Spin Coherent States, Bell States, Entanglement, Husimi Distribution, Uncertainty Relation, Bell Inequality and Bell Matrix

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Abstract. We study spin coherent states, Bell states, entanglement, Husimi distributions, uncertainty relation, Bell inequality. The distance between these states is also derived. The Bell matrix, spin coherent states and Bell states are also investigated.

Bell states [1, 2, 3, 4, 5, 6] and spin coherent states [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] play a central role in quantum computing. The Bell states are fully entangled whereas among quantum states the spin coherent states (also called atomic coherent states or Bloch coherent states) are the "most classical states". We study entanglement for these states, the Husimi distributions, the distance between the Bell states and spin coherent states. Furthermore we look at the Bell inequality and the eigenvalue problem for the Bell matrix.

Let $S_1^{(1/2)}$, $S_2^{(1/2)}$, $S_3^{(1/2)}$ be the spin matrices for spin-$\frac{1}{2}$

$$S_1^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_2^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_3^{(1/2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obeying the commutation relations

$$[S_1^{(1/2)}, S_2^{(1/2)}] = iS_3^{(1/2)}, \quad [S_2^{(1/2)}, S_3^{(1/2)}] = iS_1^{(1/2)}, \quad [S_3^{(1/2)}, S_1^{(1/2)}] = iS_2^{(1/2)}$$

and

$$(S_1^{(1/2)})^2 + (S_2^{(1/2)})^2 + (S_3^{(1/2)})^2 = \frac{3}{4}I_2.$$

Let $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis in the Hilbert space $\mathbb{C}^2$. The four Bell states are given by

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad |\Phi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \quad |\Psi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

where $|\Psi_-\rangle$ is a singlet state. The four Bell states form an orthonormal basis in the Hilbert space $\mathbb{C}^4$. The Bell states are fully entangled. Let $\sigma_1 = 2S_1^{(1/2)}$, $\sigma_2 = 2S_2^{(1/2)}$, $\sigma_3 = 2S_3^{(1/2)}$ be the Pauli spin matrices. Then we have the eigenvalue equations

$$(\sigma_1 \otimes \sigma_1)|\Phi_+\rangle = |\Phi_+\rangle, \qquad (\sigma_1 \otimes \sigma_1)|\Phi_-\rangle = -|\Phi_-\rangle,$$

$$(\sigma_1 \otimes \sigma_1)|\Psi_+\rangle = |\Psi_+\rangle, \qquad (\sigma_1 \otimes \sigma_1)|\Psi_-\rangle = -|\Psi_-\rangle,$$

$$(\sigma_2 \otimes \sigma_2)|\Phi_+\rangle = -|\Phi_+\rangle, \qquad (\sigma_2 \otimes \sigma_2)|\Phi_-\rangle = |\Phi_-\rangle,$$

$$(\sigma_2 \otimes \sigma_2)|\Psi_+\rangle = |\Psi_+\rangle, \qquad (\sigma_2 \otimes \sigma_2)|\Psi_-\rangle = -|\Psi_-\rangle,$$

$$(\sigma_3 \otimes \sigma_3)|\Phi_+\rangle = |\Phi_+\rangle, \qquad (\sigma_3 \otimes \sigma_3)|\Phi_-\rangle = |\Phi_-\rangle,$$

$$(\sigma_3 \otimes \sigma_3)|\Psi_+\rangle = -|\Psi_+\rangle, \ (\sigma_3 \otimes \sigma_3)|\Psi_-\rangle = -|\Psi_-\rangle.$$

From the Bell states we can form the Bell matrix B

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = 2\sqrt{2}S_1^{(1/2)} \otimes S_1^{(1/2)} + \sqrt{2}S_3^{(1/2)} \otimes I_2$$

where I_2 is the 2×2 identity matrix. The Bell matrix is hermitian and unitary with $B^2 = I_4$, $\mathrm{tr}(B) = 0$, $\det(B) = 1$. Hence the eigenvalues are -1 (twice), $+1$ (twice). The Bell matrix can be written as $B = \exp(K)$ with the skew hermitian matrix

$$K = i\pi \begin{pmatrix} \frac{1}{(2(\sqrt{2}+2))} & 0 & 0 & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} \\ 0 & \frac{1}{(2(\sqrt{2}+2))} & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3)} & 0 \\ 0 & -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} & \frac{(10+7\sqrt{2})}{(4(2\sqrt{2}+3))} & 0 \\ -\frac{(4+3\sqrt{2})}{(4(2\sqrt{2}+3))} & 0 & 0 & \frac{(10+7\sqrt{2})}{(4(2\sqrt{2}+3))} \end{pmatrix}$$

with the trace equal to $2\pi i$. With $K = iH$ we find a hermitian matrix H. The eigenvalues of K are 0 (twice) and $i\pi$ (twice). From the Bell matrix we can form the projection operators $\Pi_+ = (I_4 + B)/2$, $\Pi_- = (I_4 - B)/2$ with $\Pi_+\Pi_- = 0_4$ and $\Pi_+ - \Pi_- = B$. The hermitian matrix

$$\frac{\hat{H}}{\hbar\omega} = S_1^{(1/2)} \otimes S_1^{(1/2)} + \gamma S_3^{(1/2)} \otimes S_3^{(1/2)}$$

with $\gamma > 0$ admits the Bell states $|\Psi_-\rangle$, $|\Psi_+\rangle$, $|\Phi_-\rangle$, $|\Phi_+\rangle$ as eigenvectors with the corresponding eigenvalues

$$\lambda_{\Psi-} = \frac{1}{4}(-1-\gamma), \quad \lambda_{\Psi+} = \frac{1}{4}(1-\gamma), \quad \lambda_{\Phi-} = \frac{1}{4}(-1+\gamma), \quad \lambda_{\Phi+} = \frac{1}{4}(1+\gamma).$$

From the Bell states we can form the four density matrices

$$\rho_{\Phi_+} = |\Phi_+\rangle\langle\Phi_+|, \quad \rho_{\Phi_-} = |\Phi_-\rangle\langle\Phi_-|, \quad \rho_{\Psi_+} = |\Psi_+\rangle\langle\Psi_+|, \quad \rho_{\Psi_-} = |\Psi_-\rangle\langle\Psi_-|.$$

The reduced density matrices (taking the partial trace) is the same for all four Bell states, namely

$$\rho_R = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Hence the von Neumann entropy is given by $S(\rho) = 1$ which indicates that the Bell states are fully entangled.

Starting point in the construction of the spin coherent states are the spin matrices $S_1^{(s)}$, $S_2^{(s)}$, $S_3^{(s)}$ and $S_+^{(s)} := S_1^{(s)} + iS_2^{(s)}$, $S_-^{(s)} := S_1^{(s)} - iS_2^{(s)}$. Then

$$K^{(s)}(\theta,\phi) := \frac{1}{2}\theta e^{i\phi} S_-^{(s)} - \frac{1}{2}\theta e^{-i\phi} S_+^{(s)}$$

is a skew hermitian matrix with $\mathrm{tr}(K^{(s)}(\theta,\phi)) = 0$ and $0 \le \theta < \pi$, $0 \le \phi < 2\pi$. It follows that $\exp(K^{(s)}(\theta,\phi))$ is a unitary matrix with $\det(\exp(K^{(s)}(\theta,\phi))) = 1$. Then the spin coherent states are given by

$$\exp(K^{(s)}(\theta,\phi)) \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T \quad \text{with} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T \in \mathbb{C}^{2s+1}.$$

The unitary matrix $\exp(K^{(s)}(\theta,\phi))$ describes a rotation through an angle θ about an axis $\mathbf{n} = (\sin(\phi), -\cos(\phi), 0)$. Note that the eigenvalues of $K^{(s)}(\theta,\phi)$ do not depend on ϕ and the eigenvectors do not depend on θ. This is true for all spin s. The overlap between two spin coherent states is given by

$$\langle\Omega_1|\Omega_2\rangle = (\cos(\theta_1/2)\cos(\theta_2/2) + e^{i(\phi_1-\phi_2)}\sin(\theta_1/2)\sin(\theta_2/2))^{2s}.$$

For $s = 3/2$ the eigenvalues of $K^{(3/2)}(\theta,\phi)$ are $\lambda_1 = -3i\theta/2$, $\lambda_2 = -i\theta/2$, $\lambda_3 = i\theta/2$, $\lambda_4 = 3i\theta/2$ with the corresponding normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3}ie^{i\phi} \\ -\sqrt{3}e^{2i\phi} \\ -ie^{3i\phi} \end{pmatrix}, \quad \mathbf{v}_2 = \frac{\sqrt{3}}{2\sqrt{2}} \begin{pmatrix} 1 \\ ie^{i\phi}/\sqrt{3} \\ e^{2i\phi}/\sqrt{3} \\ ie^{3i\phi} \end{pmatrix},$$

$$\mathbf{v}_3 = \frac{\sqrt{3}}{2\sqrt{2}} \begin{pmatrix} 1 \\ -ie^{i\phi}/\sqrt{3} \\ e^{2i\phi}/\sqrt{3} \\ -ie^{3i\phi} \end{pmatrix}, \quad \mathbf{v}_4 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{3}ie^{i\phi} \\ -\sqrt{3}e^{2i\phi} \\ ie^{3i\phi} \end{pmatrix}.$$

The four vectors form an orthonormal basis in the Hilbert space $\mathbb{C}^4$. The unitary matrix $\exp(K^{(3/2)})$ is given by

$$\exp(K^{(3/2)}) = e^{\lambda_1}\mathbf{v}_1\mathbf{v}_1^* + e^{\lambda_2}\mathbf{v}_2\mathbf{v}_2^* + e^{\lambda_3}\mathbf{v}_3\mathbf{v}_3^* + e^{\lambda_4}\mathbf{v}_4\mathbf{v}_4^*.$$

The spin coherent state for spin $s = \frac{3}{2}$ can also be written as

$$|z\rangle = \frac{1}{(1+z\overline{z})^{3/2}} \begin{pmatrix} 1 \\ \sqrt{3}z \\ \sqrt{3}z^2 \\ z^3 \end{pmatrix}$$

where $z = e^{i\phi}\tan(\theta/2)$ $(0 \leq \theta < \pi, 0 \leq \phi < 2\pi)$. With the standard basis

$$|3/2, -3/2\rangle = (0 \quad 0 \quad 0 \quad 1)^T, \quad |3/2, -1/2\rangle = (0 \quad 0 \quad 1 \quad 0)^T,$$

$$|3/2, 1/2\rangle = (0 \quad 1 \quad 0 \quad 0)^T, \quad |3/2, 3/2\rangle = (1 \quad 0 \quad 0 \quad 0)^T$$

we can write setting $|\Omega\rangle \equiv |\theta, \phi\rangle$

$$|\Omega\rangle = \sum_{m=-3/2}^{3/2} \sqrt{\binom{3}{3/2+m}} (\cos(\theta/2))^{3/2+m} (\sin(\theta/2)e^{i\phi})^{3/2-m} |3/2, m\rangle$$

i.e.

$$|\Omega\rangle = \begin{pmatrix} \cos^3(\theta/2) \\ \sqrt{3}e^{i\phi}\cos^2(\theta/2)\sin(\theta/2) \\ \sqrt{3}e^{2i\phi}\cos(\theta/2)\sin^2(\theta/2) \\ e^{3i\phi}\sin^3(\theta/2) \end{pmatrix}.$$

We note that the spin-$\frac{1}{2}$ coherent state is given by

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}.$$

Then the Kronecker product [21] provides the normalized state in the Hilbert space $\mathbb{C}^4$

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta/2) \\ e^{i\phi}\cos(\theta/2)\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2)\cos(\theta/2) \\ e^{2i\phi}\sin^2(\theta/2) \end{pmatrix}.$$

Let $N = 1 + 3z\overline{z} + 3(z\overline{z})^2 + (z\overline{z})^3 \equiv (1+x^2+y^2)^3$ with $z = x + iy$ $(x, y \in \mathbb{R})$. The density matrices are given by

$$|z\rangle\langle z| = \frac{1}{N}\begin{pmatrix} 1 & \sqrt{3}\overline{z} & \sqrt{3}\overline{z}^2 & \overline{z}^3 \\ \sqrt{3}z & 3z\overline{z} & 3z\overline{z}^2 & \sqrt{3}z\overline{z}^3 \\ \sqrt{3}z^2 & 3z^2\overline{z} & 3(z\overline{z})^2 & \sqrt{3}z^2\overline{z}^3 \\ z^3 & \sqrt{3}z^3\overline{z} & \sqrt{3}z^3\overline{z}^2 & (z\overline{z})^3 \end{pmatrix}$$

and

$$|\Omega\rangle\langle\Omega| =$$

$$\begin{pmatrix} \cos^6(.) & \sqrt{3}e^{-i\phi}\cos^5(.)\sin(.) & \sqrt{3}e^{-2i\phi}\cos^4(.)\sin^2(.) & e^{-3i\phi}\cos^3(.)\sin^3(.) \\ \sqrt{3}e^{i\phi}\cos^5(.)\sin(.) & 3\cos^4(.)\sin^2(.) & 3e^{-i\phi}\cos^3(.)\sin^3(.) & \sqrt{3}e^{-2i\phi}\cos^2(.)\sin^4(.) \\ \sqrt{3}e^{2i\phi}\cos^4(.)\sin^2(.) & 3e^{i\phi}\cos^3(.)\sin^3(.) & 3\cos^2(.)\sin^4(.) & \sqrt{3}e^{-2i\phi}\cos(.)\sin^5(.) \\ e^{3i\phi}\cos^3(.)\sin^3(.) & \sqrt{3}e^{2i\phi}\cos^2(.)\sin^4(.) & \sqrt{3}e^{i\phi}\cos(.)\sin^5(.) & \sin^6(.) \end{pmatrix}$$

with $(.) = (\theta/2)$. Utilizing the partial trace the reduced density matrices are

$$|z\rangle\langle z|_R =$$

$$\frac{1}{N}\begin{pmatrix} 1 + 3(x^2+y^2) & \sqrt{3}(x-iy)^2(1+x^2+y^2) \\ \sqrt{3}(x+iy)^2(1+x^2+y^2) & (x^2+y^2)^2(3+x^2+y^2) \end{pmatrix}$$

and

$$|\Omega\rangle\langle\Omega|_R = \begin{pmatrix} \sin^4(\theta/2)(2\sin^2(\theta/2)-3)+1 & \sqrt{3}e^{2i\phi}\sin^2(\theta/2)\cos^2(\theta/2) \\ \sqrt{3}e^{-2i\phi}\sin^2(\theta/2)\cos^2(\theta/2) & \sin^4(\theta/2)(-2\sin^2(\theta/2)+3) \end{pmatrix}.$$

Obviously the trace is given by 1 for the two matrices. The determinant for $|\Omega\rangle\langle\Omega|_R$ is given by

$$\det(|\Omega\rangle\langle\Omega|_R) = \frac{\sin^6(\theta)}{16}.$$

Hence the determinant is equal to 0 for $\theta = 0$ and $1/16$ for $\theta = \pi/2$. The eigenvalues of $|\Omega\rangle\langle\Omega|_R$ are given by

$$\lambda_1(\theta) = \frac{1}{2} + \frac{\sqrt{4-\sin^6(\theta)}}{4}, \qquad \lambda_2(\theta) = \frac{1}{2} - \frac{\sqrt{4-\sin^6(\theta)}}{4}.$$

For $\theta = 0$ we have the eigenvalues $\lambda_1(0) = 1$ and $\lambda_2(0) = 0$ and the state is not entangled. For $\theta = \pi/2$ we have $\lambda_1(\pi/2) = 1/2 + \sqrt{3}/4$, $\lambda_2(\pi/2) = 1/2 - \sqrt{3}/4$ and the state is entangled but not fully entangled.

Consider now the Bell states and the spin coherent states for spin$-\frac{3}{2}$ and the Husimi distribution. We obtain

$$|\langle\Phi_+|\Omega\rangle|^2 = \frac{1}{2}(\sin^6(\theta/2)+\cos^6(\theta/2)) + \cos(3\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Phi_-|\Omega\rangle|^2 = \frac{1}{2}(\sin^6(\theta/2)+\cos^6(\theta/2)) + \cos(3\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Psi_+|\Omega\rangle|^2 = \frac{3}{2}\cos^2(\theta/2)\sin^2(\theta/2) + 3\cos(\phi)\cos^3(\theta/2)\sin^3(\theta/2)$$

$$|\langle\Psi_-|\Omega\rangle|^2 = \frac{3}{2}\cos^2(\theta/2)\sin^2(\theta/2) - 3\cos(\phi)\cos^3(\theta/2)\sin^3(\theta/2).$$

Hence we find

$$0 \le |\langle\Phi_+|\Omega\rangle|^2 \le 1/2, \quad 0 \le |\langle\Phi_-|\Omega\rangle|^2 \le 1/2,$$

$$0 \le |\langle\Psi_+|\Omega\rangle|^2 \le 3/4, \quad 0 \le |\langle\Psi_-|\Omega\rangle|^2 \le 3/4.$$

From these results it follows that the distance between the Bell states and the spin coherent states cannot be 0

$$\||\Phi_+\rangle - |\Omega\rangle\|^2 > 0, \qquad \||\Phi_-\rangle - |\Omega\rangle\|^2 > 0,$$

$$\||\Psi_+\rangle - |\Omega\rangle\|^2 > 0, \qquad \||\Psi_-\rangle - |\Omega\rangle\|^2 > 0.$$

The shortest distance for $\||\Phi_+\rangle - |\Omega\rangle\|^2$, $\||\Phi_-\rangle - |\Omega\rangle\|^2$ is $2-\sqrt{2}$, the shortest distance for $\||\Psi_+\rangle - |\Omega\rangle\|^2$ is $2-\sqrt{3}$ and the shortest distance for $\||\Psi_-\rangle - |\Omega\rangle\|^2$ is $2-8/(3\sqrt{7})$.

Let $S_1^{(s)}$, $S_2^{(s)}$, $S_3^{(s)}$ be the spin matrices with spin-s and the commutation relations

$$[S_1^{(s)}, S_2^{(s)}] = iS_3^{(s)}, \quad [S_2^{(s)}, S_3^{(s)}] = iS_1^{(s)}, \quad [S_3^{(s)}, S_1^{(s)}] = iS_2^{(s)}.$$

Let $|\psi\rangle$ be a normalized state in the Hilbert space $\mathbb{C}^{2s+1}$. One defines the variance as

$$\Delta S_j^{(s)} := \sqrt{\langle\psi|(S_j^{(s)})^2|\psi\rangle - (\langle\psi|S_j^{(s)}|\psi\rangle)^2}, \quad j = 1,2,3.$$

The uncertainty relation is

$$(\Delta S_1^{(s)})^2 \cdot (\Delta S_2^{(s)})^2 \ge \frac{1}{4}|\langle\psi|[S_1^{(s)}, S_2^{(s)}]|\psi\rangle|^2 \equiv \frac{1}{4}|\langle\psi|iS_3^{(s)}|\psi\rangle|^2.$$

With $\langle\psi|S_3^{(3/2)}|\psi\rangle = 0$ for all four Bell states we find for $|\Phi_+\rangle$, $|\Phi_-\rangle$ that $3/4 > 0$. For $|\Psi_+\rangle$ and $|\Psi_-\rangle$ we find $\sqrt{27}/4 > 0$. For $s = 3/2$ and the spin coherent states we find

$$\frac{9}{16}(\cos^2(\theta) + \sin^4(\theta)\sin^2(\phi)\cos^2(\phi)) \ge \frac{9}{16}\cos^2(\theta).$$

Note that the standard form of the uncertainty relation we obtain by taking the square root on both sides. Note that the nonnegative term

$$\sin^4(\theta)\sin^2(\phi)\cos^2(\phi)$$

takes a maximum for $\theta = \pi/2$, $\phi = \pi/4$, namely $1/4$. For $\theta = 0$ and ϕ arbitrary we obtain an equality for the uncertainty relation.

Consider now the Bell inequality, Bell states and spin coherent states. Let σ_1, σ_2 be the Pauli spin matrices and

$$\hat{A}_1 = \sigma_1, \quad \hat{B}_1 = \sigma_2, \quad \hat{B}_1 = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2), \quad \hat{B}_2 = \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_2).$$

Note that $(\hat{A}_1)^2 = (\hat{A}_2)^2 = (\hat{B}_1)^2 = (\hat{B}_2)^2 = I_2$, where I_2 is the 2×2 identity matrix. Find

$$|\langle\psi|\hat{A}_1\otimes\hat{B}_1|\psi\rangle + \langle\psi|\hat{A}_1\otimes\hat{B}_2|\psi\rangle + \langle\psi|\hat{A}_2\otimes\hat{B}_1|\psi\rangle - \langle\psi|\hat{A}_2\otimes\hat{B}_2|\psi\rangle|$$

with $|\psi\rangle$ replaced by the Bell states. We obtain the well-known results

$$|\langle\Phi_+|\hat{A}_1\otimes\hat{B}_1|\Phi_+\rangle + \langle\Phi_+|\hat{A}_1\otimes\hat{B}_2|\Phi_+\rangle + \langle\Phi_+|\hat{A}_2\otimes\hat{B}_1|\Phi_+\rangle - \langle\Phi_+|\hat{A}_2\otimes\hat{B}_2|\Phi_+\rangle| = 0$$
$$|\langle\Phi_-|\hat{A}_1\otimes\hat{B}_1|\Phi_-\rangle + \langle\Phi_-|\hat{A}_1\otimes\hat{B}_2|\Phi_-\rangle + \langle\Phi_-|\hat{A}_2\otimes\hat{B}_1|\Phi_-\rangle - \langle\Phi_-|\hat{A}_2\otimes\hat{B}_2|\Phi_-\rangle| = 0$$
$$|\langle\Psi_+|\hat{A}_1\otimes\hat{B}_1|\Psi_+\rangle + \langle\Psi_+|\hat{A}_1\otimes\hat{B}_2|\Psi_+\rangle + \langle\Psi_+|\hat{A}_2\otimes\hat{B}_1|\Psi_+\rangle - \langle\Psi_+|\hat{A}_2\otimes\hat{B}_2|\Psi_+\rangle| = \sqrt{2}\cdot 2$$
$$|\langle\Psi_-|\hat{A}_1\otimes\hat{B}_1|\Psi_-\rangle + \langle\Psi_-|\hat{A}_1\otimes\hat{B}_2|\Psi_-\rangle + \langle\Psi_-|\hat{A}_2\otimes\hat{B}_1|\Psi_-\rangle - \langle\Psi_-|\hat{A}_2\otimes\hat{B}_2|\Psi_-\rangle| = \sqrt{2}\cdot 2.$$

If we find θ, ϕ such that

$$|\langle\Omega|\hat{A}_1\otimes\hat{B}_1|\Omega\rangle + \langle\Omega|\hat{A}_1\otimes\hat{B}_2|\Omega\rangle + \langle\Omega|\hat{A}_2\otimes\hat{B}_1|\Omega\rangle - \langle\Omega|\hat{A}_2\otimes\hat{B}_2|\Omega\rangle| > 2$$

then the Bell inequality is violated. For the absolute value we obtain

$$12\cdot\sqrt{2}\sqrt{\cos^2(\phi)}\cos^3(\theta/2)\sin^3(\theta/2).$$

With $\phi = 0$, $\theta/2 = \pi/4$ we obtain $3/\sqrt{2} \approx 2.121$ and the Bell inequality is violated for this value. The normalized state for $\phi = 0$, $\theta/2 = \pi/4$ is given by

$$\begin{pmatrix} 1/(2\sqrt{2}) \\ \sqrt{3}/(2\sqrt{2}) \\ \sqrt{3}/(2\sqrt{2}) \\ 1/(2\sqrt{2}) \end{pmatrix}.$$

This vector cannot be written as a Kronecker product of two vectors in $\mathbb{C}^2$. Note that the eigenvalues of the reduced density matrix for this vector are $1/2+\sqrt{3}/4$, $1/2-\sqrt{3}/4$.

Consider the Bell matrix B, Bell states and the spin coherent state for spin-$\frac{3}{2}$. We have

$$\langle\Phi_+|B|\Phi_+\rangle = \frac{1}{\sqrt{2}}, \qquad \langle\Phi_-|B|\Phi_-\rangle = -\frac{1}{\sqrt{2}},$$

$$\langle\Psi_+|B|\Psi_+\rangle = \frac{1}{\sqrt{2}}, \qquad \langle\Psi_-|B|\Psi_-\rangle = -\frac{1}{\sqrt{2}}.$$

For the spin coherent state for spin-$\frac{3}{2}$ $|z\rangle$ and $f(z,\overline{z}) = f(x,y) = \langle z|B|z\rangle$ we obtain

$$f(x,y) = \frac{1+8x^3-x^6+3y^2-3x^4(1+y^2)-y^4(3+y^2)-3x^2(-1+2y^2+y^4)}{\sqrt{2}(1+x^2+y^2)^3}.$$

We find two critical points. The first critical point is $x=\frac{1}{2}(\sqrt{5}-1)$, $y=0$. The second critical point is $x=-\frac{1}{2}-\frac{\sqrt{5}}{2}$, $y=0$. Then for the first critical point we obtain $\frac{3}{\sqrt{10}}$ and for the second critical point we obtain $-\frac{3}{\sqrt{10}}$. Together with the two eigenvalues of B, namely -1 and $+1$ we have

$$-1 < -\frac{3}{\sqrt{10}} < -\frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} < \frac{3}{\sqrt{10}} < +1.$$

Extensions to higher dimensions are obvious, for example $\mathbb{C}^6 = \mathbb{C}^{2s+1}$ $(s=5/2)$ and the orthonormal basis

$$\frac{1}{\sqrt{2}}(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1)^T,\ \frac{1}{\sqrt{2}}(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1)^T,\ \frac{1}{\sqrt{2}}(0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0)^T,$$

$$\frac{1}{\sqrt{2}}(0\ \ 1\ \ 0\ \ 0\ \ -1\ \ 0)^T,\ \frac{1}{\sqrt{2}}(0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0)^T,\ \frac{1}{\sqrt{2}}(0\ \ 0\ \ 1\ \ -1\ \ 0\ \ 0)^T$$

and spin coherent states. Here Hamilton operators [22] can be written as Kronecker products of spin matrices $S_j^{(1/2)}$ and $S_k^{(1)}$ together with the 2×2 and 3×3 identity matrices. In the Hilbert space $\mathbb{C}^8 = \mathbb{C}^{2s+1}$ $(s=7/2)$ the GHZ-state and spin coherent states can be studied.

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