The generalized smooth functions
Poisson brackets and cohomology

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January 31, 2021

Abstract
We define the generalized smooth functions and Poisson brackets. We propose a cohomology theory for the generalized functions.

1 The Poisson brackets and cohomology

1.1 Poisson brackets
The Poisson brackets are defined for a symplectic manifold \((M, \omega)\) [BG] by the formula:
\[
\{f, g\} = \omega(df, dg)
\]
where \(f, g \in C^\infty(M)\) are smooth functions over the manifold \(M\). Due to the fact that the symplectic form is closed, we have the Jacobi identities:
\[
\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}
\]

1.2 Cohomology
For a manifold \(M\), it is possible to define the cohomology \([G]\) of the differential forms over \(M\) with help of the differential \(d\):
\[
H^\ast(M, \mathbb{R}) = H^\ast(\Lambda^\ast(TM), d)
\]

2 The generalized functions

2.1 Definition
The generalized functions are defined like the commutative algebra \(A(I)\):
\[
A(I) = C^\infty(M)[X_1, X_2, \ldots, X_k]/I
\]
where \(I\) is an ideal [LB] of the polynomials over the smooth functions
\[
C^\infty(M)[X_1, X_2, \ldots, X_k]
\]
such that \(A(I)\) is of finite type over \(C^\infty(M)\).
2.2 Example
If I is generated by elements which don’t depend on M, then A(I) is a tensor product by the smooth functions and is a trivial fiber bundle in algebras.
\[ A(I) = \mathbb{R}[X_1, X_2, \ldots, X_k]/\overline{I} \otimes_{\mathbb{R}} C^\infty(M) \]

3 The Poisson brackets
The Poisson brackets for A(I) are defined by the formula:
\[ \{ a, a' \} = \omega(da, da') \]
where \( a, a' \in A(I) \) and \( \omega \in \Lambda^2(TA(I)) \) is a symplectic form, \( TA(I) = \text{Der}(A(I)) \) are the derivations of \( A(I) \).

4 Connections over the generalized functions
A connection \( \nabla \) over \( A(I) \) is a linear application such that:
\[ \nabla_X(f.a) = X(f).a + f.\nabla_X(a) \]
with \( a \in A(I) \) and \( f \in C^\infty(M) \), \( X \) is a vector field of \( M \).

5 Cohomology of the generalized functions
The cohomology \([G]\) of the algebra of the generalized smooth functions is:
\[ H^*(M, I, \mathbb{R}) = H^*(A(I), \mathbb{R}) \]
It is a functor from the pairs \((M, I)\), with values in the category of algebras.

References