Vorticity Lie-invariant decomposition theorem for barotropic real Schur flows

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Abstract

We formulate the problem of Lie-invariant decomposition of vorticity and establish the theorem for any ideal barotropic real Schur flows in $d$-dimensional Euclidean space $\mathbb{E}^d$ for any integer $d \geq 3$. The fact that there are two Helmholtz theorems of the complementary components of vorticity found recently in 3-space real Schur flows is not coincidental, but underlied by our fundamental decomposition Theorem, thus essential.

1 Introduction

The Taylor columns are ubiquitous in rotating fluid systems (c.f., e.g., Ref. [1] for a historical account). A ‘column’ indicates the two-dimensional structure, and, when referring to the velocity field in the rotation plane, it means the total 3-space velocity has a gradient matrix uniformly of the real Schur form (RSF), particularly for the formal Taylor-Proudman limit of compressible rotating flows[2]. On the other hand, any real matrices are similar to the Schur form, and, to our best knowledge, Li, Zhang & He[3] first used the (canonical) RSF of the local velocity gradient for a kinetic analysis of flows. Keylock[4] then resorted instead to the complex Schur form (but with remarks on RSF). Many other works also have emerged [S. Tian et al., private communication (2017); W.-N. Zou, private communication (2020)]; see, e.g., Xu et al. [5] for even more recent and comprehensive bibliography. Ref. [5] also made particular clarifications, among others, on the left/right-eigenvector and the corresponding left/right real Schur forms briefly remarked in Ref. [3], the latter mainly focusing on dynamical aspects of the case of global RSF, i.e., the real Schur flow (also RSF).

Due to fundamental duality of locality and globality discussed in the above, the study of the properties of RSF is in some sense resembling the situation of exploring special relativity (SR), since the general relativity is supported by locally flat space-time where SR physical laws apply (accord-
ing to Einstein’s equivalence principle). In particular, the compressible RSF deserves further theoretical attention. For example, a helical one may play the role of base flow for understanding the helicity effect on the compressibility of a turbulence\[^7\]. Here the focus is the Lie invariances. But, instead of the conventional interests on the invariants of different/higher orders (e.g., Refs. \[^8, 12, 10, 11, 9\]), we investigate which and how many independent components of the vorticity 2-form are invariant, which offers the knowledge of the dynamics of the finer structures of the geometrical objects.

One direct motivation is to answer the questions: whether the two Helmholtz/frozen-in laws found in Ref. \[^6\] is merely a coincidence, or there is something deeper or more essential underlying it; and, how about the equipartition of helicity by the decomposition made there? By geometrical formulation of the problem and by considering general barotropic ideal flows in \(d\)-dimensional Euclidean space \(\mathbb{E}^d\) for \(d \geq 3\\[^14\]\, we will establish a Lie-invariant decomposition Theorem of the vorticity, together with the velocity, to give a positive answer of essentiality to the former questions, while the last question concerning the global invariants appears to be of a larger topic connecting various aspects of more involved techniques beyond the scope of this note.

2 Vorticity Lie-invariant decomposition of the real Schur flow

It is natural to use the language of differential forms to describe fluid dynamics, especially for the frozen-in law of vorticity which is of geometrical nature (c.f., Ref. \[^1\] for a historical account and many modern references). Differential forms are special covariant tensors and the vorticity 2-form is simply the antisymmetric second order one which can also be represented with a matrix. For example, the vorticity exterior derivative of the 1-form \(U := \sum_{i=1}^{d} u_i dx_i\), corresponding to the velocity vector \(\mathbf{u} := \{u_1, u_2, \ldots, u_d\}\), reads, with the index ‘\(i\)’ ↔ ‘\(\partial_{x_i}\)’, for \(d = 4\) in co-ordinate form

\[
\begin{align*}
\text{d}U &= (u_{2,1} - u_{1,2}) dx_1 \wedge dx_2 + (u_{3,1} - u_{1,3}) dx_1 \wedge dx_3 + \\
&\quad + (u_{4,1} - u_{1,4}) dx_1 \wedge dx_4 + (u_{3,2} - u_{2,3}) dx_2 \wedge dx_3 + \\
&\quad + (u_{4,2} - u_{2,4}) dx_2 \wedge dx_4 + (u_{4,3} - u_{3,4}) dx_3 \wedge dx_4,
\end{align*}
\]

whose matrix representation, for the components associated to the bases \(dx_i \wedge dx_j\), writes

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{u_{2,1} - u_{1,2}}{2} & 0 & \frac{u_{1,3} - u_{3,1}}{2} & \frac{u_{1,4} - u_{4,1}}{2} \\
\frac{u_{3,2} - u_{2,3}}{2} & 0 & \frac{u_{2,4} - u_{4,2}}{2} & \frac{u_{3,4} - u_{4,3}}{2} \\
\frac{u_{4,3} - u_{3,4}}{2} & 0 & \frac{u_{4,2} - u_{2,4}}{2} & 0
\end{pmatrix}.
\]
The matrix representation of $\nabla \mathbf{u}$ in $\mathbb{R}^d$ is

$$G = \begin{pmatrix}
    u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} & \ldots & u_{d,1} \\
    u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} & \ldots & u_{d,2} \\
    u_{1,3} & u_{2,3} & u_{3,3} & u_{4,3} & u_{5,3} & u_{6,3} & \ldots & u_{d,3} \\
    u_{1,4} & u_{2,4} & u_{3,4} & u_{4,4} & u_{5,4} & u_{6,4} & \ldots & u_{d,4} \\
    u_{1,5} & u_{2,5} & u_{3,5} & u_{4,5} & u_{5,5} & u_{6,5} & \ldots & u_{d,5} \\
    u_{1,6} & u_{2,6} & u_{3,6} & u_{4,6} & u_{5,6} & u_{6,6} & \ldots & u_{d,6} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    u_{1,d} & u_{2,d} & u_{3,d} & u_{4,d} & u_{5,d} & u_{6,d} & \ldots & u_{d,d}
\end{pmatrix}.$$  \hspace{1cm} (3)

The three-dimensional (3D) case corresponds to the left-upper $3 \times 3$ block, the RSF of which is arranged to have its left-upper $2 \times 2$ block corresponding to the two conjugate complex eigenvalues of $G$, thus two vanishing left-lower elements (in blue color): when all eigenvalues are real, $u_{1,2}$ also vanishes, which is a stronger condition but which in general does not lead to stronger results in our discussions, thus will not be particularly discussed; similarly is for $d > 3$. [The RSF is nonunique, depending on how the order of the eigenvalues or the corresponding coordinates are arranged, which however is not essential.] $G$ can be decomposed into symmetric and anti-symmetric parts, $D = (G + G^T)/2$ and $A = (G - G^T)/2$, the latter, given in Eq. (2) for $d = 4$, may be viewed as a representation of the vorticity 2-form $\Omega = dU$.

### 2.1 Formulation

Let’s start with the Navier-Stokes equation for the barotropic RSF in $\mathbb{R}^3$:

$$\begin{align*}
    \partial_t \mathbf{u}_h + \mathbf{u}_h \cdot \nabla \mathbf{u}_h &= -\nabla_h \Pi + \nu \nabla^2 \mathbf{u}_h, \\
    \partial_t \mathbf{u}_3 + \mathbf{u}_h \cdot \nabla \mathbf{u}_3 + u_{3,3,3} &= -\Pi_{3,3} + \nu \nabla^2 u_3,
\end{align*}$$

where $x_1$ and $x_2$ are the ‘horizontal’ coordinates and the corresponding $\mathbf{u}_h := \{u_1, u_2\}$ is independent of the ‘vertical’ coordinate $x_3$ (i.e., $\partial_{x_3} \mathbf{u}_h = 0$), and, where $\nabla \Pi = (\nabla p)/\rho$. Higher-dimensional case can be similarly formulated, and such flow can of course be realized. For example, isothermal $p = c^2 \rho$ results in $\Pi = c^2 \ln \rho$, where $c$ is the sound speed. In Ref. [4], the author noticed that the incompressible RSR would have very restrictive structures, allowing even no periodic solution, but richer structures can present in the compressible one. Indeed, starting from some initial random data, we successfully simulated the above system together with the continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$ in a cyclic box (with periodic boundary condition forbidden by the incompressible RSF), as presented in Fig. 1 for the field structure of the three components of velocity. As can be clearly seen, cylindrical condition is realized to have two-dimensional $\mathbf{u}_h$, but
the \(u_3\) is three-dimensional, a characteristic termed as two-component-two-dimensional coupled with one-component-three-dimensional (2C2Dcw1C3D) in Ref. [6]. Specific analysis of compressible RSF, especially the turbulence dynamics, however belongs to another communication for different purpose (manuscript in preparation). The selective cylindrical condition also presents in the Taylor-Proudman limit of fast rotating compressible flows [2], as remarked earlier, thus such a 2C2Dcw1C3D flow is quite of physical sense instead of being purely artificial.

![Figure 1](image)

**Figure 1:** A typical snapshot of the structure of the three components \((u \rightarrow u_1, v \rightarrow u_2\text{ and } w \rightarrow u_3)\) of the velocity of a 3D RSF. For clarity, only a few values of the amplitudes, as shown by the legend for \(u_3\), are coded with colors (and only one side of each periodic/repeated boundary of the box is shown, to avoid redundancy in data processing).

In terms of differential forms, the inviscid \([\nu = 0\text{ in Eqs. (4,5)}, \text{ say}]\) equations of the horizontal and total velocities read,

\[
\begin{align*}
\partial_t U_h + L_{u_h} U_h &= -d_h(\Pi - u_{h}^2/2), \\
\partial_t U + L_{u} U &= -d(\Pi - u^2/2),
\end{align*}
\]

which also applies for general *ideal* real Schur flows in \(\mathbb{R}^d\) with \(d \geq 3\), the precise meaning of \(U_h\) for \(d > 3\) to be further clarified below. Thus, with the
interchanges of the exterior derivative with the (partial) time derivative and
the Lie derivative, and, with the replacement of \( L_u \) by \( L_u \) (to be explained
later), we have the Lie-invariance laws for the vorticity 2-forms \( \Omega_h = dU_h \)
and \( \Omega = dU \),

\[
\begin{align*}
\partial_t \Omega_h + L_u \Omega_h &= 0, \\
\partial_t \Omega + L_u \Omega &= 0.
\end{align*}
\]

We already can see that the two frozen-in laws of decomposed vorticities
(for which the spatial derivatives of \( u_h \) and \( u_z \) are respectively responsible),
found in Ref. 6 for \( d = 3 \), correspond to Eq. (8) and the substraction by
it of Eq. (9), but now also for compressible barotropic RSF. [Such a result
however does not guarantee a naive extension to \( d > 3 \), because, as said, in
the latter case the precise meaning of the index \( h \) needs special clarification,
which is actually our main task.] The objective of frozen-in decomposition
is then translated to Lie-invariant decomposition which can be formulated
for general \( d \) as the following:

**Definition 1** A Lie-invariant decomposition of a (Lie-invariant) vorticity
2-form \( \Omega \) into \( M \geq 1 \) components of a barotropic ideal flow in \( \mathbb{R}^d \) is that

\[
\Omega = \sum_{i=1}^{M} \Omega_i, \text{ with } (\partial_t + L_u)\Omega_i = 0,
\]

where \( \Omega_i \)'s are linearly independent.

**Remark 1** Obviously, \( M \leq d(d-1)/2 \) (no larger than the number of the
elements of the \( A \) matrix), and when \( M = 1, \Omega_1 = \Omega \). Actually, we will see
that \( M \geq \lceil \frac{d+1}{2} \rceil \), where \( \lceil ... \rceil \) denotes the integer part.

2.2 Observation

We first see from the left-upper \( 3 \times 3 \) RSF block (with \( u_{1,2} = u_{1,3} = 0 \),
in blue color), designated with single underline and single right wall, of the
matrix in Eq. (3) that the vorticity 2-form, corresponding to the anti-
symmetric part of the left-upper \( 2 \times 2 \) block, of the horizontal velocity \( u_h \)
is Lie-carried by the latter, and thus also by the whole 3-space \( u \). Since
the whole \( \Omega = dU \) is Lie invariant respect to \( u \), we immediately see that
vorticity 2-form component corresponding to anti-symmetric part of the
right column of the matrix is also accordingly Lie invariant, from simple
substraction: this corresponds to the remark immediately after Eq. (8).

Now in the 4-space with the left-upper \( 4 \times 4 \) RSF block, designated with
double underlines and double right walls, with extrally \( u_{1,4} = u_{2,4} = 0 \), thus
also in blue color, the vorticity 2-form component corresponding to the anti-
symmetric part of the left-upper \( 2 \times 2 \) block is Lie-carried by the 4-space \( u \).
Thus, the vorticity 2-form component corresponding to the anti-symmetric part of the rest two columns on the right, but not the third column as in 3-space, is also accordingly Lie invariant, again, from simple substraction.

We have already seen from the above an inductive procedure. For example, in the 5-space, similar to the 3-space case, with the left-upper $5 \times 5$ RSF block with extrally $u_{1,5} = u_{2,5} = u_{3,5} = u_{4,5} = 0$. We see both the vorticity 2-form components, corresponding respectively to the anti-symmetric parts of the left-upper $2 \times 2$ block and to the anti-symmetric part of the third and fourth columns, and their sum as a component of the 5-space vorticity 2-form, are all Lie-carried by $u$ in $E^5$; then, still from simple substraction, the vorticity 2-form component corresponding to the anti-symmetric part of the fifth column is also Lie invariant. And, similar to the 4-space case, the 6-space RSF flow carries three linearly independent Lie-invariant vorticity 2-form components, corresponding respectively to the anti-symmetric parts of the first, second and third two columns of the $G$ matrix.

The above analysis indicates for general $d$ the following

**Theorem 1** With $d+1$ understood to be adding an extra spatial dimension the velocity component of which is constant, i.e., appending an extra column and row of zeros to the bottom and right of the $d$-space velocity gradient matrix $G$, there are $M = \left\lfloor \frac{d+1}{2} \right\rfloor$ linearly independent Lie-invariant (with respect to the ideal barotropic flow in $E^d$) vorticity 2-form components, each of which subsequently corresponding to the anti-symmetric part of the two columns of $G$ associated to $\Omega_i = dU_i$ for

$$U_i := u_{2i-1}dx_{2i-1} + u_{2i}dx_{2i}. \quad (11)$$

**Remark 2** When $i \neq d/2$, each $U_i$ and $\Omega_i$ are respectively perpendicular and parallel to the $d$th coordinate, thus the notations $U_i$, and $\Omega_i$, corresponding to Eqs. (6), would also be justified.

2.3 Proof

The induction made in the observations in Sec. 2.2 already constitutes the majority of the proof, except that we actually already used

**Lemma 1** The vorticity 2-form (component) Lie-carried by a $k$-space velocity is also invariant when the space is trivially extended to $k+1$ dimensions, where ‘trivially’ refers to the property that the velocity components responsible for the vorticity 2-form do not depend on the extended spatial coordinate.

The proof of Lemma 1 is straightforward by simple calculation with $L_u = \iota_u d + du$, where $\iota_u$ is the interior product, with $u$ changed from a $k$-space velocity to a $(k+1)$-space one.
The linear independence of the \([\frac{d+1}{2}]\) components in Eq. (10) is obvious by the fact that the numbers of bases, \(dx_m \wedge dx_n\), involved in different \(\Omega_i\)s are different, thus the complete proof follows.

**Remark 3** Any linear combinations of \(\Omega_i\)s are also Lie-invariant. It is very tempting to say that \(\Omega_i\)s constitute the linearly independent Lie-invariant basis of the invariant vorticity (component) of RSF, but there can be more vanishing components than those left-lower corners of RSF \(G\), which then may lead to some very special Lie-invariant vorticity component(s) not representable by those \(\Omega_i\)s: that’s why we said in Remark 1 that \(M\) could be larger than \([\frac{d+1}{2}]\).

### 3 Further discussions

The conventional construction of higher-order Lie invariants then can be applied to these objects; that is, we have the following

**Corollary 1** Any linear-sum and wedge-product combinations of \(\Omega_i\)s are also locally invariant.

Other invariants can also be constructed from transforming \(U_i\) into a local 1-form (a special one in the coset \([U_i]\)) in the way of constructing local helicities by Oseledets \[15\] and Gama & Frisch \[8\], which we can but do not bother to formulate seriously into a corollary. Thus, as yet another informal corollary, various new Cauchy invariants equations associated to the decomposed components, in addition to those established by Besse & Frisch \[11\], follow.

Using matrix decomposition or transformation techniques, one may perform various decompositions of the flow and/or the vorticity, which for lack of the consideration on the dynamics (such as the Lie invariance respect to the flow discussed in the above) may be called ‘kinetic’. For example, besides the well-known decomposition of \(G\) into the symmetric and antisymmetric parts (respectively, \(D\) and \(A\)), the RSF \(G\) can be further transformed into the canonical form and then decomposed into the shear part \(S\) and the canonical part \(N\), the latter further being composed of the dilation part \(E\), the part for the strain rate along some eigenvector \(Z\) and the rotation part \(\Psi\) \[3\]. For the antisymmetric \(A\) representing the vorticity, its canonical form is (block) diagonal with some \(2 \times 2\) antisymmetric block(s) and all other elements vanishing: \[16\], each \(2 \times 2\) antisymmetric block represents the (rate of) rotation in the plane (c.f., equation 23 in Ref. \[2\] and the discussion following it), and, if more than one, all the rotation planes are orthogonal to each other. We call such rotations ‘pure’. For example, if the left-lower and right-upper \(2 \times 2\) blocks of the matrix \(A\) are zeros, then the (pure)
rotations are respectively in the \( x_1-x_2 \) and \( x_3-x_4 \) planes in the corresponding canonical coordinate frame. However, the \( A \) of RSF \( G \) is in general not in the canonical form, thus containing ‘entangled’ rotations. For example, \( G \) being of RSF though, none of the nondiagonal elements of matrix (2) is indicated to be vanishing, thus the possibility of simultaneous rotations in \( x_1-x_2, x_3-x_4, x_1-x_3 \) and \( x_2-x_4 \) planes in the corresponding coordinate frame.

The entanglement of the rotations may lead to ambiguity and even confusions about vorticity, as already presents in 3D case: more than one rotation planes are in general involved in the vorticity, thus very complicated flow pattern from the latter, unless in the canonical frame where there always is only one plane for the pure rotation.

We conclude by returning to remark on the ‘dynamical’ decomposition: the study of compressibility reduction of helical turbulence in Ref. [7] has made use of helical RSF as the chiral base flow but has not yet exploited the ideal vorticity frozen-in decomposition, however we expect that the latter may be intrinsic to the fundamental mechanisms of other issues of fully \( d \)-dimensional flows (not RSF), including 3D incompressible turbulence.

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The Euclidean metric is needed for defining energy, among others, thus we use $E^d$ instead of $E^d$.

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