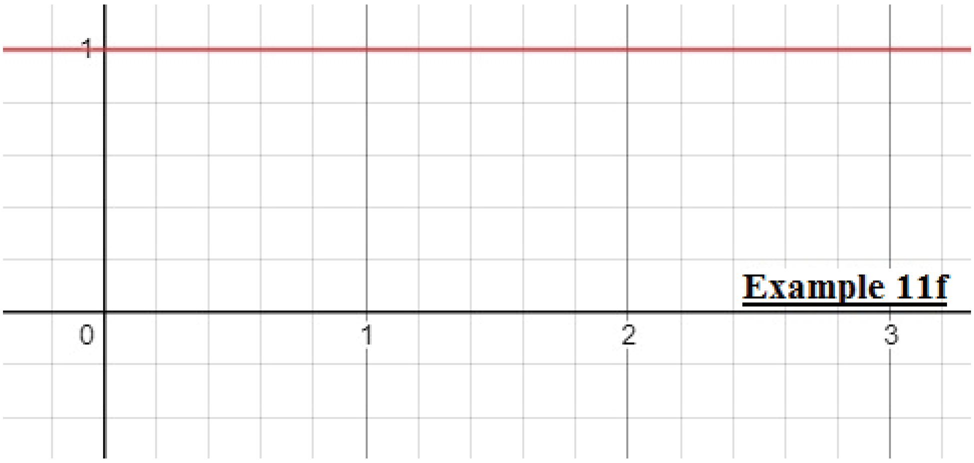
Example 11f:

$$\frac{\frac{2}{x}-1}{\frac{2}{x}-1}$$

This example will result in $\frac{\eta_0}{\eta_0}$. This is essentially $\frac{0}{0}$ as its value as an i-multiple. The value will resolve to $\bigoplus 1$ which is positive 1 where it originates.

$$f(2) \ = \frac{\frac{2}{\eta_0}-1}{\frac{2}{\eta_0}-1} = \frac{\frac{2}{\eta_0}-\frac{0}{0}}{\frac{2}{\eta_0}-\frac{0}{0}} = \frac{\eta_0}{\eta_0} \ \equiv \ \frac{0}{0} = \bigoplus 1 \ \longrightarrow \ 1$$

Graph 11f:



Examples 11g and 11h:

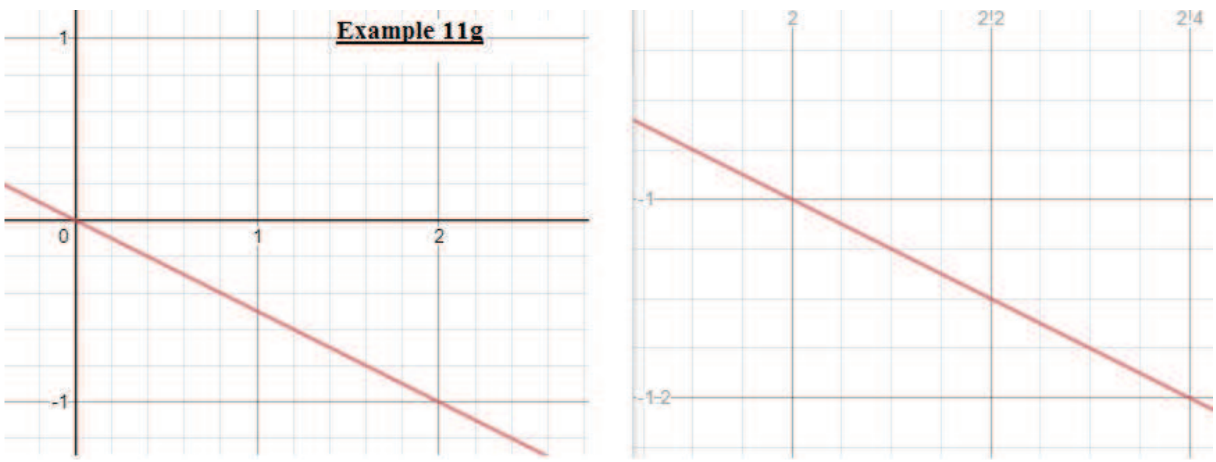
Ex. 11g $\frac{\frac{x}{2}-1}{\frac{2}{x}-1}$ Ex. 11h $\frac{\frac{2}{x}-1}{\frac{x}{2}-1}$

Both examples will result in the feedback solution $\frac{0}{0}$ at $x = 2$. Both expressions will result in -1 at $x = 2$. Both will require the final step be treated as multiplication. Their graphs confirm this.

Example 11g:

$$f(2) = \frac{\frac{\eta_0}{2}-1}{\frac{2}{\eta_0}-1} = \frac{-1}{\frac{2}{\eta_0}-\frac{0}{0}} = \frac{-1}{\eta_0} \equiv -1 \cdot \left(\frac{1}{\eta_0}\right) = -1 \cdot \eta_0 = -1$$

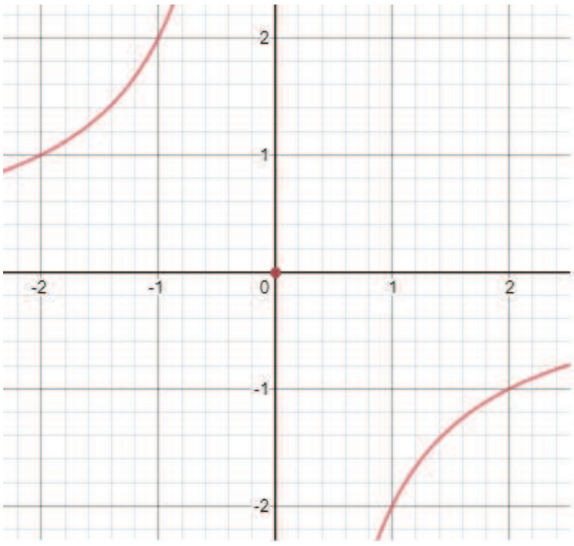
Graph 11g

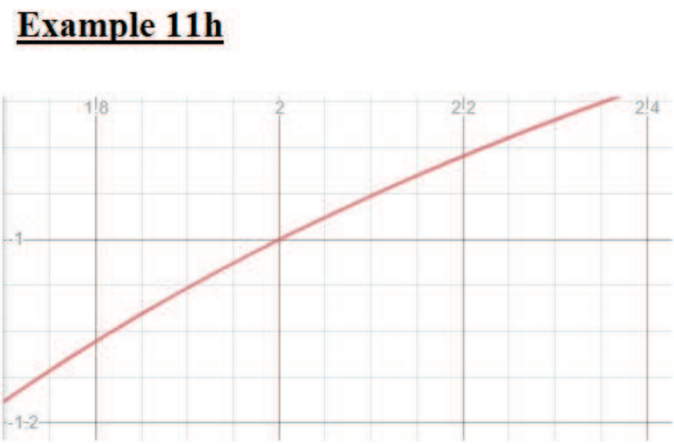


Example 11h:

$$f(2) = \frac{\frac{2}{\eta_0}-1}{\frac{\eta_0}{2}-1} = \frac{\frac{2}{\eta_0}-\frac{0}{0}}{0-1} = \frac{\eta_0}{-1} \equiv \eta_0 \cdot \left(\frac{1}{-1}\right) = \eta_0 \cdot -1 = -1$$

Graph 11h:





Finally consider the following examples in which multiple instances of naught are occurring simultaneously without other real numbers to interact with.

Examples 11i and 11j:

Ex. 11i $\frac{\frac{2}{x}-\frac{2}{x}}{(x-2)}$ Ex. 11j $\frac{\frac{2}{x}+\frac{2}{x}}{(x-4)}$

Figure 11i can be solved by interpreting the values of $\frac{x}{2}$ in three different ways. The feedback value is obtained at $x = 2$. The application of naught to resolve this provides the following:

$$\frac{\frac{2}{\eta_0}-\frac{2}{\eta_0}}{(\eta_0-2)} = \begin{cases} \frac{0}{(-2)} = 0 \\ \frac{\eta_0-\eta_0}{(-2)} = \frac{0}{(-2)} = 0 \\ \frac{\frac{0}{\eta_0}}{(-2)} = \frac{0}{(-2)} = 0 \end{cases}$$

In the first solution (*top*) we note that having the same denominator, $\frac{2}{\eta_0} - \frac{2}{\eta_0}$ is already 0 in the numerator. Therefore using null math operations the solution is 0 for all x. At $x = 2$ this is zero as η_0 but still graphed as 0. The second solution (*middle*) each instance of $\frac{2}{\eta_0}$ will result to naught (*i.e. 2 divided by* η_0 *equals* η_0). Reduced then to $\eta_0 - \eta_0 = 0$ you have $\frac{0}{(-2)}$ which equals 0. Lastly (*bottom*) we have $\frac{2}{\eta_0} - \frac{2}{\eta_0} = \frac{2-2}{\eta_0} = \frac{0}{\eta_0}$. This value will again reduce to 0 leaving the final step as $\frac{0}{(-2)} = 0$.

Example 11j with two fractions added together will not result in a feedback solution. However there remains an instance of occurrence naught when $x = 0$ in the numerator $\frac{2}{x} + \frac{2}{x}$. Likewise there will be division by 0 in the denominator of the expression when $x = 4$. Both instances are easily resolvable with naught.

For $x = 0$:

$$\frac{\frac{2}{0} + \frac{2}{0}}{0 - 4} = \frac{\eta_0 + \eta_0}{-4} = \frac{\eta_0}{-4} = 0$$

Why did we not finish this example with $\left(\eta_0 \cdot -\frac{1}{4}\right) = -\frac{1}{4}$? The requirement to treat a final naught in the numerator or denominator of a traction during resolution requires only to situations which arise in the feedback value. This example has not feedback value at this or any input value. This this final step is concluded as normal where naught, a kind of zero, is divided by a real number and resolves to 0.

For $x = 4$:

$$\frac{\frac{2}{4} + \frac{2}{4}}{4 - 4} = \frac{1}{0} = \eta_0 \doteq 0$$

2.c.7—Completing the Square:

Another method of factoring polynomials is by means of completing the square. Not all polynomials are immediately factorable but can be massaged into a form which is factorable provided a couple of rules are adhered to. Firstly this applies to quadratic expressions. The coefficient in front of the x^2 component must be 1 and it may be necessary to perform some mathematical process to make it so. The example below has all components which are divisible by 4 in order to illustrate this without generating fractions.

$$4x^2 + 8x - 36 \quad \longrightarrow \quad x^2 + 2x - 9$$

Add out the final term. Take half the coefficient of the x term, then square it, and add it to both sides and solve.

$$\frac{2}{2} = 1 \quad , \quad 1^2 = 1$$

$$x^2 + 2x + 1 = 9 + 1 \qquad \longrightarrow \qquad x^2 + 2x + 1 = 10$$

$$(x+1)(x+1) = 10$$

$$\sqrt{(x+1)^2} = \sqrt{10}$$

$$x + 1 = \pm 3.1622776601683793319988935444327$$

$$x = \begin{cases} 2.162 \\ -4.162 \end{cases}$$

Either of these values will satisfy the equation:

$$(x+1)(x+1) \quad = \quad x^2 + 2x + 1 \quad = \quad 10$$

$x = 2.162$	$x = -4.162$
$(2.162)^2 + 2(2.162) + 1$	$(-4.162)^2 + 2(-4.162) + 1$
$4.674 + 4.324 + 1 = 10$	$17.322 - 8.324 + 1 = 10$

Decimal points were heavily truncated and rounded to the thousandths place. Attempting this example will yield values close to 10 due to the truncation. Extending the input values out to further decimal places without truncation will yield values closer to 10.

2.c.7i—Completing the Negative Square:

What happens when this process results in taking a negative root? Traditional math provides the number i is required. This will occur with null algebra as well but with the fully resolved values. Consider the same example used above but with the final term added rather than subtracted in the expression.

$$4x^2 + 8x + 36 \quad \longrightarrow \quad x^2 + 2x + 9$$

The requirements are the same to complete the square. Again everything in this example is divisible by 4 to illustrate how to mathematically manipulate the equation into a proper format with the x^2 coefficient being 1. Dividing by 4 will provide a coefficient of $+1$ for the x^2 term. You still separate out the single coefficient, in this case by subtracting it out. Finally the coefficient with the x term is divided by 2 and then squared before adding it to both sides of the equation.

$$x^2 + 2x + 9 = 0 \quad \longrightarrow \quad x^2 + 2x = -9$$

$$\frac{2}{2} = 1 \qquad 1^2 = 1 \qquad x^2 + 2x + 1 = -9 + 1 \qquad \longrightarrow \qquad x^2 + 2x + 1 = -8$$

This final stage is now factorable. Even though this particular example has resulted in roots of negative numbers we can still complete the square and resolve the resulting i multiples.

$$(x + 1)(x + 1) = -8$$

$$(x + 1)^2 = -8$$

$$\sqrt{(x + 1)^2} = \sqrt{-8}$$

$$x + 1 \quad = \quad \pm(\oplus\, 2.828) \quad = \quad \pm 2.828427124716i \quad = \quad \pm\hat{2}.828427124716$$

Traditional math provides $x = -1 \pm 2.828i$. Either value will satisfy the equation:

$x = -1 + 2.828i$ $([-1 + 2.828i] + 1)^2 = -8$	$x = -1 - 2.828i$ $([-1 - 2.828i] + 1)^2 = -8$

The number $(2.828427124716)i$, being an i multiple will resolve to 2.828427124716 on the plane which it originates and -2.828427124716 on the adjoining subspace plane. Thus when resolving the values we are left with:

$$x + 1 = \pm\hat{2}.828$$

$$x = \begin{cases} \bar{1}.828 \\ -\bar{3}.828 \end{cases}$$

If we return now to the original example equation $(x + 1)^2 = y^2$. These will apply in two preliminary example applications the solution values. We already know already that $y^2 = -8$. In the first we will plug in the values as they appear once resolved $(1.828 \text{ and } -3.828)$. In the second we shall aknowledge that these values are actually resolved i-multiples $(\bar{1}.828 \text{ and } -\bar{3}.828)$. See that this alone, in each example does not produce the value expected. Note the decimal values for the inputs are heavily truncated to save space.

<u>Example with 1.828 and -3.828:</u>

$y^2 = (1.828 + 1)(1.828 + 1) = 8$	$y^2 = (-3.828 + 1)(-3.828 + 1) = 8$
$(x + 1)(x + 1) = -8$ $(1.828 + 1)(1.828 + 1) \neq -8$ $(1.828)^2 + (1.828 \cdot 1) + (1.828 \cdot 1) + (1 \cdot 1) \neq -8$ $3.341584 + 1.828 + 1.828 + 1 \neq -8$ $8 \neq -8$	$(x + 1)(x + 1) = -8$ $(-3.828 + 1)(-3.828 + 1) \neq -8$ $14.6535 + (-3.828 \cdot 1) + (-3.828 \cdot 1) + 1 \neq -8$ $14.6535 - 7.656 + 1 \neq -8$ $8 \neq -8$

Example with $\overline{1}.828$ and $-\overline{3}.828$:

$y^2 = (\overline{1}.828\ +1)(\overline{1}.828\ +1) = 8$	$y^2 = (-\overline{3}.828 + 1)(-\overline{3}.828 + 1) = 8$
$(x+1)(x+1) = -8$ $(\overline{1}.828 + 1)(\overline{1}.828 + 1)\ \neq\ -8$ $(\overline{1}.828)^2 + (\overline{1}.828\ \cdot\ 1) + (\overline{1}.828\ \cdot\ 1) + (1\ \cdot\ 1) \neq -8$ $-3.341584 + 1.828 + 1.828 + 1 \neq -8$ $1.3144 \neq -8$	$(x+1)(x+1) = -8$ $(-\overline{3}.828 + 1)(-\overline{3}.828 + 1)\ \neq\ -8$ $-14.6535 + (-3.828\ \cdot\ 1) + (-3.828\ \cdot\ 1) + 1 \neq -8$ $-14.6535 - 7.656 + 1 \neq -8$ $-21.3096 \neq -8$

In the instances using the fully resolved solution values 1.828 and -3.828, we have $8 \neq -8$. Equally clear, in the use resolved values held as i-multiples $\overline{1}.828$ and $-\overline{3}.828$ we have $1.3144 \neq -8$ and $-21.3096 \neq -8$.

When using the fully resolved values, had you not known these were actually resolved i-multiples, the magnitude of the value is correct but not the sign; 8 rather than -8. In the instances of using the inputs as i-multiples the values are totally off; 1.3144 and -21.3096 respectively. Why? This is because the entire binomial itself is an i-multiplies. $(x+1)$ obtained from completing the square may contain a variable in order represent a range of values, but $(x+1)$ represents a single number. In the instance of this example, $(x+1)^2 = -8$, the $(x+1)$ is an i-multiple. This will apply to both components of the binomial, the x variable and the coefficient. You will need to treat the input values, the variables and the coefficients in this example as i multiples originating on the axis associated with this equation. Had you not known that the root of y^2 in this example is $\hat{2}.828$, you would use only its resoved value of 2.828 and never realize it is actually $2.828i$.

Go back to the first step:

$$(x+1)^2 = -8$$

By using the input and coefficient as i-multiples you obtain the correct answer of -8.

$\underline{x = \bar{1}.828 \quad \text{as} \quad \dot{2}.828 - 1}$	$\underline{x = -\dot{3}.828 \quad \text{as} \quad \dot{-}2.828 - 1}$
$(x+1)(x+1) = -8$	$(x+1)(x+1) = -8$
$(\hat{1}.828 + \hat{1})(\hat{1}.828 + \hat{1}) = -8$	$(-\hat{3}.828 + \hat{1})(-\hat{3}.828 + \hat{1}) = -8$
$-3.341584 - 1.828 - 1.828 - 1 = -8$	$-14.6535 + 3.828 + 3.828 - 1 = -8$

The realization that $x + 1 = \pm\hat{2}.828427124716$ implies both $\hat{x}$ and $\hat{1}$ may not be intuitive. We only obtain the $\pm\hat{2}.828427124716$ value when solving for x because the equation required taking the root of a negative number. In this example that is -8. If we say that $(x+1)(x+1) = (x+1)^2 = y$ then $y = -8$ and it is the squaring of values involving x which generate that negative value for y. This is only possible if the values for the input side begin squared are themselves i-multiples. Even in the original expression $(x+1)(x+1) = -8$, the values for x and 1 are both even here already i-multiples. It's just that they are in their resolved state.

Note that the decimal values of these components have been heavily truncated to save space. The value does equal -8 but to see that occur you'll need to use closer to decimal places like as follows:

$$\hat{x} = \begin{cases} \hat{1}.82842712474619009760337744841 94 \\ -\hat{3}.82842712474619009760337744841 94 \end{cases}$$

The more exact the value input the closer the value comes to the -8.

There is one more form for Completing the Negative Square which is useful—the Difference of negative squares. Given we have now identified that completing the negative square provides the binomials solution components are i-multpiles we make use of the subspace planes on which they are separated from each other. Consider again the two resolved values for x from the equation:

$$(x+1)^2 = -8 \quad \longrightarrow \quad x = \begin{cases} \dot{1}.828 \\ -\dot{3}.828 \end{cases}$$

Each value comes from the root, from its ± 2.828 minus 1. If we accept the value of x is an i-multiple then it is just as correct to say each of these values is representative of a separate half of $\oplus x = (\dot{x} + \dot{1})$. The positive resolved i-multiple of $(x+1)^2 = -8$ is $x = \dot{1}.828$. This value exists on the plane of occurrence. We will use V_n here to prevent confusing the two values. V_1 will represent the positive resolved component.

$$\hat{x} = \hat{1}.828 \quad \longrightarrow \quad \dot{x} = \dot{1}.828$$

$$\hat{V}_1 = (\dot{1}.828 + \dot{1})$$

$$\dot{V}_1 = (\dot{1}.828 + \dot{1})$$

The negative resolved i-multiple of $(x+1)^2 = -8$ is $x = -\dot{3}.828$. This value will occur on a coadjoining subspace plane and can be set as the $\check{x}$ down component.

$$\hat{x} = -\hat{3}.828 \quad \longrightarrow \quad x = -\dot{3}.828$$

$$\check{V}_2 = (-\dot{3}.828 + \dot{1})$$

$$-\dot{V}_2 = (-\dot{3}.828 + \dot{1})$$

$$\dot{V}_2 = (\dot{3}.828 - \dot{1})$$

In squaring these binomials which are representative of $(x+1)^2$ component of the example equation can be literally thought of as multiplying the originating plane's value with that of the coadjoining plane.

$$(x+1)^2 = -8$$

$$(x+1)(x+1) = -8$$

$$(\dot{V}_1)(\dot{V}_2) = -8$$

$$(\dot{1}.828 + \dot{1})(\dot{3}.828 - \dot{1}) = -8$$

$$-7 + 1.828 - 3.828 + 1 = -8$$

Thus we may use the following summarization for completing the negative square—Given a polynomial of the form $(x+a)^2 = -b$ and let $\dot{c} = \pm\sqrt{-b}$ then the following are true:

$([\dot{c}-a]+\dot{a})^2=-b$	$([-\dot{c}-a]+\dot{a})^2=-b$	$([\dot{c}-a]+\dot{a})([\dot{c}+a]-\dot{a})=-b$

When doing this with an equation it may not be possible to simply pull the singular coefficient out to one side of the equals sign. Consider the following equation:

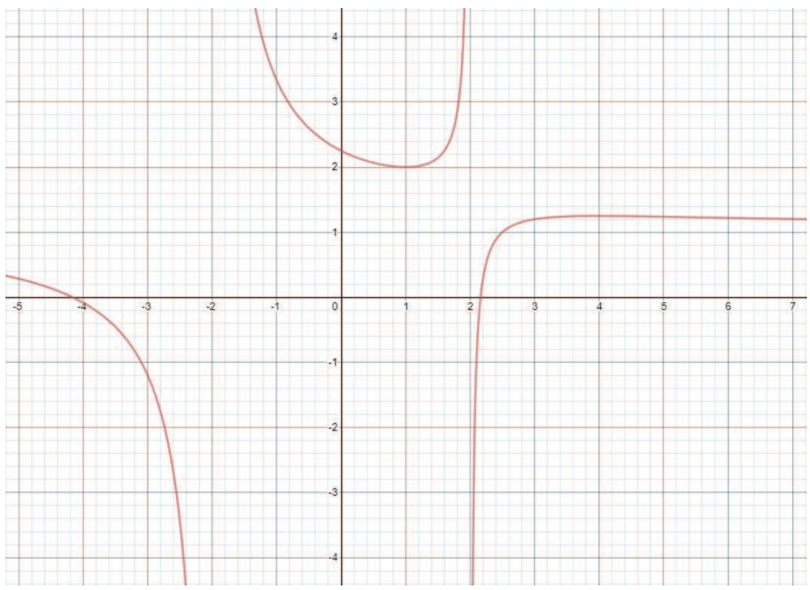
$y=\frac{x^2+2x+9}{x^2-4}=\frac{x^2+2x+9}{(x+2)(x-2)}$	$y=\frac{x^2+2x-9}{x^2-4}=\frac{x^2+2x-9}{(x+2)(x-2)}$

In such an instances we can still use completing the square allow a factorable portion of the numerator but we must adjust the processes. Instead of adding the squared value of half of the x coefficient too both sides of an equals sign, instead you will add and subtract it from the numerator.

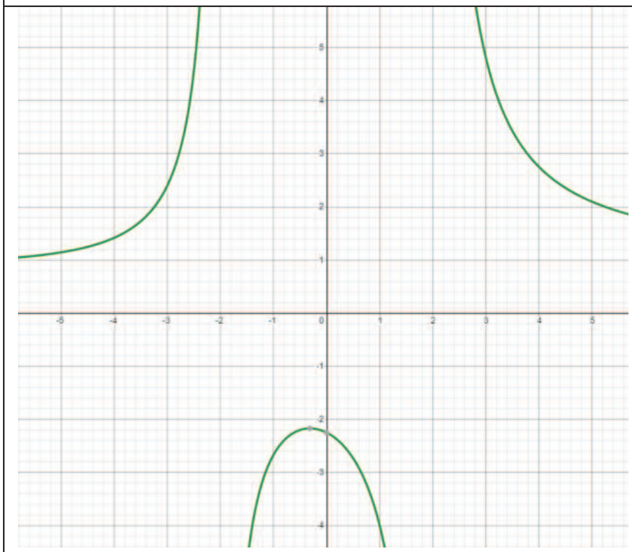
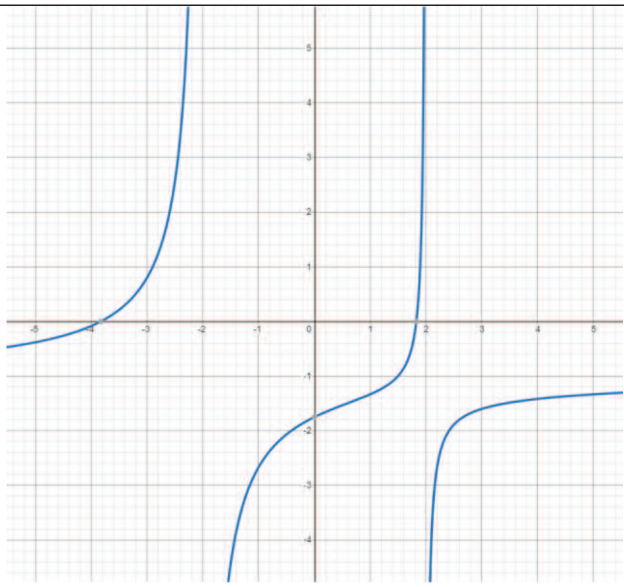
$y=\frac{x^2+2x+9}{(x+2)(x-2)}$ $y=\frac{(x^2+2x+1)+9-1}{(x+2)(x-2)}$ $y=\frac{(\dot{x}+\dot{1})^2+8}{(x+2)(x-2)}$	$y=\frac{x^2+2x-9}{(x+2)(x-2)}$ $y=\frac{(x^2+2x+1)-9-1}{(x+2)(x-2)}$ $y=\frac{(x+1)^2-10}{(x+2)(x-2)}$

In the instance of $y=\frac{(x+1)^2-10}{(x+2)(x-2)}$ the numerator and denominator are neither expected nor appear to contain any i-multiples. We can graph the equation easily:

Complete The Square—Non i-multiple: $y=\frac{(x+1)^2-10}{(x+2)(x-2)}$

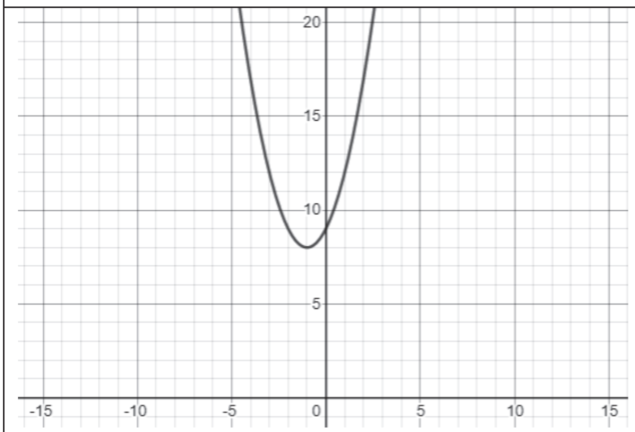
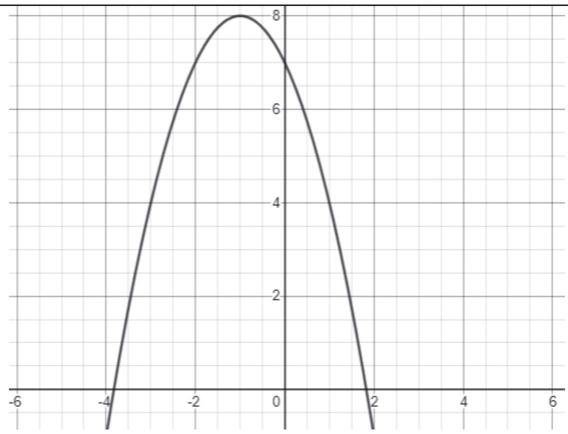


In the example which we know causes a root of negative number when completing the square it's a little more complicated. It essentially represents two equations. In you weren't aware of the i-multiples you would graph $y=\frac{(x+1)^2+8}{(x+2)(x-2)}$. If however you were aware these were actually i-multiples inside the squared parentheses you would graph this with the dotted resolved values as $y=\frac{(\dot{x}+\dot{\imath})^2+8}{(x+2)(x-2)}$. Since graphing utilities likely don't understand the presence of the resolved values you can instead graph the fully squared equation $y=\frac{(-x^2-2x-1)+8}{(x+2)(x-2)}=\frac{-x^2-2x+7}{(x+2)(x-2)}$.

Completing Negative Square: Ignored i-multiples $y=\frac{(x+1)^2+8}{(x+2)(x-2)}$	Completing Negative Square: Resolved i-multiples $y=\frac{(\dot{x}+\dot{\imath})^2+8}{(x+2)(x-2)}=\frac{-x^2-2x+7}{(x+2)(x-2)}$
	

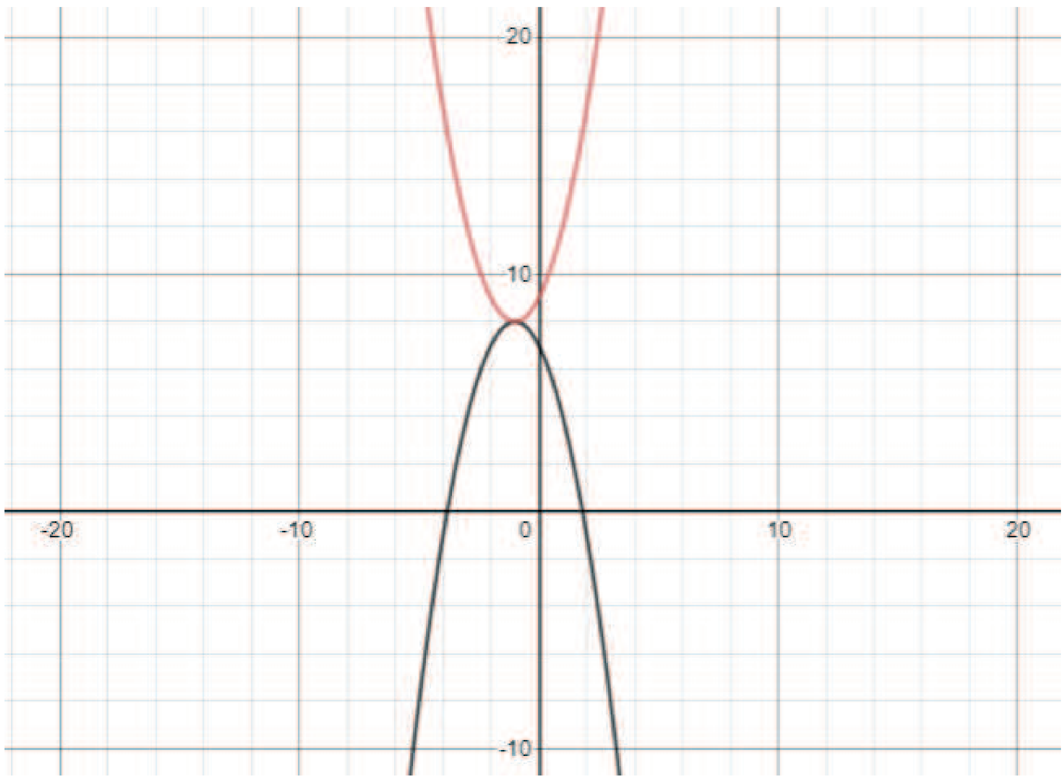
2.c.7ii—Subspace Equations and Completing the Square:

Return to the equation $y = (x+1)^2 + 8$. Recall this is the completed square of the original equation $y = x^2 + 2x + 9$. Using just this equation we have a form in which we ignore the presence of the i-multiples and a form in which we include them:

Completing Negative Square: Ignored i-multiples Ex 1 $y = (x+1)^2 + 8$	Completing Negative Square: Resolved i-multiples Ex1 $y = (\dot{x} + \dot{1})^2 + 8 = -x^2 - 2x + 7$
	

You can see that these two graphs are vertical mirrors of each other.

$\boldsymbol{y = (x+1)^2 + 8}$ $\quad y = (\dot{x} + \dot{1})^2 + 8 = -x^2 - 2x + 7$



$y = (\dot{1}.828 + \dot{1})^2 + 8 = -\overline{3}.342 - \overline{3}.656 + \overline{7} = 0$

$y = (-\dot{3}.828 + \dot{1})^2 + 8 = -\overline{14}.654 + \overline{7}.656 + 7 = 0$

From the completed square / completed negative square equation $y = (x+1)^2 + 8$ there are three equations which correspond to the adjoining and coadjoining subspaces:

SY-Plane:

$$y = \left(-\tfrac{1}{s} + 1\right)^2 + 8 \qquad\qquad y = \left(-\tfrac{1}{\dot{s}} + \dot{1}\right)^2 + 8$$

$$y = \left(\tfrac{1}{s^2} - \tfrac{2}{s} + 1\right) + 8 \qquad\qquad y = \left(-\tfrac{1}{s^2} + \tfrac{2}{s} - 1\right) + 8$$

$$y = \tfrac{1}{s^2} - \tfrac{2}{s} + 9 \qquad\qquad y = -\tfrac{1}{s^2} + \tfrac{2}{s} + 7$$

The set of equations on the left is what you would obtain if you ignored, or didn't realize, the values with in the parentheses are in fact both i-multiples. This is because the entire value of $\left(-\frac{1}{s} + 1\right)$ is a binomial, but still representative of a single number. A number which in this example is a squared i-multiple. The equations on the right are the values obtained when we keep track that these are actually resolved i-multiples. The solutions obtained for the zeros in completing the negative square will apply here as well. Using the equation $y = \left(-\frac{1}{\dot{s}} + \dot{1}\right)^2 + 8$ which keeps track of the i-multiples, you set the equation equal to 0 and solve:

$$y = \left(-\tfrac{1}{\dot{s}} + \dot{1}\right)^2 + 8 \qquad \left(-\tfrac{1}{\dot{s}} + \dot{1}\right)^2 = -8 \qquad -\tfrac{1}{\dot{s}} + \dot{1} = \sqrt{-8}$$

$$-\tfrac{1}{\dot{s}} = \begin{cases} \dot{2}.828 - 1 = \bar{1}.828 \\ -\dot{2}.828 - 1 = -\bar{3}.828 \end{cases} \qquad \dot{s} = \begin{cases} -\frac{1}{\bar{1}.828} = -\bar{0}.547 \\ \frac{1}{\bar{3}.828} = \bar{0}.261 \end{cases}$$

Note decimals are heavily truncated to save space. Using the truncated values will result in values close that that shown. To get exact answers requires using more exact decimal values.

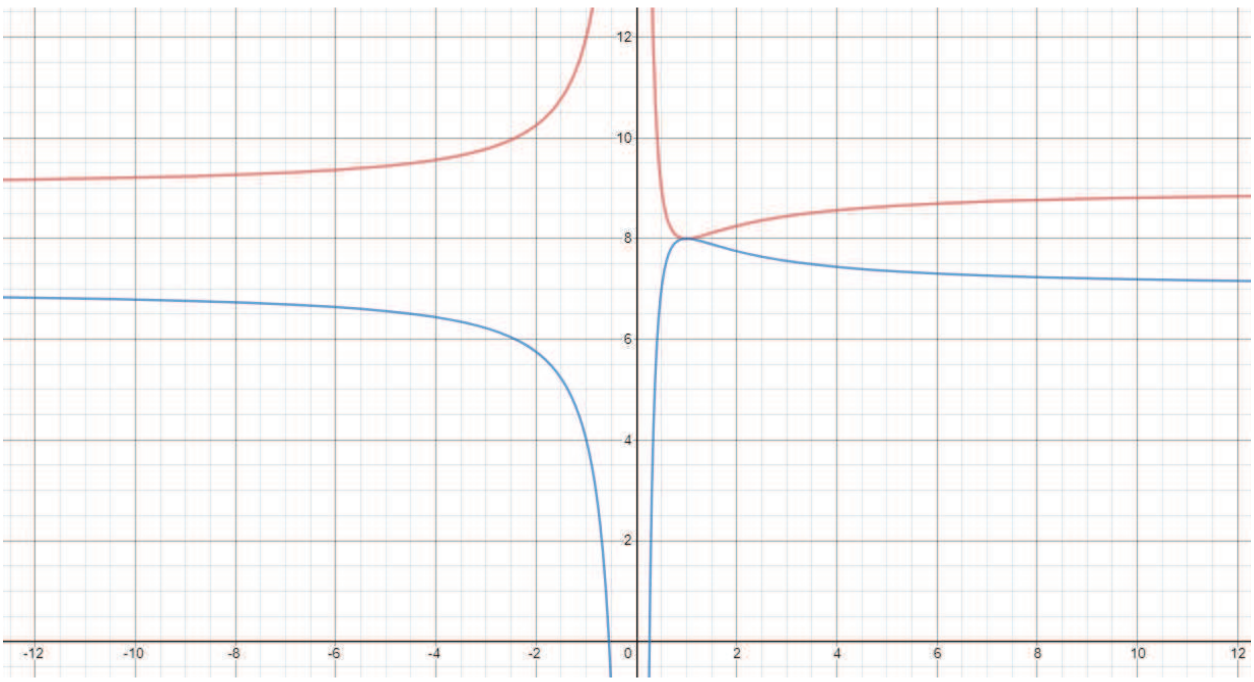
$\dot{s} = -\bar{0}.547$ $y = \left(-\frac{1}{\dot{s}} + \dot{1}\right)^2 + 8$ $y = \left(-\frac{1}{-\bar{0}.547} + \dot{1}\right)^2 + 8$ $y = \left(-\frac{1}{0.29926} - \frac{2}{0.547} - 1\right) + 8$	$\dot{s} = \bar{0}.261$ $y = \left(-\frac{1}{\dot{s}} + \dot{1}\right)^2 + 8$ $y = \left(-\frac{1}{\bar{0}.261} + \dot{1}\right)^2 + 8$

$y = (-3.341 - 3.6563 - 1) + 8$ $y = 0$	$y = \left(-\frac{1}{\bar{0}.0681} + \frac{2}{\bar{0}.261} - 1\right) + 8$ $y = (-14.684 + 7.6628 - 1) + 8$ $y = 0$

The combined graphs of the two $y = f(s)$ functions are shown below and are again mirrors of eachother. This is again due to the negative generated by squaring i-multiples.

Completing Negative Square: Ignored i-multiples	Completing Negative Square: Resolved i-multiples
$\boldsymbol{y = \left(-\frac{1}{s} + 1\right)^2 + 8 \;=\; \frac{1}{s^2} - \frac{2}{s} + 9}$	$\boldsymbol{y = \left(-\frac{1}{\dot{s}} + \dot{i}\right)^2 + 8 \;=\; -\frac{1}{\dot{s}^2} + \frac{2}{\dot{s}} + 7}$

Ex2



Using $\dot{s} = -\bar{0}.547$ and $\dot{s} = \bar{0}.261$

$$y = \left[\left(-\frac{1}{-\bar{0}.547} + \dot{i}\right)\left(-\frac{1}{-\bar{0}.261} - \dot{i}\right)\right] + 8$$

$$y = [-7.004 + 1.828 - 3.831 + 1] + 8$$

$$y = 0$$

A final note here on the negative square solution. This value will produce a zero with the given values as i-multiples. However, if you attempt to multiply out the values as a generalized equation using the variable s you get an equation different that the one you started with. Consider this using the current example:

Given $y = \left(-\frac{1}{\dot{s}} + \dot{1}\right)^2 + 8$ we know that $y = -\frac{1}{\bar{s}^2} + \frac{2}{s} + 7$. If you multiply out $y = \left[\left(-\frac{1}{\dot{s}} + \dot{1}\right)\left(\frac{1}{\dot{s}} - \dot{1}\right)\right] + 8$, the form of the implied by the above negative square solution, you will get $y = \frac{1}{\bar{s}^2} - \frac{2}{s} + 9$. The reason the negative square solution is viable is that it considers not just the variable inputs but rather each binomial itself as half of the i-multiple being squared. The form of the equation seen is till the resolved version of the equation. To use the continuing example this is:

$$y = \left(\dot{x} + \dot{1}\right)^2 + 8 \quad \equiv \quad (\dot{x} + \dot{1})(\dot{x} + \dot{1}) + 8$$

Both of these equations when multiplied out directly from there resolved values, which maintain identification of these values are i-multiples, will provide $y = -x^2 - 2x + 7$. The difference is the component marked in red, $(\dot{x} + \dot{1})$, is itself considered the *up* component of two resolved i-multiples. The *down* component of the resolved i-multiple pair is the one marked here in blue, $(\dot{x} + \dot{1})$. For the sake of multiplication of these terms in the form of the Completed Negative Square, requires application of signs on these terms as:

$$V_1 = \hat{V} = +V = (\dot{x} + \dot{1})$$

$$V_2 = \check{V} = -V = -(\dot{x} + \dot{1}) = (-\dot{x} - \dot{1})$$

This format maintains the form of the equation itself which when multiplied out will provide

$$y = \left(\dot{x} + \dot{1}\right)^2 + 8 \quad = \quad -x^2 - 2x + 7$$

Yet the application of the values which were solved for to obtain zeros, $x = \dot{1}.828$ and $-\dot{3}.828$ when applied to the form of the Completed Negative Square will still provide zeroes. Decimals her are heavily truncated.

$$(\dot{x} + \dot{1})(-\dot{x} - \dot{1}) + 8 \quad = \quad -6.997584 + 1.828 - 3.828 + 9 \quad = \quad 0$$

XU-Plane: $u = -\frac{1}{(x+1)^2+8}$ $u = -\frac{1}{(\dot{x}+\dot{1})^2+8}$

$u = -\frac{1}{(x^2+2x+1)+8}$ $u = -\frac{1}{(-x^2-2x-1)+8}$

$u = -\frac{1}{x^2+2x+9}$ $u = -\frac{1}{-x^2-2x+7}$

$u = \frac{1}{x^2+2x-7}$

Solving for the zeros provides the same values as before for the $\left(\dot{x}+\dot{1}\right)^2+8$ equaiton. Using the denominator alone we have:

$-8 = (x+1)^2$ $x = \sqrt{-8}-1$ $x = \pm 2.8284x - 1$

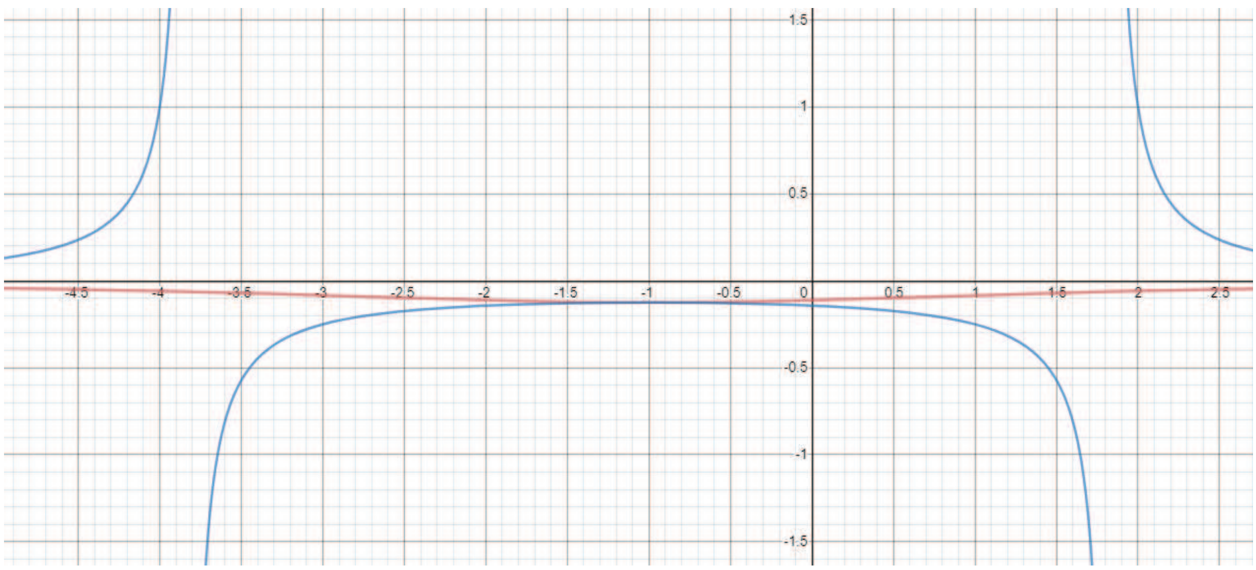
$$x = \begin{cases} \hat{1}.8284 \\ -\hat{3}.8284 \end{cases}$$

Ignoring i-multiples $x = 1.828$	Maintaining i-multiples $x = \hat{1}.828$
Ex 3 $u = -\frac{1}{(x+1)^2+8}$ $u = -\frac{1}{((1.828)^2+2(1.828)+1)+8}$ $u = -\frac{1}{3.342+3.656+9}$ $u = -\frac{1}{3.342+3.656+9}$ $u = -\frac{1}{16}$	Ex3 $u = -\frac{1}{\left(\dot{x}+\dot{1}\right)^2+8}$ $u = -\frac{1}{(-\bar{3}.342-\bar{3}.656-1)+8}$ $u = -\frac{1}{0} = -\eta_0 \doteq 0$
Ignoring i-multiples $x = -3.828$	Maintaining i-multiples $x = -\hat{3}.828$
Ex 3 $u = -\frac{1}{(x+1)^2+8}$ $u = -\frac{1}{((-3.828)^2+2(-3.828)+1)+8}$ $u = -\frac{1}{14.654-7.656+9}$	Ex3 $u = -\frac{1}{\left(\dot{x}+\dot{1}\right)^2+8}$ $u = -\frac{1}{(-\overline{14}.654+\bar{7}.665-1)+8}$

$u = -\frac{1}{16}$	$u = -\frac{1}{0} = -\eta_0 \doteq 0$ $u = -\frac{1}{0} = -\eta_0 \doteq 0$
Also at $x = \pm\infty$, $u \doteq 0$	

Ex3

$u = -\frac{1}{(x+1)^2+8}$ $\qquad\qquad$ $u = -\frac{1}{(\dot{x}+\dot{1})^2+8}$



SU-Plane:

Solving for the zeros in the denominator of the $u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2+8}$ equation will again provide $\dot{s} = -\bar{0}.547$ and $\dot{s} = \bar{0}.261$.

$u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2+8}$ $\quad \longrightarrow \quad$ solving for the denominator

$\sqrt{-8} = -\frac{1}{s} + \dot{1}$ $\qquad$ $\pm\hat{2}.828 - 1 = -\frac{1}{s}$ $\qquad$ $s = \begin{cases} \frac{1}{3.8284} = \hat{0}.2612 \\ \frac{1}{-1.828} = -\hat{0}.54704 \end{cases}$

Ignoring i-multiples $s = -0.547$

Ex 4

$$u = -\frac{1}{\left(-\frac{1}{s}+1\right)^2 + 8}$$

$$u = -\frac{1}{\frac{1}{0.2992}+\frac{2}{0.547}+9}$$

$$u = -\frac{1}{3.342+3.656+9}$$

$$u = -\frac{1}{16}$$

Maintaining i-multiples $s = -\hat{0}.547$

Ex4

$$u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2 + 8}$$

$$u = -\frac{1}{-\frac{1}{0.2992}-\frac{2}{0.547}+7}$$

$$u = -\frac{1}{(-\bar{3}.342-\bar{3}.656-1)+8}$$

$$u = -\frac{1}{0} = -\eta_0 \doteq 0$$

Ignoring i-multiples $s = 0.261$

Ex 4

$$u = -\frac{1}{\left(-\frac{1}{s}+1\right)^2 + 8}$$

$$u = -\frac{1}{\frac{1}{0.0681}-\frac{2}{0.261}+9}$$

$$u = -\frac{1}{14.684-7.663+9}$$

$$u = -\frac{1}{16}$$

Maintaining i-multiples $s = \hat{0}.261$

Ex4

$$u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2 + 8}$$

$$u = -\frac{1}{-\frac{1}{0.0681}+\frac{2}{0.261}+7}$$

$$u = -\frac{1}{-14.684+7.662+7}$$

$$u = -\frac{1}{0} = -\eta_0 \doteq 0$$

Also at $s = 0$

$$u = -\frac{1}{\left(-\frac{1}{s}+1\right)^2+8}$$

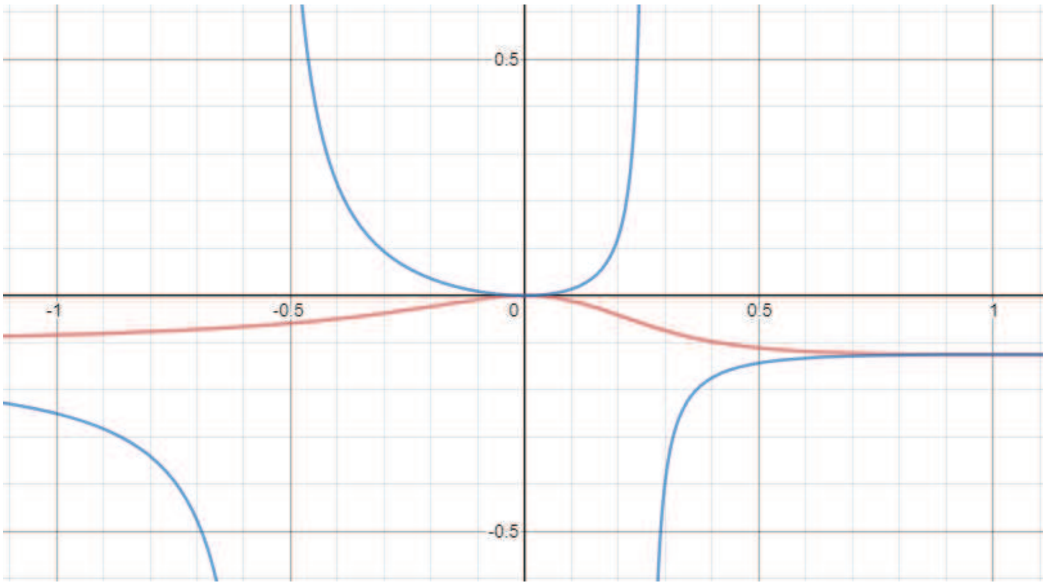
$$u = -\frac{1}{\left(-\frac{1}{0}+1\right)^2+8} = -\frac{1}{\infty} \doteq \eta_0 \doteq 0$$

$$u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2+8}$$

$$u = -\frac{1}{\left(-\frac{1}{\dot{0}}+\dot{1}\right)^2+8} = \frac{1}{\infty} \doteq \eta_0 \doteq 0$$

Ex4

$\boldsymbol{u = -\frac{1}{\left(-\frac{1}{s}+1\right)^2+8}}$ $\qquad$ $\boldsymbol{u = -\frac{1}{\left(-\frac{1}{\dot{s}}+\dot{1}\right)^2+8}}$



2.c.8—The Quadratic Formula:

The Quadratic Formula will provide the exact value for the input which will zero a quadradic equation. The Quadratic Formula is as follows:

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \text{for} \qquad ax^2+bx+c$$

The only difference will be for a negative radical which will resolve to the positive i-multiple component of the root. Consider the following examples:

Example 1:

$$x^2+4 \qquad \longrightarrow \qquad ax^2+bx+c \qquad \longrightarrow \qquad (1)x^2+(0)x+4$$

$$x^2+4 \qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \longrightarrow \quad \frac{\sqrt{-4(1)(4)}}{2} \quad = \quad \frac{\widehat{4}}{2} \quad = \quad \dot{2}$$

With the solution value applied: $\quad (\dot{2})^2+4 \quad = \quad -4+4 \quad = \quad 0$

Example 2:

$2x^2 - 9x + 5$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \longrightarrow \quad \frac{9 \pm \sqrt{(-9)^2 - 4(2)(5)}}{2(2)} \quad \longrightarrow \quad \frac{9 \pm \sqrt{81 - 40}}{4} \quad \longrightarrow \quad \frac{9 \pm \sqrt{41}}{4}$$

$$x = \frac{9 \pm 6.403}{4} = \begin{cases} 3.851 \\ \\ 0.649 \end{cases}$$

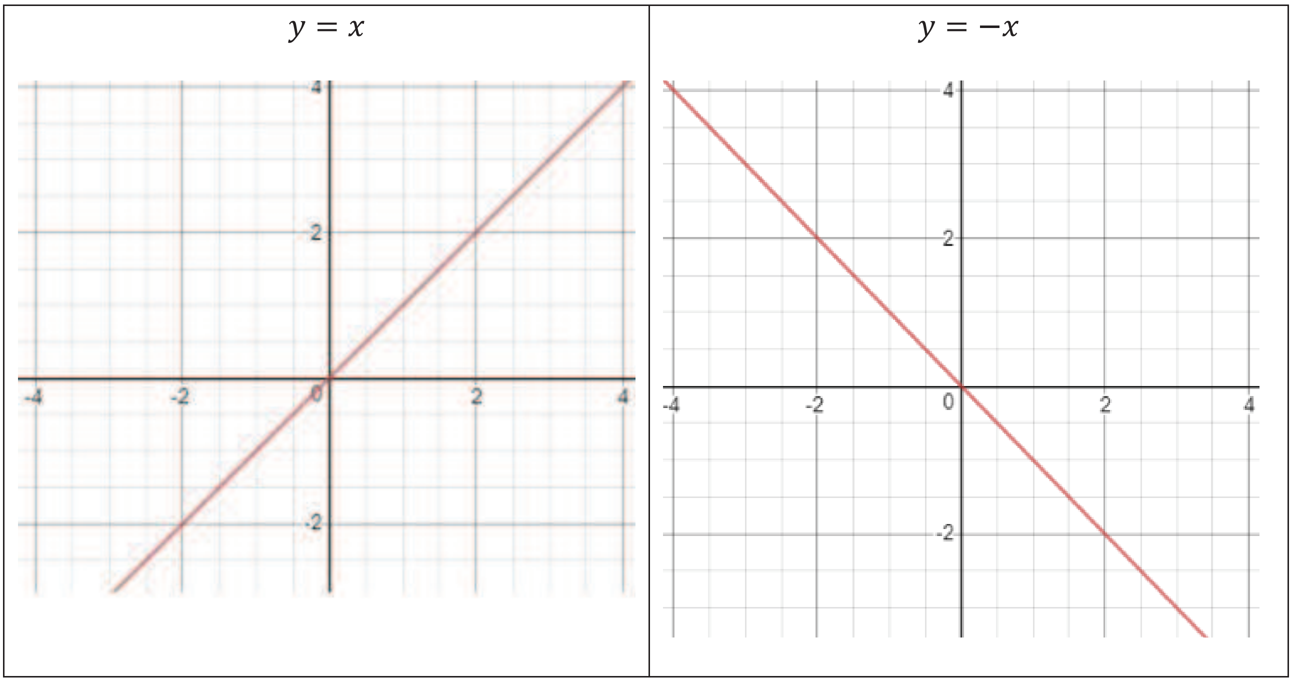
Chapter 3

Common Graphs

3.a—Graphs of $\frac{0}{0}$ Some graphs are important to be able to recognize. They represent very common ratios as well as rates of change. Some of these change slightly when we consider the inclusion of plus-and-minus, subspace, numbers.

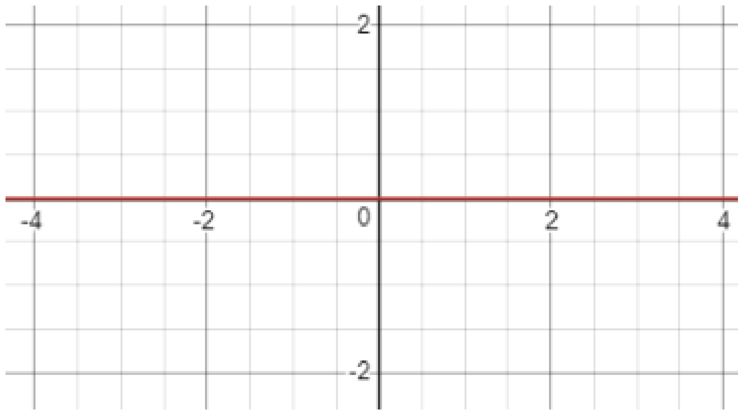
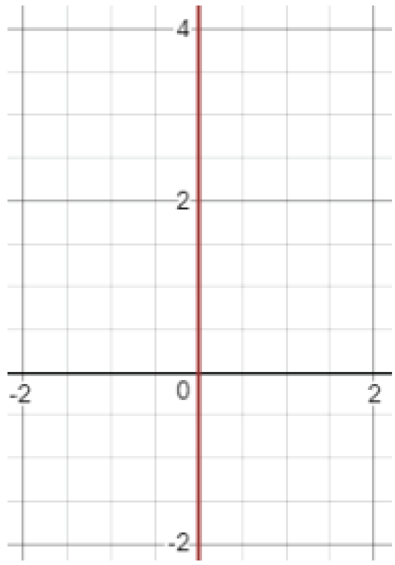
The first four graphs shown will revisit $\frac{0}{0}$. The most basic linear graphs are $y = x$ and $y = -x$. The slopes of the graph are 1 and -1 respectively.

Figure 18:



The second set of these simple graphs are those of the Cartesian plane axis, $y = 0$ and $x = 0$. The graph of the x-axis, $y = 0$, has 0 slope as there is no change in rise for any all change in run. The graph of the y axis, $x = 0$, is traditionally described as having an undefined slope. The slope is a change in rise over zero change in run. This is *naught*, the absence of slope and therefore resolves to 0; non-existent slope (*no slope*) rather than 0 slope. This is just like that of $y = 0$ but turned vertical.

Figure 19:

$y = 0$	$x = 0$
	

These four graphs are simple but display clearly positive slope, negative slop, and two different types of zero slope. Traditionally these four approaches on the Cartesian plane are used as an argument against the ability to resolve $\frac{0}{0}$ to any specific value, an argument which is flawed. Within the axioms of Null Algebra these issues are resolved and viable solutions result. Here they will be used as a teaching point on how the supposed trouble points are resolved.

The argument presented is to consider a graph of $\frac{y}{x}$. Traditional mathematics uses the results of this expression to show $\frac{0}{0}$ is impossible to set to any single reasonable value. By setting the values of x or y to 0 or others values you can force the equation to equal either 0 or 1 at will. Again using this reason to argue the expression as $\frac{0}{0}$ is indeterminate is flawed. This section will show how $\frac{0}{0}$ resulting from such an expression $\frac{y}{x}$ approaches definite values.

First note that the expression $\frac{y}{x}$ is meaningless on its own; it is just an expression. For it to have meaning it must either equal some constant or be set equal to another variable. To keep things in the standard Cartesian XY-Plane we will begin by examining instances of this expression which match either $\pm 1 = \frac{y}{x}$ and $C = \frac{y}{x}$ where C is some constant. In the first instant we are examining a known function $y = x$ where the x has simply been divided out making it $\pm 1 = f(x\,, y)$. In the second situation we have more generally $C = f(x\,, y)$.

Notice how we arrive at the form of $\pm 1 = \frac{y}{x}$. This equation represents the two most basic linear relationships: $y = x$ and $y = \ -x$. Divide out x in these equations to get $1 = \frac{y}{x}$ and $-1 = \frac{y}{x}$. The forms of the equations have changed but they are still $y = x$ and $y = \ -x$. So you have to maintain that relationship otherwise it's a different equation. In other words you only really have one free variable, in this case $\pm x$, and y must equal whatever value it takes.

The substitution for $y = x$ and $y = -x$ into $1 = \frac{y}{x}$ and $-1 = \frac{y}{x}$ provides respectively $1 = \frac{x}{x}$ and $-1 = \frac{-x}{x} = -\frac{x}{x}$. This is exactly the one-to-one ratio of $y = x$ and $y = -x$. At $x = 0, y = 0$ with the equations $1 = \frac{0}{0}$ and $-1 = \frac{0}{0}$. This is the first instance which is argued for inconsistency. Obviously $1 \neq -1$ so how can this be? This issue comes from the fact that traditional mathematics ignores zero being positive-and-negative, claiming in error that it is instead neither. Whether we mark a positive or negative sign next to a zero, traditional mathematics claims the value is just plain 0.

Null mathematics requires we observe trending on the function in addition to using 0 as a positive-and-negative number. The topic of trending was mentioned earlier. These expressions are always in a one-to-one ratio and will equal 1 and -1 respectively for all values of x. The resulting $\frac{0}{0}$ comes directly from the equation as a single value, not interacting with any other expressions. Let us consider $1 = \frac{y}{x}$ first. The equation is trending toward 1 and the resulting $\frac{0}{0}$ is not a feedback value. The value is $\frac{0}{0} = \oplus\, 1$ which resolves to 1 on the originating axis. So the relationship of $1 = \frac{y}{x} = \frac{0}{0}$ does equal 1 which matches $y = x \quad \longrightarrow \quad 0 = 0$.

What about $= -x \quad \longrightarrow \quad -1 = \frac{y}{x}$? Start the problem off the same way. Begin by substituting the value of $y = -x$. You will get $-1 = \frac{y}{x} = \frac{-x}{x} = -\frac{x}{x}$. Now when you substitute for $x = 0$ you get an equation which is trending to -1 with the following solution:

$$-1 = -\frac{x}{x} = -\left(\frac{0}{0}\right) = -(\oplus\, 1) = -(1) = -1$$

Again the one-to-one relationship is preserved resulting in -1 for all values of x in the equation $y = -x \quad \longrightarrow \quad -1 = \frac{y}{x}$.

The next set of issues come when traditional mathematics suggests the same $\frac{0}{0}$ expression will equal either 0 or infinity. This is not an issue any more than the simple resolution of the value in the two instances above. This results from instances where the constant the expression equals is not set statically to 1 or -1. Instead we have an expression such that $C = \frac{y}{x}$ where both y and x are free variables. Obviously C will vary drastically.

We ask then what happens when either x or y is held at 0 and the other variable is allowed to approach 0 ? The value it will approach is again $\frac{0}{0}$ and you must evaluate the equation. Lets begin with $C = \frac{0}{x}$, where y is held fixed at 0 and x is allowed to vary. If we stated $x \neq 0$ traditioanl mathematics says $C = 0$ for all x. The equation is clearly trending toward 0. What it fails to tell you is whether that is 0 as in zero cookies or 0 as in no value, i.e. η_0. It could be either. Recall from the properties of naught that $0 = \eta_0$ but $0 \not\equiv \eta_0$.

When $x = 0$ we have $C = \frac{0}{0}$. This value does not originate from a constant one-to-one expression or from a negative radical (*the plus-and-minus component of an* i*-multiple*). It cannot be evaluated as $\oplus$ 1. This is a fixed 0 being divided by a number which just happens to also be 0, and results in a feedback value. The solution is obtained by substituting with naught.

$$C = \frac{0}{x} = \frac{0}{\eta_0}$$

This equation asks us how many times we can take 0 cookies and divide them up into non-existence. Well the cookies are still out there somewhere. You don't have any and may be sad about that but you cannot make them non-existent. You cannot do this, not even 0 times. The output itself is naught η_0. Yet this value is a kind of 0 and graphable as such so in the end its still 0.

$$C = \frac{0}{x} = \frac{0}{\eta_0} = \eta_0 \doteq 0$$

This is essentially a graph of the x-axis. If you solve out for y you find $Cx = y = 0$. This can only be so if the value of the constant is itself 0, which the above expression shows to be true.

What then if we hold $x = 0$ and let y vary to obtain $= \frac{y}{0}$? This carries similar logic to the last example. When $y = 0$ we get $C = \frac{0}{0}$. The value does not result from a constant 1 to 1 ratio nor is it the plus-and-minus component of i-multiple resulting from a negative radical. This value is a feedback value and must be solved with naught. Note that even in the form $C = \frac{y}{0}$ traditional mathematics states this value is undefined claiming it is a limit that attempts to reach infinity. Infinity is impossibly far away and therefore naught, η_0, a kind of 0. So again the expression is trending toward 0 which will see in a moment.

By solving with naught for the feedback value we get:

$$C = \frac{\eta_0}{0}$$

This time we are being asked if we have no value at all, naught, how many times we can invoke from nothing the presence of some value which holds 0 units. You can do this 0 times because you cannot evoke something from nothing. Even if you were to gather massive amounts of raw energy and run $E = mc^2$ in reverse to create matter it came not from the void but rather transumutated raw energy. The answer is again 0.

$$C = \frac{\eta_0}{0} = 0$$

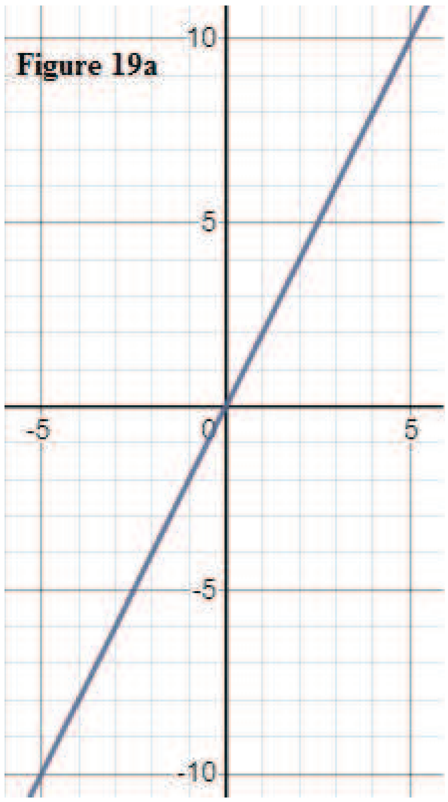
Here again we can solve for x and get:

$$\frac{1}{C} = \frac{0}{y} \quad \longrightarrow \quad \frac{y}{C} = 0 = x$$

The only way this can be true is if $C = 0$ to result in naught, resolvable to 0 for all x. This is the graph of the y-axis for which x is always 0 for all values of y.

This same situation works for any constant C. Consider fixed value for $C = 2$ in the equation $y = 2x$. This equation holds that $2 = \frac{y}{x}$. You will find the value of y is always twice x. A graph like this looks just like the graph of $y = x$ but with an incline, or slope, which is steeper or shallower depending on the factor multiplying the x input.

Graph of $2 = \frac{y}{x}$



You are still left with a choice to hold one input at a given value while the other varies. When one of the variables equals 0 the other must as well. See below how this is resolved:

$2 = \frac{y}{x} = \frac{0}{x} \quad \longrightarrow \quad 2x = 0$ $x = 0$	$2 = \frac{y}{x} = \frac{y}{0} \quad \longrightarrow \quad 2 \cdot 0 = y$ $y = 0$

Traditional math can be used to solve this problem. The value of x must be 0. If you let x vary, when it reaches 0 this is resolved by way of naught to solve for the value of the Constant. $$C = \frac{0}{x} = \frac{0}{\eta_0} = \eta_0 \doteq 0$$ Now use this value for the constant when $x = 0$ and you'll find the value for y is identical to that found from traditional math. $$y = c \cdot x \qquad y = \eta_0 \cdot 0 = 0$$ The multiplication of naught shows that no multiplication takes place leaving $y = x = 0$	Again we see that traditional math shows y will equal 0 when x equals 0. Now if you instead hold that x equal 0 and let y vary, when it equals 0 you will solve the feedback value with naught for the constant. $$C = \frac{y}{0} = \frac{\eta_0}{0} = 0$$ Now use this value for the constant when $y = 0$ and you find the value for x is identical to that found from traditional math. $$y = c \cdot x \qquad \frac{y}{c} = x$$ Where y was held to be 0. Regarless the value of the constant the expression is trending toward 0 for x. Using the value of the constant as 0 will require resolving it with naught and will still provide $x = 0$. This is the same value we get for x above.

(*See division by zero in Chapter 1, obtaining values for slope of some equations and in Chapter 2 on factoring for more*).

3.b—Other Common Graphs:

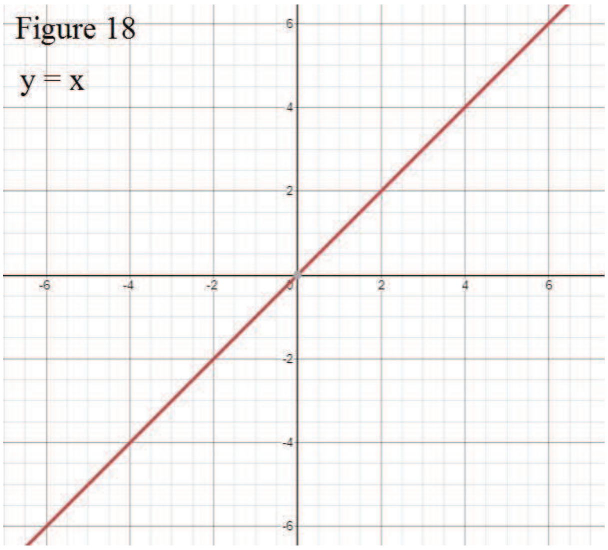
3.b.1—Graph of $y = x$

This is the basic linear graph. The rise to run is a simple 1 to 1 ratio for any two variables set equal to each other. Though the axis of the graphs will be labeled differently the graphs of the following equations all look the same.

$$y = x \qquad u = s$$

Etc.

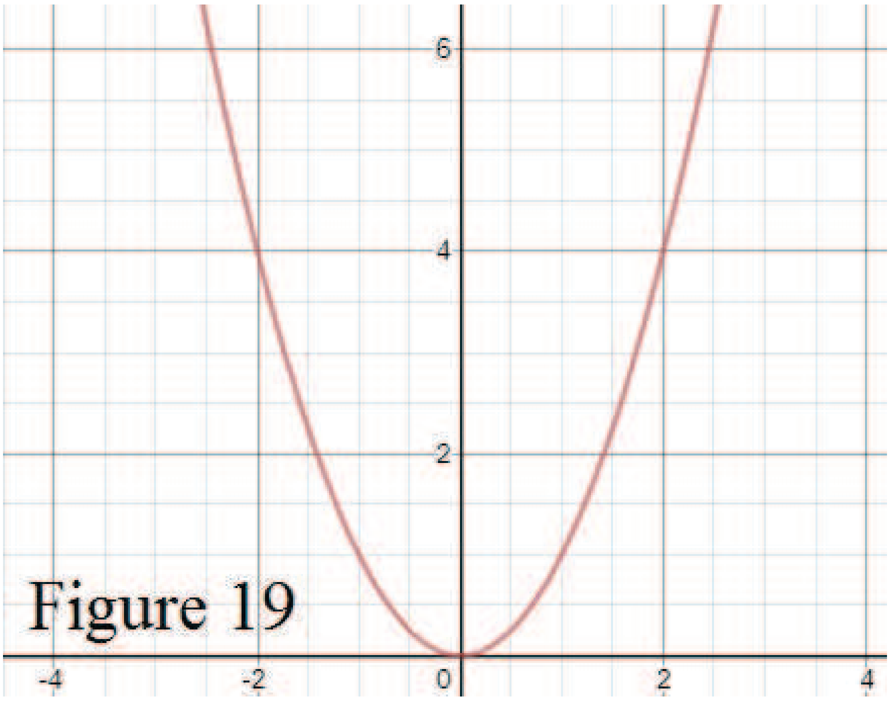
Figure 18:



3.b.2—Graph of $y = x^2$:

This is the basic quadratic equation. The graph is a parabola. The range of y extends from 0 positive without bound for all values of $x = \mathbb{R}$.

Figure 19:



3.b.3—Graph of $y = (\oplus\, x)^2$:

The inclusion of the subspace plus-and-minus numbers will essentially invert the graph over the x-axis. The graph of $y = (\oplus\, x)^2$ will result in $-y$ for all plus-and-minus values of x from $y = 0$ to $y = -\infty$ without bound. The value $\oplus\, x$ will resovle to $+x$ on the x-axis of this equation. See Chapter 2 section on i-multiples if you need a review on why the square of these values results in a negative number.

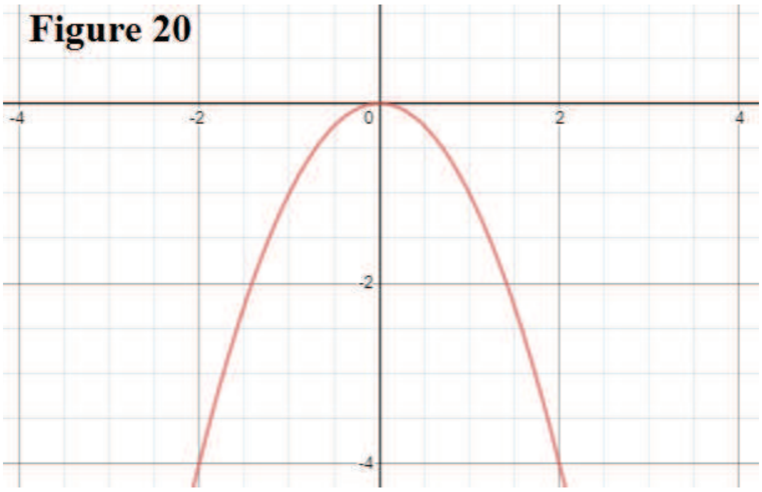
Example: $y = (\oplus\, x)^2 = \hat{x}^2$

$x = 2 \qquad y = \hat{2}^2 = (2^2)(\oplus 1^2) = -4$

$x = -2 \qquad y = -\hat{2}^2 = (-2^2)(\oplus 1^2) = -4$

Figure 20 below shows $y = (\oplus\, x)^2 \;=\; \hat{x}^2$ for $x = \mathbb{R}$

Figure 20:

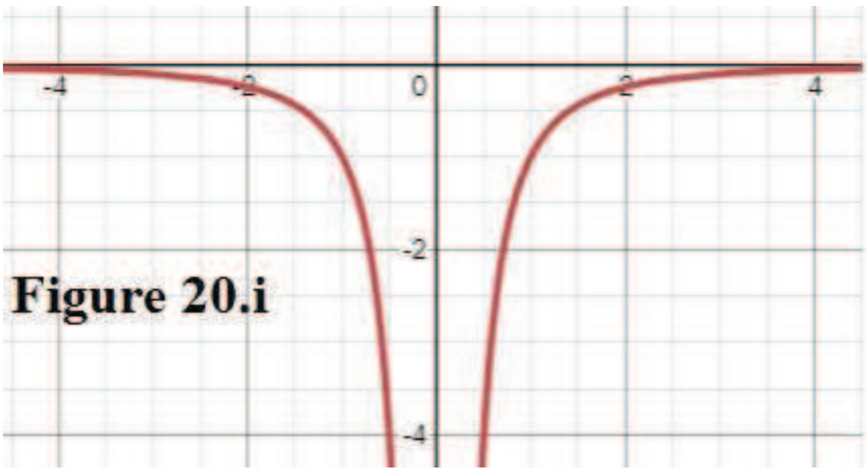


Graphable on a graphing utility as $y = -(x^2)$

Note there is no restrictions on whether or not the variable x can have negative values. The *up* indicated specifies this is the positive magnitude component of an i-multiple. But we can still enter negative values for the variable x. The $\hat{x}$ will resolve to x at its place of occurrence and whether or not we maintain the presence of the up indicator it can take both positive and negative values. The magnitude of the value will square to a positive number whilst the up will square as $(\oplus 1)^2$, resulting in a final negative value. The adjoining subspace SY-Plane is defined by the equation:

$$y = \hat{x}^2 = \left(-\frac{1}{\check{s}}\right)^2 = \frac{1}{\check{s}^2}$$

The $\check{s}$ will resolve to $-s$ but again there is no restriction on whether the value for s as a variable is positive or negative. The graph below in Figure 20.i shows the graph of $y = \frac{1}{\check{s}^2}$



Graphable as $y = -\left(\frac{1}{s^2}\right)$

The graph of $y = f(x\,, s)$ is given by $y = \hat{x}^2 + \frac{1}{\check{s}^2}$

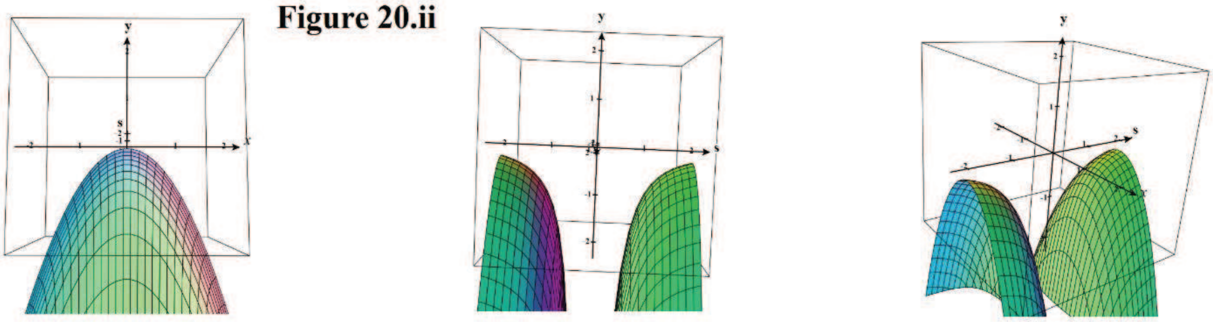


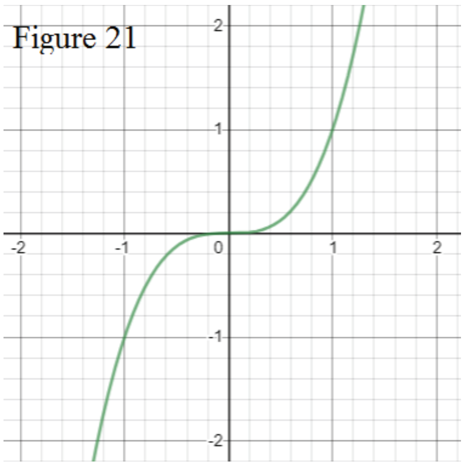
Figure 20.ii

Graphable as $y = -(x^2) - \left(\frac{1}{s^2}\right)$

<u>3.b.4—Graph of $y = x^3$:</u>

Domain and Range include all real numbers. Figure 21 shows the graph of $y = x^3$ for positive or negative numbers including 0.

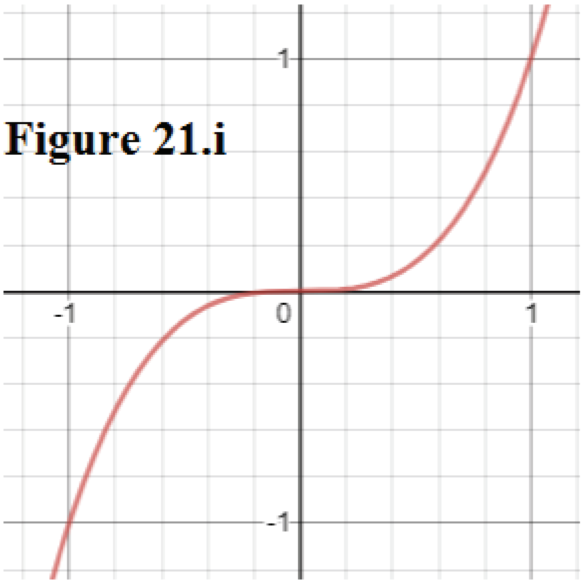
<u>Figure 21:</u>



<u>3.b.4.i—Graph of $y = (\bigoplus x)^3$:</u>

The graph of $y = (\bigoplus x)^3 = \hat{x}^3$ looks identical to the graph of $y = x^3$. The value of $\hat{x}^3$ is the cube of the magnitude of the i-mutiple times the cube of $\bigoplus 1$. The cube of the $\bigoplus 1$ is $+1$. The graph of $y = \hat{x}^3$ is shown in figure 21.i below.

Figure 21.i:

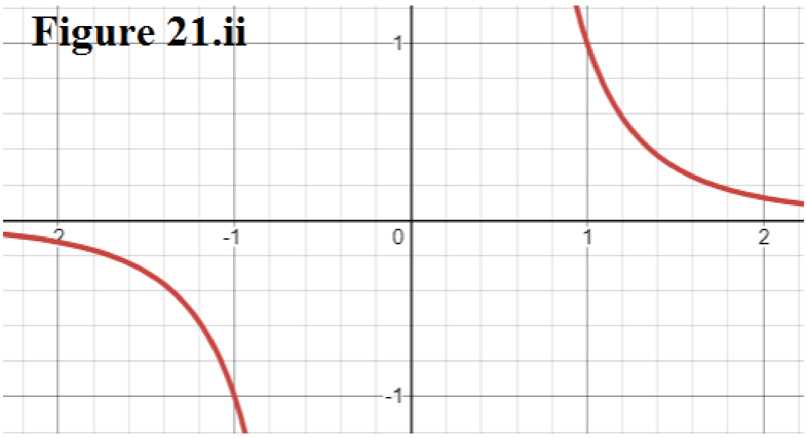


The SY-Plane equation of the adjoining subspace plane is defined as:

$$y = \hat{x}^3 = \left(-\frac{1}{\check{s}}\right)^3 = -\frac{1}{\check{s}^3}$$

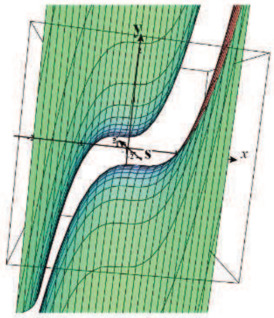
The $\check{s}^3$ will resolve to $(-s)^3$. But s can still take positive and negative values. Figure 21.ii shows the graph of $y = -\frac{1}{\check{s}^3} = -\frac{1}{(-s)^3}$.

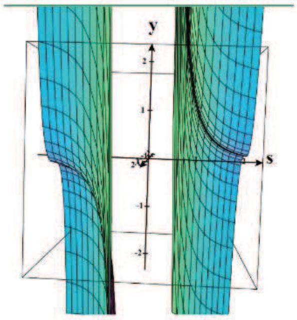
Figure 21.ii:

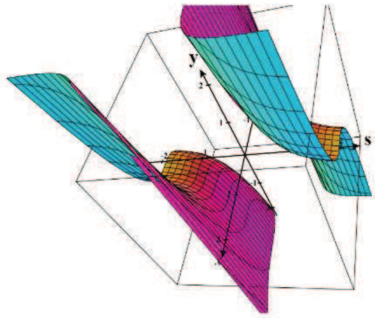


The $y = f(x\,, s)$ equation is defined as $y = \hat{x}^3 - \frac{1}{\check{s}^3}$

Figure 21.iii





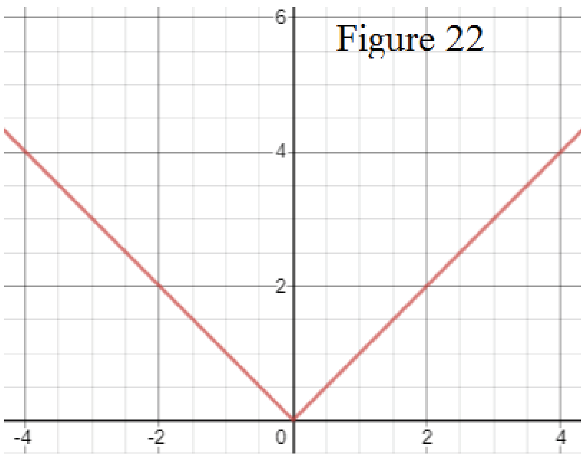


Graphable as $y = x^3 - \frac{1}{(-s)^3}$

<u>3.b.5—Graph of $y = |x|$:</u>

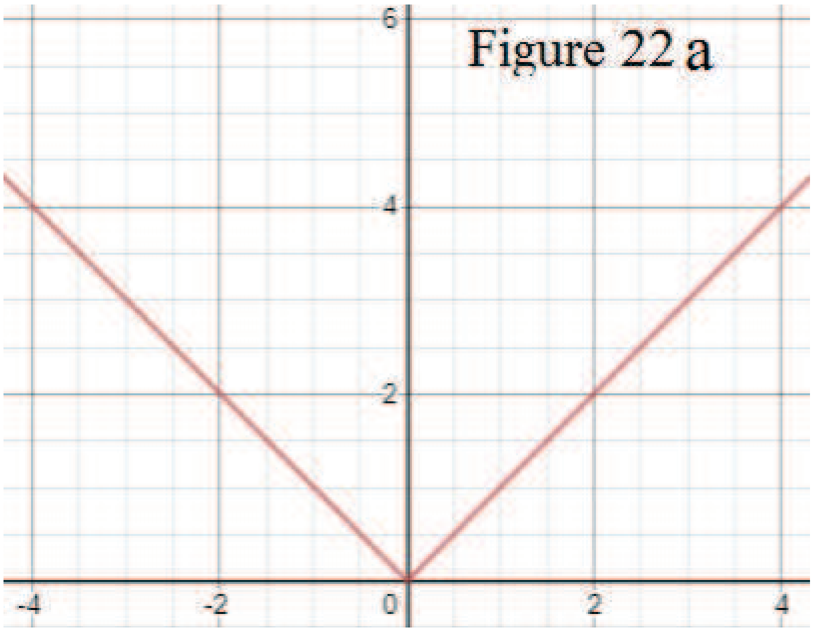
The absolute value graph will provide a positive output regardless of whether the input value x is positive or negative. The graph, Figure 22 makes a distinctive v-shape.

<u>Figure 22:</u>



The graph of $y = |\oplus x|$ is virtually identical. The value of $\oplus x$ resolves to positive x where the plus-and-minus value originates. Yet this is still a variable, x, and there is no restriction upon the value it may take. Figure 22.a shows this equation, $y = |\oplus x| \quad \longrightarrow \quad y = |\hat{x}|$

Figure 22.a:

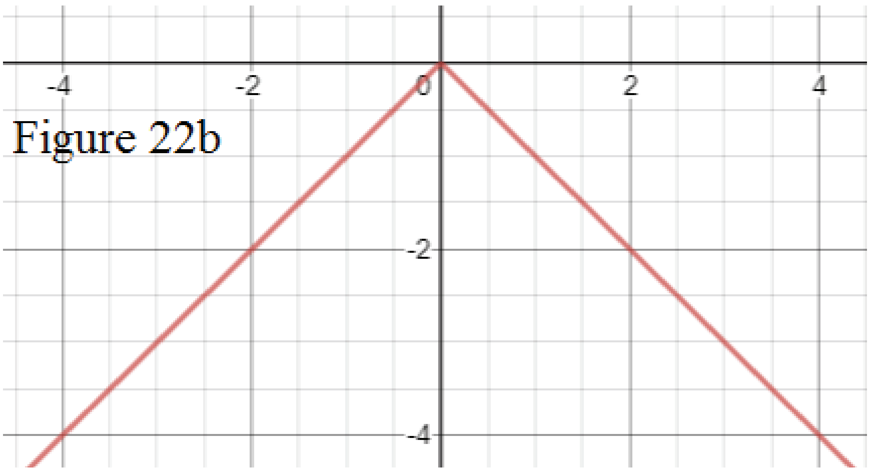


The u equations comes from the subspace transformation.

$$\Xi y = \frac{\Xi}{\varsigma u}|\hat{x}| \quad \longrightarrow \quad u = -\frac{1}{|\hat{x}|} = -\frac{1}{\left|-\frac{1}{\check{s}}\right|} = -|-\check{s}| = -|s|$$

$\check{s}$ resolves to $-s$ which leaves use with $-|s|$. Again though the variable s can take both positive and negative values. Figure 22.b below shows the graph of $u = -|-\check{s}| = -|s|$.

Figure 22.b:



The $y(s)$ equation can be found by simple substitutions on the $u(s)$ equation.

$$u = -|-\check{s}| \quad \leftrightarrow \quad -\frac{1}{y} = -|-\check{s}| \quad \leftrightarrow \quad y = \frac{1}{|-\check{s}|} = \frac{1}{|s|}$$

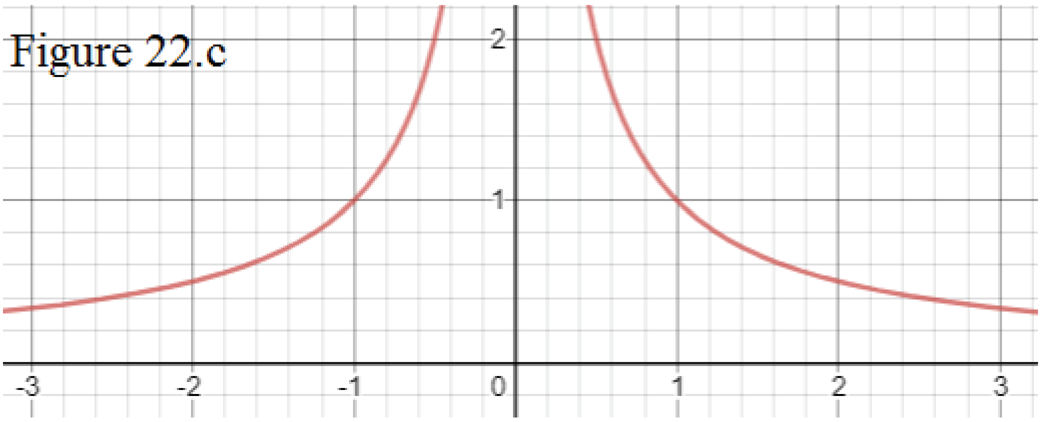


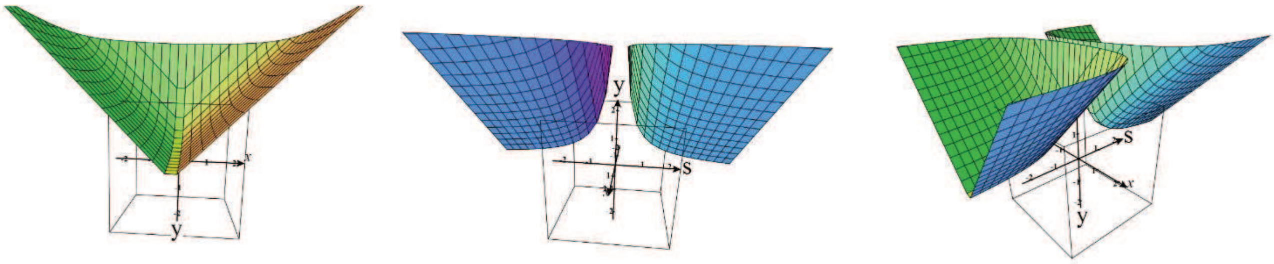
Figure 22.c

The combined graph of $y = f(x, s)$ is shown below in Figure 22.d.

$y = |\hat{x}| + \frac{1}{|-\check{s}|}$

This is done by plotting on the graphing utility $y = x + \frac{1}{s}$.

Figure 22.d



3.b.6—Graph of $y = \sqrt{x}$:

The root function has the Range of $y = \mathbb{R}$ and Domain $x \geq 0$ within traditional mathematics. This is shown in the graph of Figure 23.

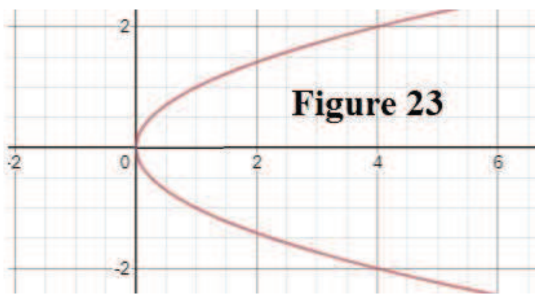


Figure 23

However, this is not the end of the graph. Within traditional mathematics the negative values of x are not included in the domain of the function. Within Null Mathematics they are part of the domain. Notice for any $-x$ value the roots are positive-and-negative numbers, subspace value i-multiples.

Where a is the root of the absolute value of any given number— $a = \sqrt{|b|}$

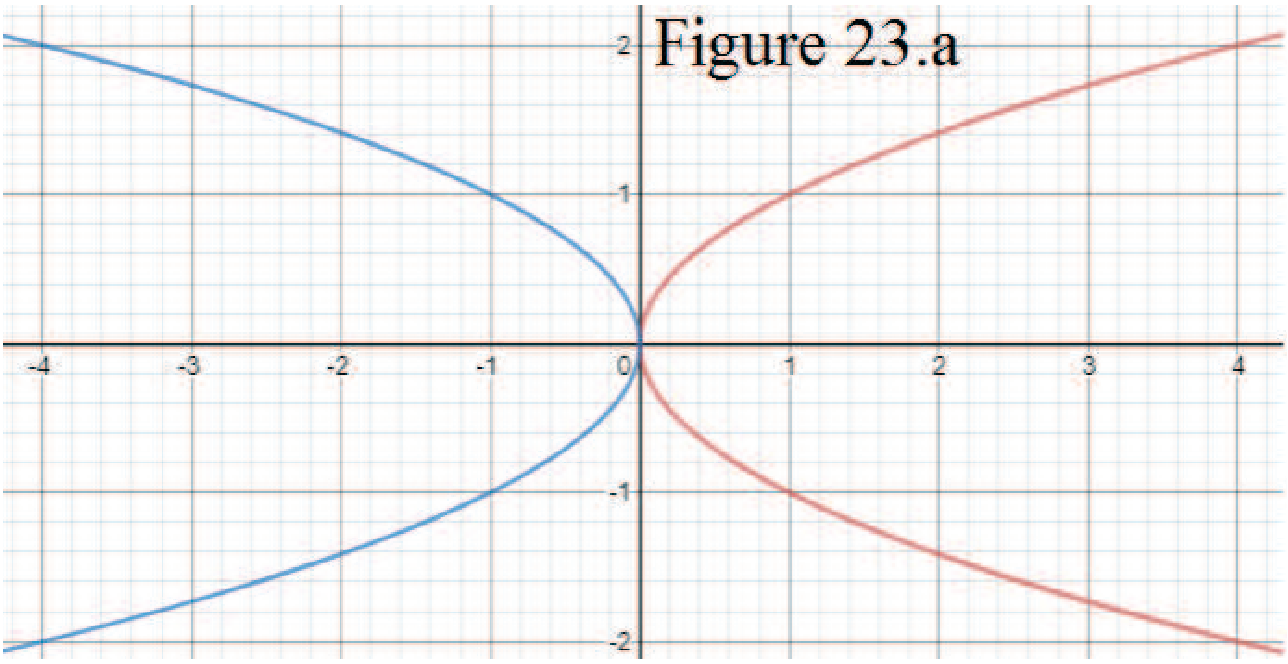
Then for any negative number $-b$ we have:

$$y = \sqrt{-b} = \pm(\oplus\, a) \quad = \quad \pm\hat{a} \quad = \quad \pm a$$

Thus for the equation $y = \sqrt{x}$ the negative domain of x has a defined range which is essentially a mirrored over the y-axis. Figure 23.a shows this graph.

For $x \geq 0$:	For $x < 0$:
Given $x = a^2$ and $\sqrt{a^2} = \pm n$ $y = \sqrt{x} = \pm n$ Red graph below for positive x inputs	Given $-x = a^2$ and $\sqrt{a^2} = \pm(\oplus\, n)$ $y = \sqrt{x} = \pm(\oplus\, n) = \pm\hat{n} = \pm n$ Blue graph below for negative x inputs

Figure 23.a:



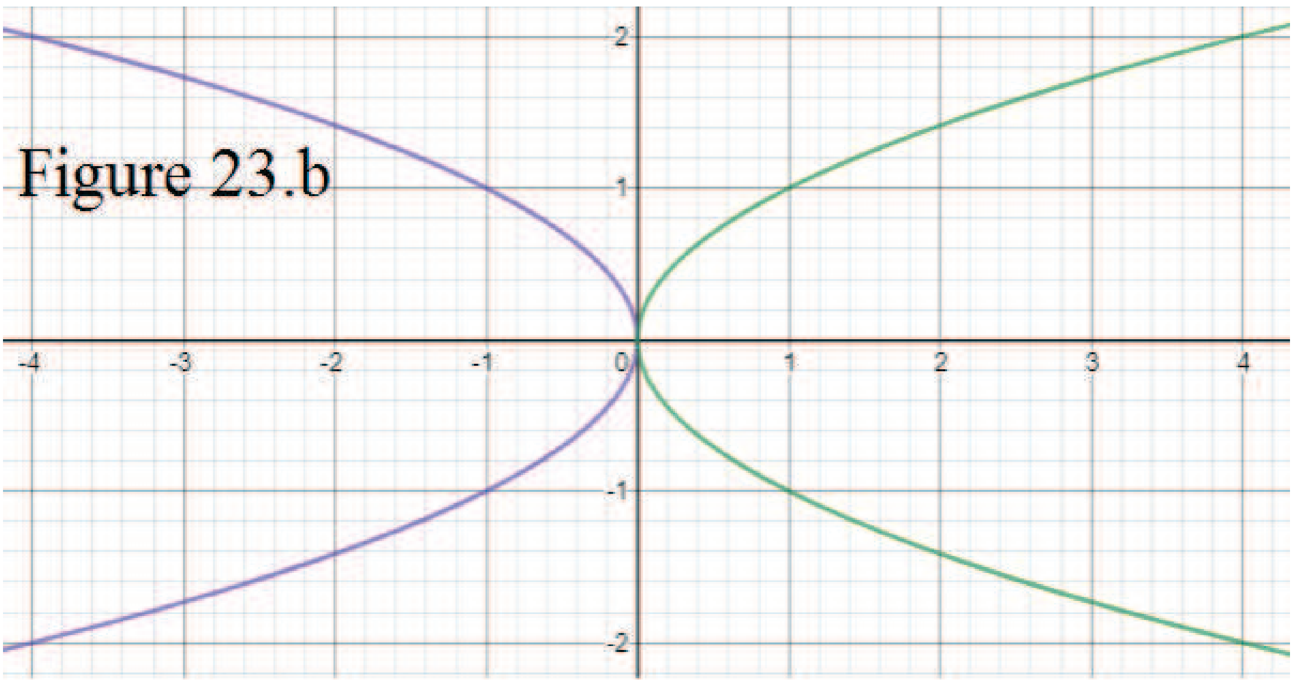
Though the root of a number is a $\pm$ result most graphing utilities only show the top, positive portion. This is to maintain that this is a function which passes the vertical line test, in addition to the fact that in most real-world applications you will find only one of the results applies. The u equation is found by subspace transformation just like any other equation.

$$\Xi y = \frac{\Xi}{\varsigma u}\sqrt{x} \quad \longrightarrow \quad u = -\frac{1}{\sqrt{x}} = -\frac{1}{\sqrt{-\frac{1}{s}}}$$

For $s > 0$:	For $s \leq 0$:
Given $-s = a^2$ and $\sqrt{a^2} = \pm(\oplus\, n)$ $u = -\frac{1}{\sqrt{-\frac{1}{s}}} = \pm\frac{1}{\frac{1}{\oplus n}} = \pm\check{n} = \mp n$ Green graph below for positive s inputs	Given $-(-s) = a^2$ and $\sqrt{a^2} = \pm n$ $u = -\frac{1}{\sqrt{-\frac{1}{s}}} = \pm\frac{1}{\frac{1}{n}} = \pm n$ Purple graph below for negative s inputs

Note the because this equation originates on the XY-Plane and the US-Plane equation is a subspace the positive s *values are the negative* i*-multiple compoents which correspond to those of the negative* x *inputs, the positive* i*-multiple components. That is shown here in the resolution of the plus-and-minus number to first plus or minus n-down and finally to minus or plus n.*

Figure 23.b show this graph for $u = -\frac{1}{\sqrt{-\frac{1}{s}}}$.



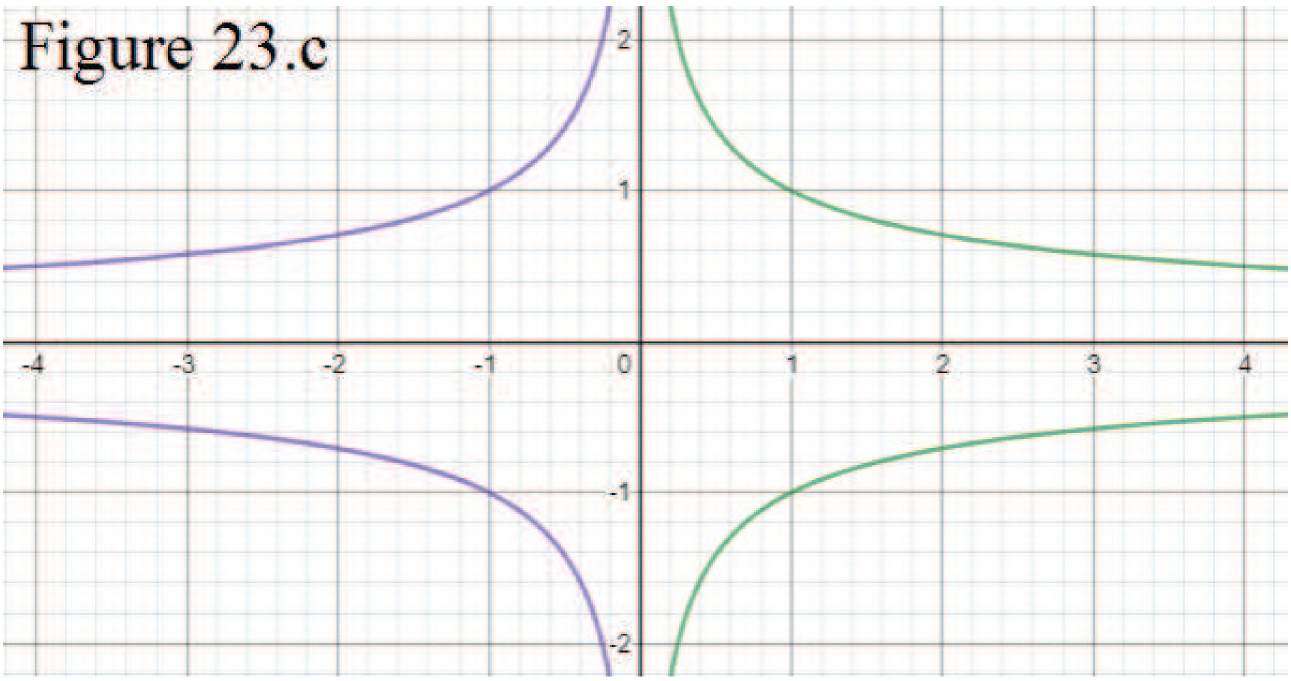
The graph looks identical to that of the YX-plane. The difference you'll see in most graphing utilities immediately is the tendency to graph only the positive output of the root to maintain a graph which passes the vertical line test. Due to the negative sign on the outside of the radical bar this will be the downward negative curve of the purple graph on the left side.

The graph of $y = f(s)$ is again found by simple substitution:

$$y = \sqrt{x} \quad \longrightarrow \quad y = \sqrt{-\frac{1}{s}}$$

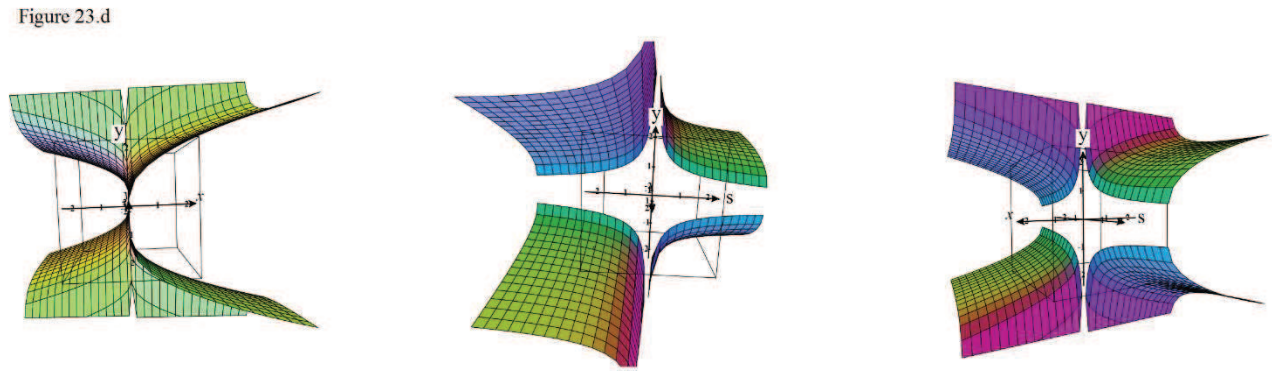
For $s > 0$:	For $s \leq 0$:
Given $-s = a^2$ and $\sqrt{a^2} = \pm(\oplus\, n)$	Given $-(-s) = a^2$ and $\sqrt{a^2} = \pm n$
$y = \sqrt{-\frac{1}{s}} = \pm\frac{1}{\oplus n} = \pm\frac{1}{\breve{n}} = \mp\frac{1}{n}$	$y = \sqrt{-\frac{1}{s}} = \pm\frac{1}{n}$
Green graph below for positive s inputs	Purple graph below for negative s inputs

Figure 23.c below shows the graph of $y = \sqrt{-\frac{1}{s}}$:



The combined graph of $y = f(x\,, s)$ is given by $y = \sqrt{x} + \sqrt{-\frac{1}{s}}$. Figure 23.d shows the graph here below.

Figure 23.d:



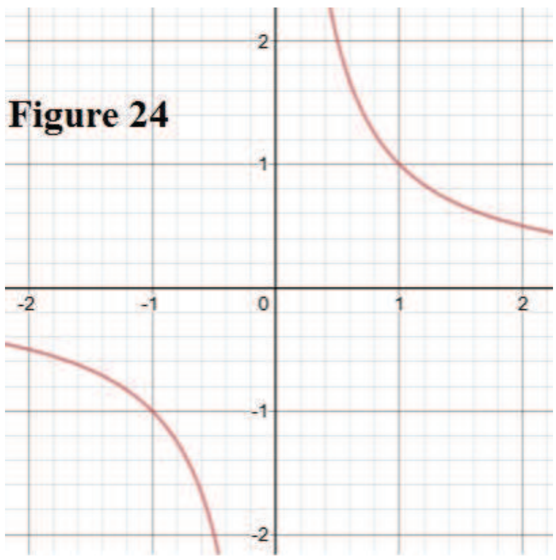
3.b.7—Graph of $y = \frac{1}{x}$:

Traditional mathematics holds the Domain cannot include 0 whilst the Range includes all real numbers.

Domain: $-\infty \leq x < 0$ and $0 < x \leq \infty$

Range: $y = \mathbb{R}$

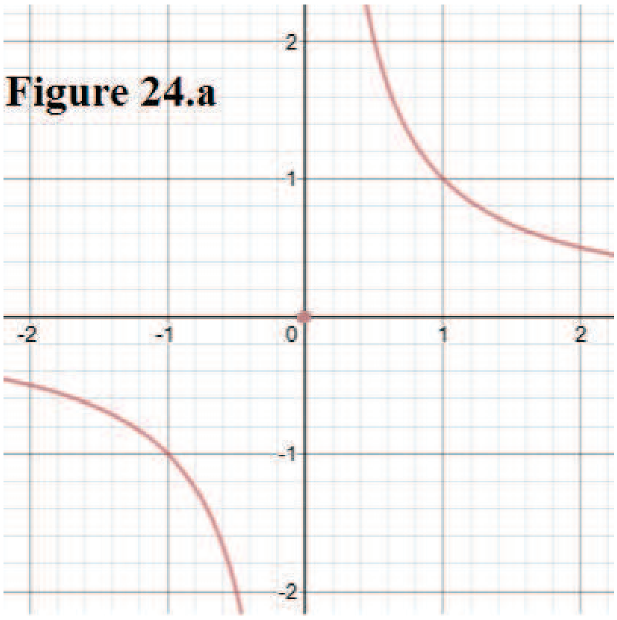
Figure 24:



The value will resolve to zero as η_0 when $x = 0$. See Figure 24.a.

Figure 24.a

Graph of $y = \frac{1}{x}$ with $y = \eta_0$ when $x = 0$.

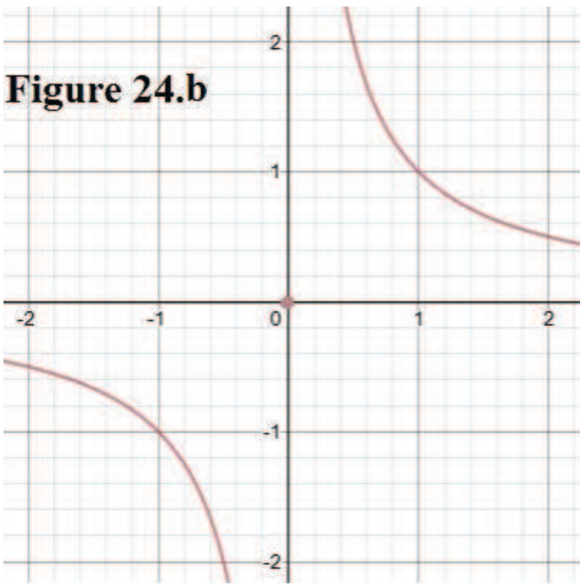


The $u(s)$ equations is:

$$\Xi y = \frac{\Xi}{\varsigma u}\frac{1}{x} \quad \longrightarrow \quad \oplus 1 = \frac{u}{x} \quad \longrightarrow \quad u = -x = \frac{1}{s}$$

The graph of $u = \frac{1}{s}$ is shown in figure 24.b and looks virtually identical to the graph in Figure 24.a.

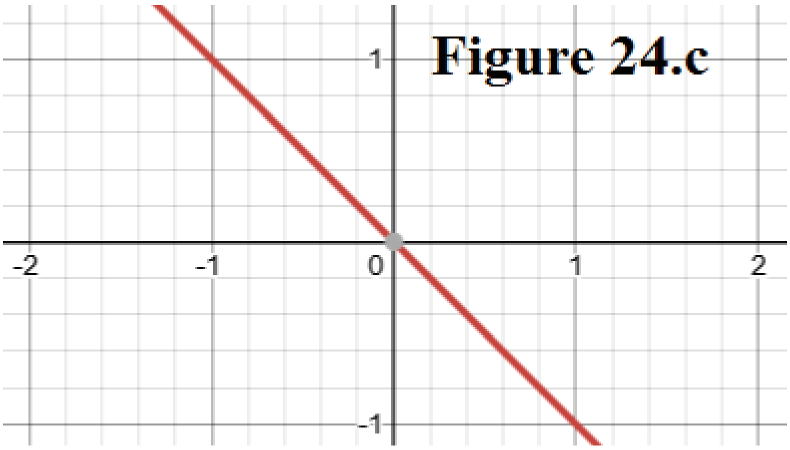
Figure 24.b



The $y(s)$ equation is shown below with its graph in Figure 24.c

$$u = \frac{1}{s} \quad \longrightarrow \quad -\frac{1}{y} = \frac{1}{s} \quad \longrightarrow \quad y = -s$$

Figure 24.c

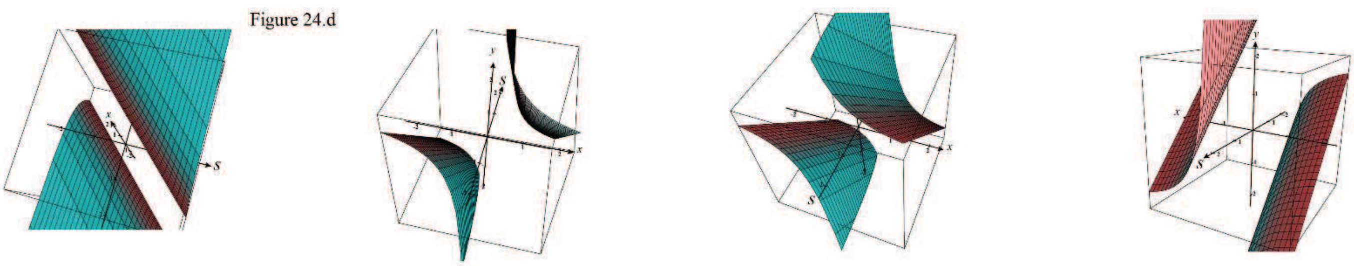


The full graph of $y(x, s)$ is show below in Figure 24.d.

$y = -s$ and $y = \frac{1}{x}$. They are combined as:

$$y = \frac{1}{x} + (-s) \quad \longrightarrow \quad y = \frac{1}{x} - s$$

Figure 24.d:



3.c—Trigonometry:

Circular Trigonometric functions are periodic, meaning the pattern of the graph and values they represent will repeat after some position. The horizontal spacing for which this repetition occurs is called the period and is usually represented as either 360 degrees or 2π radians. As a result of the periodic repetition there will exist multiple input values which generate the same output; such input values are called conterminal angles. Trigonometric functions are normally measured in either degrees or radians. The two measurements systems are not identical so it's important to make a note of which is being used before beginning calculations to avoid errors.

There are three standard trigonometric functions; Sine, Cosine, and Tangent. These can be used to define respectively the Cosecant, Secant and Cotangent functions. All of which are based on the following relations for circular trigonometry:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta}\cdot e^{-i\theta} = 1 = \cos^2\theta + \sin^2\theta$$

This is what is called circular trigonometry. All of the relations correspond to the equation $1 = x^2 + y^2$. The Pythagorean Identity $1 = \cos^2\theta + \sin^2\theta$ fits this pattern as $\cos\theta$, and $\sin\theta$ are respectively the x and y componets of a point position on the Cartesian plane at a distance r from the origin, where $r = 1$. Figure 25 below shows the graph of both $1 = x^2 + y^2$ and $1 = \cos^2\theta + \sin^2\theta$.

Figure 25:

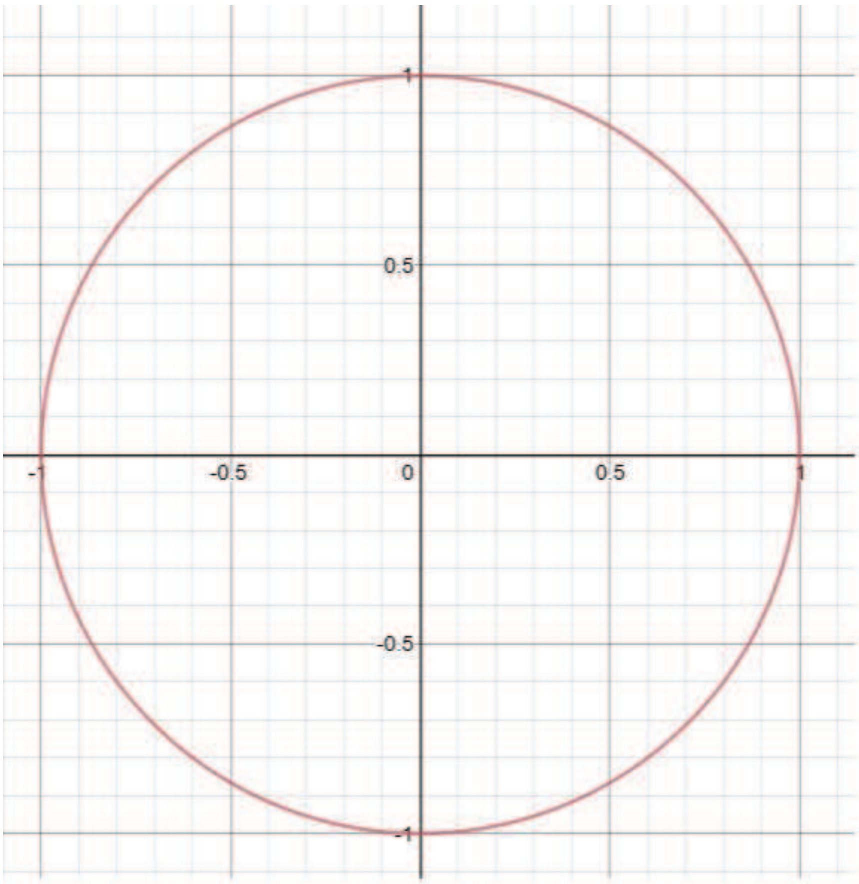


Figure 25
$1 = x^2 + y^2$
$1 = \cos^2\theta + \sin^2\theta$

Re-examine for a moment the following relations:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

Both of these identities contain i-multiples. They have to otherwise their products would not be a sum of perfect squares: $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta$. The i-multiples in both components originate naturally on the y-axis: $y = i\sin\theta$. This i-multiple can be resolved using the solutions provided in chapter 2. In this instance the magnitude of the i-multiple is $\sin\theta$. It will take the positive (*up*) value where it originates and pair with its negative (*down*) counterpart on an adjoining subspace axis. If we chose to not include the up-circumflex identifying this component as an i-multiple we obtain the following:

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$e^{\theta} = \cosh\theta + \sinh\theta \qquad e^{-\theta} = \cosh\theta - \sinh\theta$$

$$e^{\theta}\cdot e^{-\theta} = 1 = \cosh^2\theta - \sinh^2\theta$$

These look very similar to the identities listed on the previous page but define a very different graph. These are the hyperbolic trigonometric functions. Theta is usually replaced with omega which runs from 0 to infinity and there are no conterminal angles.

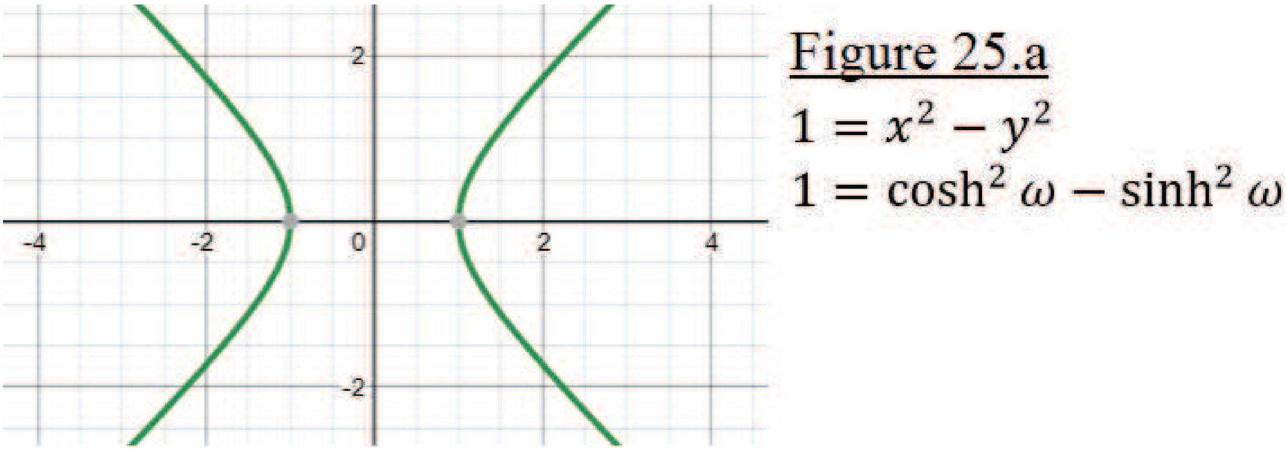
$$\cosh\omega = \frac{e^{\omega}+e^{-\omega}}{2} \qquad \sinh\omega = \frac{e^{\omega}-e^{-\omega}}{2}$$

$$e^{\omega} = \cosh\omega + \sinh\omega \qquad e^{-\omega} = \cosh\omega - \sinh\omega$$

$$e^{\omega}\cdot e^{-\omega} = 1 = \cosh^2\omega - \sinh^2\omega$$

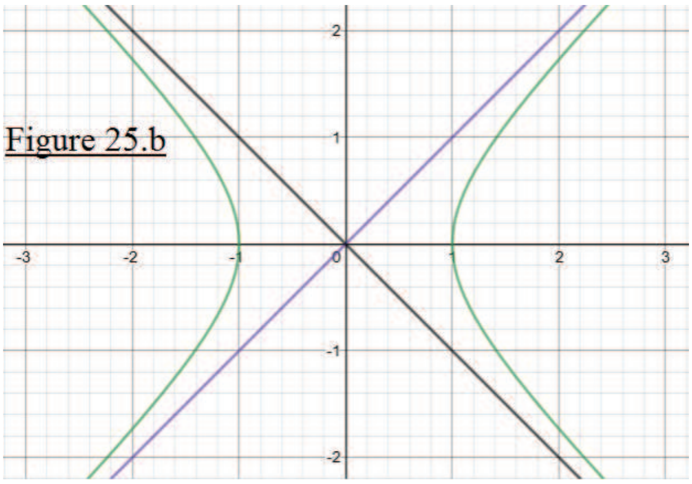
The hyperbolic identities follow the pattern of the equation $1 = x^2 - y^2$. The graph for both $1 = x^2 - y^2$ and $1 = \cosh^2\omega - \sinh^2\omega$ is shown below in Figure 25.a.

Figure 25.a:



Notice there are two asymptotes on the graph, the lines described by $y = x$ and $y = -x$. These are shown in Figure 25.b:

Figure 25.b:



We can draw several conclusions from these two sets of identities. The first is obvious and was just discussed; that resolution of the circular trigonometric i-multiples to subspace up-values results in hyperbolic trigonometry.

Circular Trigonometry	$e^{i\theta} \cdot e^{-i\theta} = 1 = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$
i-multiples resolved to positive-*up* Theta components. Still circular trigonometry as the i-multiples are kept track of.	$e^{\hat{\theta}} \cdot e^{-\hat{\theta}} = 1 = \left(\cos\theta + \sin\hat{\theta}\right)\left(\cos\theta - \sin\hat{\theta}\right)$
If you did not know to include the i-multiples (*did not know you began with an equation describing a circle*) you	

would write the equation without the *up*-circumflex. This results in hyperbolic trigonometry.	$e^{\theta} \cdot e^{-\theta} = 1 = (\cosh\omega + \sinh\omega)(\cosh\omega - \sinh\omega)$ $1 = \cosh^2\omega - \sinh^2\omega$

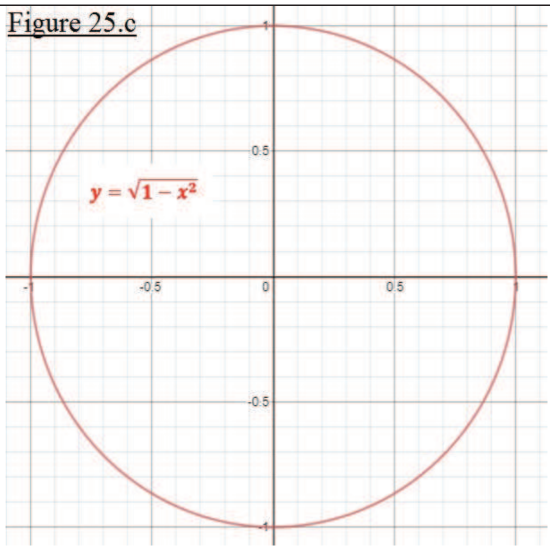
This implies the second conclusion: Instances of hyperbolic trigonometry can be converted to circular trigonometry by including the i-multiple on the sinh component and both e compoentns. Lastly it is also possible that an instant of hyperbolic trigonometry, such as $1 = \cosh^2\omega - \sinh^2\omega$ is actually describing circular trigonometry by removing the presence of subspace i-multiple notation thereby making the mathematics circular. A similar situation was described in the section on Completing the Square using the example $(x+1)^2 = -8$. Remember that i-multipes may be present in an equation and not immediately identified. Unless you have taken the root of a negative number you will not immediately be able to identify their presence until you encounter sign opposition like in the aforementioned mentioned example.

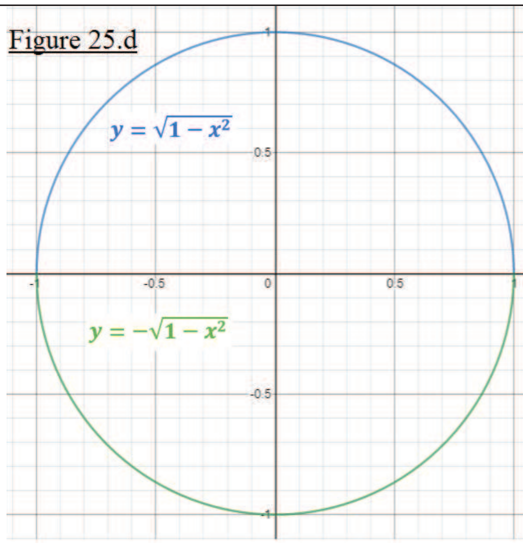
In reexamining the equation $1 = x^2 + y^2$ we can derive the following to equation:

$$y = \sqrt{1 - x^2}$$

This is still a circle. For values in the domain of $-1 \leq x \leq 1$ defined by traditional mathematics you will get $\pm$ values for y; i.e. you still get a circle. Note some graphing utilities will only show the positive values giving you just the top half of the circle. Don't be fooled by this. If you want to force it graph the bottom half of the circle in such an event you could plot $y = -\sqrt{1 - x^2}$. See figures 25.c and 25.d below:

Figure 25.c:





This also illustrates a critical point, that in $y = \sqrt{1 - x^2}$, we must acknowledge the result of the radical is $\pm y$. We only see the full circle this equation represents when both the positive and negative output value for y are included. So we actually have two equations which are simultaneously implied and which must both be accounted for:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}$$

This is straight forward enough. For trigonometric polar equations with a radius of $r = 1$ we have coordinates $x = \cos\theta$ and $y = \sin\theta$. These can be substituted into $y = \sqrt{1 - x^2}$ directly giving $\sin\theta = \sqrt{1 - \cos^2\theta}$. This equation is also found directly from the trigonometric Pythagorean identity for sines and cosines. If we square both sides we easily arrive at the Pythagorean Identity $1 = \cos^2\theta + \sin^2\theta$.

Where did the i go? Recall above when factoring either $1 = \cos^2\theta + \sin^2\theta$ or $1 = x^2 + y^2$ we get an i attached to the variable in the second position. The i is still there even in the equation solved for y. It is shown below here for both equations of y.

$yi = \sqrt{1 - x^2}$	$yi = -\sqrt{1 - x^2}$

The root still applies a $\pm$ value to the y. If we resolve the i on the y side of the equation it will leave it in a form identical to that before it was included in the equation.

$y \cdot \hat{1} = \sqrt{1-x^2}$ $y = \sqrt{1-x^2}$	$y \cdot \hat{1} = -\sqrt{1-x^2}$ $y = -\sqrt{1-x^2}$

We can also apply the i multiple to the x side of the equation. It will again leave it unchanged from its original form when its value is resolved to $\hat{1}$.

$yi = \sqrt{1-x^2}$ $y = \frac{1}{i} \cdot \sqrt{1-x^2}$	$yi = -\sqrt{1-x^2}$ $y = \frac{1}{i} \cdot \left(-\sqrt{1-x^2}\right)$
Where $i = \frac{0}{0}$ then $\frac{1}{i} = \frac{1}{0/_0} = \frac{0}{0} = i$	
$y = i\sqrt{1-x^2}$	$y = i\left(-\sqrt{1-x^2}\right)$
$y = \hat{1} \cdot \sqrt{1-x^2}$	$y = \hat{1} \cdot \left(-\sqrt{1-x^2}\right)$
$y = \sqrt{1-x^2}$	$y = -\sqrt{1-x^2}$

So we're still getting the top and bottom of the circle. Instead of resolving the i we could chose to include it within the radical.

$y = i\sqrt{1-x^2}$	$y = -i\sqrt{1-x^2}$
$y = \sqrt{-1+x^2}$ $y = \sqrt{x^2-1}$	$y = -\sqrt{-1+x^2}$ $y = -\sqrt{x^2-1}$

This should look familiar. If we square either equation we get $1 = x^2 - y^2$. This is hyperbolic trigonometry. The function $y = \sqrt{x^2 - 1}$ still describes a $\pm y$ value because of the root and so graphs out the expected hyperbolic curves. It remains though a representation of two simultaneously implied equations:

$$y = \sqrt{x^2 - 1} \quad \text{and} \quad y = -\sqrt{x^2 - 1}$$

If your graphing utility will only display the positive y components you can graph both functions together in order to see the entire hyperbolic curve. Figures 25.e and 25.f below show the full curve.

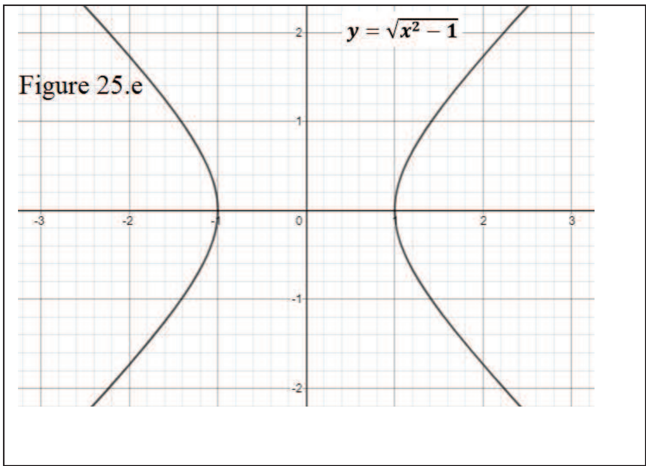


Figure 25.e

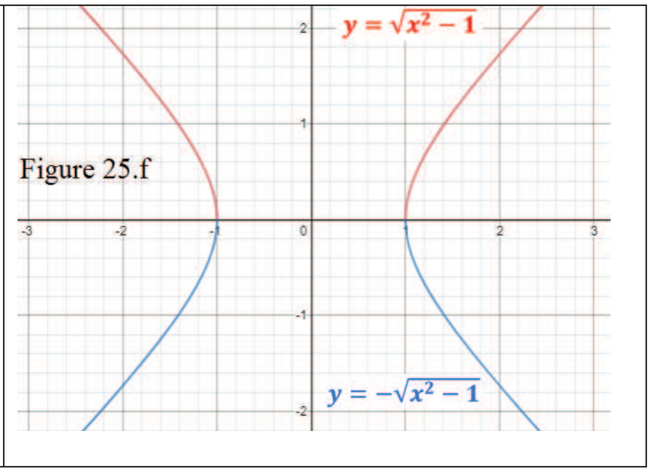


Figure 25.f

A curious thing happens when you extend the domain of x to include values which are traditionally forbidden. For Circular Trigonometry following the pattern of $1 = x^2 + y^2$ the domain is traditionally held to be $-1 \leq x \leq 1$. If the domain of x is expanded to $x < -1 \cap 1 < x$ the value inside the radical will be negative. Due to the radical the value of y is still $\pm$. So we must apply this extension of the domain of x to both $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

If $a^2 = 1 - x^2$

For $x < -1 \cap 1 < x$ $\qquad y = \sqrt{1 - x^2} = \hat{a} \qquad y = -\sqrt{1 - x^2} = -\hat{a}$

This can be graphed as $y = \sqrt{-(1 - x^2)}$ and $y = -\sqrt{-(1 - x^2)}$. You'll find this is identical to the graphs of $y = \sqrt{x^2 - 1}$ and $y = -\sqrt{x^2 - 1}$ which describe the hyperbolic trigonometry.

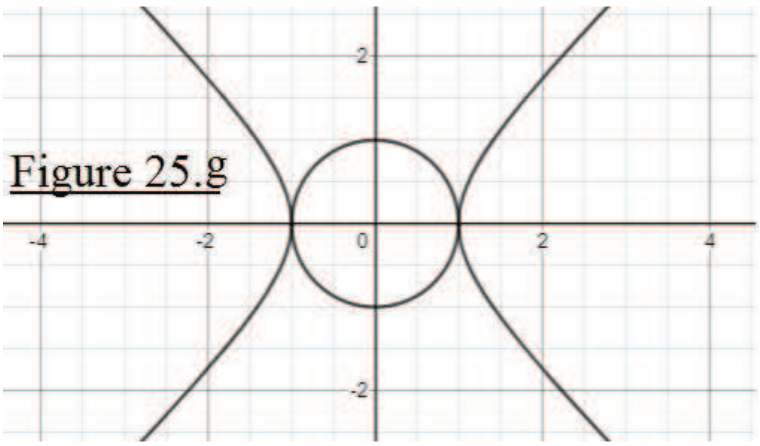
The same situation will occur with equations for hyperbolic trigonometry. For Hyperbolic Trigonometry following the pattern of $1 = x^2 - y^2$ the domain is traditionally held to be $x < -1 \cap 1 < x$. If the domain of x is expanded to include $-1 \leq x \leq 1$ the value inside the radical will be negative. Due to the radical the value of y is still $\pm$. So we must apply this extension of the domain of x to both $y = \sqrt{x^2 - 1}$ and $y = -\sqrt{x^2 - 1}$.

If $a^2 = x^2 - 1$

For $-1 \leq x \leq 1$ $\qquad y = \sqrt{x^2 - 1} = \hat{a} \qquad y = -\sqrt{x^2 - 1} = -\hat{a}$

This can be graphed as $y = \sqrt{-(x^2 - 1)}$ and $y = -\sqrt{-(x^2 - 1)}$. Here we find the result is identical to $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ which describe circular trigonometry. So regardless of whether using circular or hyperbolic trigonometry, input values which are outside the traditionally defined domains will by Null mathematics describe the other trigonometric discipline. The full graph of both $y = \sqrt{1 - x^2}$ and $y = \sqrt{x^2 - 1}$ is shown below in Figure 25.g.

Figure 25.g



Now let's consider the subspace values which correspond to the circular and hyperbolic functions. The subspace equation which corresponds to circular equation $1 = x^2 + y^2$ can be found by performing a conversion on x and y as inputs, or against y as an output in the form of $y = \sqrt{1 - x^2}$ and then resolving x in terms of s. Here we use a transformation against y.

$$\Xi y = \frac{\Xi}{\varsigma u}\sqrt{1 - x^2} \quad \longrightarrow \quad u = -\frac{1}{\sqrt{1 - x^2}} \quad \longrightarrow \quad u = -\frac{1}{\sqrt{1 - \left(-\frac{1}{s}\right)^2}} \quad \longrightarrow$$

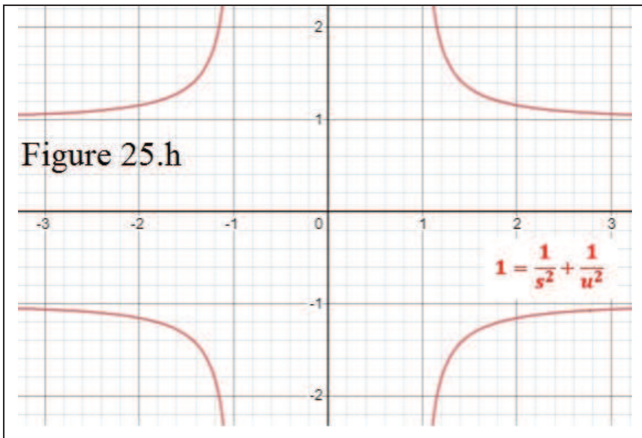
$$u = -\frac{1}{\sqrt{1 - \frac{1}{s^2}}} \quad \longrightarrow \quad u = -\frac{1}{\sqrt{\frac{s^2 - 1}{s^2}}}$$

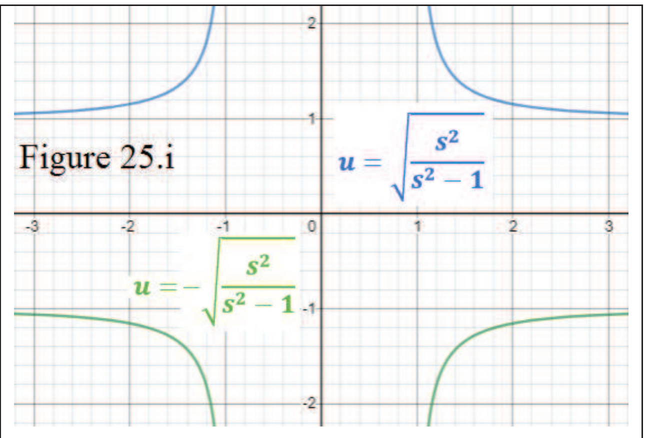
You can continue with the equation in this form but it's messy. This state of the equation will still have positive or negative value for u as a result of $\pm$ outputs from the radical defining two simultaneous equations. To clean it up square the entire equation. You many then reciprocate the fraction which forms the denominator of the s input side of the equation. Finally reapply the roots and solve for *u*.

$$u = -\frac{1}{\sqrt{\frac{s^2 - 1}{s^2}}} \quad \longrightarrow \quad u^2 = \left(-\frac{1}{\sqrt{\frac{s^2 - 1}{s^2}}}\right)^2 \quad \longrightarrow \quad u^2 = \frac{1}{\frac{s^2 - 1}{s^2}} \quad \longrightarrow$$

$$u^2 = \frac{s^2}{s^2 - 1} \quad \longrightarrow \quad u = \sqrt{\frac{s^2}{s^2 - 1}}$$

Again the presence of the radical implies for output you have $\pm u$. Both equations are simultaneously present and must both be accounted for as $u = \sqrt{\frac{s^2}{s^2-1}}$ and $u = -\sqrt{\frac{s^2}{s^2-1}}$. Figure 25.h shows the graph of this function. Figure 25.i shows the graphs indicating the upper and lower halves. Again note that $u = \sqrt{\frac{s^2}{s^2-1}}$ is identical to $1 = \frac{1}{s^2} + \frac{1}{u^2}$. This is the circular trigonometry of the subspace equation which corresponds to $1 = x^2 + y^2$. If your graphing utility does not show the top and bottom half when plotting $u = \sqrt{\frac{s^2}{s^2-1}}$ you may simply plot it along with $u = -\sqrt{\frac{s^2}{s^2-1}}$.





This graph has two vertical asymptotes:

$$s = \pm 1 \quad \longrightarrow \quad u = \sqrt{\frac{1}{0}} = \frac{1}{0} = \eta_0 = 0$$

It also has a horizontal asymptote at $u = \pm 1$. The values of $-1 \leq s \leq 1$ will result in negative roots. Traditional mathematics shows the domain of s to be $D: s < -1 \cap 1 < s$. Note that the Traditional Math domain in $1 = \frac{1}{s^2} + \frac{1}{u^2}$ equation is exactly that which the same mathematical discipline denotes is forbidden for the equation of the circle, $1 = x^2 + y^2$. Or put another way where the domain is defined by traditional mathematics in both the $u(s)$ and $y(x)$ equations, it is considered undefined in the other.

For $u = \sqrt{\frac{s^2}{s^2-1}}$ the extension of the domain of s into $-1 \leq s \leq 1$ will create negative radicals such that:

For $-1 \leq s \leq 1$ we have $-(a^2) = \frac{s^2}{s^2-1}$.

Then $u = \sqrt{\frac{s^2}{s^2-1}} = \hat{a}$ and $\quad u = -\sqrt{\frac{s^2}{s^2-1}} = -\hat{a}$

This is equivalent to graphing $u = \sqrt{-\left(\frac{s^2}{s^2-1}\right)}$ and $u = -\sqrt{-\left(\frac{s^2}{s^2-1}\right)}$. Note this is identical to $1 = \frac{1}{s^2} - \frac{1}{u^2}$ which describes the hyperbolic trigonometric relations for the subspace equation. Figures 25.j and 25.k show graphs of these equations.

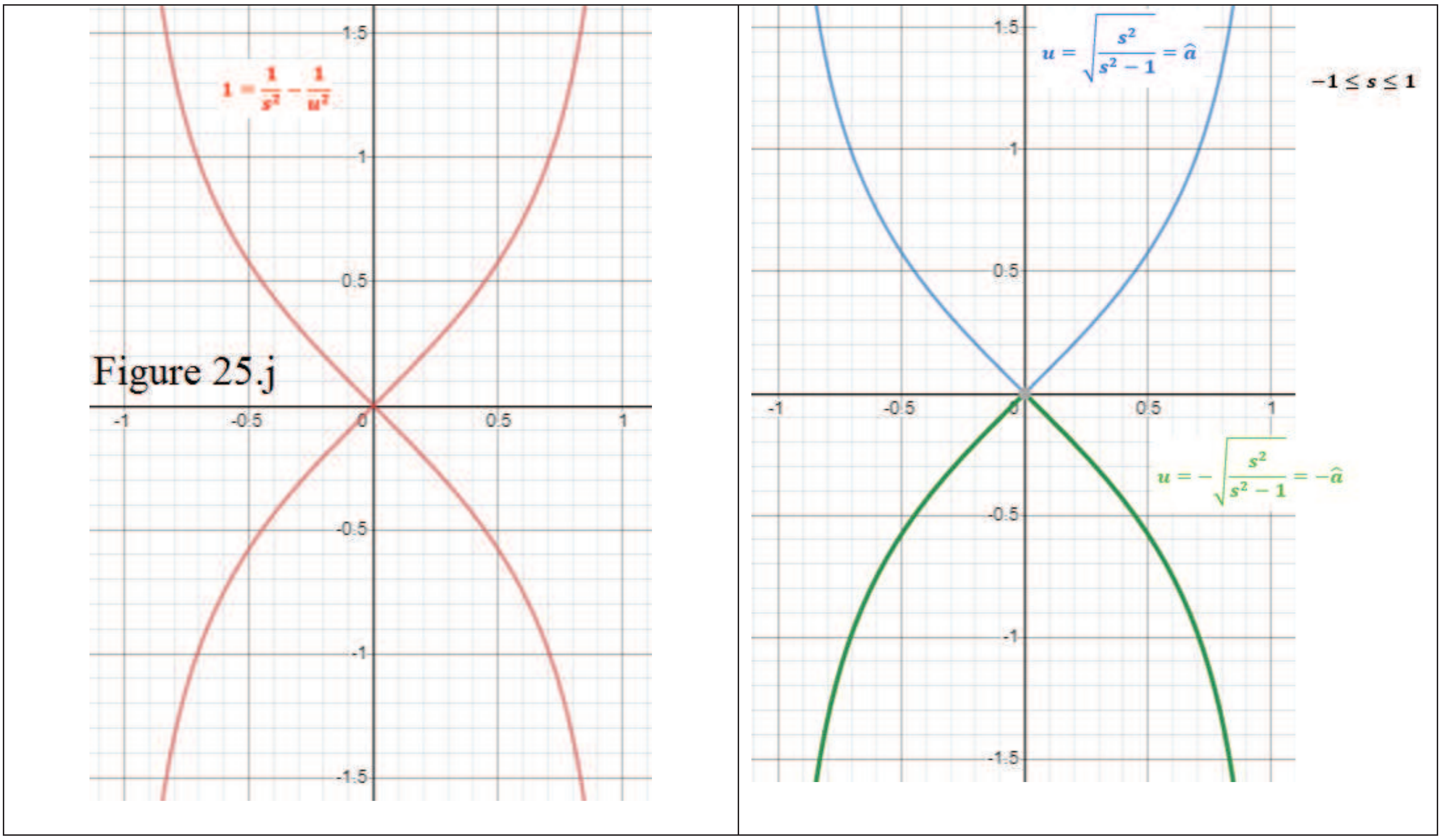


Figure 25.j

By the same caveat if we begin with $1 = \frac{1}{s^2} - \frac{1}{u^2} \equiv u = \sqrt{\frac{s^2}{1-s^2}}$ we still have two simultaneously implied equations due to the $\pm u$ values generated by the radical sign. This is $u = \sqrt{\frac{s^2}{1-s^2}}$ and $u = -\sqrt{\frac{s^2}{1-s^2}}$. It is identical to what is displayed in Figures 25.j and 25.k shown here above and are the hyperbolic trigonometric relations of the subspace equation. It has the traditionally defined domain of $-1 \leq s \leq 1$. If we extend the domain of s to include $s < -1 \cap 1 < s$ we obtain negative values within he radical which provides the following:

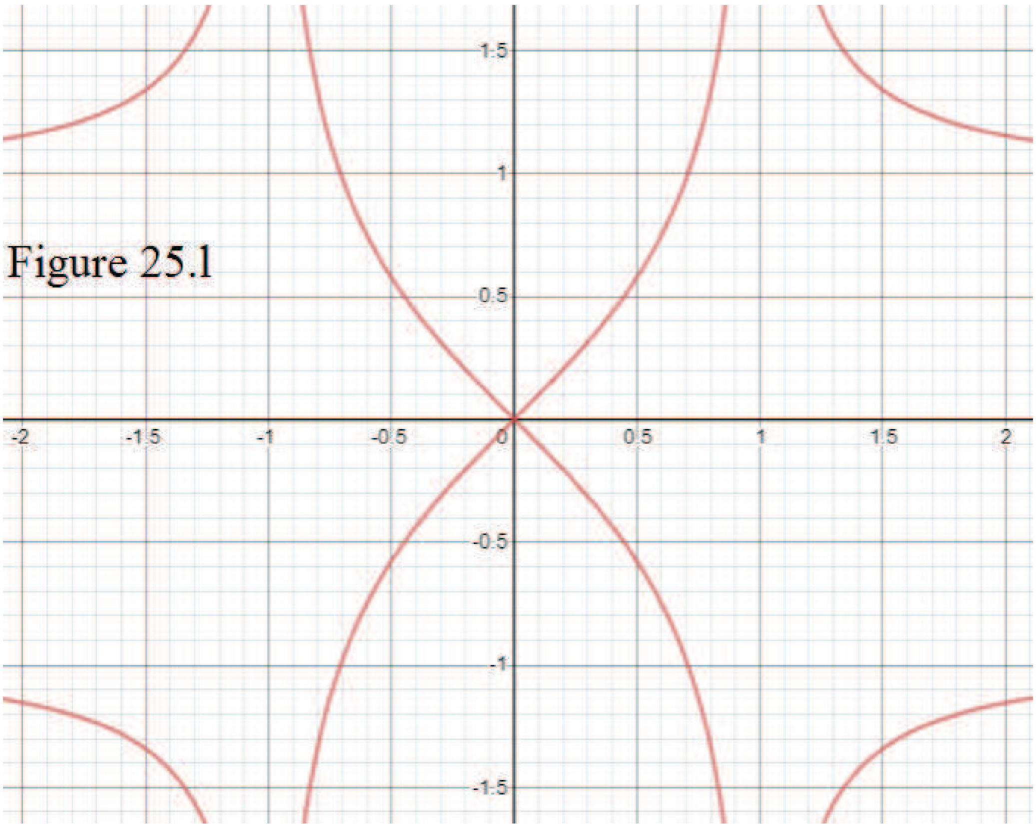
For $s < -1 \cap 1 < s$ and $-(a^2) = \frac{s^2}{1-s^2}$

$u = \sqrt{\frac{s^2}{1-s^2}} = \hat{a}$ and $u = -\sqrt{\frac{s^2}{1-s^2}} = -\hat{a}$

This is graphable as $u = \sqrt{-\left(\frac{s^2}{1-s^2}\right)}$ and $u = -\sqrt{-\left(\frac{s^2}{1-s^2}\right)}$. These are in fact identical to the circular trigonometric relations for the subspace equations $1 = \frac{1}{s^2} + \frac{1}{u^2} \equiv u = \sqrt{\frac{s^2}{s^2-1}}$.

Thus whether examining $1 = \frac{1}{s^2} + \frac{1}{u^2}$ or $1 = \frac{1}{s^2} - \frac{1}{u^2}$ the graph utilizing the full domain of s will appear as that shown in Figure 25.1:

Figure 25.1:



In keeping with the pattern thus far we will also look at the equation and graph of $y(s)$ and $y(x, s)$. The easiest way to obtain the equation for $y(s)$ is to begin with the $u(s)$ equation. For this example we have $u = \sqrt{\frac{s^2}{s^2-1}}$ for the circular trigonometric relations. From here we can convert to the $y(s)$ equation.

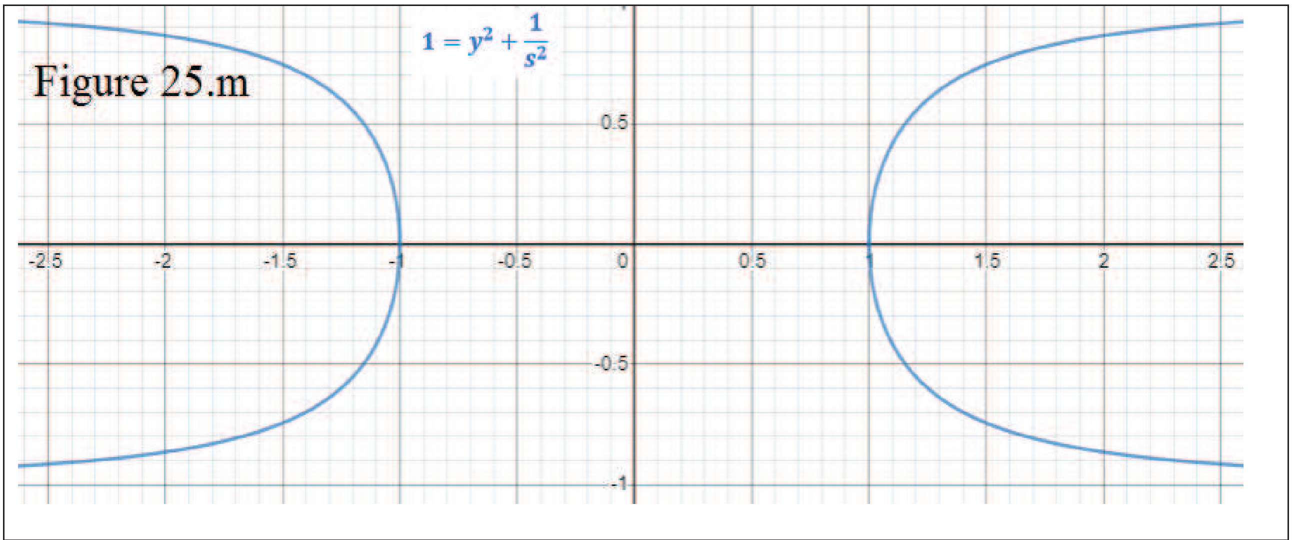
$$u = \sqrt{\frac{s^2}{s^2 - 1}} \quad \longrightarrow \quad -\frac{1}{y} = \sqrt{\frac{s^2}{s^2 - 1}} \quad \longrightarrow \quad \frac{1}{y^2} = \frac{s^2}{s^2 - 1}$$

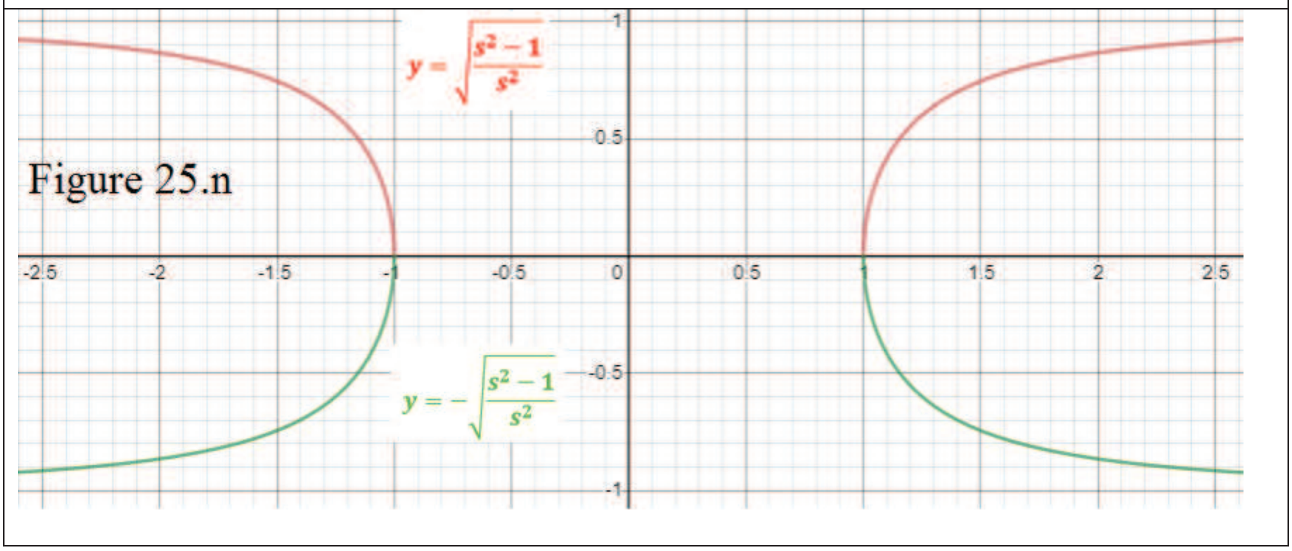
$$y = \sqrt{\frac{s^2 - 1}{s^2}}$$

As before we have the values $\pm y$ as a result of the radical giving us two simultaneous equations which must both be accounted for:

$$y = \sqrt{\frac{s^2-1}{s^2}} \qquad \text{and} \qquad y = -\sqrt{\frac{s^2-1}{s^2}}$$

These equations are derived from $1 = \frac{1}{s^2} + y^2$. The graphs defined by these equations are shown below in figures 25.m and 25.n.





The values for the domain of s are clearly $s \leq -1 \cap 1 \leq s$. Like in the earlier examples the extension of the domain of s to $-1 < s < 1$ will take us away from a sort-of inside out circular trigonometric set of relations to an even more skewed hyperbolic relation.

For $-1 < s < 1$

With $-(a^2) = \frac{s^2-1}{s^2}$

$$y = \sqrt{\frac{s^2-1}{s^2}} = \hat{a} \qquad \text{and} \qquad y = -\sqrt{\frac{s^2-1}{s^2}} = -\hat{a}$$

This is graphable as $y = \sqrt{-\left(\frac{s^2-1}{s^2}\right)}$ and $y = -\sqrt{-\left(\frac{s^2-1}{s^2}\right)}$ These equations are equivalent to the hyperbolic $y(s)$ relations $1 = \frac{1}{s^2} - y^2$ and $\pm y = \sqrt{\frac{1-s^2}{s^2}}$. They are shown below in Figures 25.o and 25.p.

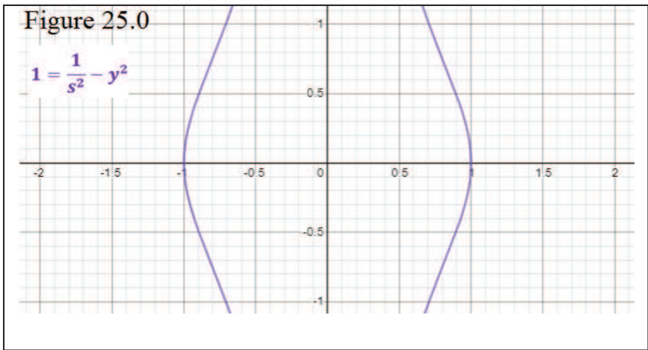


Figure 25.0

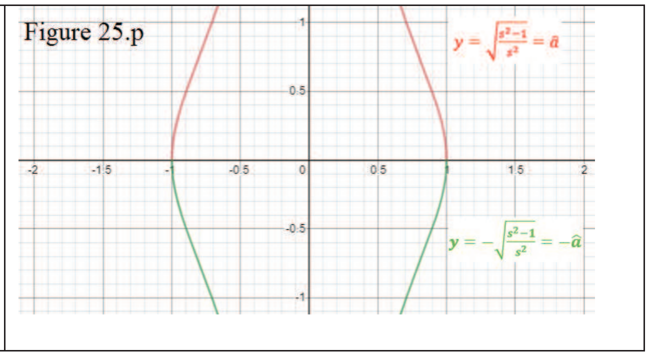


Figure 25.p

As with the earlier examples, if we begin with the hyperbolic relations for the $y(s)$ relations $1 = \frac{1}{s^2} - y^2$ and $\pm y = \sqrt{\frac{1-s^2}{s^2}}$, we have the traditional domain of $-1 < s < 1$, with graph which appear identical to those of Figures 25.0 and 25.p respectively. The extension of the domain of s into $s \leq 1 \cap 1 \leq s$ will result in graphs which are identical to the circular relations for the $y(s)$ equations with graphs which are identical to Figures 25.m and 25.n.

Lastly we will examine the combination of these components in the $y = f(x, s)$ equation.

By combining the circular $y = \sqrt{1 - x^2}$ and $y = \sqrt{\frac{s^2-1}{s^2}}$ (*which is identical to* $y = \sqrt{1 - \frac{1}{s^2}}$) we get the following equation:

$y=\sqrt{1-x^2}+\sqrt{1-\frac{1}{s^2}}$ $y^2=1-x^2+1-\frac{1}{s^2}$	
$2=x^2+y^2+\frac{1}{s^2}$	$y=\sqrt{2-x^2-\frac{1}{s^2}}$

The final two equations on the bottom are equivalent. The one on the right will be easier to graph. Whenever the domain of x enters into $x<-1 \cap 1<x$ and s enters into $-1 \leq s \leq 1$ there will be an automatic crossover to values consistent with hyperbolic relations between the output of y and the corresponding input.

The Figures below show theses aspects. In order to graph these you must recognize like in the other examples the radical implies two simultaneous equations:

$$y=\sqrt{2-x^2-\tfrac{1}{s^2}} \qquad \text{and} \qquad y=-\sqrt{2-x^2-\tfrac{1}{s^2}}$$

A graphing utility may not automatically display both. In this case simply plot both of these equations show here. Additionally the graphing utility will not know what to do when the values of these circular equation's inputs entered into their traditionally defined forbidden domains. Instead of showing the hyperbolic values they correspond to the graphing utility will likely not display anything or show an error. You can get it to show these correlations by plotting two additional equations which are equivalent to what they should be displaying:

$$y=\sqrt{-\left(2-x^2-\tfrac{1}{s^2}\right)} \qquad \text{and} \qquad y=-\sqrt{-\left(2-x^2-\tfrac{1}{s^2}\right)}$$

Figure 25.q shows only the circular relations of $y = \sqrt{2 - x^2 - \frac{1}{s^2}}$.

Figure 25.q

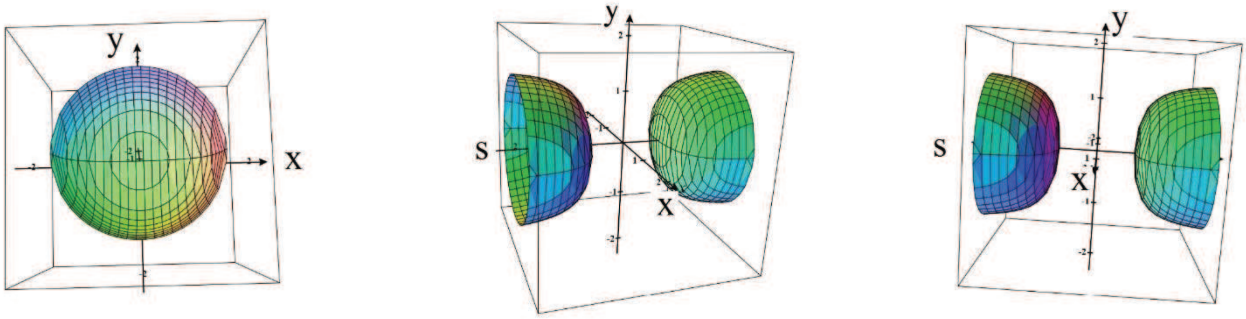


Figure 25.r shows only the hyperbolic relations of $y = \sqrt{2 - x^2 - \frac{1}{s^2}}$.

Figure 25.r

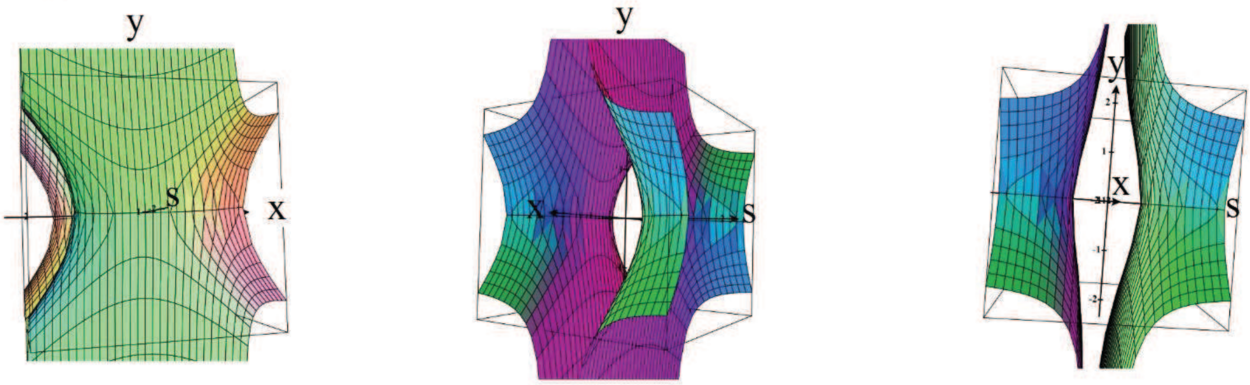
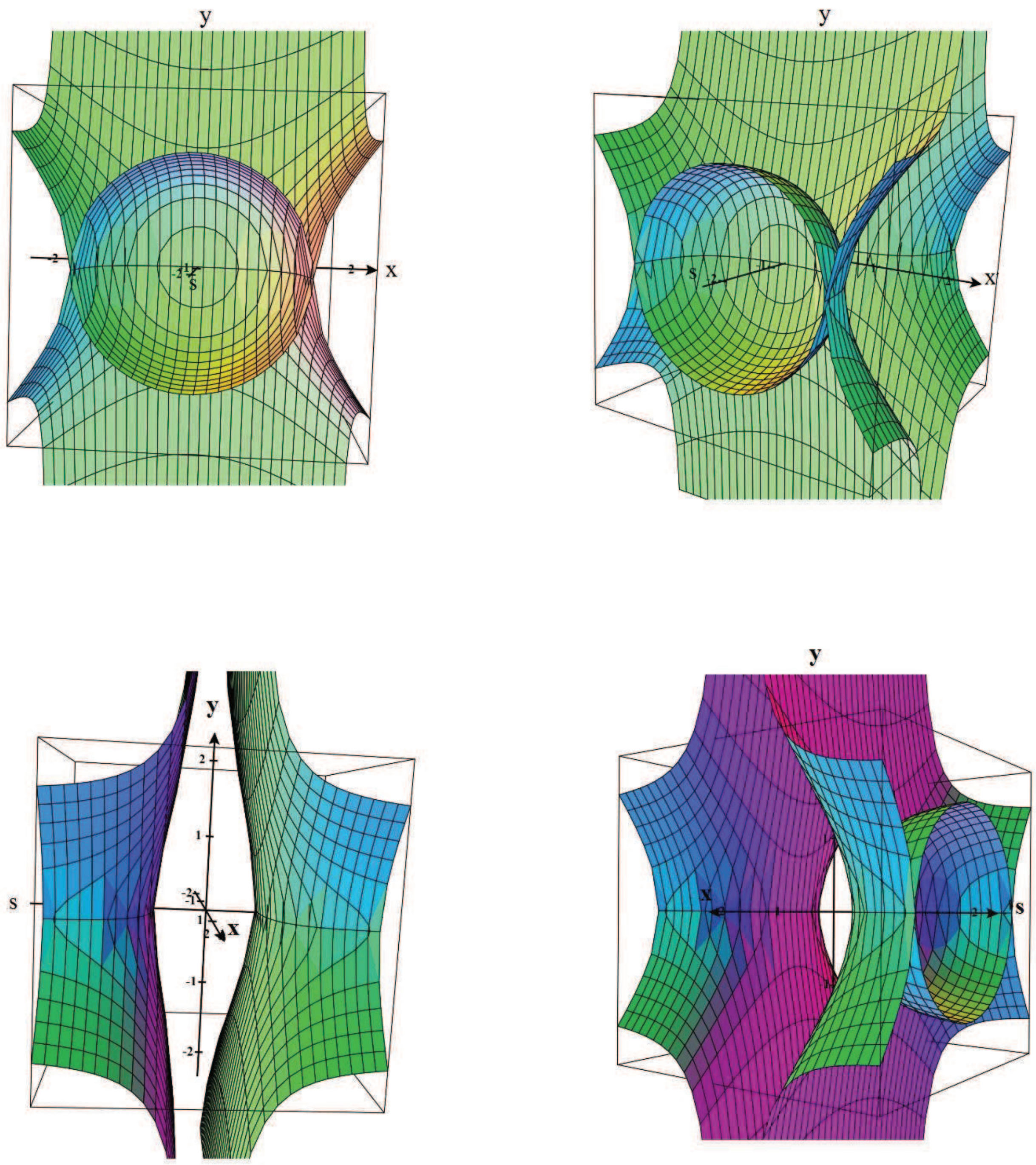


Figure 25.s shows the combined full set of relations of $y = \sqrt{2 - x^2 - \frac{1}{s^2}}$.

Figure 25.s



3.c.1—Basic Trigonometric Functions:

The following section will use the Sine function to consider how trigonometric functions apply to angles of 0 and η_0. The principles covered here can be extended to any of the trigonometric functions.

3.c.1.i—The Sine Function:

Sine Function: $y = \sin(x)$

Subspaces:

SY: $y = \sin\left(-\frac{1}{s}\right) = -\sin\left(\frac{1}{s}\right)$

XU: $u = -\frac{1}{\sin(x)} = -\csc(x)$

SU: $u = -\csc\left(-\frac{1}{s}\right) = \csc\left(\frac{1}{s}\right)$

Properties:

Period— 2π radians

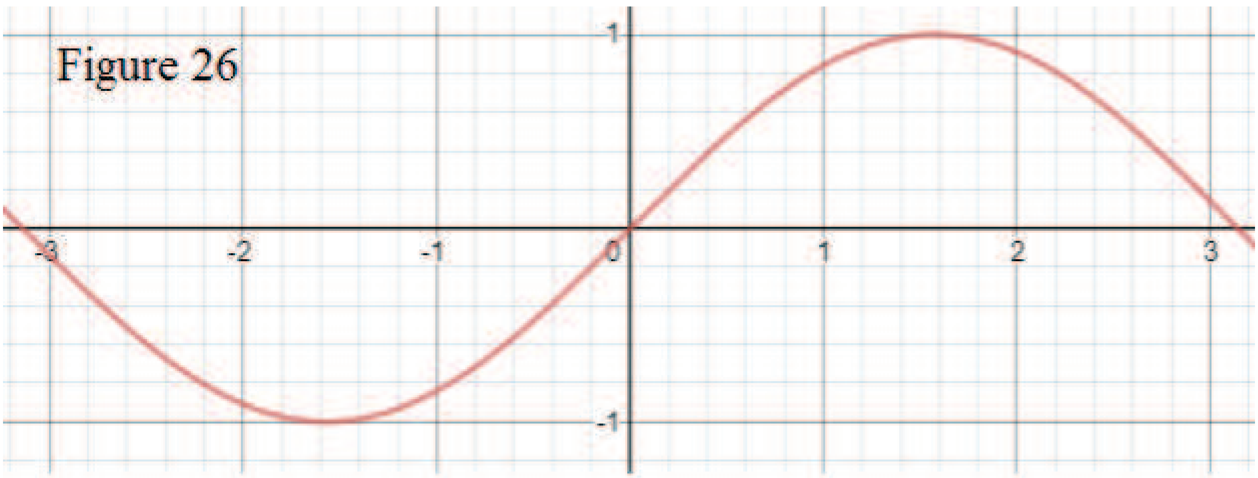
Domain— $\mathbb{R}$

Range— $-1 \leq y \leq 1$

Zeros— At 0 and all whole number multiples of Pi radians, $n\pi$

Figure 26:

Figure 26 shows one complete cycle of the Sine function (*a distance of* 2π *radians*) from $-\pi$ to π.



So we must ask ourselves what are the sine values of things like η_0, i, $\frac{0}{0}$ and any $\oplus$ number.

The best way be examine this is to examine the unit circle where on the Trigonometric functions are defined. Figure 26a below shows the unit circle. Figure 26b shows a zoom-in on the unit circle radius.

Figure 26.a

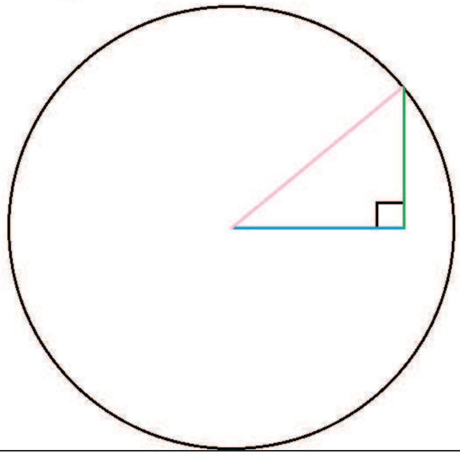
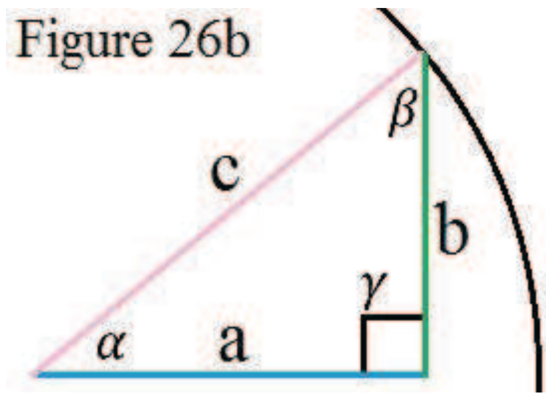


Figure 26b



The unit circle displays what is commonly called right-angle trigonometry. The red line in the images above is the diameter of the circle. We can measure an extension along the x-axis (assuming of course we have placed the center of the circle on the origin of the XY-Plane), and a distance vertically, parallel to the y-axis. They form a triangle. As we rotate the line representing the radius(C in the images) about the full 360 degrees of the circle the lengths of the base (a) and height (b) will change from 0 units in length to a some upper limit.

A precept from Euclidean Geometry provides *Two* ***triangles are*** *said to be* ***similar*** *if their corresponding* ***angles are congruent*** *and the corresponding sides* ***are*** *in proportion.* Without proving this Geometric principle, what it says is that no matter how much bigger one triangle is than another, if their sides remain in proportion they will have identical internal angles.

Because of this feature it is convenient to use a circle whose radius is 1 unit in length. Hence the name, The Unit Circle. The smaller size is easier to work with and the ratios discussed will apply to a circle with a radius of any size. The six basic Trigonometric functions are calculated as follows:

Given an angle $n \in \{0 , 360$ degrees

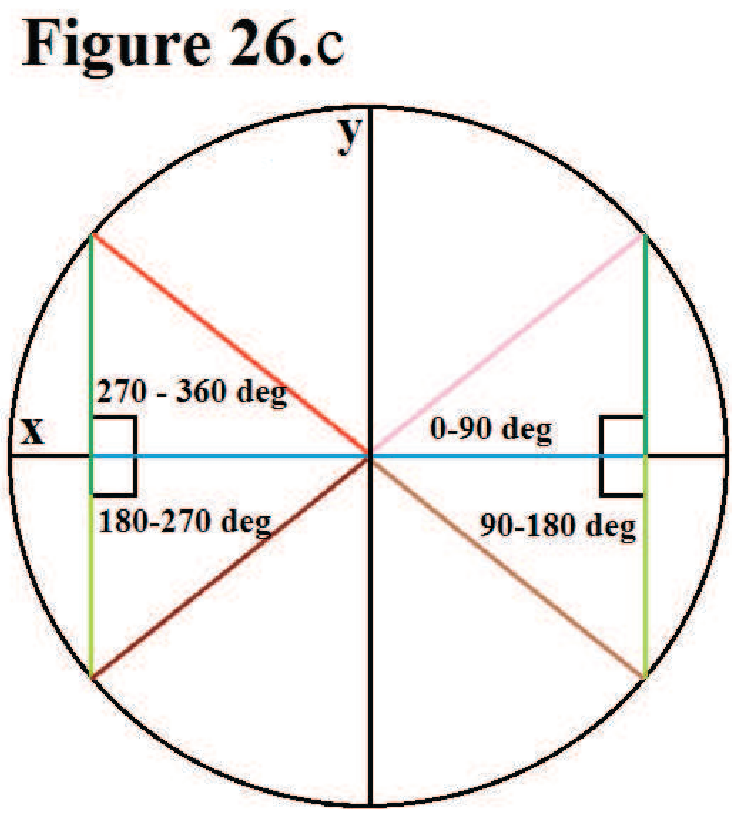
$\sin n = \frac{opp}{hyp}$	$\csc n = \frac{hyp}{opp}$
$\cos n = \frac{adj}{hyp}$	$\sec n = \frac{hyp}{adj}$

$\tan n = \frac{opp}{adj}$	$\cot n = \frac{adj}{opp}$

If we rotate the radius (*the hypotenuse in figures 26.a and 26.b*) about the circle we find the angle γ will remain 90 degrees while angles marked by α and β repeat through a range of 0 to 90 degrees. Obviously whenever α or β reaches 0 degrees the other angle really doesn't matter anymore as the triangle is gone. We are left with a straight line. The line, geometrically, has an angle of 180 degrees. So mathematically we can maintain a triangle which can have three angles, 0, 90 and 90.

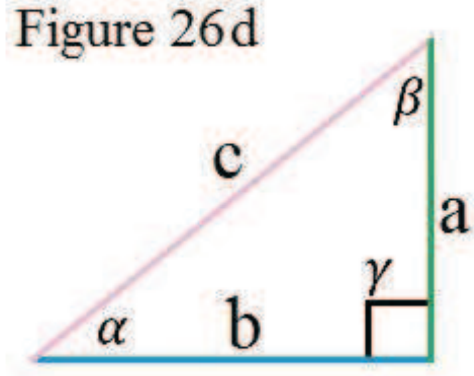
As the radius rotates about the circle, we always have it forming a right-triangle in some quadrant of the Cartesian plane. Figure 26.c shows the triangles formed as the radius line is rotated about the circle.

Figure 26.c:



So no matter what quadrant of the circle we are in, we can set up a triangle to work right-angle trigonometry. Recall from above that the sign of an angle n is defined as the ratio of the $\frac{opp}{hyp}$.

This means you can take the sine (*and any other trigonometric function*) of angle α, β and γ from the unit circle. It just depends on which angle you are measuring. Look again at the unit triangle and consider the various trigonometric functions for each of the angles as right angle trigonometry. Note, the angles in the below diagrams are being measured from the positive x-axis instead of from the positive y-axis where 0 degrees is usually placed. This is just to make the triangles more visually appealing.



$$\sin\alpha = \frac{opp}{hyp} = \frac{a}{c}$$

$$\sin\beta = \frac{opp}{hyp} = \frac{b}{c}$$

$$\sin\gamma = \frac{opp}{hyp} = \frac{c}{c}$$

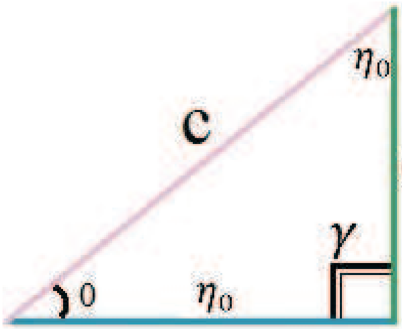
Using this reasoning we can now begin to examine trigonometric functions of the more exotic arguments. Let's begin by looking at a triangle which is defined mathematically as having an angle of 0 degrees for angle α.

An angle $\alpha = 0^{\circ}$ is a straight line of length C which itself is 180 degrees.

Figure 26.e

We can still draw this as a triangle mathematically

Figure 26.f



In the triangle on the right as angle α approaches 0, angle β will approach 90 degrees. This all still works out as a regular (though be it a very narrow) triangle, right up unto $\alpha = 0$. When this happens side $a = 0$. Additionally angle β is now coexistant in the exact same space as angle γ.

This isn't as simple as saying they are both there. Rather they are the same angle, and there values are merged into one 180 degree angle. At the point $\alpha = 0$ we label angle $\beta = \eta_0$ and allow angle γ to represent the entire value—we use the symbol ⊓ to represent a double-90 angle, 180 degrees. Lastly, because side $a = 0$, side c will coexist with side b. Side b does not equal the length of side c. Rather side c is the hypotenuse and the base. It is not side b though either. Instead, it displaces side b which will have the length of η_0. Lastly side a is held to be 0 only in keeping with the idea that this is still mathematically a triangle. *In this state the 0 is really a resolved naught. The value which should be used in the calculation is $a = \eta_0$.*

Had we chose to zero out angle β instead of α we would get the same type of odd triangle. The values listed above for sides a and b will swap as well as those of angles α and β. The law of Sines will still apply. Applying the values from the above triangle in figure 26.f (*the straight line of figure 26.e*) we have the following.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{\sin\alpha}{a} = \frac{\sin 0}{0} = \frac{\sin 0}{\eta_0} = \frac{0}{\eta_0} = 0$$

$$\frac{\sin\beta}{b} = \frac{\sin\eta_0}{\eta_0} = \frac{\sin 0}{\eta_0} = \frac{0}{\eta_0} = 0$$

$$\frac{\sin\gamma}{c} = \frac{\sin(90+90)}{c} = \frac{\sin 180}{c} = \frac{0}{c} = 0$$

Now because the unique changes in the nature of this triangle, first we need to make the substitution that side b is now also side c. Also, even though γ is now 180 degrees, it still occupies the base of the triangle, which in a normal triangle is held be 90 degrees. In this situation, since it is in reality a line, as there is no opposite side for angle γ. Thus we must use η_0 for its Opposite. So we get the following:

$$\sin\alpha = \frac{opp}{hyp} \quad \longrightarrow \quad \sin 0 = \frac{\eta_0}{c} = \frac{0}{c} \quad \longrightarrow \quad 0 = 0$$

$$\frac{\sin\alpha}{a} = \frac{\sin 0}{\eta_0} = \frac{\frac{0}{c}}{\eta_0} = \frac{0}{\eta_0} = \eta_0 = 0$$

$$\frac{\sin\beta}{b} = \frac{\sin\eta_0}{\eta_0} = \frac{\frac{\eta_0}{c}}{\eta_0} = \frac{0}{\eta_0} = 0$$

$$\frac{\sin\gamma}{c} = \frac{\sin 180}{c} = \frac{\frac{\eta_0}{c}}{c} = \frac{0}{c} = 0$$

The component for $\frac{\sin\gamma}{c} = \frac{\frac{\eta_0}{c}}{c}$ *may need some clarification. Since this is now really a line, with sides a = 0 and b =* η_0 *there is no side opposite to angle* γ*. So that value of opposite must be occupied by* η_0*. Had this been a real 90 degree angle instead of the double-90 of the hypothetical triangle above, the opposite side would be the hypotenuse. Sine of 90 = 1 while sine of 180 equals 0.*

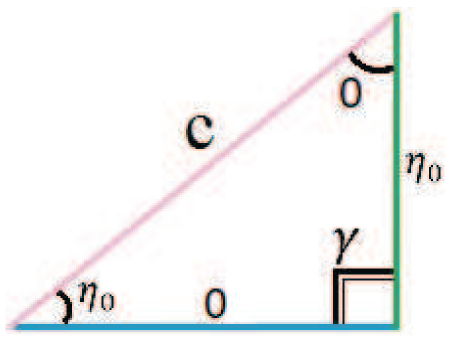
The component for $\frac{\sin\alpha}{a}$ *having side a, can be represented as either* $a = \eta_o$ *or as its resolved value* $a = 0$*. If you use* $a = 0$ *you'll get* $\frac{\sin\alpha}{a} = \frac{\sin 0}{0} = \frac{0}{0}$*. The* $\frac{0}{0}$ *did not result from a constant one-to one ratio. Neither did it result from the root of a negative number so it cannot be resolved to* $\hat{1}$*. Instead the numerator was always 0 and by trending 0 divided by any number is 0. Likewise if you instead hold the denominator to have always been 0, then again by trending any number including this 0 in the numerator, when divided by 0 will trend toward infinity, naught and then resolve to 0.*

Also note the $\sin\gamma$*, the sine of 180 degrees will equal 0 even by traditional mathematics. From this we see the Laws of Sines are satisfied for this use of angles having both 0 and* η_0*.*

See the properties for naught if you need a refresher.

The value of η_0 is similar to 0. As said above if we were to allow angle $\alpha = \eta_0$ we get same triangle like above but swap-outs on sides a and b, as well as angles α and β. The image of that triangle is marked up below in figure 26.g. The math on it will be analogous to that on figure 26.f above.

Figure 26.g



$$\frac{\sin\alpha}{a} = \frac{\sin\eta_0}{\eta_0} = \frac{\frac{\eta_0}{c}}{\eta_0} = \frac{0}{\eta_0} = 0$$

$$\frac{\sin\beta}{b} = \frac{\sin 0}{0} = \frac{\frac{0}{c}}{\eta_0} = \frac{0}{\eta_0} = \eta_0 = 0 \text{ resolves to } 0$$

$$\frac{\sin\gamma}{c} = \frac{\sin 180}{c} = \frac{\frac{\eta_0}{c}}{c} = \frac{0}{c} = 0$$

We will return to these types of triangles below as we continue to explore the other trigonometric functions. For the moment we will turn to the Sine of plus-and-minus values. A value $\oplus n$ can be resolved several ways. The value $\oplus n = \sqrt{-(n^2)}$. At the location of its occurrence it can be resolved to $\oplus n = ni = \hat{n}$. Although ni and $\hat{n}$ are equivalent they are not identical and will result in different values for a function when used as inputs. Respectively they are the differences between examining values on the complex plane and their resolved values in real space.

Numbers on the complex plane have the form of $a \pm bi$. Just like in chapter 2, we can examine only the i-multiple component by setting $a = 0$. If $b = 1$ we are left with i only. Further we can specify that $i = \oplus 1 = \hat{1}$. In the paragraph just before is stated that ni and $\hat{n}$ are equivalent they are not identical. This is of particular importance with trigonometric functions which take these values as arguments. The Sine function here has two parts, the function Sin(x) and the argument which takes the place of (x). In this case that is a very special constant, a number $\oplus n$.

Though we could resolve the value $\oplus n$ to $\hat{n}$ were it alone, here it is the argument of a trigonometric function. That function does not seek to provide the Sine of $\hat{n}$ or $\check{n}$. Instead it seeks to provide the Sine of $\oplus n = ni$. This means you have to actually take the Sine of the unresolved term before resolving any i-multiples.

If you were to try and resolve the value first you will get an entirely different and incorrect answer.

Don't fall into this trap. *If you are given*

$$\sin\left(\sqrt{-(n^2)}\right) = \sin(\oplus n)$$

You must resolve the plus-and-minus number to its i-multiple configuration, $\sin(ni)$ *, in order to solve. NOT* $\sin(\hat{n})$*. If you use the complex number a + bi where a = 0 and b = 1, the you have* $\sin(i)$*. This* $\sin(ni)$ *is identical to* $\sin(i)$ *where n = 1.*

If we were to resolve the argument $\sin(i)$ *to* $\sin(\hat{1})$ *it is indistinguishable from normal circular trigonometry* $\sin(1)$*. Assuming the value is in degrees the sine of an* i*-multiple's positive up component with a magnitude of 1 is:*

$$\sin(\hat{1}) \approx 0.017$$

You would further assume its negative-down component lied on the adjoing subspace plane. The diagram below shows what this would look like with triangles drawn from the unit circle.

If the original $\sin(\hat{1})$ *occurred on the XY-Plane for the positive-up component, the negative-down component will appear in the SY-Plane.*

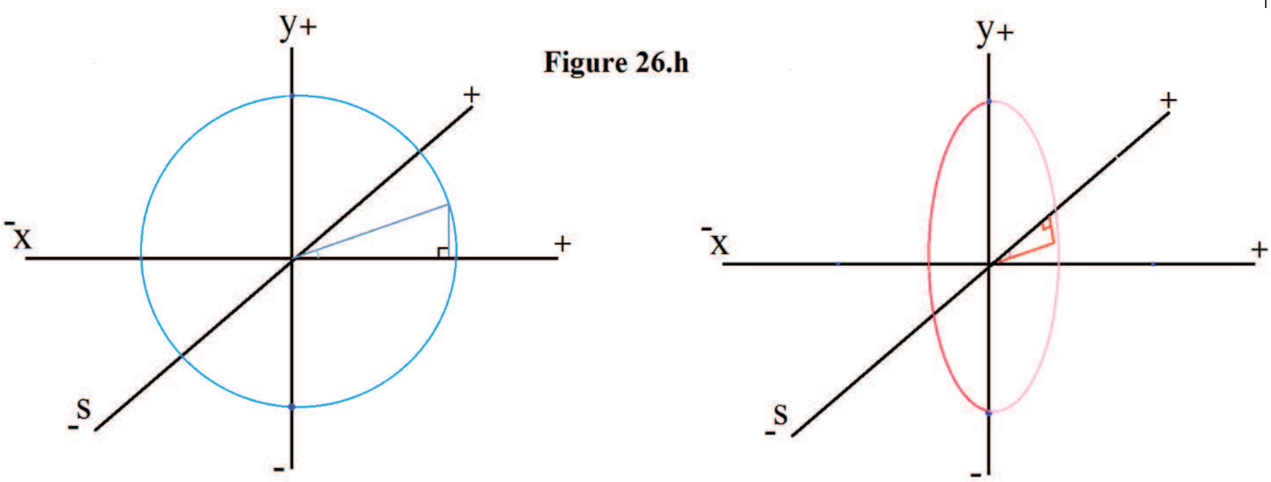
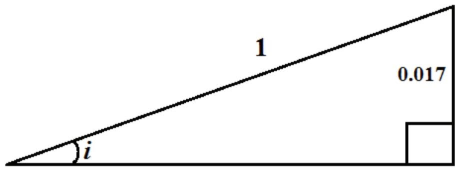
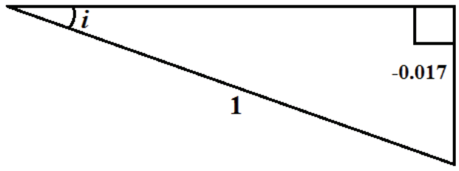


Figure 26.h



$$\theta = i = \oplus 1 = \begin{cases} \hat{1} = 1 \\ \check{1} = -1 \end{cases}$$



As you will see below this is not the answer you should obtain. You must keep the i-multiple until after calculating the Sine. This will require use of a trigonometric identity and hyperbolic trigonometric functions.

The unresolved value $\sin(i)$ is new to us as a trigonometric function argument. This not just a value of x such that $y = \sqrt{1 - x^2}$ with an x value resulting in a negative root. Instead the initial argument itself is an i-multiple being acted upon by a cyclic function. The sine function is essentially acting upon the root, of the n multiple of a negative number. In the form of a complex number you need to use trigonometric properties to rewrite its form in order to solve. First consider that the $\sin(i)$ can be written $\sin(a + bi)$ where $a = 0$ and $b = 1$.

From $\sin(a + bi)$ we can get the following:

$$\sin(a + bi) \;=\; \sin(a)\cosh(b) + i\cos(a)\sinh(b)$$

All that's left now is to fill in the values for a and b.

$$\sin(0)\cosh(1) + i\cos(0)\sinh(1)$$

$$(0) + i(1)\sinh(1)$$

$$i\sinh(1)$$

$$\approx 1.1752i$$

This purely hyperbolic interpretation still leaves an i-multiple which may now be resolved direly to

$$\sin(i) \approx 1.1752i \approx \hat{1}.1752$$

The hyperbolic resolution of the value can be generalized as:

$$\sin(ni) = i\sinh(n) = \hat{1} \cdot \sinh(n)$$

So for $\sin\left(\sqrt{-1}\right) = \sin(i)$ we have the following:

$$\sin(i) = \sinh 1 \approx 1.17$$

The negative-*down* component of the i-multiple argument is used in the coadjoining subspace plane. Assuming the original equation is on the XY-Plane such that $y = \sin(i)$ the y value is from the Sine function. The input is the x-axis value itself. The given equation, as was already covered, is resolved as:

$$y = \sin(i) \longrightarrow i\sinh(1) \longrightarrow \hat{\imath}\sinh(1) \approx 1.17$$

The equation $y = i\sinh(1)$ is the implied equation given by $y = \sin(i)$ and it is in terms of $y = f(x)$. The coadjoining subspace plane, the XU-Plane equation may be calculated from $y = i\sinh(x)$.

$$\Xi y = \frac{\Xi}{\varsigma u} i\sinh(x) \quad \longrightarrow \quad u = -\frac{1}{i\sinh(x)}$$

The adjoining SY subspace plane is represented by:

$$y = i\sinh\left(-\frac{1}{s}\right)$$

In this equation the s-axis input which cooresponds to the x-axis value is $s = -\frac{1}{x}$. In the equation $y = \sin(i) = i\sinh(x)$. This was resolved to $y = \hat{\imath} \cdot \sinh(1)$ taking the *up* value with $x = 1$. In the $y = \sin\left(-\frac{1}{s}\right)$ the value of s when $x = 1$ will be $s = -\frac{1}{1} = -1$. Lastly is the resolution of the i. In the original equation it takes the positive *up* value. Here it will take the negative *down* value.

$$y = i\sinh\left(-\frac{1}{s}\right) \quad \longrightarrow \quad y = \check{\imath}\sinh(-1) \quad \longrightarrow \quad y \approx -(-1.17) \quad \longrightarrow \quad y \approx 1.17$$

Likewise should you encounter a hyperbolic Sine of i you must use a trigonometric property to resolve the equation into a form without an i as its argument, a process equivalent to taking the hyperbolic sine of the i before resolving it.

Given $\sinh(i)$, where $a + bi$ is $0 + (1)i$, and a*ssuming this is the XY-Plane* you will need the property which follows:

$$\sinh(ix) = \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i\sin(x)$$

Thus for $\sinh(i)$ you get:

$$\sinh(i) \quad \longrightarrow \quad i\sin(1) \quad \longrightarrow \quad \hat{1}\sin(1) \approx 0.017$$

In the same fashion above the down component on the SY-Plane will be given by:

$$\check{1}\sin(-1) = -\sin(-1) \approx 0.017$$

<u>3.c.2—The Cosine Function:</u>

The last thing we'll look at in this text is the Cosine function with an i-multiple argument. With this information combined with the other listed trigonometric information on the Sine function you can extrapolate the properties for the other standard trigonometric functions, namely, the Cosecant, Secant, Tangent and Cotangent functions.

<u>Cosine:</u>

$y = \cos(x) = \frac{1}{\sec(x)}$ Is a *Co-function* of the Sine function

The graph of the Cosine function is shown below.

Properties: Period— 2π radians

Subspaces: SY: $y = \cos\left(-\frac{1}{s}\right) = \cos\left(\frac{1}{s}\right)$ XU: $u = -\frac{1}{\cos(x)} = -\sec(x)$

SU: $u = -\sec\left(-\frac{1}{s}\right) = -\sec\left(\frac{1}{s}\right)$

Cosine of i-multiple arguments: $y = \cos(ni) = \cosh(n)$

Laws of Cosines: $a^2 = b^2 + c^2 - 2\cos\alpha$

$$b^2 = a^2 + c^2 - 2\cos\beta$$

$$c^2 = a^2 + b^2 - 2\cos\gamma$$

Property:	Traditional mathematics:	Null Algebra:
Domain	$x = \mathbb{R}$	$x = \mathbb{R}$
Range	$-1 \leq y \leq 1$	$-1 \leq y \leq 1$
Zeros	$x = n\frac{\pi}{2}$	$x = n\frac{\pi}{2}$
		Where *n* is any whole integer value

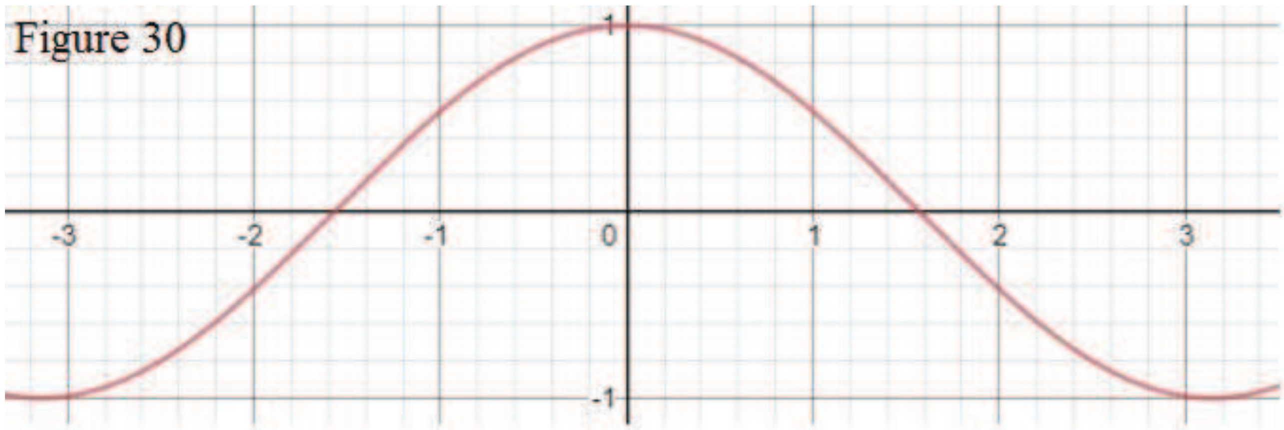


Figure 30
1
-1
-3
-2
-1
0
1
2
3

Afterword

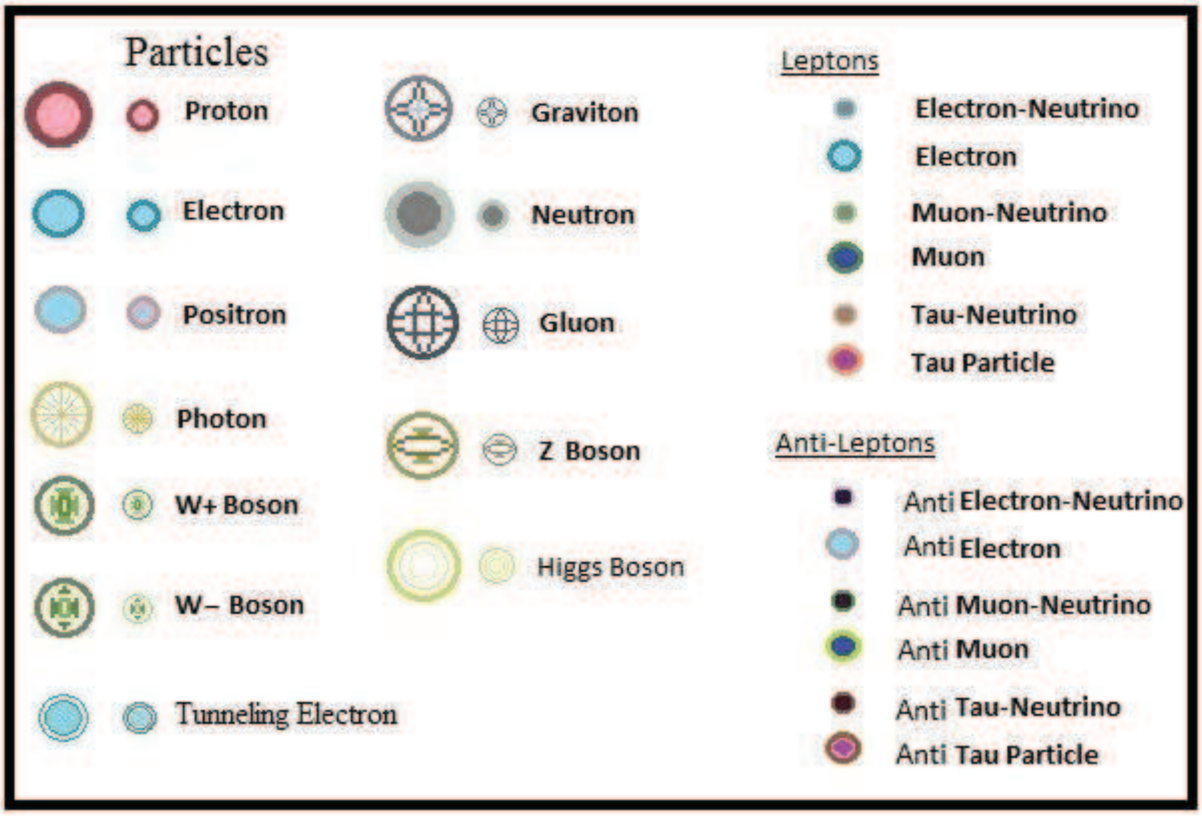
I began writing my first draft of this text between 2012 and 2013. Even now in 2021 its not finished and I suspect it never really can be. Math and science are constantly evolving disciplines. This one is in its infancy and my approach is not the only method being attempted to solve the unknowns of dividing by zero and negative radicals. It will take time for these concepts to be understood and accepted in traditional mathematics.

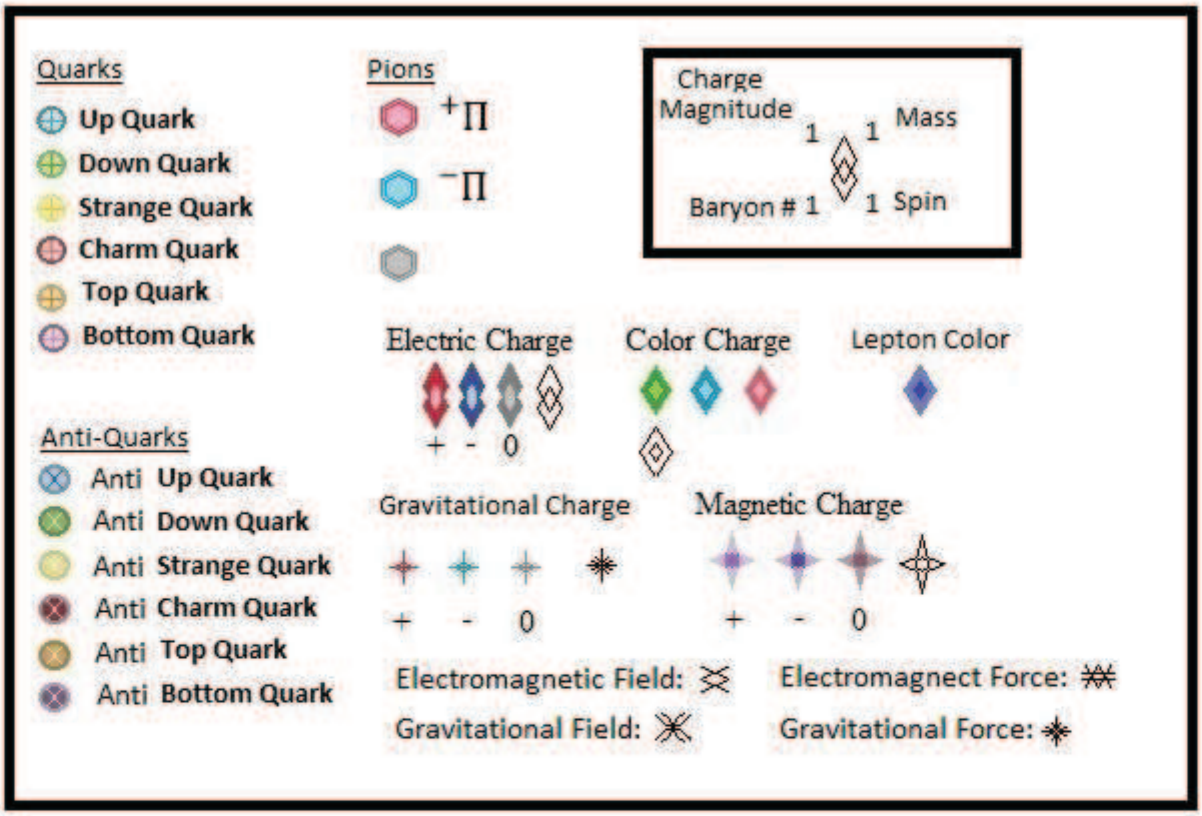
Where to go from here? Though trigonometry itself is slightly beyond the scope of this text, the principles laid out here need extended to the remainder of the standard circular and hyperbolic trigonometric functions. It should also be extended to Calculus, Differential Equations and Quantum Mechanics where instances of division by 0 and negative radicals will have a profound impact when resolved to real number values.

Peer review is process, like the pursuit of knowledge, is also never really finished. It's best completed when new theory is put to use by the community it is intended for. There can be no better way of doing this than publication. Though this text has not reached the level of completion I hoped for I believe it is correct and presented in the way which makes it easy to understand by anyone having a solid foundation in Algebra.

Any new theory has without doubt not accounted for all instances for use. I suspect there will be instances encountered which have not been covered in this text and which do not fit the methods listed herein. With patience and examination I also suspect you find this theory simply incomplete, needing either adjustment or addition consistent with expectations in a developing theory.

What is this used for? I remember reading Benjamin Franklin, while detailing his experiments in electricity, was asked precisely this; what good is it? His response was "What good is a newborn baby?" Indeed. The impact of this extension to mathematics won't be fully known till its applied and tested over time to all manner of disciplines. It is my hope this material will be found useful by others and allow many new discoveries in the fields of science and mathematics. All herein is my own work, including the symbols found below. Provided proper citation is used and credit for the work acknowledged all may feel free to use it in their own efforts to drive forward the progress of man.


Particles
Proton
Electron
Positron
Photon
W+ Boson
W– Boson
Tunneling Electron
Graviton
Neutron
Gluon
Z Boson
Higgs Boson
Leptons
Electron-Neutrino
Electron
Muon-Neutrino
Muon
Tau-Neutrino
Tau Particle
Anti-Leptons
Anti Electron-Neutrino
Anti Electron
Anti Muon-Neutrino
Anti Muon
Anti Tau-Neutrino
Anti Tau Particle


Quarks
Up Quark
Down Quark
Strange Quark
Charm Quark
Top Quark
Bottom Quark
Anti-Quarks
Anti Up Quark
Anti Down Quark
Anti Strange Quark
Anti Charm Quark
Anti Top Quark
Anti Bottom Quark
Pions
+Π
−Π
Charge Magnitude
1
1
Mass
Baryon # 1
1
Spin
Electric Charge
+ - 0
Color Charge
Lepton Color
Gravitational Charge
+ - 0
Magnetic Charge
+ - 0
Electromagnetic Field:
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Gravitational Field:
Gravitational Force: