Affirmative resolve of the Riemann Hypothesis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 170 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I try to solve the proposition about the Mobius function equivalent to the Riemann Hypothesis. First, the non-trivial formula for Mobius function is proved in theorem 1 and theorem 2. In theorem 4, I get upper bound for the sum of the mobius functions (for meaning of R.H. See theorem 4).

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known

Theorem .

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove Left hand formula.

Lemma 1.

$$\sum_{n|m} \mu(n) = 1(m=1), \sum_{n|m} \mu(n) = 0(m \neq 1)$$

Proof. First, if $m=1$, it is $\sum_{n|m}\mu(n)=\mu(1)=1$. Second case. There is a little explanation for this. Let m's prime factorization be $m=p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_k^{n_k}$. Then it becomes $\sum_{n|m}\mu(n) ={}_k C_0 -{}_k C_1 +{}_k C_2 -{}_k C_3 + \cdots {}_k C_k = (1-1)^k = 0$. □

Theorem 1.

$$\sum_{n\le m}\mu(n)[\frac{m}{n}]=1$$

Proof. $\sum_{m'=1}^{m}\sum_{n|m'}\mu(n)=1$ is from Lemma 1

$$1=\sum_{m'=1}^{m}\sum_{n|m'}\mu(n)=(\mu(1))+(\mu(1)+\mu(2))+(\mu(1)+\mu(3))$$

$$+(\mu(1)+\mu(2)+\mu(4))+\cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $[\frac{m}{2}]$ times that is a multiple of 2 less than m. In general, the number of occurrences of $\mu(n)(n<m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. I get $\sum_{n\le m}\mu(n)[\frac{m}{n}]=1$. □

example
$m=10$ case, $10-5-3-2+1-1+1=1$.$m=13$ case, $13-6-4-2+2-1+1-1-1=1$ etc..

Theorem 2.

$$|\sum_{n=1}^{m}\mu(n)-\sum_{n=1}^{m}\frac{m}{n}\mu(n)|=O(\log m)$$

Proof. Remark

$$|\sum_{n=1}^{m}\mu(n)-[\sum_{n=1}^{m}\frac{m}{n}\mu(n)]|=O(\log m)$$

So,

$$\sum_{n=1}^{m}\mu(n)-[\sum_{n=1}^{m}\frac{m}{n}\mu(n)]\approx\sum_{n=1}^{m+1}\mu(n)-[\sum_{n=1}^{m+1}\frac{m+1}{n}\mu(n)]$$

If

$$\sum_{n=1}^{m}\mu(n)-\sum_{n=1}^{m}\frac{m}{n}\mu(n)=O(\log m) \tag{1}$$

then

$$\sum_{n=1}^{m}\frac{m}{n}\mu(n)+\frac{1}{m+1}(\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)-[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)])-\sum_{n=1}^{m}\mu(n)=O(\log m) \tag{2}$$

. Because

$$\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)+\mu(m+1)-\frac{1}{m+1}[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)]-[\sum_{n=1}^{m}\frac{m}{n}\mu(n)]-k\log m-\mu(m+1)$$

$$\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)+\mu(m+1)-[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)]-k\log m-\mu(m+1)$$

$$\sum_{n=1}^{m+1}\frac{m+1}{n}\mu(n)-\sum_{n=1}^{m+1}\mu(n)$$

$k\log m$ is vanished by $\sum_{n=1}^{m}\mu(n)-[\sum_{n=1}^{m}\frac{m}{n}\mu(n)]\approx\sum_{n=1}^{m+1}\mu(n)-[\sum_{n=1}^{m+1}\frac{m+1}{n}$ $\mu(n)]$
(I think the case $|\sum_{n\le m}\frac{m}{n}\mu(n)-[\sum_{n\le m}\frac{m}{n}\mu(n)]|<\frac{1}{2}$. If $|\sum_{n\le m}\frac{m}{n}\mu(n)-[\sum_{n\le m}\frac{m}{n}\mu(n)]|\ge\frac{1}{2}$ then $|\sum_{n\le m}\frac{m}{n}\mu(n)-[\sum_{n\le m}\frac{m}{n}\mu(n)]\pm 1|<\frac{1}{2}$. I select $[\sum_{n\le m}\frac{m}{n}\mu(n)]$ or $[\sum_{n\le m}\frac{m}{n}\mu(n)]\pm 1$ implicitly. $|\sum_{n\le m}\frac{m}{n}\mu(n)-[\sum_{n\le m}\frac{m}{n}\mu(n)]$ $<\epsilon_0$, then for $M, m_0<M<m, \sum_{n=1}^{M}\frac{M}{n}\mu(n)-[\sum_{n=1}^{M}\frac{M}{n}\mu(n)]<\frac{1}{2}$.)
From (1) to (2) Left side increases $|\frac{1}{m+1}(\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)-[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)])|<\frac{1}{(m+1)^{1+\epsilon'}}$. Take integral of $\frac{1}{x^{1+\epsilon'}}$ 1 to m.

$$\int_{x=1}^{m}\frac{1}{x^{1+\epsilon'}}=-\frac{1}{\epsilon'}m^{-\epsilon'}+\frac{1}{\epsilon'},|\frac{1}{\epsilon'}-\frac{1}{\epsilon'}m^{-\epsilon'}|<F(m)$$

$|\frac{1}{m+1}(\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)-[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)])|<\frac{1}{2(m+1)}$ or $|\frac{1}{m+1}(\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)-[\sum_{n=1}^{m}\frac{m+1}{n}\mu(n)]\pm 1)|<\frac{1}{2(m+1)}$ is got.
$\frac{1}{2m}=\frac{1}{m^{1+\epsilon'}}\Rightarrow$

$$m=1024\Rightarrow\epsilon'=\frac{1}{10}\Rightarrow\frac{1}{\epsilon'}=10, F(m)\le 10$$

$$m=4096\Rightarrow\epsilon'=\frac{1}{12}\Rightarrow\frac{1}{\epsilon'}=12, F(m)\le 12$$

Generally,

$$F(m)\le\log_2 m$$

I get the formula (2).

□

example:m=1000 case

$$\sum_{n=1}^{1000} \mu(n) = 2$$

$$\sum_{n=1}^{1000} \frac{1000}{n} \mu(n) \fallingdotseq 4.411$$

example:m=10000 case

$$\sum_{n=1}^{10000} \mu(n) = -23$$

$$\sum_{n=1}^{10000} \frac{10000}{n} \mu(n) \fallingdotseq -20.827$$

Lemma 2.

$$\sum_{n=1}^{x} \mu(n) \ changes \ sign \ \in [n_0^{(1-\epsilon'')}, n_0](n_0 < m, m > \exists m_{\epsilon''})$$

Proof. $\sum_{n=1}^{x} \mu(n)$ changes sign in the interval $[n_0^{(1-\epsilon'')}, n_0], m > \exists m_{\epsilon''}$ ([10]). □

Theorem 3.

$$|\sum_{n \le N} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < K N^{\frac{1}{2}+\epsilon} (\frac{m}{2} < \frac{2}{3} m < N < m)$$

Proof. Start with $\sum_{n \le N} \mu(n)(N > \frac{2}{3}m)$. Compare $\sum_{n \le N} \mu(n) + \sum_{n>N}^{m'} (\frac{m}{n} - [\frac{m}{n}])\mu(n)$ and $\sum_{n=1}^{m'} \mu(n).(m' < m.)$ It is assumed that m' is first increased from the original value. The former terms add up little by little. Therefore, a positive increase means that there were many positive terms. If negative terms and positive terms appear same time from first terms, then increase never occur. If only positive terms exists, this case is further easy. Therefore $\sum_{n \le N} \mu(n) + \sum_{n>N}^{m'} (\frac{m}{n} - [\frac{m}{n}])\mu(n) < \sum_{n=1}^{m'} \mu(n)$. $|\sum_{n>N}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < K(m-1)^{\frac{1}{2}+\epsilon} + 1 + K N^{\frac{1}{2}+\epsilon}$. By theorem 2, $|\sum_{n=1}^{m} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < K(m-1)^{\frac{1}{2}+\epsilon} + 1 + k \log m$. $\sum_{n \le m} (\frac{m}{n} - [\frac{m}{n}])\mu(n) = A, \sum_{n \le N} (\frac{m}{n} - [\frac{m}{n}])\mu(n) = B$, I take N as , $A > 0, B < 0$. or $A < 0, B > 0$, (By lemma 2, $\sum_{n=1}^{x} \mu(n)$ takes positive and negative. Start with small number, $\sum_{n=1}^{x} (\frac{m}{n} - [\frac{m}{n}])\mu(n)$ takes positive and negative. $x > \frac{2}{3}m$ case also that takes positive and negatve.) $A > 0 \Rightarrow A - B < \sum_{n \le m} \mu(n) + K N^{\frac{1}{2}+\epsilon}. A < 0 \Rightarrow -A + B < -\sum_{n \le m} \mu(n) + K N^{\frac{1}{2}+\epsilon}$. I get $|B| < K N^{\frac{1}{2}+\epsilon}$. So, $|\sum_{n \le N} (\frac{m}{n} - [\frac{m}{n}])\mu(n)| < k \log m + K'' N^{\frac{1}{2}+\epsilon} < K N^{\frac{1}{2}+\epsilon}$. □

Theorem 4.

$$|\sum_{n=1}^{m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

R.H. is got.

Proof. I take some constant $n = N$ that satisfies $|\sum_{n=1}^{N}(\frac{m}{n} - [\frac{m}{n}])\mu(n)| < KN^{\frac{1}{2}+\epsilon}(N \geq \frac{2}{3}m)$, by theorem 3. I think

$$|\sum_{n\geq N}^{m''}(\frac{m}{n} - [\frac{m}{n}])\mu(n)|(m'' < m)$$

and this formula takes maximum absolute value at m''. I get

$$\frac{1}{2}|\sum_{n\geq N}^{m''}\mu(n)| > |\sum_{n\geq N}^{m''}(\frac{m}{n} - [\frac{m}{n}])\mu(n)|$$

$|\frac{m}{n} - [\frac{m}{n}]|$ is less than $\frac{1}{2}$ and gradually decreases as m increases. $|\sum_{n\geq N}^{m''}(\frac{m}{n} - [\frac{m}{n}])\mu(n)|$ is less than $\frac{1}{2}|\sum_{n\geq N}^{m''}\mu(n)|$. Like in theorem 3, $\sum_{n\leq N}(\frac{m}{n} - [\frac{m}{n}])\mu(n)$ and $\sum_{n\leq m}(\frac{m}{n} - [\frac{m}{n}])\mu(n)$ are opposite sign. I express $\sum_{n\leq N}(\frac{m}{n} - [\frac{m}{n}])\mu(n) = B, \sum_{n\leq m}(\frac{m}{n} - [\frac{m}{n}])\mu(n) = A, \sum_{n\leq N}\mu(n) = B'$. $|\sum_{n\leq m}(\frac{m}{n} - [\frac{m}{n}])\mu(n)| < |\frac{A-B'}{2} + B|$. $\frac{A-B'}{2}$ and B takes opposite sign. If $\frac{A-B'}{2}$ takes large absolute value, then this is less than $\frac{K(m-1)^{\frac{1}{2}+\epsilon}+k\log m+KN^{\frac{1}{2}+\epsilon}}{2}$. If $|\frac{A-B'}{2}|$ is very small, then this is less than $|B| < KN^{\frac{1}{2}+\epsilon}$. I used theorem 3. By theorem 2

$$|\sum_{n\leq m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

□

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