Computation of the n digits of pi in O (n) iterations

Xingfeng Chen

Email: chenxingfeng2020@126.comg

February 2021

Abstract

A method for computing the n decimal digits of π in O(n) iterations with O (n) computation complexity and O (1) memory. The result of $3.1415926 < \pi < 3.1415927$ by Zu Chongzhi can be calculated in 10 iterations or 12288 polygons with 9 bit significant decimal intermediate precision. The 1000th bit of π can be calculated at 1660th iteration with 1005 bit significant intermediate precision and the 10000th bit can be calculated at 16608th iteration with 10005 bit significant intermediate precision. The key through is that the area of a circle equals to the sum of the difference of areas between polygons inside one circle. This method is similar to Liu Hui's method, but there's a little bit of a difference. Guess that Zu Chongzhi may used this method to obtain higher accuracy.

Keywords: digit of π; 3.1415926;The Nine Chapters on the Art of Mathematics;Zu Chongzhi;Liu hui;Li Chunfeng

1 Introduction

1.1 Backgroud

As discussed in reference [10],π is a constant that finds it's origin from far antiquity. The calculation method has gone through three eras:Geometrical era, Calculus era and Modern era and computers. Geometric methods are almost no longer used in modern times due to accuracy and complexity. This article intends to introduce a new geometry based calculation method.

1.2 Definition of Symbols

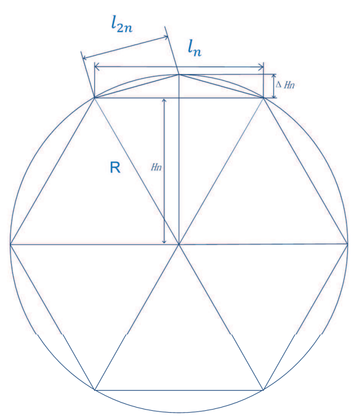

Figure 1: Symbols definition of inscribed polygon and circle

As shown in Figure1,let R be the radius of a circle, $l_n$be the edge length of N-polygon,l_{2n}be the edge length of 2N-polygon,Ln be the perimeter of N-polygon, Hn is the distance from circle's center to edge of N-polygon, S_n be the area of N-polygon inscribed in a circle,S_{2n} be the area of

2N-polygon inscribed in a circle, $\triangle S_{2n-n}$ be the area of difference of S_{2n} and S_n in formula (8), $\triangle Hn$ is the difference of R and Hn in formula (3),and S_r be the area of the circle.

1.3 Description of the method

The area of a circle with R=1 is $S_r = \pi R^2 = \pi . S_6$ is the area of a regular hexagon inscribed in a circle. S_{12} is the area of a regular 12 polygon inscribed in a circle. $\triangle S_{12-6} = S_{12} - S_6$ is the area of difference of S_{12} and S_6.As shown in Figure2, we can get $S_{12} < \pi < S_{12} + \triangle S_{12-6}$. For more common case, we can get $S_{2n} < \pi < S_{2n} + \triangle S_{2n-n}$.

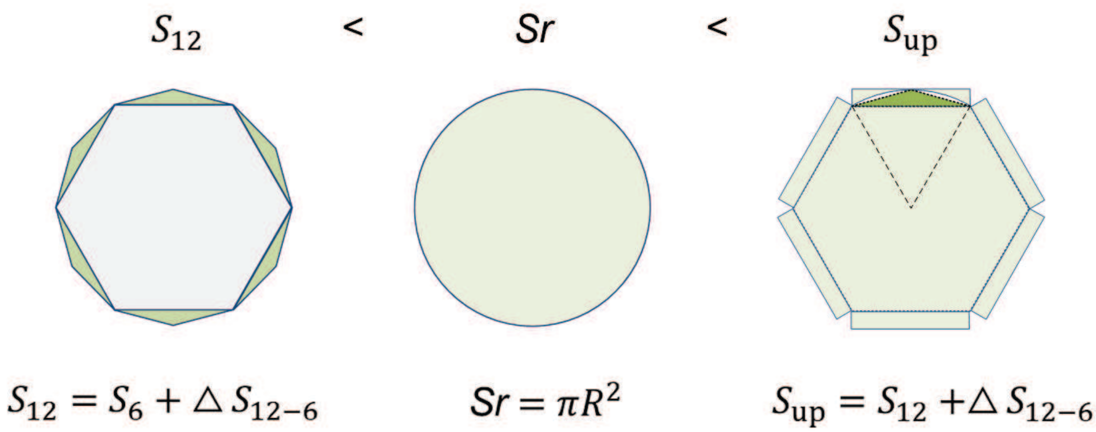

Figure 2: The upper and lower bounds of π using hexagon and 12 polygon

The formulas of this method are listed below:

$$l_6 = R = 1 \tag{1}$$

$$\text{Hn} = \sqrt{R - \left(\frac{l_n}{2}\right)^2} \tag{2}$$

$$\triangle\text{Hn} = R - Hn \tag{3}$$

$$l_{2n} = \sqrt{\left(\frac{l_n}{2}\right)^2 + (\triangle\text{Hn})^2} \tag{4}$$

$$\text{L}n = n \times \text{ln} \tag{5}$$

$$S_{12} = \frac{1}{2} \times l_6 \times R = 3 \tag{6}$$

$$\triangle S_{2n-n} = \frac{1}{2} \times \text{Ln} \times \triangle Hn \tag{7}$$

$$S_{2n} = S_n + \triangle S_{2n-n} \tag{8}$$

$$S_{2n} < \pi < S_{2n} + \triangle S_{2n-n}, n = 3 \times 2^i, i = 2, 3, 4 \longrightarrow \infty \tag{9}$$

A simple summary with a formula is as the following:

$$\pi = S_{12} + \sum \triangle S_{2n-n} \tag{10}$$

2 Method Comparison

The great ancient Chinese mathematician Liu hui recorded the detail steps of calculating π in his book of “The Nine Chapters on the Art of Mathematics”,and another great Tang Dynasty mathematician Li Chunfeng added comments to this book.Thanks for Li Chunfeng’s record in “The Nine Chapters” and other books, we know that Zu Chongzhi got a better π calculation result. It is a huge regret that the detailed calculation method and calculation steps of Zu Chongzhi were not recorded.Some people think that Zu did not change the algorithm method of Liu, yet some other people disagree.Liu hui got the result of 3.1416 or 3927/1250 for π using polygon. Liu hui’s two methods are named Liu’s method1 and Liu’s method2 in this artical. The main through of the method in this artical is similar to of Liu’s methods,and the difference is how to calculate S_n. Table 1 shows the detail differences.

Table 1: The differences of this method and Liuhui’s methods

Liu’s method1	Liu’s method2	This method
$l_6 = R = 1$ $Hn = \sqrt{R - \left(\frac{l_n}{2}\right)^2}$ $\triangle\mathrm{Hn} = R - Hn$ $l_{2n} = \sqrt{\left(\frac{l_n}{2}\right)^2 + (\triangle Hn)^2}$ $\mathrm{L}n = n \times ln$ $\mathrm{S}_{2n} = \frac{1}{2} \times \mathrm{L}n \times R$ $\pi \approx \mathrm{S}_{2n}$ $n = 3 \times 2^i, n \to 3072$	$l_6 = R = 1$ $Hn = \sqrt{R - \left(\frac{l_n}{2}\right)^2}$ $\triangle\mathrm{Hn} = R - Hn$ $l_{2n} = \sqrt{\left(\frac{l_n}{2}\right)^2 + (\triangle Hn)^2}$ $\mathrm{L}n = n \times \ln$ $\mathrm{S}_{2n} = \frac{1}{2} \times \mathrm{L}n \times R$ $\Delta\mathrm{S}_{2n-n} = \mathrm{S}_{2n} - \mathrm{S}_n$ $\pi \approx \mathrm{S}_{192} + \frac{36}{105}\Delta\mathrm{S}_{192-96}$	$l_6 = R = 1$ $Hn = \sqrt{R - \left(\frac{l_n}{2}\right)^2}$ $\triangle\mathrm{Hn} = R - Hn$ $l_{2n} = \sqrt{\left(\frac{l_n}{2}\right)^2 + (\triangle Hn)^2}$ $\mathrm{L}n = n \times \ln$ $S_{12} = \frac{1}{2} \times l_6 \times R = 3$ $\triangle S_{2n-n} = \frac{1}{2} \times \mathrm{Ln} \times \triangle\mathrm{Hn}$ $S_{2n} = S_n + \triangle S_{2n-n}$ $S_{2n} < \pi < S_{2n} + \triangle S_{2n-n}$ $n = 3 \times 2^i, i = 2, 3, 4 \to \infty$

3 Result and Error Analyses

3.1 Python test code

The method is tested using python 3.8. An example code of 12 bit intermediate precision with 20 iteration is as the following.

```python
import math
from decimal import *
getcontext().prec = 12            #precision parameter
getcontext().rounding = ROUND_HALF_UP      #rounding parameter
def l2N_F(ln):  # calc next l
    L2N_Temp0 = Decimal(ln)*Decimal(ln)
    L2N_Temp1 = Decimal(4-L2N_Temp0).sqrt()
    l2N = Decimal(2-L2N_Temp1).sqrt()
    return l2N
l = Decimal(1)        #start from hexagon
N = Decimal(6)
S_Sum_Min = Decimal(3)
R = Decimal(1)      # R= 1
for i  in range(20):  #20 is the iteration number
    l=l2N_F(l)      #calc next l
    N =  N * 2      #next 2N
    Delta_H_HALF = (R-((Decimal(R-l*l/4)).sqrt()))/2
    L = N*l
    S_Hui = L/2
    print("gon number is ",N)
    Delta_S = L*Delta_H_HALF
    if Delta_S == 0:
        break
    S_Sum_Min = S_Sum_Min + Delta_S
    S_Sum_Max = S_Sum_Min + Delta_S
    print("the iteration num is ",i)
    print("ln is",l)
    print("Ln is",L)
    print("Delta S2n-n is",Delta_S)
    print("HUI PI Method1 is",S_Hui)
    print("pi min is ", S_Sum_Min)
    print("pi max is ", S_Sum_Max)
```

The “getcontext().prec = 12” is for the significant decimal intermediate precision definition, and “range(20)” is the iteration control number. We can change getcontext().prec and range(20) for higher accuracy

3.2 Result

The result of $3.14159261 < \pi < 3.14159271$ can be calculated in 10 iterations or 12288 polygon with 9 bit significant intermediate precision. The 1000th bit of π can be calculated at 1660th iteration with 1005 bit significant intermediate precision. The 995-1000th bits are “1989”. The 10000th bit can be calculated at 16608th iteration with 10005 bit significant intermediate precision. The 9995-10000th bits are “5678”. The results are checked with the number references [7]. Higher precision testing also shows that it is effective.

3.3 Error Analyses and Comparison

Table 2 shows the errors of different methods.The table is calculated using 12 bit Intermediate precision. We can get 3.14159665505 using Liu’s method2 at 192-polygon and 3.14159128005 using Liu’s method1 at 3072-polygon. We can get $3.14159210598 < \pi < 3.14159374875$ at 3072-polygon and $3.14159261935 < \pi < 3.14159272203$ at 12288-polygon using the method of this article. It shows better accuracy.

Table 2: The errors of different methods

N_polygon	l_n	Ln	$\triangle S_{2n-n}$	π(Liu's method1)	π(Liu's method2)	π_min(this method)	π_max(this method)
n=6	1	6	—	3	—	—	—
n=12	0.517638090204	6.21165708245	0.10582854123	3.10582854123	—	3.10582854123	3.21165708246
n=24	0.261052384437	6.26525722649	0.0268000720501	3.13262861325	—	3.13262861328	3.15942868533
n=48	0.130806258451	6.27870040565	0.00672158976391	3.13935020283	—	3.13935020304	3.14607179280
n=96	0.065438165622	6.28206389973	0.00168174784110	3.14103194987	—	3.14103195088	3.14271369872
n=192	0.032723463295	6.28290495258	0.000420521397253	3.14145247629	3.14159665505	3.14145247228	3.14187299368
n=3072	0.002045306823	6.28318256010	0.00000164276949625	3.14159128005	—	3.14159210598	3.14159374875
n=6144	0.00102265341147	6.28318256007	4.10693944856E-07	3.14159128004	—	3.14159251667	3.14159292736
n=12288	0.000511331594956	6.28324263882	1.02677609582E-07	3.14162131941	—	3.14159261935	3.14159272203

We can also get Zu Chongzhi's π using Liu's method1 but it needs 15 bit intermediate precision rather than 9 bit intermediate precision using in this method. And the result of Liu's method1 is not stable when n increases. For example, the result of 786432 polygon with 15 bit intermediate precision using Liu's method1 is 3.14154730190912. While using this method, the result is stable.

4 Proof

$$l_{2n} = \sqrt{\left(\frac{l_n}{2}\right)^2 + (\triangle Hn)^2} = \sqrt{\left(\frac{l_n}{2}\right)^2 + \left(1 - \sqrt{1 - \left(\frac{l_n}{2}\right)^2}\right)^2} = \sqrt{2 - \sqrt{4 - \mathrm{ln}^2}} \tag{11}$$

$$\triangle \mathrm{Hn} = \mathrm{R} - \mathrm{Hn} = 1 - \sqrt{1 - \left(\frac{l_n}{2}\right)^2} = \frac{1}{2}\left(2 - \sqrt{4 - \mathrm{ln}^2}\right) = \frac{1}{2} l_{2n}^2 \tag{12}$$

$$\frac{\triangle S_{4n-2n}}{\triangle S_{2n-n}} = \frac{\frac{1}{2} L_{2n} \times \triangle H_{2n}}{\frac{1}{2} L_n \times \triangle H_n} = \frac{L_n \times l_{2n}^2}{L_{2n} \times l_{4n}^2} = \frac{2n \times l_{2n} \times l_{4n}^2}{n \times l_n \times l_{2n}^2} = \frac{2 \times l_{2n} \times l_{4n}^2}{1 \times l_n \times l_{2n}^2} \tag{13}$$

As shown in figure 3,when n increases, from a geometric point of view, l_{2n} is getting closer and closer to l_n, with an extreme value of 1/2. From formula (12), we get an interesting result that $\triangle \mathrm{Hn} = \frac{1}{2} l_{2n}^2$.For $\frac{l_{12}}{l_6} \approx 0.5176$,we can get $\frac{1}{2} < \frac{l_{2n}}{l_n} < 0.52 (\mathrm{n} >= 6)$, then we put it in formula (13).The result is $\frac{1}{4} < \frac{\triangle S_{4n-2n}}{\triangle S_{2n-n}} < 0.2812 < \frac{1}{3} (\mathrm{n} >= 6)$.

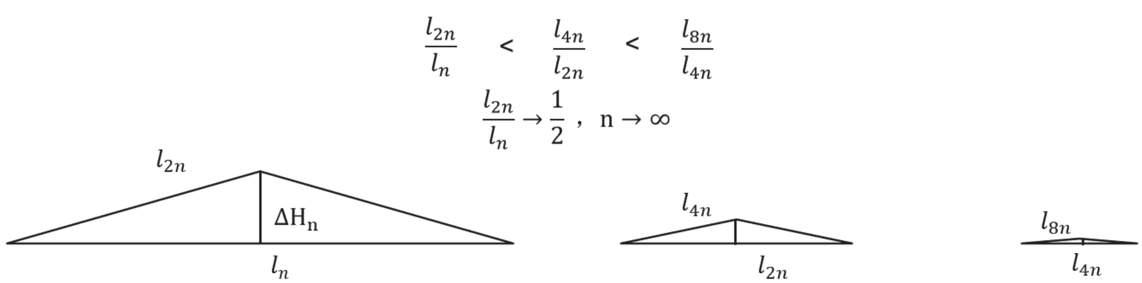

Figure 3: The decrease ratio of l_n

If we do not consider the computing numerical error, because π meet formula (9), if $\triangle S_{2n-n}$ can meet formula(14), we can get the kth digit of π. Because $\triangle \mathrm{S}_{24-12} \approx 0.10582854123 < \frac{1}{3}$,we get the formula (15). If we can find i which can meet formula (16), i can also meet formula (14). The value of i is calculated in formula (17),and it can be the iteration number. So the computation complexity is O(k) and the memory needed for l_n and S_{2n} and $\triangle S_{2n-n}$ is O(1).

$$\triangle \mathrm{S}_{2n-n} < 10^{-(k+1)} \tag{14}$$

$$\triangle \mathrm{S}_{2n-n} < \frac{1}{3^i} \triangle \mathrm{S}_{24-12} < \frac{1}{3^{(\mathrm{i}+1)}} \quad , i = \log_2\left(\frac{2n}{3}\right) - 3,\ \mathrm{n} > 12 \tag{15}$$

$$\frac{1}{3^{(i+1)}} < 10^{-(k+1)} \tag{16}$$

$$\Rightarrow \ \mathrm{i} \ > \log_3 10^{(n+1)} - 1 = \mathrm{k}\log_3 10 + \log_3 10 - 1 \tag{17}$$

Another important error is the numerical error in the iterations. Let SA_{2n} be the accurate result of S_{2n}, SC_{2n} be the calculated numerical result of S_{2n}, $\delta S_{2n} = SA_{2n} - SC_{2n}$ be the numerical error of S_{2n}, Let $\delta\triangle S_{2n-n}$ be numerical error of $\triangle S_{2n-n}$, and the total error is in formula (18). There is no numerical error for $S_{12} = 1$, so $\delta S_{12} = 0$.

$$\delta \mathrm{S_{2n}} = \delta S_{12} + \sum \delta\Delta S_{2n-n} = \sum \delta\Delta S_{2n-n} \tag{18}$$

We need more bits to keep the accuracy of $\triangle S_{2n-n}$. The accurate extra numbers needed is for future study. Thanks for decrease trend of $\triangle S_{2n-n}$, and the decrease rate is about 1/4, we get more accuracy when we use same significant bits when n increases.

5 Future work and Conjecture

5.1 Future work

The proof section just calculates the number of iteration. The exact digital number of middle result needs more strict proof, because when the iteration increase the error of l_n may spread.

The thought can be extended to three and more demission ball as shown in formula (19). V_m is the accurate volume result of m demission ball, V is a cube volume inside the ball, and $\sum \triangle V$ the volume between the cube and the ball.For m = 3, $V = \frac{4}{3}\pi R^3$, and it can also used to calulate π. The difficulty is to find an easy or symmetric method to divide a ball and it is not visible for high dimension.

$$V_m = V + \sum \triangle V \tag{19}$$

5.2 Conjecture

There is no detail steps recoded of Zu Chongzhi's method. He wrote his method in his math book of "缀术",but his book was lost in the Tang Dynasty. Zu Chongzhi maybe used this method to obtain higher accuracy. Compared with Liu's methods,the complexity is similar，but it can achieve more accuracy. Li Chunfeng recorded zucongzhi's result and brief describe in "隋书"as shown in the appendix.There are different explanations about Li Chunfeng's record. $\triangle S_{2n-n}$ is called as "cha mi" or "差幂" in the book of "The Nine Chapters on the Art of Mathematics" written by Liu hui. Li's brief description is that "又设开差幂，开差立，兼以正圆参之" in "隋书". This subject can be further studied .

References

[1] 郭书春.九章算术新校. 中国科学技术大学出版社. ISBN:9787312033599. 2014.

[2] 曹纯. 九章算术译注.上海三联书店.ISBN:9787542663405.2018.

[3] 吴文俊.《九章算术》与刘徽.北京师范大学出版社.1982.

[4] Shen Kangshen, John N. Crossley and Anthony W.-C. Lun.The nine chapters on the mathematical art: companion and commentary.Oxford University Press/Science Press.1999.

[5] Simon Plouffe. On the computation of the n'th decimal digit of various transcendental numbers. November 1996

[6] Fabrice Bellard. Computation of the n'th digit of pi in any base in O(n2). unpublished (1997).https://bellard.org/pi/pi_n2/pi_n2.html

[7] Fabrice Bellard.Some Pi digits in base 10.https://bellard.org/pi/pi2700e9/pidigits.html

[8] Robin wilson.The story of π. 2007.
https://s3-eu-west-1.amazonaws.com/content.gresham.ac.uk/data/binary/1809/thestoryofpicorrected.pdf

[9] Xavier Gourdon.Computation of the n-th decimal digit of π with low memory.2003.
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.187.6569

[10] Boris Gourévitch.Pi-Story or the history of the number Pi.http://www.pi314.net/eng/histoire.php

[11] 白尚恕. 《九章算术》注释. 北京：科学出版社.1983.

Appendices

A

《南史 祖冲之传》，唐李延寿撰
冲之解钟律博塞，当时独绝，莫能对者。以诸葛亮有木牛流马，乃造一器，不因风水， 施机自运，不劳人力。又造千里船，于新亭江试之，日行百余里。于乐游苑造水碓磨， 武帝亲自临视。又特善算。永元二年卒，年七十二。著《易老庄义》，释《论语》、 《孝经》，注《九章》，造《缀术》数十篇。

B

《隋书 律历志》，唐李淳风撰
古之九数，圆周率三，圆径率一，其术疏舛。自刘歆、张衡、刘徽、王蕃、皮延宗之徒， 各设新率，未臻折衷。宋末，南徐州从事史祖冲之，更开密法，以圆径一亿为一丈，圆周盈 数三丈一尺四寸一分五厘九毫二秒七忽，数三丈一尺四寸一分五厘九毫二秒六忽，正数在 盈二限之间。密率，圆径一百一十三，圆周三百五十五。约率，圆径七，周二十二。又设 开差幂，开差立，兼以正圆参之。指要精密，算氏之最者也。所著之书，名为《缀术》， 学官莫能究其深奥，是故废而不理