# ARBITRARY $l$-SOLUTIONS OF THE SCHRODINGER EQUATION IN ARBITRARY DIMENSIONS FOR THE ENERGY DEPENDENT GENERALIZED INVERSE QUADRATIC YUKAWA POTENTIAL.

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## ABSTRACT

Within the framework of Nikiforov-Uvarov method, we obtained an approximate solution of the Schrodinger equation for the Energy Dependent Generalized inverse quadratic Yukawa potential model. The bound state energy eigenvalues for was computed for various vibrational and rotational quantum numbers. Special cases were considered when the potential parameters were altered, resulting into Energy Dependent Kratzer and Kratzer potential, Energy Dependent Kratzer fues and Kratzer fues potential, Energy Dependent Inverse quadratic Yukawa and Inverse quadratic Yukawa Potential, Energy Dependent Yukawa (screened Coulomb) and Yukawa (screened Coulomb) potential, and Energy Dependent Coulomb and Coulomb potential, respectively. Their energy eigenvalues expressions and numerical computations agreed with the already existing literatures.

## 1. INTRODUCTION

Wave equations with energy dependent potentials have been in existence for over 80 years. They can be seen in relativistic quantum mechanics considering particle in an external electromagnetic field. [1-3]. Energy-dependent potential has been studied in nonrelativistic and relativistic quantum mechanics [4-10]. Recently, researchers have showed renewed interest in the study of Energy Dependent Potential (in both relativistic and non-relativistic regime), some of the study amongst others are; [11] studied the Schrödinger equation in D-dimensions for an energy-dependent Hamiltonian that linearly depends on energy and quadratic on the relative distance using the Nikiforov-Uvarov formalism.[12] showed the influence of the modification of the scalar product, found in the problems of the energy-dependent potential, on the physical properties of the harmonic oscillator in one dimension. More so, they discussed the effect of this change on the thermodynamic properties of the oscillator. [13] solved the Dirac equation for the energy-

dependent pseudo-harmonic and Mie-type potentials under the pseudospin and spin symmetries using the supersymmetry quantum mechanics. [4] solved the Dirac equation for the energy-dependent Yukawa potential including a tensor interaction term within the framework of the pseudospin and spin symmetry limits with arbitrary spin-orbit quantum number using the Nikiforov–Uvarov method. Ikot et al. [14] solved the energy dependent Kratzer potential within the framework of non-relativistic quantum mechanics. [15] generalized Schrodinger equations that include the position-dependent mass was solved for systems featuring energy-dependent potentials. [16] studied the Dirac equation for an energy-dependent potential in the presence of spin and pseudospin symmetries with arbitrary spin-orbit quantum number $\kappa$. [17] also solved the Dirac equation for the energy-dependent Coulomb (EDC) potential including a Coulomb-like tensor (CLT) potential was studied in the presence of spin and pseudospin symmetries with arbitrary spin–orbit quantum number k by [16]. This solution was achieved within the framework of the asymptotic iteration method. It is worthy to state here that to the best of our knowledge, no one can give a solid answer to the question ″what is the most useful form for an Energy Dependent Potential(EDP)?″ [16].

The generalized inverse quadratic Yukawa potential (GIQYP) is a superposition of the inverse quadratic Yukawa (IQY) and the Yukawa potential. It is asymptotic to a finite value as $r \to \infty$ and becomes infinite at $r = 0$ [18]. This potential has been solved within the framework of the Proper Quantisation Rule [19] and Eigenfunction was obtained via the Formula Method[20].

The Generalized inverse quadratic Yukawa potential model is of the form [18]

$$V(r) = -V_0 \left(1 + \frac{e^{-\alpha r}}{r}\right)^2 \tag{1a}$$

It has been noted that differences do not exist between the behavior of the modified Yukawa potential and the inversely quadratic Yukawa potential [21, 22] or the Yukawa potential [23]. Its application to diverse areas of physics has been of great interest concern in recent times [24-25]. In addition, several quantum mechanical models have been studied extensively both in the relativistic and non-relativistic terrain by several authors[26-34].

The Energy Dependent Generalized inverse quadratic Yukawa potential model is of the form

$$V(r, E_{n,l}) = -\frac{A(1+\eta E_{n,l})e^{-2\alpha r}}{r^2} - \frac{B(1+\eta E_{n,l})e^{-\alpha r}}{r} - (1+\eta E_{n,l})C \quad (1b)$$

$$A = C = V_0 \text{ and } B = 2V_0$$

The Generalized inverse quadratic Yukawa potential reduces to a constant potential when $A = B = 0$.

The study of dimensions plays an important role in many areas of physics and the extension of physical problems to higher dimensional space is of great interest. [34] noted that the exact solutions of both the relativistic and nonrelativistic wave equation with certain physical potential in higher dimensions are remarkably important not only in physics and chemistry, but also in pure and applied mathematics.

The organization of the work is as follows: In the next section, we, the review of the NU In Sect. 3, this method is applied method obtain the bound state solutions. . In Sect. 4, we obtain numerical results while in the final section. In Sect. 5 we discuss some special cases and in Sect. 6, we give the concluding remark.

## 2. REVIEW OF NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov (NU) method is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions [35]. The main equation which is closely associated with the method is given in the following form [36-37]

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \quad (2)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second-degree, $\tilde{\tau}(s)$ is a first-degree polynomial and $\psi(s)$ is a function of the hypergeometric-type.

The exact solution of Eq. (2) can be obtained by using the transformation

$$\psi(s) = \phi(s)y(s) \quad (3)$$

This transformation reduces Eq. (2) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \quad (4)$$

The function $\phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{5}$$

where $\pi(s) = \frac{1}{2}[\tau(s) - \tilde{\tau}(s)]$ (5a)

with $\pi(s)$ being at most a first-degree polynomial. The second $\psi(s)$ being $y_n(n)$ in Eq. (3), is the hypergeometric function with its polynomial solution given by Rodrigues relation

$$y^{(n)}(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n \rho(s)] \tag{6}$$

Here, $B_n$ is the normalization constant and $\rho(s)$ is the weight function which must satisfy the condition

$$\big(\sigma(s)\rho(s)\big)' = \sigma(s)\tau(s) \tag{7}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{8}$$

It should be noted that the derivative of $\tau(s)$ with respect to $s$ should be negative. The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter $\lambda$, respectively:

$$\pi(s) = \frac{\sigma'(s)-\tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{9}$$

where $k = \lambda - \pi'(s)$ (10)

The value of $k$ can be obtained by setting the discriminant of the square root in Eq. (9) equal to zero. As such, the new eigenvalue equation can be given as

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s),\ n = 0,1,2,\ldots \tag{11}$$

## 3. BOUND STATE SOLUTION WITH ENERGY DEPENDENT GENERALIZED INVERSE QUADRATIC YUKAWA POTENTIAL IN D DIMENSION

The radial Schrodinger equation in $D$ dimension can be written as [38-39]:

$$\left[\frac{d^2R_{nl}}{dr^2} - \frac{2\mu V(r)}{\hbar^2} - \frac{(D+2\ell-1)(D+2\ell-3)}{4r^2} + \frac{2\mu E_{nl}}{\hbar^2}\right]R_{nl}(r) = 0 \tag{12}$$

where $\mu$ is the reduced mass, $E_{nl}$is the energy spectrum,$\hbar$ is the reduced Planck's constant and $n\ and\ l$ are the radial and orbital angular momentum quantum numbers respectively (or vibration-rotation quantum number in quantum chemistry). Substituting equation (1) into equation (12) gives:

$$\left[\frac{d^2R_{nl}}{dr^2}-\frac{2\mu}{\hbar^2}\left(-V_0\left(1+\frac{e^{-\alpha r}}{r}\right)^2\left(1+\eta E_{n,l}\right)\right)-\frac{(D+2\ell-1)(D+2\ell-3)}{4r^2}+\frac{2\mu E_{nl}}{\hbar^2}\right]R_{nl}(r)=0 \quad (13)$$

Simplifying further equation 13 becomes;

$$\left[\frac{d^2}{dr^2}-\ \frac{2\mu}{\hbar^2}\left(-\frac{A\left(1+\eta E_{n,l}\right)e^{-2\alpha r}}{r^2}-\frac{B\left(1+\eta E_{n,l}\right)e^{-\alpha r}}{r}-\left(1+\eta E_{n,l}\right)C\right)-\frac{(D+2\ell-1)(D+2\ell-3)}{4r^2}+\frac{2\mu E_{nl}}{\hbar^2}\right]R_{nl}=0 \quad (15)$$

Employing the Pekeris type approximation scheme [40], which is given by

$$\frac{1}{r^2}=\frac{4\alpha^2e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2}\ \text{and}\ \frac{1}{r}=\frac{2\alpha e^{-\alpha r}}{(1-qe^{-2\alpha r})} \quad (14)$$

equation 14 becomes;

$$\frac{d^2R_{n\ell}(r)}{dr^2}+\frac{1}{(1-e^{-2\alpha r})^2}\left[\frac{2\mu\left(E_{nl}+C\left(1+\eta E_{n,l}\right)\right)}{\hbar^2}(1-e^{-2\alpha r})^2+\frac{4\mu B\alpha\left(1+\eta E_{n,l}\right)e^{-2\alpha r}}{\hbar^2}(1-e^{-2\alpha r})+\frac{8\mu A\alpha^2\left(1+\eta E_{n,l}\right)e^{-4\alpha r}}{\hbar^2}-\frac{(D+2\ell-1)(D+2\ell-3)4\alpha^2e^{-2\alpha r}}{4}\right]R_{n\ell}(r) \quad (16)$$

Eq. (16) can be simplified into the form and introducing the following dimensionless abbreviations

$$\left\{\begin{array}{c}-\varepsilon_n=\frac{\mu\left(E_{nl}+C\left(1+\eta E_{n,l}\right)\right)}{2\hbar^2\alpha^2}\\ \beta=\frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}\\ \chi=\frac{\mu B\left(1+\eta E_{n,l}\right)}{\hbar^2\alpha}\\ \gamma=\frac{(D+2\ell-1)(D+2\ell-3)}{4}\end{array}\right\} \quad (17)$$

And using the transformation $s=e^{-2\alpha\, r}$ so as to enable us apply the NU method as a solution of the hypergeometric type

$$\frac{d^2R_{n\ell}(r)}{dr^2} = 4\alpha^2 s^2 \frac{d^2R_{n\ell}(s)}{ds^2} + 4\alpha^2 s \frac{dR_{n\ell}(s)}{ds} \tag{18}$$

$$\frac{d^2R_{n\ell}(s)}{ds^2} + \frac{1-qs}{s(1-qs)}\frac{dR_{n\ell}(s)}{ds} + \frac{1}{s^2(1-qs)^2}[-s^2(\varepsilon_n - \beta + \chi) + s(2\varepsilon_n + \chi - \gamma) - \varepsilon_n]R_{n\ell}(s) = 0 \tag{19}$$

Comparing Eq. (19) and Eq. (2), we have the following parameters

$$\left.\begin{matrix} \tilde{\tau}(s) = 1 - s \\ \sigma(s) = s(1-s) \\ \tilde{\sigma}(s) = -s^2(\varepsilon_n - \beta + \chi) + s(2\varepsilon_n + \chi - \gamma) - \varepsilon_n \end{matrix}\right\} \tag{20}$$

Substituting these polynomials into Eq. (9), we get $\pi(s)$ to be

$$\pi(s) = -\frac{qs}{2} \pm \sqrt{(a-k)s^2 + (b+k)s + c} \tag{21}$$

where

$$\left.\begin{matrix} a = \frac{1}{4} + \varepsilon_n - \beta + \chi \\ b = -(2\varepsilon_n + \chi - \gamma) \\ c = \varepsilon_n \end{matrix}\right\} \tag{22}$$

To find the constant $k$, the discriminant of the expression under the square root of Eq. (21) must be equal to zero. As such, we have that

$$k\pm = -(\chi - \gamma) \pm 2\sqrt{\varepsilon_n\left(\frac{1}{4} - \beta + \gamma\right)} \tag{23}$$

Substituting Eq. (23) into Eq. (21) yields

$$\pi(s) = -\frac{s}{2} \pm \left[\left(\sqrt{\varepsilon_n} + \sqrt{\left(\frac{1}{4} - \beta + \gamma\right)}\right)s - \sqrt{\varepsilon_n}\right] \tag{24}$$

From the knowledge of NU method, we choose the expression $\pi(s)_-$ which the function $\tau(s)$ has a negative derivative. This is given by

$$k_{-} = -(\chi - \gamma) - 2\sqrt{\varepsilon_n\left(\frac{1}{4} - \beta + \gamma\right)} \tag{25}$$

with $\tau(s)$ being obtained as

$$\tau(s) = 1 - 2s - 2\left[\left(\sqrt{\left(\frac{1}{4} - \beta + \gamma\right)} + \sqrt{\varepsilon_n}\right)s - \sqrt{\varepsilon_n}\right] \tag{26}$$

Referring to Eq. (10), we define the constant $\lambda$ as

$$\lambda = -\frac{1}{2} - \left(\sqrt{\left(\frac{1}{4} - \beta + \gamma\right)} + \sqrt{\varepsilon_n}\right) + (\gamma - \chi) - 2\sqrt{\varepsilon_n\left(\frac{1}{4} - \beta + \gamma\right)} \tag{27}$$

Substituting Eq. (27) into Eq. (11) and carrying out simple algebra, where

$$\tau'(s) = -2\left(1 + \left(\sqrt{\left(\frac{1}{4} - \beta + \gamma\right)} + \sqrt{\varepsilon_n}\right)\right) < 0 \tag{28}$$

and

$$\sigma''(s) = -2 \tag{29}$$

We have

$$\varepsilon_n = \frac{1}{4}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \beta + \gamma}\right)^2 + \beta - \chi}{\left(\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \beta + \gamma}\right)\right)}\right]^2 \tag{30}$$

Substituting Eqs. (17) into Eq. (30) yields the energy eigenvalue equation of the Energy Dependent Generalized Inverse Quadratic Yukawa Potential in $D$ dimension in the form

$$E_{n\,\ell} = -C\left(1 + \eta E_{n,l}\right) - \frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2 + \frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2} - \frac{\mu B\left(1+\eta E_{n,l}\right)}{\hbar^2\alpha}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2} + \frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{31}$$

The corresponding wave functions can be evaluated by substituting $\pi(s)$_ *and* $\sigma(s)$ from Eq. (24) and Eq. (20) respectively into Eq. (5) and solving the first order differential equation. This gives

$$\Phi(s) = s^{\sqrt{\varepsilon_n}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\beta+\gamma}} \tag{32}$$

The weight function $\rho(s)$ from Eq. (7) can be obtained as

$$\rho(s) = s^{2\sqrt{\varepsilon_n}}(1-s)^{2\sqrt{\frac{1}{4}-\beta+\gamma}} \tag{33}$$

From the Rodrigues relation of Eq. (6), we obtain

$$y_n(s) \equiv N_{n,l} P_n^{\left(2\sqrt{\varepsilon_n},\ 2\sqrt{\frac{1}{4}-\beta+\gamma}\right)}(1-2s) \tag{34}$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial.

Substituting $\Phi(s)\ and\ y_n(s)$ from Eq. (32) and Eq. (34) respectively into Eq. (3), we obtain

$$\psi_n(s) = N_{n,l}\, s^{\sqrt{\varepsilon_n}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\beta+\gamma}} P_n^{\left(2\sqrt{\varepsilon_n},\ 2\sqrt{\frac{1}{4}-\beta+\gamma}\right)}(1-2s) \tag{35}$$

## 4. Special Cases (Deductions from Eq. (31))

In this section, we take some adjustments of constants in Eq. (1a and b) to have the following cases:

- **Kratzer Potential**

If $\alpha \to 0$ and if set $A = -V_1$, $B = 2V_1$ and $C = -V_1$ Equation 1c reduces to

$$V(r) = \frac{A\left(1+\eta E_{n,l}\right)}{r^2} - \frac{B\left(1+\eta E_{n,l}\right)}{r} + C\left(1+\eta E_{n,l}\right) \tag{36}$$

Equation(31) becomes the energy dependent Kratzer Potential in $D$ dimesnsions;

$$E_{n\,\ell} = C\left(1+\eta E_{n,l}\right) - \frac{\mu\, B^2\left(1+\eta E_{n,l}\right)^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2} \tag{37}$$

If we set $\eta = 0$, the energy equation reduces to energy equation for Kratzer potential in $D$ dimesnsions;

$$E_{n\,\ell} = C - \frac{\mu B^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu A}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2} \tag{38}$$

**Comments**; Equation(38) is the Energy eigenvalue equation for the Kratzer potential in $D$ dimensions. If $D = 3$ reduces to energy equation for Kratzer potential in 3D, Eq. (45) is very consistent with the result obtained in Eq. (125) of Ref.[37]

- **Inversely Quadratic Yukawa Potential**

If $B = C = 0$ the potential (Equation 1a) reduces to the Inverse Quadratic Yukawa Potential[41].

$$V(r) = -\frac{A\left(1+\eta E_{n,l}\right)e^{-2\alpha r}}{r^2} \tag{39}$$

$$E_{n\,\ell} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2+\frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{2\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{40}$$

If we set $\eta = 0$, the energy equation reduces to energy equation for Inverse Quadratic Yukawa Potential in $D$ dimesnsions;

$$E_{n\,\ell} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{2\mu A}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2+\frac{2\mu A}{\hbar^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{2\mu A}{\hbar^2}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{41}$$

**Comments**; Equation (41) is the energy equation for the Inverse Quadratic Yukawa Potential in $D$ Dimensions. If $D = 3$ , equation (41) reduces to the energy equation in 3D, which is identical to the results in; Eq. (37) of ref. [42], Eq.(18) of ref.[43] and Eq.(47) of ref.[44].

- **Yukawa Potential**

If  and $A = C = 0$ the potential (Equation 1c) reduces to the Yukawa Potential[4]. [4] solved the Dirac equation for the energy-dependent Yukawa potential including a tensor interaction term within the framework of the pseudospin and spin symmetry limits with arbitrary spin-orbit quantum number. The limiting cases of the model was reduced to the energy-dependent Yukawa and Coulomb potentials, respectively.

$$V(r) = -\frac{B\left(1+\eta E_{n,l}\right)e^{-\alpha r}}{r} \tag{42}$$

$$E_{n\,\ell} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2-\frac{\mu B\left(1+\eta E_{n,l}\right)}{\hbar^2\alpha}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{43}$$

If we set $\eta = 0$, the energy equation reduces to energy equation for Yukawa Potential  in $D$ Dimensions;

$$E_{n\,\ell} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2-\frac{\mu B}{\hbar^2\alpha}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)}\right]^2 \tag{44}$$

**Comments**; Eq.(44) is the energy eigenvalue equation for the Yukawa potential in $D$ dimensions. If $D = 3$, eq. (44) becomes identical with eqs.(87) and (15) reported in ref.[45] and [46] respectively.

- **Kratzer-Feus Potential**

if we set $C = 0$, $\alpha \to 0$ , eq.(1c) reduces to the  Coulomb plus Inverse-Square Potential[47-48]

$$V(r) = -\frac{B}{r} + \frac{A}{r^2} \tag{45}$$

$$A = C = -V_1 \text{ and } B = 2V_1$$

$$E_{n\,\ell} = -\frac{2\mu B^2\left(1+\eta E_{n,l}\right)^2}{\hbar^2\left(2n+1+\sqrt{\frac{8\mu A\left(1+\eta E_{n,l}\right)}{\hbar^2}+(D+2\ell-1)^2}\right)} \tag{46}$$

If we set $\eta = 0$, the energy equation reduces to energy equation for Kratzer-Fues Potential  in $D$ dimesnsions;

$$E_{n\,\ell} = \frac{-2\mu B^2}{\hbar^2\left(2n+1+\sqrt{\frac{8\mu A}{\hbar^2}+(D+2\ell-2)^2}\right)^2} \tag{47}$$

**Comments**; Eq. (47) is also known as the Kratzer-Feus potential, this potential was studied by ref [48] in arbitrary dimensions. If we set $\hbar = \mu = 1$,eq. (47) fully agrees with the result reported in eq. (28) of ref. [48]. Eq. (37) is also consistent with the result obtained in Eq. (15) of Ref [47].

- **Coulomb Potential**

If $A = C = 0$, $\alpha \to 0$ the potential (Equation 1c) reduces to the Coulomb Potential[34]

$$V(r) = -\frac{B}{r} \tag{48}$$

$$E_{n\,\ell} = -\frac{\mu B^2\left(1+\eta E_{n,l}\right)^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2} \tag{49}$$

If we set $\eta = 0$, the energy equation reduces to energy equation for Coulomb Potential in $D$ dimesnsions;

$$E_{n\,\ell} = -\frac{\mu B^2}{2\hbar^2\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(D+2\ell-1)(D+2\ell-3)}{4}}\right)^2} \tag{50}$$

**Comments**; Eq.(50) is the energy equation for Coulomb potential in $D$ Dimensions. This result is consistent with the result obtained in Eq. (7.14) of Ref. [34].

## 5. DISCUSSION

In our study, the energy eigenvalues of the Energy Dependent Generalized Inverse Quadratic Yukawa Potential (EDGIQYP) model were computed using Eq. (31), for different values of the energy slope parameters given$(\eta)$ which are presented in Table 1-4 in 3D. When $\eta = 0$ , the energy equation(31) reduces to the Energy for Generalized Inverse Quadratic Yukawa Potential (EDGIQYP) model, and the corresponding numerical eigenvalues presented in Tables1-4 for $\eta = 0$ agrees perfectly with the result presented in Table 1 of ref.[18] in the absence of the energy dependence.

We have plotted the shape of the Energy Dependent Generalized Inverse Quadratic Yukawa Potential (EDGIQYP) model in Figs. 1-2. This figure gives an insight into the behaviour of the potential. Also, the variation of the energy eigenvalues with different parameters such as Dimensions $(D)$ , coupling strength $V_0$ and particle mass $(\mu)$ are shown in Figs. 7 – 12

respectively, for various values of $n$ *and* $\ell$. In these figures, there is a decrease in energy eigenvalues as the potential strength increases (quasi asymptotic) in the ground state energy level.There was an increase in energy as dimensions' increases for both the the ground state and first excite energy levels respectively. It's evident also that as the energy slope parameter $(\text{for } \eta \geq 0)$ increases the energy values increase too.

In Figs. 3 and 4, The variation of the ground state $(n = 0)$ energy level and first excited state $(n = 1$ state for various values of the energy slope parameter$(\eta)$ as a function of $l$ were plotted. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$. It was observed that as the rotational quantum number increases the energy increases for various values of $\eta$. In Figs. 5 and 6, the variation of the ground state $(n = 0)$ and first excited state $(n = 1$ states energy level for the different values of the energy slope parameter( for $\eta \leq 0)$ as a function of $l$. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$. In the first excited it is observed that the energy becomes more negative (more attractive) for increasing $\eta$ and $l$. In the ground state, there existed an irregular behaviour as some points broke off due to the presence complex values which are unacceptable for bound state solutions.

## 6. CONCLUSION

In this study, the approximate bound state solutions of the Schrodinger equation with Energy Dependent Generalized Inverse Quadratic Yukawa Potential (EDGIQYP) model was obtained, via the Nikiforov-Uvarov method. The energy eigenvalues of was computed and special cases considered. Our results were consistent with the results in available literature. The shape of the potential model was plotted and this gives a better understanding to the behaviour of the potential model. The variation of the combined energy eigenvalue with the potential parameters $(D, V_0 \text{ and } \mu)$ were also plotted. It was discovered that the energy eigenvalues decrease as the various potential strength $(V_0)$ increase in the ground state. The present study can be extended to scrutinize the thermodynamic properties of energy dependent systems [49-50].

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**Table 1. The bound state energy levels (in units of fm−1) of the GIQYP for various values of $n, l$ and for $\hbar = \mu = 1$, $V_0 = 0.5$ $and$ $\alpha = 0.001$**

| $n$ | $l$ | $E_{nl}$ $\eta = 0$ | $E_{nl}$ $\eta = 0.3$ | $E_{nl}$ $\eta = 0.4$ |
|---|---|---|---|---|
| 0 | 1 | -0.68960354 | -0.540881135 | -0.506890535 |
| 1 | 1 | -0.571807619 | -0.478344718 | -0.45428128 |
| 2 | 1 | -0.537127791 | -0.45790939 | -0.436746066 |
| 3 | 1 | -0.522410094 | -0.44889584 | -0.428947535 |
| 0 | 2 | -0.563050854 | -0.474126328 | -0.450862611 |
| 1 | 2 | -0.53372401 | -0.456181624 | -0.435330535 |
| 2 | 2 | -0.520749321 | -0.448032007 | -0.42823602 |
| 3 | 2 | -0.513895979 | -0.443670831 | -0.424428249 |
| 0 | 3 | -0.532601875 | -0.455583817 | -0.434834803 |
| 1 | 3 | -0.520190696 | -0.447728815 | -0.427983558 |
| 2 | 3 | -0.513578774 | -0.4434969 | -0.424283086 |
| 3 | 3 | -0.509646332 | -0.440963924 | -0.422065022 |
| 0 | 4 | -0.519902567 | -0.44757064 | -0.427851451 |
| 1 | 4 | -0.513414329 | -0.443405797 | -0.424206841 |
| 2 | 4 | -0.509543853 | -0.440906841 | -0.422017192 |
| 3 | 4 | -0.507053026 | -0.439293102 | -0.42060214 |

**Table 2. The bound state energy levels (in units of fm−1) of the GIQYP for various values of $n, l$ and for $\hbar = \mu = 1, V_0 = 0.5\ and\ \alpha = 0.01$**

| $n$ | $l$ | $E_{nl}$ $\eta = 0$ | $E_{nl}$ $\eta = 0.3$ | $E_{nl}$ $\eta = 0.4$ |
|---|---|---|---|---|
| 0 | 1 | -0.677413346 | -0.533276163 | -0.500165487 |
| 1 | 1 | -0.561940037 | -0.471450998 | -0.448062163 |
| 2 | 1 | -0.528190997 | -0.451485229 | -0.430921372 |
| 3 | 1 | -0.514145048 | -0.442922673 | -0.423531854 |
| 0 | 2 | -0.55338674 | -0.467294042 | -0.444686873 |
| 1 | 2 | -0.524911823 | -0.449816583 | -0.429554613 |
| 2 | 2 | -0.512594774 | -0.442124652 | -0.422877789 |
| 3 | 2 | -0.506388292 | -0.438268817 | -0.419543693 |
| 0 | 3 | -0.523833592 | -0.449241354 | -0.42907784 |
| 1 | 3 | -0.512076165 | -0.441846493 | -0.422647408 |
| 2 | 3 | -0.506112992 | -0.43812357 | -0.419424336 |
| 3 | 3 | -0.502880256 | -0.43616801 | -0.417753812 |
| 0 | 4 | -0.511809303 | -0.441701803 | -0.42252723 |
| 1 | 4 | -0.505970901 | -0.438047907 | -0.41936201 |
| 2 | 4 | -0.502802325 | -0.43612858 | -0.41772203 |
| 3 | 4 | -0.501087669 | -0.435171326 | -0.416932685 |

**Table 3. The bound state energy levels (in units of fm−1) of the GIQYP for various values of $n, l$ and for $\hbar = \mu = 1, V_0 = 1\ and\ \alpha = 0.001$**

| $n$ | $l$ | $E_{nl}$ $\eta = 0$ | $E_{nl}$ $\eta = 0.3$ | $E_{nl}$ $\eta = 0.4$ |
|---|---|---|---|---|
| 0 | 1 | -2.9940045 | -1.095985476 | -0.953730354 |
| 1 | 1 | -1.4970045 | -0.909932541 | -0.820368029 |
| 2 | 1 | -1.2197845 | -0.843534551 | -0.771214559 |
| 3 | 1 | -1.122760125 | -0.814219641 | -0.749065228 |
| 0 | 2 | -1.302201872 | -0.883072695 | -0.80268238 |
| 1 | 2 | -1.155363939 | -0.831952923 | -0.763421879 |
| 2 | 2 | -1.093938232 | -0.808367119 | -0.745076568 |
| 3 | 2 | -1.062548532 | -0.795710385 | -0.735161319 |
| 0 | 3 | -1.143685817 | -0.828799823 | -0.761180195 |
| 1 | 3 | -1.088310737 | -0.806742392 | -0.743910796 |
| 2 | 3 | -1.059418944 | -0.794772607 | -0.73448474 |
| 3 | 3 | -1.042468019 | -0.787587976 | -0.728806789 |
| 0 | 4 | -1.085664968 | -0.805938767 | -0.743328245 |
| 1 | 4 | -1.057929862 | -0.794305782 | -0.734144741 |
| 2 | 4 | -1.041548733 | -0.787294122 | -0.728592128 |
| 3 | 4 | -1.031076397 | -0.782753693 | -0.724989228 |

**Table 4. The bound state energy levels (in units of fm−1) of the GIQYP for various values of $n, l$ and for $\hbar = \mu = 1, V_0 = 1\ and\ \alpha = 0.01$**

| $n$ | $l$ | $E_{nl}$ $\eta = 0$ | $E_{nl}$ $\eta = 0.3$ | $E_{nl}$ $\eta = 0.4$ |
|---|---|---|---|---|
| 0 | 1 | -2.940450000 | -1.085469075 | -0.945254024 |
| 1 | 1 | -1.470450000 | -0.899255077 | -0.811503393 |
| 2 | 1 | -1.198450000 | -0.833117822 | -0.762452911 |
| 3 | 1 | -1.103512500 | -0.804156172 | -0.740561286 |
| 0 | 2 | -1.279268339 | -0.872291081 | -0.793693742 |
| 1 | 2 | -1.135367365 | -0.821569833 | -0.754665745 |
| 2 | 2 | -1.075445609 | -0.798394601 | -0.73663523 |
| 3 | 2 | -1.045120125 | -0.786188347 | -0.727102695 |
| 0 | 3 | -1.123949057 | -0.818442092 | -0.752437387 |
| 1 | 3 | -1.069983263 | -0.796805959 | -0.735495805 |
| 2 | 3 | -1.042124745 | -0.785295278 | -0.726461699 |
| 3 | 3 | -1.026091933 | -0.778625634 | -0.721241813 |
| 0 | 4 | -1.067417989 | -0.796021807 | -0.734927697 |
| 1 | 4 | -1.040702385 | -0.784852132 | -0.726140704 |
| 2 | 4 | -1.025236123 | -0.778359523 | -0.721050146 |
| 3 | 4 | -1.015671687 | -0.774402898 | -0.717976939 |

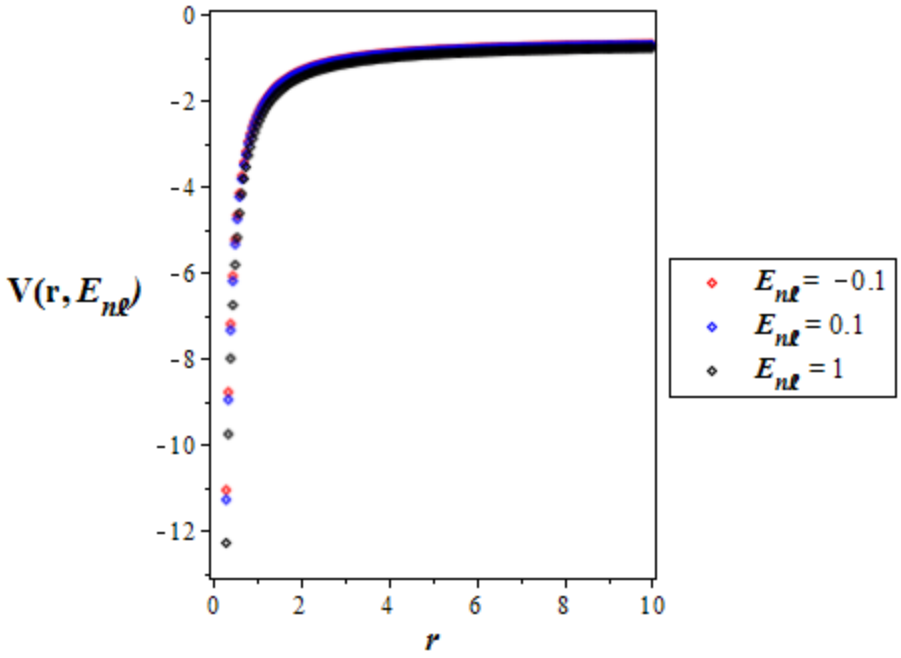


Figure 1: The variation of the Potential energy for various values of the energy $(E_{nl})$ as a function of $r$. We choose $V_0 = 0.5$ ,$\eta = 0.3$ and $\alpha = 0.001$.

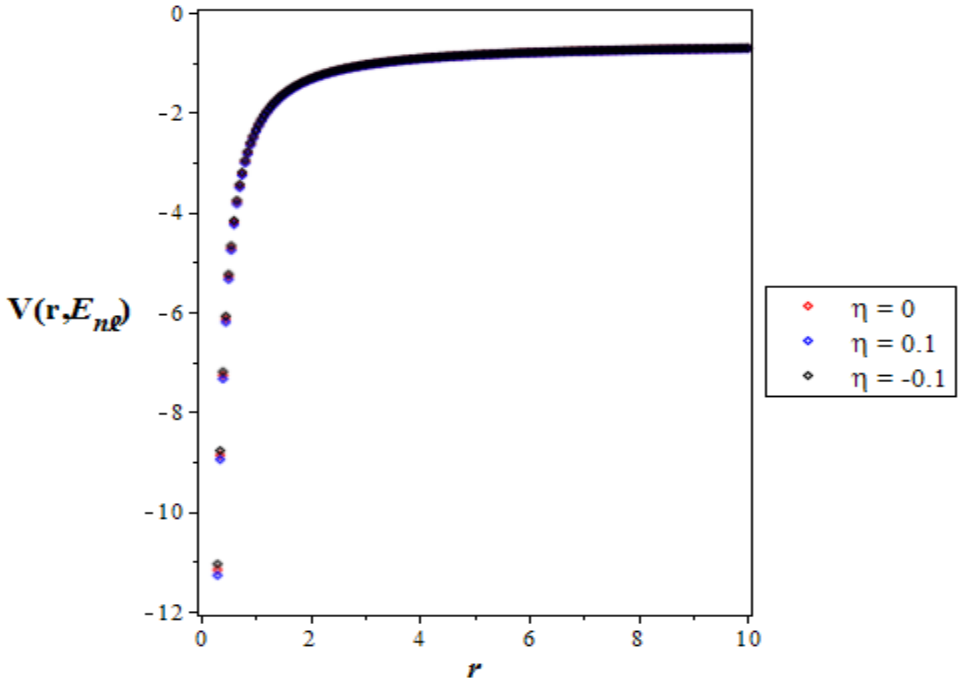


Figure 2: The variation of the Potential energy for various values of the energy slope parameter $(\eta)$ as a function $(r)$. We choose $V_0 = 0.5$ ,$E_{nl} = 0.1$ and $\alpha = 0.001$.

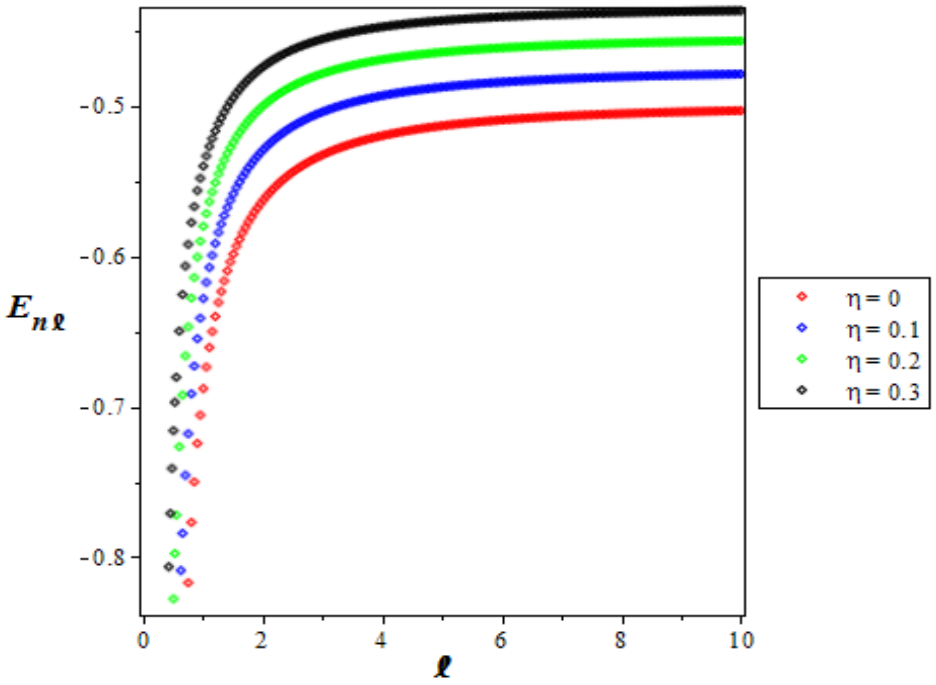


Figure 3: The variation of the ground state ($n = 0$) energy level for various values of the energy slope parameter($\eta$) as a function of $l$. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$.

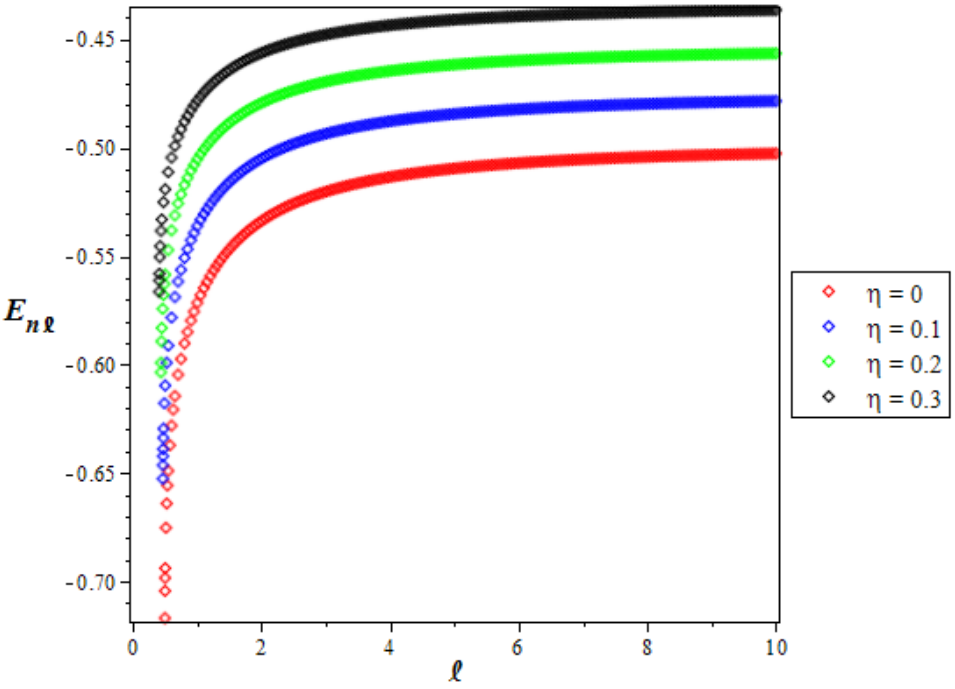


Figure 4: The variation of the first excited state ($n = 1$) energy level for various values of the energy slope parameter($\eta$) as a function of $l$. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$.

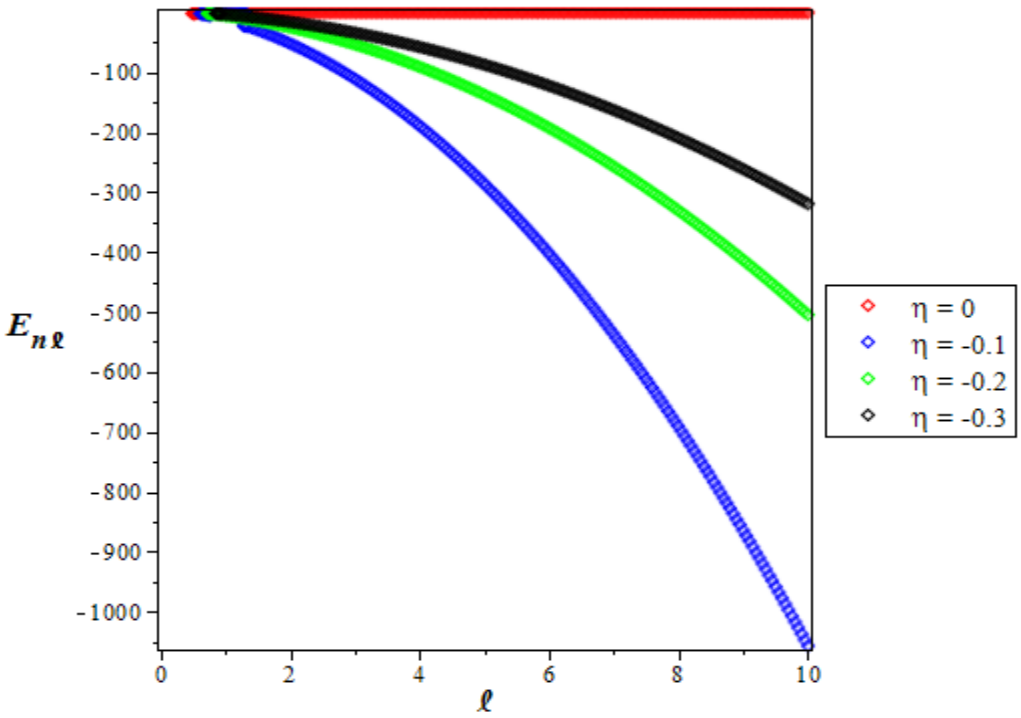


Figure 5: The variation of the ground state ($n = 0$) energy level for various values of the energy slope parameter($\eta \leq 0$) as a function of $l$. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$.

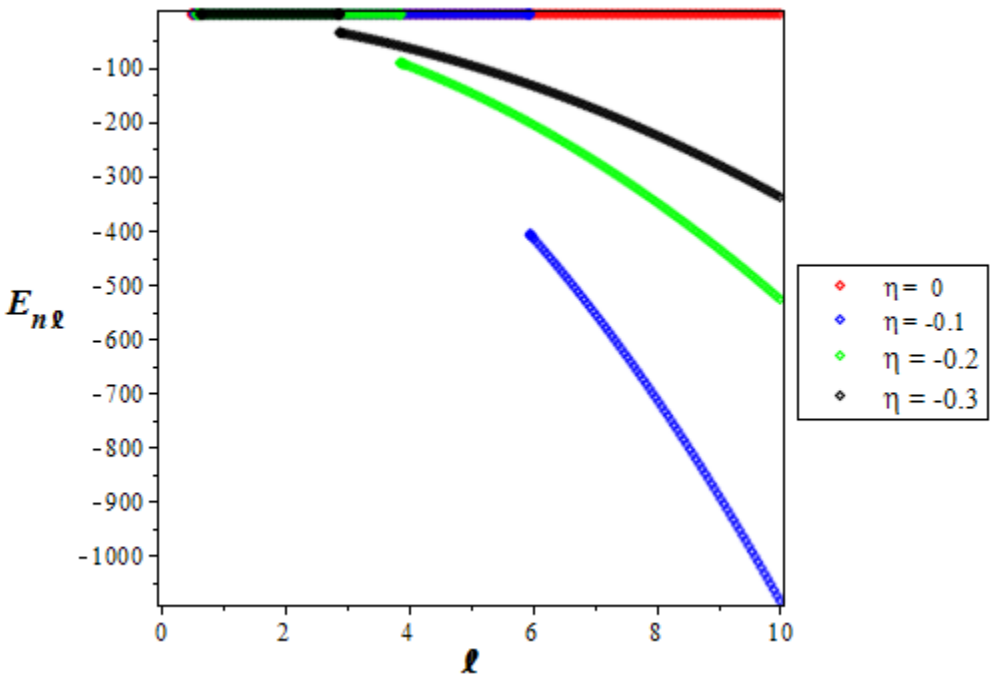


Figure 6: The variation of the first excited state ($n = 1$) energy level for various values of the energy slope parameter($\eta \leq 0$) as a function of $l$. We choose $V_0 = 0.5$ , and $\alpha = 0.001$ in $3D$.

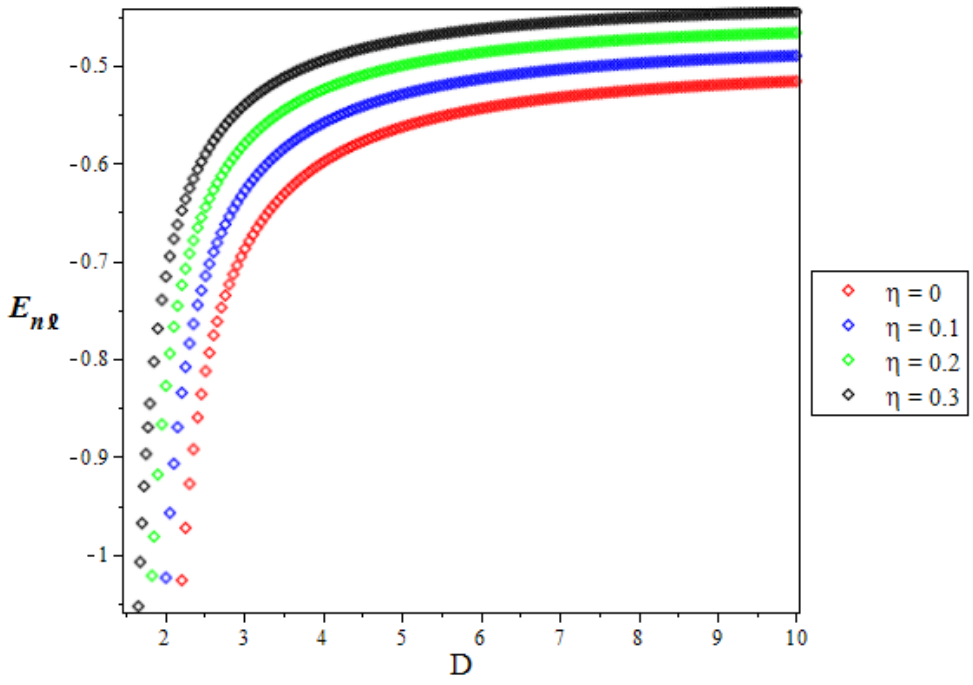


Figure 7: The variation of the ground state ($n = 0$) energy level for values of the energy slope parameter($\eta$) as a function of $D$. We choose $l = 1$ , $V_0 = 0.5$ , and $= 0.001$ .

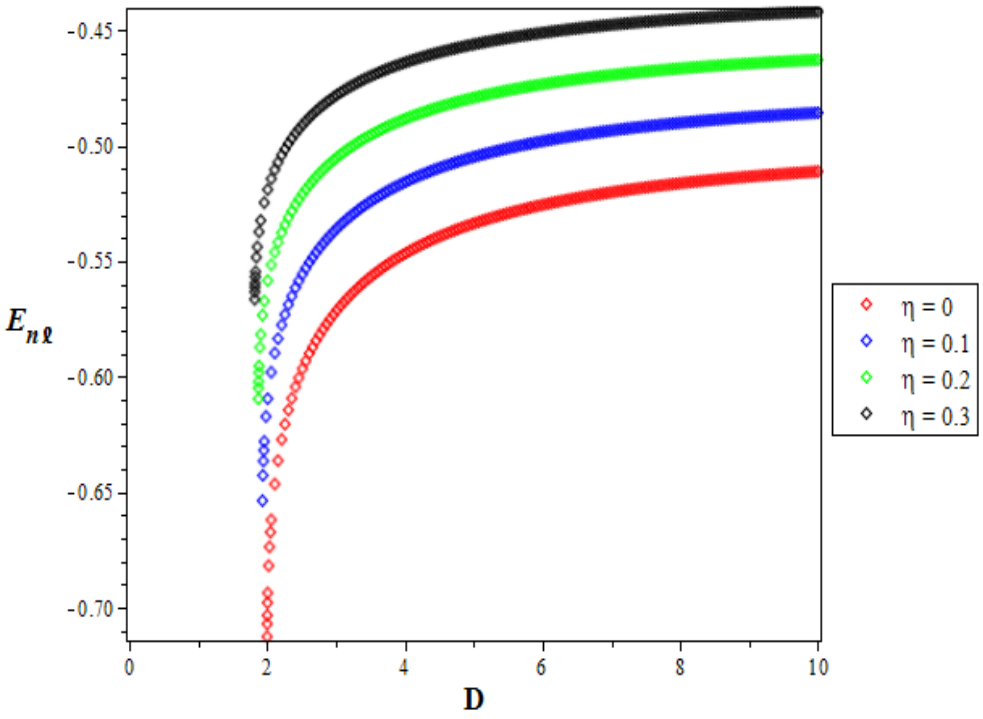


Figure 8: The variation of the First excited state ($n = 1$) energy level for values of the energy slope parameter($\eta$) as a function of $D$. We choose $l = 1$ , $V_0 = 0.5$ , and $\alpha = 0.001$

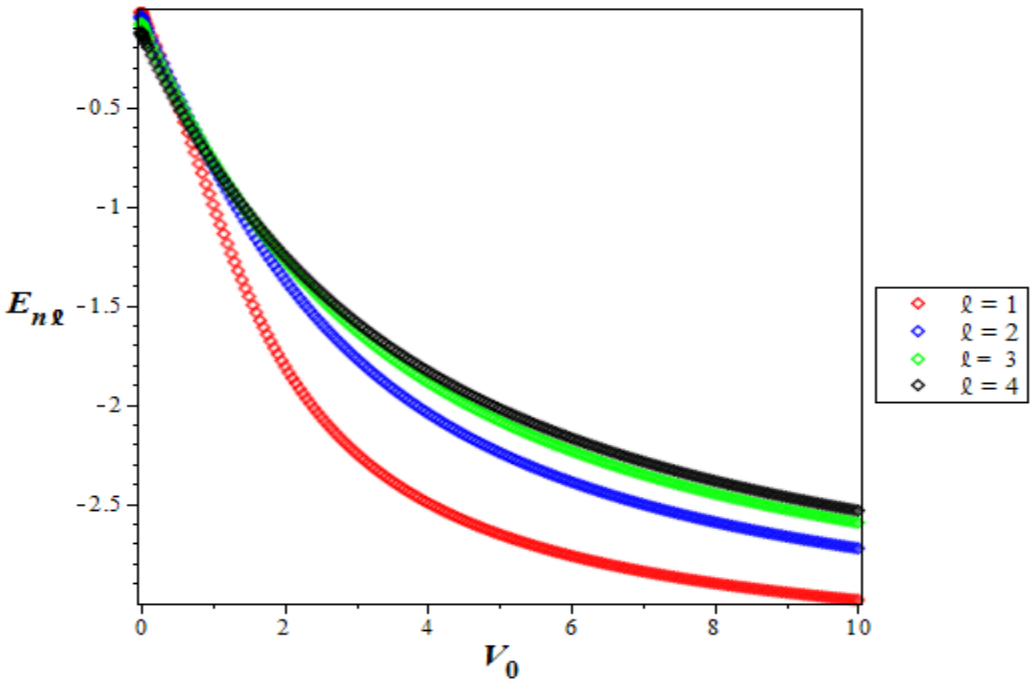


Figure 9: The variation of the ground state $(n = 0)$ energy level for various values of the Rotational Quantum Number$(l)$ as a function of $V_0$. We choose $= 0.3$ , and $\alpha = 0.001$ in $3D$.

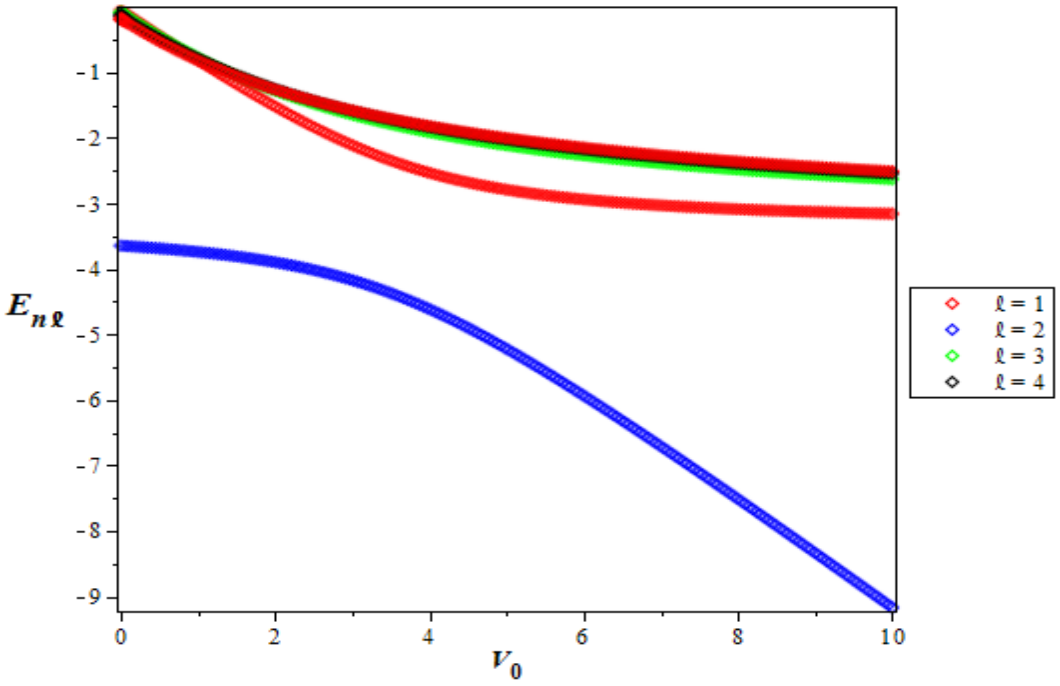


Figure 10: The variation of the First excited state $(n = 1)$ energy level for various values of the Rotational Quantum Number$(l)$ as a function of $V_0$. We choose $= 0.3$ , and $\alpha = 0.001$ in $3D$.

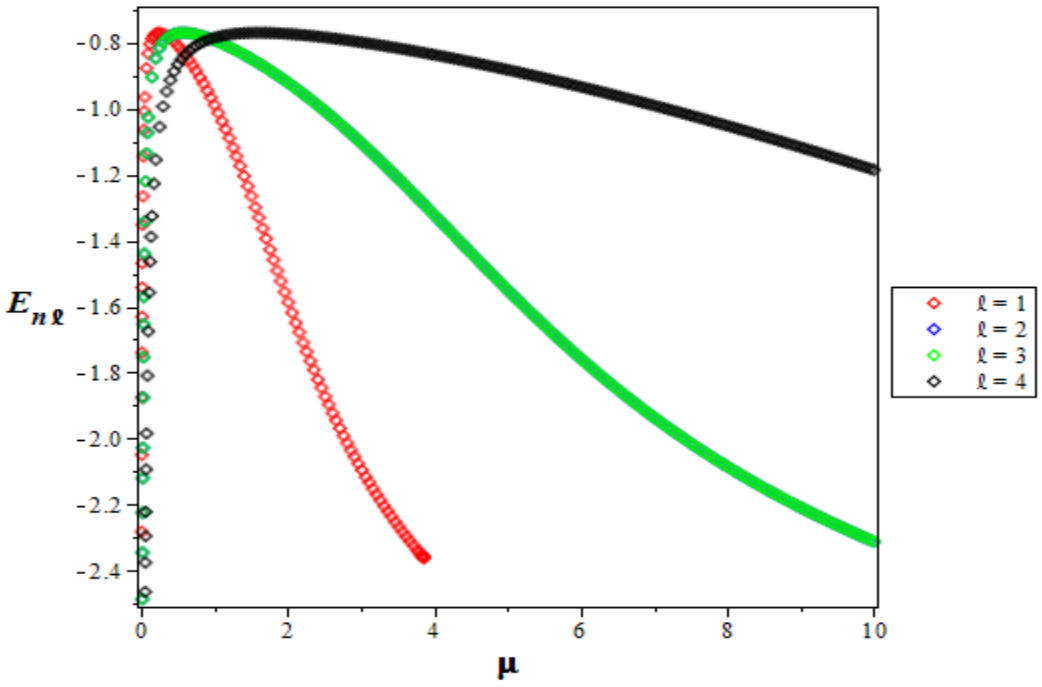


Figure 11: The variation of the ground state ($n = 0$) energy level for various values of the Rotational Quantum Number($l$) as a function of $\mu$. We choose $= 0.3$ , $V_0 = 0.5$ and $\alpha = 0.001$ in $3D$.

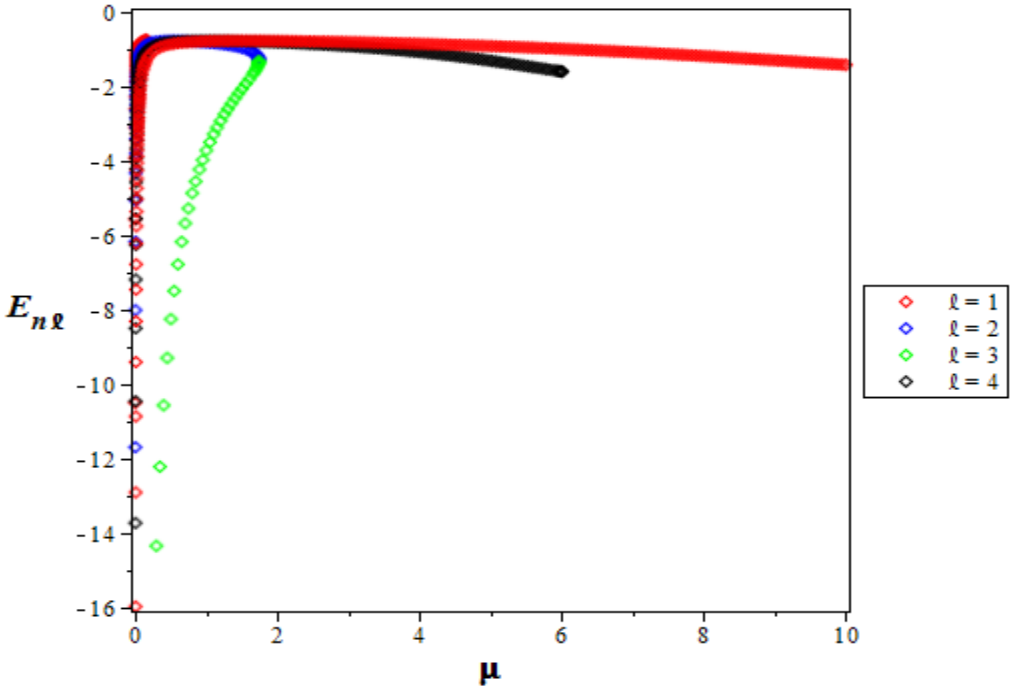


Figure 12: The variation of the First excited state ($n = 1$) energy level for various values of the Rotational Quantum Number($l$) as a function of $\mu$. We choose $= 0.3$ , $V_0 = 0.5$ and $\alpha = 0.001$ in $3D$