Affirmative resolve of the Riemann Hypothesis

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Abstract

In this paper, we prove the proposition about the Mobius function equivalent to the Riemann Hypothesis.

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as $R.H$, and the Mobius function as $\mu(n)$.

Next theorem is well-known

Theorem

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2} + \epsilon}) \iff R.H$$

I will prove Left hand formula.

Lemma 1.1

$$\sum_{n|m} \mu(n) = 1 (m = 1), \sum_{n|m} \mu(n) = 0 (m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m} \mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let $m$’s prime factorization be $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$. Then it becomes $\sum_{n|m} \mu(n) = k C_0 - k C_1 + k C_2 - k C_3 + \cdots + k C_k = (1 - 1)^k = 0$.

Theorem 1

$$\sum_{n \leq m} \mu(n)\left\lceil \frac{m}{n} \right\rceil = 1$$

Proof. $\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$ is from Lemma 1.1

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$
See $\mu(n)$ in this expression as a character. $\mu(1)$ appears $m$ times in the expression. $\mu(2)$ appears \left\lceil \frac{m}{2} \right\rceil$ times that is a multiple of 2 less than $m$. In general, the number of occurrences of $\mu(n)(n < m)$ in this expression is the number \left\lfloor \frac{m}{n} \right\rfloor that is a multiple of $n$ below $m$. I get $\sum_{n \leq m} \mu(n)\left\lfloor \frac{m}{n} \right\rfloor = 1$. 

example

$m = 10$ case, $10 - 5 - 3 - 2 + 1 - 1 + 1 = 1.m = 13$ case, $13 - 6 - 4 - 2 + 2 - 1 + 1 - 1 - 1 = 1$ etc..

**Theorem 2**

$$\sum_{n=1}^{x} \frac{m}{n} \mu(n) \text{ changes sign at } n_0 \in [m^{\frac{1}{2}}(1-\epsilon'), m^{\frac{1}{2}}](m > \exists m_{\epsilon'})$$

**Proof.** $\sum_{n=1}^{x} \frac{m}{n} \mu(n) \text{ changes sign in the interval } [m^{\frac{1}{2}}(1-\epsilon'), m^{\frac{1}{2}}], m > \exists m_{\epsilon'} ([1])$. 

**Lemma 3.1**

$$-1 < f(n) < 1, (n = 1, \cdots m) \Rightarrow \left| \sum_{n=1}^{m} f(n) \right| < m$$

**Proof.** negative terms sum is (I call it $F_1$) satisfy $|F_1| < m$. positive terms sum is (I call it $F_2$) satisfy $|F_2| < m$. $|F_1 + F_2| < m$. In other words, summation of $m$ elements that is absolute value 1 or less is less than $m$. 

**Theorem 3**

$$\left| \sum_{n=1}^{m} \mu(n) \right| < K m^{\frac{1}{2} + \epsilon}$$

and R.H. is true.

**Proof.** From theorem 1

$$\sum_{n \leq n_0} \mu(n)\left\lfloor \frac{m}{n} \right\rfloor + \sum_{n_0 < n \leq m} \mu(n)\left\lfloor \frac{m}{n} \right\rfloor = 1$$

($n_0 < \sqrt{m}$ is the sign change point of $\sum_{n \leq x} \frac{m}{n} \mu(n)$.) By lemma 3.1, Using $\sum_{n \leq n_0} \mu(n)\left\lfloor \frac{m}{n} \right\rfloor$ and $\sum_{n \leq n_0} \mu(n)\frac{m}{n}$. These are $n_0(< \sqrt{m})$ terms, so the difference of size is less than $\sqrt{m}$.

The following is obtained by calculation for $\sum_{n_0 < n \leq m} \mu(n)\left\lfloor \frac{m}{n} \right\rfloor$. $\sqrt{m}$ term
is sum of all terms satisfy \([\frac{m}{n}] = \lfloor \sqrt{m} \rfloor - 1\), \(m/\sqrt{m} = \sqrt{m} \geq \lfloor \sqrt{m} \rfloor\) and \(m/(m/(\sqrt{m} - 1)) = \sqrt{m} - 1 \geq \lfloor \sqrt{m} - 1 \rfloor\), \((m/m/(\sqrt{m} - 1) + 1) = m/(\sqrt{m} - 1)/m/(\sqrt{m} - 1) < \sqrt{m} - 1\), so the range is \(\sqrt{m} - m/(\sqrt{m} - 1)\). Next term is sum of all terms satisfy \([\frac{m}{n}] = \lfloor \sqrt{m} \rfloor - 2\), \(m/(m/(\sqrt{m} - 2)) \geq \lfloor \sqrt{m} - 2 \rfloor\). The range is \(m/(\sqrt{m} - 1)\) to \(m/(\sqrt{m} - 2)\). The last term satisfy \([\frac{m}{n}] = 1\), that is \(\frac{m}{2}\) to \(m\).

\[
\sum_{n_0 < n \leq m} \mu(n)\lfloor \frac{m}{n} \rfloor = (\lfloor m/(n_0) \rfloor - 1) \times \sum_{m/(\lfloor m/(n_0) \rfloor) < n \leq m/(\lfloor m/(n_0) \rfloor) - 1} \mu(n) + \cdots + \left(\lfloor \sqrt{m} \rfloor \times \sum_{m/(\lfloor \sqrt{m} \rfloor + 1) < n \leq m/\sqrt{m}} \mu(n) + ([\sqrt{m}]-1) \times \sum_{\sqrt{m} < n \leq m/(\sqrt{m} - 1)} \mu(n) + ([\sqrt{m}]-2) \times \sum_{m/(\sqrt{m} - 1) < n \leq m/(\sqrt{m} - 2)} \mu(n) + \cdots + 1 \times \sum_{m/2 < n \leq m} \mu(n)\right)
\]

By induction, there are "almost correct" formulas. \(|N \times \sum_{m/(N+1) < n \leq m/N} \mu(n)| < \frac{N}{N+1} K(m/N)^{1/2(1+\epsilon')} 1 \times \sum_{m/2 < n \leq m} \mu(n)| < \frac{1}{2}(K(m-1)^{1/2(1+\epsilon')} + 1)\)

example: \(m = 10000\) case,

\(-95 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0-91

\(+0+0-88-87+86+0+0+84 \times 2 + \cdots\)

\((84 \times 2\) means \(\mu(118) = 1, [10000/118] = 84, \mu(119) = 1, [10000/119] = 84)\)

will transform

\(-95+85-84 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0

\(-91+0+0-88-87+86+0+85+84 \times 2 - 84 + \cdots\)

\((\lfloor \sqrt{m} \rfloor - 1) \times \sum_{\sqrt{m} < n \leq m/(\sqrt{m} - 1)} \mu(n)\)

From here. (It might be 0.) \(|\sum_{\sqrt{m} < n \leq m/(\sqrt{m} - 1)} \mu(n)|\) is less than \(\frac{1}{\lfloor \sqrt{m} \rfloor - 1} K(m/n_0)\)

Later, I calculate real example.

example: \(m = 100\) case.

\(-6 = 10 - 9 + 0 + 6 - 5 \times 2 - 4 - 3 + 2 + 1 \times 4\)

\(1 \times 4-4+2-3 = 4-1+1-1 = 3\) give the almost value of \(\sum_{[100/n_0]} \mu(n)\).

Actually, \(\sum_{9<n\leq100} \mu(n) = 2\). This gives \(|\sum_{[100/n_0]} \mu(n)| < K[100/n_0] = 1/4\).
$K \times 10$, 
example: $m = 10000$ case.

$-95 = 107 + 106 + 105 + 0 - 103 + 0 + 0 - 109 - 98 - 97 + 0 - 95 + 94 - 93 + 0 - 91$

$+ 0 + 0 - 88 - 87 + 86 + 0 + 84 \times 2 + 0 + 81 \times 2 + 0 + 0 - 78 + 77 - 76 \times 2 + 75$

$+ 74 + 72 \times 2 - 71 + 70 \times 2 + 69 + 68 \times 2 - 67 - 66 + 0 + 0 + 62 \times 2 - 61 + 0 - 59$

$- 58 - 57 \times 2 + 56 \times 2 - 55 \times 2 + 54 + 0 - 52 \times 2 - 51 - 50 \times 2 + 49 \times 3 + 48 \times 2 + 47 + 46$

$\times 4 + 45 \times 2 - 44 - 43 \times 3 + 0 - 41 \times 2 + 40 + 0 - 38 + 0 - 36 \times 2 - 35 \times 3 - 34 + 33 \times 6$

$- 32 - 31 + 30 \times 4 + 29 \times 3 - 28 \times 4 - 27 + 0 + 25 \times 3 + 0 - 23 \times 6 + 22 - 21 \times 2 + 0 + 19$

$\times 4 + 18 \times 7 + 17 - 16 \times 9 - 15 \times 9 + 14 \times 9 + 0 + 11 \times 2 + 10 \times 3 - 9 \times 15 + 8 \times 10 + 7$

$\times 12 - 6 \times 20 + 5 \times 16 - 4 \times 6 + 3 \times 18 - 2 \times 15 - 1 \times 25$

$- 1 \times 25 + 3 \times 8 - 2 \times 15 + 3 \times 10 , - 25 + 8 - 15 + 10 = -22$ gives the almost value of $\sum_{94<n\leq10000} \mu(n) = -22$. This gives $|\sum_{100/n_0<n\leq100} \mu(n)| < K[100/n_0] = K \times 107$.

$$\frac{1}{4} K \left( m - 1 - \frac{1}{2} \right) A \times 2 + B \times 1 \times A + B < K m \frac{1}{2} (1 + \epsilon) - m \frac{1}{2} + \epsilon$$

$$\frac{1}{2} K \left( (m - 1) \frac{1}{2} (1 + \epsilon) + 1 \right) A \frac{1}{2} + B \left( 1 - \frac{1}{2} \right) \frac{1}{3} A + B < K m \frac{1}{2} (1 + \epsilon) - m \frac{1}{2} + \epsilon$$

These formulas hold. If there is needed 4 or more terms, "almost correct" formula can be taken as flexible. It becomes to 3 terms case.

$$|\sum_{n \leq m} \mu(n)| < K m \frac{1}{2} + \epsilon + K \left[ m \frac{1}{2} (1 + \epsilon) \right] - K m \frac{1}{2} + \epsilon$$

$$|\sum_{n \leq m} \mu(n)| < K m \frac{1}{2} + \epsilon$$

So R.H. is got.  

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References

[1] Oscillatory properties of arithmetical functions. I Kaczorowski and Pintz  